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Extended Branciari quasi-b-distance spaces, implicit relations and application to nonlinear matrix equations

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Abstract

This study introduces extended Branciari quasi-b-distance spaces, a novel implicit contractive condition in the underlying space, and basic fixed-point results, a weak well-posed property, a weak limit shadowing property and generalized Ulam–Hyers stability. The given notions and results are exemplified by suitable models. We apply these results to obtain a sufficient condition ensuring the existence of a unique positive-definite solution of a nonlinear matrix equation (NME) $\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{X})\mathcal{A}_i$, where $\mathcal{Q}$ is an $n \times n$ Hermitian positive-definite matrix, $\mathcal{A}_1$, $\mathcal{A}_2, \dots, \mathcal{A}_m$ are $n \times n$ matrices, and $\mathcal{G}$ is a nonlinear self-mapping of the set of all Hermitian matrices that are continuous in the trace norm. We demonstrate this sufficient condition for the NME $\mathcal{X} = \mathcal{Q} + \mathcal{A}_1^* \mathcal{X}^{1/3}\mathcal{A}_1 + \mathcal{A}_2^* \mathcal{X}^{1/3}\mathcal{A}_2 + \mathcal{A}_3^* \mathcal{X}^{1/3}\mathcal{A}_3$, and visualize this through convergence analysis and a solution graph.

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1 Generalized metric spaces

Denote $\mathbb{R}$:= the set of real numbers, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{N}$:= the set of natural numbers, and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

Many researchers, for example, [11–13], have proposed and expanded the distance idea in metric fixed-point theory in a variety of ways. Bakhtin [3] developed the concept of a b-metric space, which Czerwik utilized in [6, 7]. Kamran et al. [11] developed the concept of an extended b-metric space, whereas Branciari [4] expanded the concept of metric space and introduced the concept of Branciari distance by substituting the property of triangle inequality with the property of quadrilateral inequality.

Definition 1.1 Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+ \setminus (0,1)$. We say that a function $\rho_e : \Xi^2 \to \mathbb{R}_+$ is an extended b-metric (ρ_e-metric, for short) if it satisfies:

(eb1) $\rho_e(\vartheta, \nu) = 0$ if and only if $\vartheta = \nu$;

(eb2) $\rho_e(\vartheta, \nu) = \rho_e(\nu, \vartheta)$;

(eb3) $\rho_e(\vartheta,\nu) \le w(\vartheta,\nu)[\rho_e(\vartheta,\upsilon) + \rho_e(\upsilon,\nu)]$,
for all $\vartheta,\nu,\upsilon \in \Xi$. The symbol (Ξ,ρ_e) denotes a ρ_e-metric space.

Definition 1.2 Let $\Xi \neq \emptyset$ be a set and let $b : \Xi^2 \to \mathbb{R}_+$ be such that, for all $\vartheta,\nu \in \Xi$ and all $u,v \in \Xi \setminus \{\vartheta,\nu\}$,
(bd1) $b(\vartheta,\nu) = 0$ if and only if $\vartheta = \nu$ (self-distance/indistancy);
(bd2) $b(\vartheta,\nu) = b(\nu,\vartheta)$;
(bd3) $b(\vartheta,\nu) \le b(\vartheta,u) + b(u,v) + b(v,\nu)$ (quadrilateral inequality).

The symbol (Ξ,b) denotes a Branciari distance space and is abbreviated as BDS.

Abdeljawad et al. [1] recently defined an extended Branciari b-distance space by integrating an extended b-metric and a Branciari distance.

Definition 1.3 Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+ \setminus (0,1)$. We say that a function $e_b : \Xi^2 \to \mathbb{R}_+$ is an extended Branciari b-metric (e_b-metric, for short) if it satisfies:
(ebb1) $e_b(\vartheta,\upsilon) = 0$ if and only if $\vartheta = \upsilon$,
(ebb2) $e_b(\vartheta,\upsilon) = e_b(\upsilon,\vartheta)$,
(ebb3) $e_b(\vartheta,\upsilon) \le w(\vartheta,\upsilon)[e_b(\vartheta,\nu) + e_b(\nu,\varrho) + e_b(\varrho,\upsilon)]$
for all $\vartheta,\upsilon \in \Xi$ all distinct $\nu,\varrho \in \Xi \setminus \{\vartheta,\upsilon\}$. The symbol (Ξ,e_b) denotes an extended Branciari b-distance space (EBbDS, for short). For $w(\vartheta,\upsilon) = b$, (Ξ,e_b) it will be called a Branciari b-distance space (BbDS, for short).

On the other hand, in [25], a b-metric space was expanded as a quasi-b-metric space, which was further extended in [10] by establishing ideas of right- and left-quasi-b-metric spaces. Motivated by the above, we propose the concept of an extended Branciari quasi-b-distance with instances, as well as its right- and left-completeness conditions, in Sect. 2. Section 3 introduces a new implicit relation for the new space structure known as the $\mathcal{G}_w$-implicit relation. In Sect. 4, we introduce the concept of right $\mathcal{G}_w$-implicit self-mapping and demonstrate a related fixed-point result in a right-complete extended Branciari quasi-b-distance space using two examples. Sections 5 and 6 present and develop new ideas such as generalized w-Ulam–Hyers stability, a weak well-posed property, and a weak limit shadowing property in the context of an extended Branciari quasi-b-distance space. Section 7 concludes with the creation and verification of a sufficient condition insuring the existence of a unique positive-definite solution to a nonlinear matrix problem. This process is visualized using convergence analysis, three alternative initializations, and a solution graph.

The significance of this work is that the symmetry criterion is eased in proving fixed-point results in the extended Branciari quasi-b-distance spaces under a new implicit relation in the context of right-completeness. We also offer new notions in the context of underlining space, such as generalised w-Ulam–Hyers stability, a weak well-posed property, and a weak limit shadowing property, as well as related findings. There is also a novel application to nonlinear matrix equations that is graphically represented with an illustration. In doing so, we show that the requirements we utilize to ensure the existence of matrix equation solutions are "weaker", in the sense of quasinorm, than those previously derived in the literature.

2 Extended Branciari quasi-*b*-distance

Definition 2.1 Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+ \setminus (0,1)$. A function $q_e : \Xi^2 \to \mathbb{R}_+$ is called an extended Branciari quasi-b-distance, if for all $\nu, \vartheta \in \Xi$ all distinct $\mu, \upsilon \in \Xi \backslash \{\nu, \vartheta\}$:

(qeb1) $q_e(\nu,\vartheta) = 0 \Rightarrow \nu = \vartheta$;

(qeb2) $q_e(\nu,\vartheta) \leq w(\nu,\vartheta)[q_e(\nu,\mu) + q_e(\mu,\upsilon) + q_e(\upsilon,\vartheta)]$.

The triplet (Ξ, q_e, w) is then called an extended Branciari quasi-b-distance space (EBQbDS, for short) with the coefficient $w(\nu,\vartheta)$.

Definition 2.2 Let (Ξ, q_e, w) be an EBQbDS and $\{\nu_n\}$ a sequence in Ξ. Then, $\{\nu_n\}$ is said to be

(i) left-Cauchy if for every $\delta > 0$, there is an $N = N(\delta) \in \mathbb{N}$ such that $q_e(\nu_r, \nu_s) < \delta$ for all $r > s > N$,

(ii) right-Cauchy if for every $\delta > 0$, there is an $N = N(\delta) \in \mathbb{N}$ such that $q_e(\nu_r, \nu_s) < \delta$ for all $s > r > N$.

(iii) Cauchy if for every $\delta > 0$, there is an $N = N(\delta) \in \mathbb{N}$ such that $q_e(\nu_r, \nu_s) < \delta$ for all $r, s > N$.

Definition 2.3 Let (Ξ, q_e, w) be an EBQbDS. Then, (Ξ, q_e, w) is called

(i) left-complete if every left-Cauchy sequence in Ξ is convergent,

(ii) right-complete if every right-Cauchy sequence in Ξ is convergent.

(iii) complete if every Cauchy sequence in Ξ is convergent.

Example 2.4 Let $\Xi = \mathbb{R}_+$ and define

$$q_e(\vartheta,\nu) = \begin{cases} |\vartheta - \nu|^2 + \vartheta & \text{if } \vartheta \neq \nu, \\ 0 & \text{if } \vartheta = \nu, \end{cases}$$

with $w(\vartheta,\nu) = 5\vartheta + 5\nu + 3$. Then, it is clear that (Ξ, q_e, w) is an EBQbDS, but it is not an EBbDS. In fact, (qeb2) holds since

$$\begin{aligned} q_e(\vartheta,\nu) &= |\vartheta - \nu|^2 + \vartheta \\ &= |\vartheta - z + z - w + w - \nu|^2 \\ &\leq |\vartheta - z|^2 + |z - w|^2 + |w - \nu|^2 \\ &\quad + 2|\vartheta - z||z - w| + 2|z - w||w - \nu| + 2|w - \nu||\vartheta - z| + \vartheta + z + w \\ &\leq (5\vartheta + 5\nu + 3)\bigl[|\vartheta - z|^2 + |z - w|^2 + |w - \nu|^2 + \vartheta + z + w\bigr] \\ &= w(\vartheta,\nu)\bigl[q_e(\vartheta,z) + q_e(z,w) + q_e(w,\nu)\bigr], \end{aligned}$$

for all $\vartheta, \nu, z \in \Xi$, but clearly $q_e(\vartheta,\nu) \neq q_e(\nu,\vartheta)$ if $\vartheta \neq \nu$.

Example 2.5 Let $\Xi = \{\frac{1}{n} : n \in \{2,3,4,5\}\}$ and define

$$q_e(\vartheta,\nu) = \begin{cases} |\vartheta - \nu|^2 + \vartheta & \text{if } \vartheta \neq \nu, \\ 0 & \text{if } \vartheta = \nu, \end{cases}$$

with $w(\vartheta,\nu) = \vartheta + \nu + 2$. Then, it is clear that (Ξ, q_e, w) is an EBQbDS, but it is not an EBbDS. In fact, (qeb2) holds. For instance,

$$q_e\left(\frac{1}{2},\frac{1}{3}\right) = 0.5277, \qquad q_e\left(\frac{1}{2},\frac{1}{4}\right) = 0.5, \qquad q_e\left(\frac{1}{2},\frac{1}{5}\right) = 0.59,$$
$$q_e\left(\frac{1}{4},\frac{1}{5}\right) = 0.2525, \qquad q_e\left(\frac{1}{5},\frac{1}{3}\right) = 0.2177,$$

and so

$$\begin{aligned} q_e\left(\frac{1}{2},\frac{1}{3}\right) &= 0.5277 \leq 1.13225 \\ &= \left(\frac{1}{2}+\frac{1}{3}+2\right)\left[q_e\left(\frac{1}{2},\frac{1}{4}\right)+q_e\left(\frac{1}{2},\frac{1}{5}\right)+q_e\left(\frac{1}{4},\frac{1}{5}\right)+q_e\left(\frac{1}{5},\frac{1}{3}\right)\right]. \end{aligned}$$

Similarly, we can prove this for all $\vartheta, \nu, z \in \Xi$, but clearly $q_e(\vartheta,\nu) \neq q_e(\nu,\vartheta)$ if $\vartheta \neq \nu$.

Example 2.6 (Inspired from [15]) Let $\Xi = [0,5]$ and define $q_e : \Xi^2 \to \mathbb{R}$ by

$$q_e(\vartheta,\nu) = \begin{cases} 0 & \text{if } \vartheta = \nu; \\ \vartheta - \nu & \text{if } \vartheta > \nu; \\ 2(\nu - \vartheta) & \text{if } \vartheta < \nu. \end{cases}$$

Then, (Ξ, q_e, w) is a left-complete EBQbDS with $w(\vartheta,\nu) = \vartheta + \nu + 2$, but it is not an EBQbDS. In fact, (qeb2) holds. Consider $\vartheta, \nu, u, v \in \Xi$ with $\vartheta > \nu$. We show that

$$q_e(\vartheta,\nu) \leq w(\vartheta,\nu)\left[q_e(\vartheta,u) + q_e(u,v) + q_e(v,\nu)\right]$$

is true.

- Suppose $\vartheta > u > v > \nu$. Then,

  $$q_e(\vartheta,\nu) = (\vartheta - \nu) \leq w(\vartheta,\nu)(\vartheta - \nu) = w(\vartheta,\nu)\left[(\vartheta - u) + (u - v) + (v - \nu)\right]$$

  is true.
- Suppose $\vartheta < u > v > \nu$. Then,

  $$q_e(\vartheta,\nu) = (\vartheta - \nu) \leq w(\vartheta,\nu)\big(3u - (\vartheta + \nu)\big) = w(\vartheta,\nu)\left[2(u - \vartheta) + (u - v) + (v - \nu)\right]$$

  is true.
- Suppose $\vartheta < u < v > \nu$. Then,

  $$q_e(\vartheta,\nu) = (\vartheta - \nu) \leq w(\vartheta,\nu)\big(3u - (2\vartheta + \nu)\big) = w(\vartheta,\nu)\left[2(u - \vartheta) + 2(v - u) + (v - \nu)\right]$$

  is true.
- Suppose $\vartheta > u < v > \nu$. Then,

  $$q_e(\vartheta,\nu) = (\vartheta - \nu) \leq w(\vartheta,\nu)(\vartheta + 2v - 3u - \nu) = w(\vartheta,\nu)\left[(\vartheta - u) + 2(v - u) + (v - \nu)\right]$$

  is true.

3 $\mathcal{G}_w$-implicit relation

In this section, we introduce a modified version of the implicit relation discussed in [2, 20] in the context of EBQbDS. We start with a defining notion of Φ_w following [26].

Denote by Φ_w the set of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is increasing;

(ii) for a set $\Xi \neq \emptyset$, there exists $w : \Xi^2 \to \mathbb{R}_+ \setminus (0,1)$ such that $\sum_{n=1}^{\infty} \varphi^n(\zeta) \prod_{i=1}^{n} w(\vartheta_i, \vartheta_m) < \infty$, for $\zeta > 0$, $\vartheta_i \in \Xi$, $\forall m \in \mathbb{N}$, where φ^n denotes the nth iterate.

It should be noted that $\varphi(\zeta) < \zeta$ and the family $\Phi_w \neq \emptyset$.

Example 3.1 Consider the EBQbDS (Ξ, e_b), where $\Xi = [0,1]$ and $w(\nu,\vartheta) = |\nu| + |\vartheta| + \frac{5}{2}$. Define the mapping $\varphi(\zeta) = \frac{2\lambda\zeta}{9}$, where $0 < \lambda < 1$. Note that $w(\nu,\vartheta) \leq 9/2$. Then, we have $\varphi^n(\zeta) \prod_{i=1}^{n} w(\nu_i, \nu) \leq \frac{2^n\lambda^n\zeta}{9^n}.(\frac{9}{2})^n = \lambda^n\zeta$. Therefore,

$$\sum_{n=1}^{\infty} \varphi^n(\zeta) \prod_{i=1}^{n} w(\nu_i, \nu) \leq \sum_{n=1}^{\infty} \lambda^n\zeta < \infty.$$

Example 3.2 Consider the EBQbDS (Ξ, e_b), where $\Xi = [1,\infty)$ and $w(\nu,\vartheta) = 1 + \frac{2}{1+\ln(\nu+\vartheta)}$. Define the mapping $\varphi(\zeta) = \frac{\lambda\zeta}{3}$, where $0 < \lambda < 1$. Note that $1 + \frac{2}{1+\ln(\nu+\vartheta)} \leq 3$. Hence, we have

$$\varphi^n(\zeta) \prod_{i=1}^{n} w(\nu_i, \nu) \leq \frac{\lambda^n\zeta}{3^n} \cdot 3^n = \lambda^n\zeta.$$

Therefore, $\sum_{n=1}^{\infty} \varphi^n(\zeta) \prod_{i=1}^{n} w(\nu_i, \nu) < \infty$ and hence $\Phi_w \neq \emptyset$.

Now, we are in a position to design the implicit relation in the setting of EBQbDS.

Let $\mathfrak{G}$ be the set of functions $\mathcal{G} : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions: for all $\zeta, \xi, \mu \geq 0$ and some $k \geq 1$,

($\mathcal{G}_1$) $\mathcal{G}(k\zeta, \xi, \xi, \zeta, \mu, 0) \leq 0$ implies that there exists $\varphi \in \Phi_w$ such that $k\zeta \leq \varphi(\xi)$;

($\mathcal{G}_2$) if $\mathcal{G}(k\zeta, \zeta, 0, 0, \zeta, \xi) > 0$ and $\mathcal{G}(k\xi, \xi, 0, 0, \xi, \zeta) > 0$, then $\zeta = 0$, $\xi = 0$.

The following examples are inspired by [2].

Example 3.3 Let $\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1^2 - a\hbar_2^2 - b\frac{\hbar_3^2+\hbar_4^2}{\hbar_5^2+\hbar_6^2+1}$, $0 < a, b < 1$ and $a + 2b < 1$.

($\mathcal{G}_1$) : Let $\zeta, \xi, \mu \geq 0$, $k \geq 1$ and $\mathcal{G}(k\zeta, \xi, \xi, \zeta, \mu, 0) = k\zeta^2 - a\xi^2 - b\frac{(\zeta^2+\xi^2)}{1+\mu^2} \leq 0$.

Then, $k^2\zeta^2 \leq \frac{k^2(a+b)}{k^2-b}\xi^2$. Hence, $k\zeta \leq \varphi(\xi)$, where $\varphi(\xi) = h\xi, h = k\sqrt{\frac{a+b}{k^2-b}}$. It is easy to check that $0 < a, b < 1$ with $a + 2b < 1$ and $k \geq 1$ can be chosen so that $h < 1$.

($\mathcal{G}_2$) : For all $\zeta > 0, \xi > 0, \mathcal{G}(k\zeta, \zeta, 0, 0, \zeta, \xi) = (k^2-a)\zeta^2 > 0, \mathcal{G}(k\xi, \xi, 0, 0, \xi, \zeta) = (k^2-a)\xi^2 > 0$.

Example 3.4 Let $\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1 - a\hbar_2 - b\hbar_3 - c\frac{\hbar_4\hbar_5}{1+\hbar_5+\hbar_6}$, $0 < a, b, c < 1$ and $a + b + c < 1$.

($\mathcal{G}_1$) : Let $\zeta, \xi, \mu \geq 0$, $k \geq 1$ and $\mathcal{G}(k\zeta, \xi, \xi, \zeta, \mu, 0) = k\zeta - a\xi - b\xi - c\frac{\zeta\mu}{1+\mu} \leq 0$.

Then, $k\zeta \leq (\frac{k(a+b)}{k-c})\xi$. Hence, $k\zeta \leq \varphi(\xi)$, where $\varphi(\xi) = h\xi$, $h = \frac{k(a+b)}{k-c} < 1$. It is easy to check that $0 < a, b, c < 1$ with $a + b + c < 1$ and $k \geq 1$ can be chosen so that $h < 1$.

($\mathcal{G}_2$) : For all $\zeta > 0, \xi > 0$, $\mathcal{G}(k\zeta, \zeta, 0, 0, \zeta, \xi) = (k-a)\zeta^2 > 0$, $\mathcal{G}(k\xi, \xi, 0, 0, \xi, \zeta) = (k-a)\xi^2 > 0$.

4 Fixed-point results under a $\mathcal{G}_w$-implicit relation

We start with defining right $\mathcal{G}_w$-implicit mappings on EBQbDS.

Definition 4.1 Let (Ξ, q_e, w) be an EBQbDS and $\mathcal{T}:\Xi\to\Xi$ be a mapping. $\mathcal{T}$ is called a right $\mathcal{G}_w$-implicit mapping if there exists $\mathcal{G}\in\mathfrak{G}$ such that

$$\mathcal{G}\big(w(\nu,\vartheta)q_e(\mathcal{T}\nu,\mathcal{T}\vartheta),q_e(\nu,\vartheta),q_e(\nu,\mathcal{T}\nu),q_e(\vartheta,\mathcal{T}\vartheta),q_e(\nu,\mathcal{T}\vartheta),q_e(\vartheta,\mathcal{T}\nu)\big)\le 0 \tag{1}$$

holds for all $\nu,\vartheta\in\Xi$.

The set of all fixed points of a self-mapping $\mathcal{T}$ on a set $\Xi\neq\emptyset$ will be denoted by Fix($\mathcal{T}$).

Theorem 4.2 *Let (Ξ, q_e, w) be a right-complete EBQbDS with $w:\Xi^2\to\mathbb{R}_+\setminus(0,1)$ and $\mathcal{T}:\Xi\to\Xi$ be a right $\mathcal{G}_w$-implicit mapping for $\mathcal{G}\in\mathfrak{G}$. Then, Fix($\mathcal{T}$) is a singleton set, if $\mathcal{T}$ is continuous. Furthermore, for any $\vartheta_0\in\Xi$, the sequence $\{\vartheta_n\}$ satisfying $\vartheta_n=\mathcal{T}\vartheta_{n-1}$ converges to (the unique) element of Fix($\mathcal{T}$).*

Proof Starting with an arbitrary point $\vartheta_0\in\Xi$, we define a sequence $\{\vartheta_n\}$ in Ξ by $\vartheta_n=\vartheta_{n-1}$ for all $n\in\mathbb{N}$. First, suppose that $\vartheta_n=\vartheta_{n+1}$ for some $n\in\mathbb{N}^*$ – then there is nothing to prove, as $\varrho=\vartheta_n=\vartheta_{n+1}=\mathcal{T}\vartheta_n=\mathcal{T}\vartheta$. Now, assume that $\vartheta_n\neq\vartheta_{n-1}$ for all $n\in\mathbb{N}$. Since we have $w(\vartheta_0,\mathcal{T}\vartheta_0)\ge 1$, by the right $\mathcal{G}_w$-implicit condition with $\nu=\vartheta_{n-1}$ and $\vartheta=\vartheta_n$, we have

$$\mathcal{G}\begin{pmatrix} w(\vartheta_{n-1},\vartheta_n)q_e(\mathcal{T}\vartheta_{n-1},\mathcal{T}\vartheta_n),q_e(\vartheta_{n-1},\vartheta_n),q_e(\vartheta_{n-1},\mathcal{T}\vartheta_{n-1}),\\ q_e(\vartheta_n,\mathcal{T}\vartheta_n),q_e(\vartheta_{n-1},\mathcal{T}\vartheta_n),q_e(\vartheta_n,\mathcal{T}\vartheta_{n-1})\end{pmatrix}\le 0,$$

that is,

$$\mathcal{G}\begin{pmatrix} w(\vartheta_{n-1},\vartheta_n)q_e(\vartheta_n,\vartheta_{n+1}),q_e(\vartheta_{n-1},\vartheta_n),q_e(\vartheta_{n-1},\vartheta_n),\\ q_e(\vartheta_n,\vartheta_{n+1}),q_e(\vartheta_{n-1},\vartheta_{n+1}),0\end{pmatrix}\le 0.$$

It follows from $(\mathcal{G}_1)$ that there is $\varphi\in\Phi_w$ such that

$$w(\vartheta_{n-1},\vartheta_n)q_e(\vartheta_n,\vartheta_{n+1})\le\varphi\big(q_e(\vartheta_{n-1},\vartheta_n)\big),\quad \text{for all } n\in\mathbb{N},$$

and so

$$q_e(\vartheta_n,\vartheta_{n+1})\le\varphi\big(q_e(\vartheta_{n-1},\vartheta_n)\big).$$

With successive use of $(\mathcal{G}_1)$, it is easy to derive that

$$q_e(\vartheta_n,\vartheta_{n+1})\le\varphi^n\big(q_e(\vartheta_0,\vartheta_1)\big),\quad \text{for all } n\in\mathbb{N}.$$

Now, we shall prove that $\{\vartheta_n\}$ is a right-Cauchy sequence. Take $m>n$, then by $(qeb2)$, we have

$$\begin{aligned} &q_e(\vartheta_n,\vartheta_m)\\ &\quad\le w(\vartheta_n,\vartheta_m)\big[q_e(\vartheta_n,\vartheta_{n+1})+q_e(\vartheta_{n+1},\vartheta_{n+2})+q_e(\vartheta_{n+2},\vartheta_m)\big]\end{aligned}$$

$$
\begin{aligned}
&\leq w(\vartheta_n,\vartheta_m)\varphi^n\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m)\varphi^{n+1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)q_e(\vartheta_{n+2},\vartheta_m) \\
&\leq w(\vartheta_n,\vartheta_m)\varphi^n\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m)\varphi^{n+1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\big[q_e(\vartheta_{n+2},\vartheta_{n+3}) + q_e(\vartheta_{n+3},\vartheta_{n+4}) \\
&\quad + q_e(\vartheta_{n+4},\vartheta_m)\big] \\
&\leq w(\vartheta_n,\vartheta_m)\varphi^n\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m)\varphi^{n+1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\varphi^{n+2}\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m) \\
&\quad \times w(\vartheta_{n+2},\vartheta_m)\varphi^{n+3}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+2},\vartheta_m)q_e(\vartheta_{n+4},\vartheta_m) \\
&\vdots \\
&\leq w(\vartheta_n,\vartheta_m)\varphi^n\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m)\varphi^{n+1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\varphi^{n+2}\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m) \\
&\quad \times w(\vartheta_{n+2},\vartheta_m)\varphi^{n+3}\big(q_e(\vartheta_0,\vartheta_1)\big) + \cdots \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\cdots w(\vartheta_{m-2},\vartheta_m)\varphi^{m-2}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\cdots w(\vartheta_{m-2},\vartheta_m)\varphi^{m-1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\leq w(\vartheta_n,\vartheta_m)\varphi^n\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+1},\vartheta_m)\varphi^{n+1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+1},\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\varphi^{n+2}\big(q_e(\vartheta_0,\vartheta_1)\big) + w(\vartheta_n,\vartheta_m) \\
&\quad \times w(\vartheta_{n+1},\vartheta_m)w(\vartheta_{n+2},\vartheta_m)w(\vartheta_{n+3},\vartheta_m)\varphi^{n+3}\big(q_e(\vartheta_0,\vartheta_1)\big) + \cdots \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+1},\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\cdots w(\vartheta_{m-2},\vartheta_m)\varphi^{m-2}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\quad + w(\vartheta_n,\vartheta_m)w(\vartheta_{n+1},\vartheta_m)w(\vartheta_{n+2},\vartheta_m)\cdots w(\vartheta_{m-2},\vartheta_m)w(\vartheta_{m-1},\vartheta_m) \\
&\quad \times \varphi^{m-1}\big(q_e(\vartheta_0,\vartheta_1)\big) \\
&\leq \sum_{i=n}^{m-1}\varphi^i\big(q_e(\vartheta_0,\vartheta_1)\big)\prod_{j=n}^{i}w(\vartheta_j,\vartheta_m) \\
&= \sum_{i=1}^{m-1}\varphi^i\big(q_e(\vartheta_0,\vartheta_1)\big)\prod_{j=n}^{i}w(\vartheta_j,\vartheta_m) - \sum_{i=1}^{n-1}\varphi^i\big(q_e(\vartheta_0,\vartheta_1)\big)\prod_{j=n}^{i}w(\vartheta_j,\vartheta_m),
\end{aligned}
$$

which tends to 0 as $n, m \to \infty$, since $\varphi \in \Phi_w$, and hence the sequence $\{\vartheta_n\}$ is a right-Cauchy sequence.

Since (Ξ, q_e, w) is a right-complete EBQbDS, then there exists a point $\varrho \in \Xi$ such that $\vartheta_n \to \varrho$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} q_e(\vartheta_n,\varrho) = \lim_{n\to\infty} q_e(\varrho,\vartheta_n) = 0.$$

Next, we claim that $\varrho \in \mathrm{Fix}(\mathcal{T})$. Using (qeb2), we have

$$q_e(\varrho,\mathcal{T}\varrho) \leq w(\varrho,\mathcal{T}\varrho)\big[q_e(\varrho,\vartheta_n) + q_e(\vartheta_n,\vartheta_{n+1}) + q_e(\vartheta_{n+1},\mathcal{T}\varrho)\big].$$

Since $\mathcal{T}$ is continuous, on letting $n \to \infty$, we obtain $q_e(\varrho, \mathcal{T}\varrho) = 0$, that is, $\mathcal{T}\varrho = \varrho$ and hence ϱ is a fixed point of $\mathcal{T}$.

Finally, we claim that $\mathrm{Fix}(\mathcal{T})$ is a singleton set. On the contrary, assume that there exist $\varrho, \varrho_0 \in \mathrm{Fix}(\mathcal{T})$ with $\varrho \neq \varrho_0$. By the use of the right $\mathcal{G}_w$-implicit condition of $\mathcal{T}$, we obtain

$$\mathcal{G}\begin{pmatrix} w(\varrho, \varrho_0)q_e(\mathcal{T}\varrho, \mathcal{T}\varrho_0), q_e(\varrho, \varrho_0), q_e(\varrho, \mathcal{T}\varrho), \\ q_e(\varrho_0, \mathcal{T}\varrho_0), q_e(\varrho, \mathcal{T}\varrho_0), q_e(\varrho_0, \mathcal{T}\varrho) \end{pmatrix} \leq 0,$$

i.e.,

$$\mathcal{G}\big(w(\varrho, \varrho_0)q_e(\varrho, \varrho_0), q_e(\varrho, \varrho_0), 0, 0, q_e(\varrho, \varrho_0), q_e(\varrho_0, \varrho)\big) \leq 0. \tag{2}$$

Again, using the right $\mathcal{G}_w$-implicit condition of $\mathcal{T}$, we obtain

$$\mathcal{G}\begin{pmatrix} w(\varrho_0, \varrho)q_e(\mathcal{T}\varrho_0, \mathcal{T}\varrho), q_e(\varrho_0, \varrho), q_e(\varrho_0, \mathcal{T}\varrho_0), \\ q_e(\varrho, \mathcal{T}\varrho), q_e(\varrho_0, \mathcal{T}\varrho), q_e(\varrho, \mathcal{T}\varrho_0) \end{pmatrix} \leq 0,$$

i.e.,

$$\mathcal{G}\big(w(\varrho_0, \varrho)q_e(\varrho_0, \varrho), q_e(\varrho_0, \varrho), 0, 0, q_e(\varrho_0, \varrho), q_e(\varrho, \varrho_0)\big) \leq 0. \tag{3}$$

It follows from $(\mathcal{G}_2)$ with (2)–(3) that $q_e(\varrho_0, \varrho) = 0$ and $q_e(\varrho_0, \varrho) = 0$, that is, $\varrho = \varrho_0$. □

Example 4.3 Let $\Xi = \mathcal{P} \cup \mathcal{Q}$, where $\mathcal{P} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $\mathcal{Q} = [1, 2]$. Define $q_e : \Xi^2 \to \mathbb{R}_+$ as

$$\begin{aligned} &q_e\left(\frac{1}{2}, \frac{1}{3}\right) = 0.02, \qquad q_e\left(\frac{1}{2}, \frac{1}{4}\right) = 0.06, \qquad q_e\left(\frac{1}{2}, \frac{1}{5}\right) = 0.02, \\ &q_e\left(\frac{1}{3}, \frac{1}{4}\right) = 0.02, \qquad q_e\left(\frac{1}{3}, \frac{1}{5}\right) = 0.01, \qquad q_e\left(\frac{1}{5}, \frac{1}{4}\right) = 0.02, \end{aligned}$$

and $q_e(\vartheta, \nu) = (\vartheta - \nu)^2$ otherwise. Then, (Ξ, q_e) is an EBQbDS with $w(\vartheta, \nu) = \vartheta + \nu + 2$, but neither a BDS (Ξ, b) nor a metric space (Ξ, d). For instance,

$$q_e\left(\frac{1}{2}, \frac{1}{4}\right) = 0.06 \nleq 0.04 = q_e\left(\frac{1}{2}, \frac{1}{3}\right) + q_e\left(\frac{1}{3}, \frac{1}{4}\right)$$

and

$$q_e\left(\frac{1}{2}, \frac{1}{4}\right) = 0.06 \nleq 0.05 = q_e\left(\frac{1}{2}, \frac{1}{3}\right) + q_e\left(\frac{1}{3}, \frac{1}{5}\right) + q_e\left(\frac{1}{5}, \frac{1}{4}\right),$$

but

$$q_e\left(\frac{1}{2}, \frac{1}{4}\right) = 0.06 \leq 0.1375 = w\left(\frac{1}{2}, \frac{1}{4}\right)\left[q_e\left(\frac{1}{2}, \frac{1}{3}\right) + q_e\left(\frac{1}{3}, \frac{1}{5}\right) + q_e\left(\frac{1}{5}, \frac{1}{4}\right)\right].$$

Similarly, we can prove the quadrilateral inequality for all $\vartheta, \nu, \upsilon \in \Xi$, but clearly $q_e(\vartheta, \nu) \neq q_e(\nu, \vartheta)$ if $\vartheta \neq \nu$.

Consider the self-mapping $\mathcal{T}$ on Ξ given by

$$\mathcal{T}(\vartheta) = \begin{cases} \frac{1}{3} & \text{if } \vartheta \in \mathcal{P}, \\ \frac{1}{5} & \text{if } \vartheta \in \mathcal{Q}. \end{cases}$$

Considering Example 3.3, we can define $\mathcal{G} \in \mathfrak{G}$ as

$$\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1^2 - a\hbar_2^2 - b\frac{\hbar_3^2 + \hbar_4^2}{\hbar_5^2 + \hbar_6^2 + 1},$$

taking $a = \frac{1}{4}$, $b = \frac{1}{4}$ so that $a + 2b = \frac{3}{4} < 1$. Here, $k = w(\nu, \vartheta) \geq 1$. One can easily check that $h = k\sqrt{\frac{a+b}{k^2-b}} < 1$ and that $\mathcal{G}$ belongs to the set $\mathfrak{G}$. We will show that $\mathcal{T}$ is a right $\mathcal{G}_w$-implicit mapping.

Let $\nu, \vartheta \in \Xi$ be such that $\nu \in \mathcal{P}, \vartheta \in \mathcal{Q}$. Then, the inequality (1) has the form

$$\left[w(\nu, \vartheta) q_e(\mathcal{T}\nu, \mathcal{T}\vartheta)\right]^2 \leq \frac{1}{4}\left[q_e(\nu, \vartheta)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\nu, \mathcal{T}\nu)]^2 + [q_e(\vartheta, \mathcal{T}\vartheta)]^2}{1 + [q_e(\nu, \mathcal{T}\vartheta)]^2 + [q_e(\vartheta, \mathcal{T}\nu)]^2},$$

that is,

$$\left[(\nu + \vartheta + 2) q_e\left(\frac{1}{3}, \frac{1}{5}\right)\right]^2 \leq \frac{1}{4}\left[q_e(\nu, \vartheta)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\nu, \frac{1}{3})]^2 + [q_e(\vartheta, \frac{1}{5})]^2}{1 + [q_e(\nu, \frac{1}{5})]^2 + [q_e(\vartheta, \frac{1}{3})]^2}.$$

We will demonstrate that the mapping $\mathcal{T}$ verifies the above relation. We have to discuss the following cases:

Case I: Let $\nu = \frac{1}{2}, \vartheta = 1$. Then, $w(\nu, \vartheta) = \frac{7}{2}$, $h = 0.71443 < 1$ and

$$\left(\frac{7}{2}\right)^2 \cdot (0.01)^2 \leq \frac{1}{4}\left[q_e\left(\frac{1}{2}, 1\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{2}, \frac{1}{3})]^2 + [q_e(1, \frac{1}{5}]^2}{1 + [q_e(\frac{1}{2}, \frac{1}{5})]^2 + [q_e(1, \frac{1}{3})]^2}.$$

Case II: Let $\nu = \frac{1}{2}, \vartheta = 2$. Then, $w(\nu, \vartheta) = \frac{9}{2}$, $h = 0.7115 < 1$ and

$$\left(\frac{9}{2}\right)^2 \cdot (0.01)^2 \leq \frac{1}{4}\left[q_e\left(\frac{1}{2}, 2\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{2}, \frac{1}{3})]^2 + [q_e(2, \frac{1}{5})]^2}{1 + [q_e(\frac{1}{2}, \frac{1}{5})]^2 + [q_e(2, \frac{1}{3})]^2}.$$

Case III: Let $\nu = \frac{1}{3}, \vartheta = 1$. Then, $w(\nu, \vartheta) = \frac{9}{3}$, $h = 0.71519 < 1$ and

$$\left(\frac{10}{3}\right)^2 \cdot (0.01)^2 \leq \frac{1}{4}\left[q_e\left(\frac{1}{3}, 1\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{3}, \frac{1}{3})]^2 + [q_e(1, \frac{1}{5})]^2}{1 + [q_e(\frac{1}{3}, \frac{1}{5})]^2 + [q_e(1, \frac{1}{3})]^2}.$$

Case IV: Let $\nu = \frac{1}{3}, \vartheta = 2$. Then, $w(\nu, \vartheta) = \frac{13}{3}$, $h = 0.71186 < 1$ and

$$\left(\frac{13}{3}\right)^2 \cdot (0.01)^2 \leq \frac{1}{4}\left[q_e\left(\frac{1}{3}, 2\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{3}, \frac{1}{3})]^2 + [q_e(2, \frac{1}{5})]^2}{1 + [q_e(\frac{1}{3}, \frac{1}{5})]^2 + [q_e(2, \frac{1}{3})]^2}.$$

Case V: Let $\nu = \frac{1}{4}, \vartheta = 1$. Then, $w(\nu, \vartheta) = \frac{13}{4}$, $h = 0.71562 < 1$ and

$$\left(\frac{13}{4}\right)^2 \cdot (0.01)^2 \leq \frac{1}{4}\left[q_e\left(\frac{1}{4}, 1\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{4}, \frac{1}{3})]^2 + [q_e(1, \frac{1}{5})]^2}{1 + [q_e(\frac{1}{4}, \frac{1}{5})]^2 + [q_e(1, \frac{1}{3})]^2}.$$

Case VI: Let $\nu = \frac{1}{4}, \vartheta = 2$. Then, $w(\nu,\vartheta) = \frac{17}{4}$, $h = 0.71205 < 1$ and

$$\left(\frac{17}{4}\right)^2 \cdot (0.01)^2 \le \frac{1}{4}\left[q_e\left(\frac{1}{4},2\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{4},\frac{1}{3})]^2 + [q_e(2,\frac{1}{5})]^2}{1 + [q_e(\frac{1}{4},\frac{1}{5})]^2 + [q_e(2,\frac{1}{3})]^2}.$$

Case VII: Let $\nu = \frac{1}{5}, \vartheta = 1$. Then, $w(\nu,\vartheta) = \frac{16}{5}$, $h = 0.7158997 < 1$ and

$$\left(\frac{16}{5}\right)^2 \cdot (0.01)^2 \le \frac{1}{4}\left[q_e\left(\frac{1}{5},1\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{5},\frac{1}{3})]^2 + [q_e(1,\frac{1}{5})]^2}{1 + [q_e(\frac{1}{5},\frac{1}{5})]^2 + [q_e(1,\frac{1}{3})]^2}.$$

Case VIII: Let $\nu = \frac{1}{5}, \vartheta = 2$. Then, $w(\nu,\vartheta) = \frac{21}{5}$, $h = 0.71217 < 1$ and

$$\left(\frac{21}{5}\right)^2 \cdot (0.01)^2 \le \frac{1}{4}\left[q_e\left(\frac{1}{5},2\right)\right]^2 + \frac{1}{4} \cdot \frac{[q_e(\frac{1}{5},\frac{1}{3})]^2 + [q_e(2,\frac{1}{5})]^2}{1 + [q_e(\frac{1}{5},\frac{1}{5})]^2 + [q_e(2,\frac{1}{3})]^2}.$$

In all these cases, it is easy to verify that the inequalities hold true. Thus, all the conditions are fulfilled and the mapping $\mathcal{T}$ has a unique fixed point (which is $\vartheta^* = \frac{1}{3}$).

Example 4.4 Let $\Xi = [0,1]$ and define

$$q_e(\nu,\vartheta) = \begin{cases} |\nu - \vartheta|^2 + \nu & \text{if } \nu \neq \vartheta, \\ 0 & \text{if } \nu = \vartheta, \end{cases}$$

with $w(\nu,\vartheta) = \nu + \vartheta + 2$. Then, it is clear that (Ξ, q_e) is an EBQbDS with $q_e(\nu,\vartheta) \neq q_e(\vartheta,\nu)$ if $\nu \neq \vartheta$, but neither a BDS (Ξ, b) nor a metric space (Ξ, d). For instance,

$$\begin{aligned} q_e(0,1) = 1 &\nleq \frac{23}{25} = q_e\left(0,\frac{1}{5}\right) + q_e\left(\frac{1}{5},1\right), \\ q_e(0,1) = 1 &\nleq \frac{96}{100} = q_e\left(0,\frac{1}{10}\right) + q_e\left(\frac{1}{10},\frac{1}{5}\right) + q_e\left(\frac{1}{5},1\right), \end{aligned}$$

but (qeb2) holds as

$$\begin{aligned} q_e(\vartheta,\nu) &= |\vartheta - \nu|^2 + \vartheta \\ &= |\vartheta - \upsilon + \upsilon - \mu + \mu - \nu|^2 \\ &\le |\vartheta - \upsilon|^2 + |\upsilon - \mu|^2 + |\mu - \nu|^2 \\ &\quad + 2|\vartheta - \upsilon||\upsilon - \mu| + 2|\upsilon - \mu||\mu - \nu| + 2|\mu - \nu||\vartheta - \upsilon| + \vartheta + \upsilon + \mu \\ &\le (\vartheta + \nu + 2)\left[|\vartheta - |^2 + |\upsilon - \mu|^2 + \mu| - \nu|^2 + \vartheta + \upsilon + \mu\right] \\ &= w(\vartheta,\nu)\left[q_e(\vartheta,\upsilon) + q_e(\upsilon,\mu) + q_e\mu(,\nu)\right], \end{aligned}$$

for all $\vartheta, \nu, \mu, \upsilon \in \Xi$, but clearly $q_e(\vartheta,\nu) \neq q_e(\nu,\vartheta)$ if $\vartheta \neq \nu$. In particular,

$$q_e(0,1) = 1 \le 2.88 = w(0,1)\left[q_e\left(0,\frac{1}{10}\right) + q_e\left(\frac{1}{10},\frac{1}{5}\right) + q_e\left(\frac{1}{5},1\right)\right].$$

Consider the self-mapping $\mathcal{T}$ on Ξ given by $\mathcal{T}(\vartheta) = \frac{\vartheta}{16}$. Considering Example 3.4, we can define $\mathcal{G} \in \mathfrak{G}$ as

$$\mathcal{G}(\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6) = \hbar_1 - a\hbar_2 - b\hbar_3 - c\frac{\hbar_4\hbar_5}{1 + \hbar_5 + \hbar_6},$$

taking $a = \frac{1}{4}$, $b = \frac{1}{4}$, $c = \frac{1}{4}$ so that $a + b + c = \frac{3}{4} < 1$. Here, $k = w(\nu, \vartheta) \geq 1$. One can easily check that $h = \frac{k(a+b)}{k-c} < 1$ and that $\mathcal{G}$ belongs to the set $\mathfrak{G}$. We will show that $\mathcal{T}$ is a right $\mathcal{G}_w$-implicit mapping.

If $\nu, \vartheta \in \Xi$, then the inequality (1) is of the form

$$w(\nu, \vartheta)q_e(\mathcal{T}\nu, \mathcal{T}\vartheta) \leq \frac{1}{4}q_e(\nu, \vartheta) + \frac{1}{4}q_e(\nu, \mathcal{T}\nu) + \frac{1}{4} \cdot \frac{q_e(\vartheta, \mathcal{T}\vartheta)q_e(\nu, \mathcal{T}\vartheta)}{1 + q_e(\nu, \mathcal{T}\vartheta) + q_e(\vartheta, \mathcal{T}\nu)},$$

that is,

$$(\nu + \vartheta + 2)q_e\left(\frac{\nu}{16}, \frac{\vartheta}{16}\right) \leq \frac{1}{4}q_e(\nu, \vartheta) + \frac{1}{4}q_e\left(\nu, \frac{\nu}{16}\right) + \frac{1}{4} \cdot \frac{q_e(\vartheta, \frac{\vartheta}{16})q_e(\nu, \frac{\vartheta}{16})}{1 + q_e(\nu, \frac{\vartheta}{16}) + q_e(\vartheta, \frac{\nu}{16})}. \tag{4}$$

It is easy to check that (4) holds whenever $q_e(\mathcal{T}\vartheta, \mathcal{T}\nu) > 0$. As examples, we check the following two cases:

Case I: Let $\nu = 0, \vartheta = \frac{1}{2}$. Then, $w(\nu, \vartheta) = \frac{5}{2}$, $h = \frac{5}{9} < 1$ and

$$\frac{5}{2} \cdot (0.015625) \leq \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4}\left[\left(\frac{15}{32}\right)^2 + \frac{1}{2}\right] + \frac{1}{4} \cdot 0.$$

Case II: Let $\nu = \frac{1}{2}, \vartheta = 1$. Then, $w(\nu, \vartheta) = \frac{7}{2}$, $h = \frac{17}{31} < 1$ and

$$\frac{17}{6}(0.031358) \leq \frac{1}{4}.\frac{3}{4} + \frac{1}{4}\left[\left(\frac{15}{32}\right)^2 + \frac{1}{2}\right] + \frac{1}{4} \cdot \frac{((\frac{15}{32})^2 + 1)((\frac{7}{16})^2 + \frac{1}{2})}{1 + ((\frac{7}{16})^2 + \frac{1}{2}) + ((\frac{31}{32})^2 + 1)}.$$

Similarly, we can verify for other cases. Thus, all the conditions are fulfilled and the mapping $\mathcal{T}$ has a unique fixed point (which is $\vartheta^* = 0$).

By choosing $\mathcal{G} \in \mathfrak{G}$ from Examples 3.3 and 3.4, we have the following consequences.

Corollary 4.5 *Let all the conditions of Theorem 4.2 be satisfied, except that the assumption of right $\mathcal{G}_w$-implicit mapping for $\mathcal{G} \in \mathfrak{G}$ is replaced by either of the form*

(I)

$$\left[w(\nu, \vartheta)q_e(\mathcal{T}\nu, \mathcal{T}\vartheta)\right]^2 \leq a\left[q_e(\nu, \vartheta)\right]^2 + b\frac{[q_e(\nu, \mathcal{T}\nu)]^2 + [q_e(\vartheta, \mathcal{T}\vartheta)]^2}{1 + [q_e(\nu, \mathcal{T}\vartheta)]^2 + [q_e(\vartheta, \mathcal{T}\nu)]^2},$$

where $0 < a, b < 1$, $a + 2b < 1$, $w(\nu, \vartheta)\sqrt{\frac{a+b}{w(\nu,\vartheta)^2 - b}} < 1$, *or*

(II)

$$w(\nu, \vartheta)q_e(\mathcal{T}\nu, \mathcal{T}\vartheta) \leq aq_e(\nu, \vartheta) + bq_e(\nu, \mathcal{T}\nu) + c\frac{q_e(\vartheta, \mathcal{T}\vartheta)q_e(\nu, \mathcal{T}\vartheta)}{1 + q_e(\nu, \mathcal{T}\vartheta) + q_e(\vartheta, \mathcal{T}\nu)}, \tag{5}$$

where $0 < a, b, c < 1$, $a + b + c < 1$, $\frac{w(\nu,\vartheta)(a+b)}{w(\nu,\vartheta) - c} < 1$. *Then,* $\mathrm{Fix}(\mathcal{T})$ *is a singleton.*

5 Generalized w-Ulam–Hyers stability

In this section, we define generalized w-Ulam–Hyers stability (Gw-UHS) of the fixed-point problem (fpp) in EBQbDS as an extension of the b-metric space case discussed in [8, 17] (see also [14]).

Definition 5.1 Let (Ξ, q_e) be a complete EBQbDS and $\mathcal{T} : \Xi \to \Xi$ be a mapping. Then, the fixed-point equation (FPE)

$$\vartheta = \mathcal{T}\vartheta, \quad \vartheta \in \Xi \tag{6}$$

is said to be Gw-UHS in the setting of EBQbDS if there exists an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0, with $\phi(0) = 0$, such that for each $\varepsilon > 0$ and an ε-solution $\upsilon \in \Xi$, that is,

$$q_e(\upsilon, \mathcal{T}\upsilon) \leq \varepsilon,$$

there exists a solution $\vartheta^* \in \Xi$ of (6) such that

$$q_e\big(\upsilon, \vartheta^*\big) \leq \phi\big(w\big(\vartheta^*, \upsilon\big)\varepsilon\big). \tag{7}$$

If $\phi(\xi) = \alpha\xi$ for all $\xi \in \mathbb{R}_+$, where $\alpha > 0$, then FPE (6) is said to be w-UHS in the setting of EBQbDS.

Remark 5.2 If $w(\vartheta, \upsilon) = 1$, then Definition 5.1 reduces to the notion of GUHS in BDS. Also, if $\phi(\xi) = \alpha\xi$ for all $\alpha \in \mathbb{R}_+$, where $\alpha > 0$, then it reduces to the notion of UHS in BDS. Finally, if $q_e(\vartheta, \upsilon) = |\vartheta - \upsilon|$, then it reduces to the classical UHS.

Theorem 5.3 *Let (Ξ, q_e, w) be a right-complete EBQbDS with $w : \Xi^2 \to \mathbb{R}_+ \setminus (0, 1)$ and $\mathcal{T} : \Xi \to \Xi$ be a continuous mapping satisfying the contractive condition* (5)*. Then, the FPE* (6) *is Gw-UHS.*

Proof Following Theorem 4.2, we have $\mathcal{T}\varrho^* = \varrho^*$, that is, $\varrho^* \in \Xi$ is a (unique) solution of the FPE (6) with $q_e(\varrho^*, \mathcal{T}\varrho^*) = 0$. Let $\epsilon > 0$ and $\sigma^* \in \Xi$ be an ϵ-solution of (6), that is,

$$q_e\big(\sigma^*, \mathcal{T}\sigma^*\big) \leq \epsilon.$$

Since $q_e(\varrho^*, \mathcal{T}\varrho^*) = q_e(\varrho^*, \varrho^*) = 0 \leq \epsilon$, ϱ^* and σ^* are ϵ-solutions. Since we have $w(\varrho^*, \sigma^*) \geq 1$, then,

$$\begin{aligned} q_e\big(\varrho^*, \sigma^*\big) &\leq w\big(\varrho^*, \sigma^*\big)\big[q_e\big(\varrho^*, \mathcal{T}\varrho^*\big) + q_e\big(\mathcal{T}\varrho^*, \mathcal{T}\sigma^*\big) + q_e\big(\mathcal{T}\sigma^*, \sigma^*\big)\big] \\ &\leq w\big(\varrho^*, \sigma^*\big)q_e\big(\mathcal{T}\varrho^*, \mathcal{T}\sigma^*\big) + \epsilon w\big(\varrho^*, \sigma^*\big). \end{aligned} \tag{8}$$

From the contractive condition (5) for $\mathcal{T}$, we obtain

$$\begin{aligned} &w\big(\varrho^*, \sigma^*\big)q_e\big(\mathcal{T}\varrho^*, \mathcal{T}\sigma^*\big) \\ &\quad \leq a q_e\big(\varrho^*, \sigma^*\big) + b q_e\big(\varrho^*, \mathcal{T}\varrho^*\big) + c\frac{q_e(\sigma^*, \mathcal{T}\sigma^*)q_e(\varrho^*, \mathcal{T}\sigma^*)}{1 + q_e(\varrho^*, \mathcal{T}\sigma^*) + q_e(\sigma^*, \mathcal{T}\varrho^*)} \end{aligned}$$

$$\leq aq_e(\varrho^*,\sigma^*) + c\epsilon \frac{q_e(\varrho^*,\mathcal{T}\sigma^*)}{1 + q_e(\varrho^*,\mathcal{T}\sigma^*) + q_e(\sigma^*,\mathcal{T}\varrho^*)}$$
$$\leq aq_e(\varrho^*,\sigma^*) + c\epsilon.$$

Therefore, from (8), we obtain

$$\begin{aligned} q_e(\varrho^*,\sigma^*) &\leq aq_e(\varrho^*,\sigma^*) + c\epsilon + \epsilon w(\varrho^*,\sigma^*) \\ &\leq aq_e(\varrho^*,\sigma^*) + c\epsilon w(\varrho^*,\sigma^*) + \epsilon w(\varrho^*,\sigma^*), \end{aligned}$$

which implies that

$$q_e(\varrho^*,\sigma^*)(1-a) \leq (1+c)\epsilon w(\varrho^*,\sigma^*),$$

i.e.,

$$q_e(\varrho^*,\sigma^*) \leq \frac{1+c}{(1-a)} w(\varrho^*,\sigma^*)\epsilon = \phi\big(w(\varrho^*,\sigma^*)\epsilon\big) \quad \text{as } \frac{1+c}{(1-a)} > 0.$$

Thus, the inequality (7) holds and therefore, the FPE (6) is Gw-UHS. □

6 Weak well-posed property, weak limit shadowing

The notion of well-posedness of an fpp has evoked much interest from several mathematicians, for example, Popa [18, 19] and others. In the paper [5], the authors defined a weak well-posed (wwp) property in BbDS. In what follows, we extend this notion to EBQbDS.

Definition 6.1 Let (Ξ, q_e, w) be a complete EBQbDS and $\mathcal{T} : \Xi \to \Xi$ be a mapping. The fpp of $\mathcal{T}$ is said to be wwp if it satisfies:

1. $\mathrm{Fix}(\mathcal{T}) = \{\vartheta^*\}$ is a singleton set in Ξ;
2. for any sequence $\{\vartheta_p\}$ in Ξ with $\lim_{p\to\infty} q_e(\vartheta_p, \mathcal{T}(\vartheta_p)) = 0$ and $\lim_{p,r\to\infty} q_e(\mathcal{T}(\vartheta_p), \mathcal{T}(\vartheta_r)) = 0$, one has $\lim_{p\to\infty} q_e(\vartheta_p, \vartheta^*) = 0$.

To guarantee the wwp of a mapping $\mathcal{T}$, we add the following additional condition for functions $\mathcal{G} \in \mathfrak{G}$ and call the respective set $\mathfrak{G}'$:

($\mathcal{G}_3$) for all $\zeta, \xi, \mu > 0$, $k \geq 1$, $\mathcal{G}(k\zeta, \xi, 0, 0, \zeta, \mu) \leq 0$ implies that there exists $\varphi \in \Phi_w$ such that $k\zeta \leq \varphi(\xi)$.

Theorem 6.2 *Let (Ξ, q_e, w) be a right-complete EBQbDS with $w : \Xi^2 \to \mathbb{R}_+ \setminus (0,1)$ and $\mathcal{T} : \Xi \to \Xi$ be a continuous and right $\mathcal{G}_w$-implicit mapping for $\mathcal{G} \in \mathfrak{G}'$ and $\varphi \in \Phi_w$ such that* $\lim_{n\to\infty} q_e(\vartheta_n, \mathcal{T}\vartheta_n) = 0$,
$\lim_{n,m\to\infty} q_e(\mathcal{T}\vartheta_n, \mathcal{T}\vartheta_m) = 0$ *and ϑ^* is a fixed point of $\mathcal{T}$. Then, the fpp of $\mathcal{T}$ is wwp, provided $\mathcal{G} \in \mathfrak{G}'$ is continuous.*

Proof Let $\{\vartheta_n\}$ be a sequence in Ξ such that $\lim_{n\to\infty} q_e(\vartheta_n, \mathcal{T}(\vartheta_n)) = 0$ and $\lim_{n,m\to\infty} q_e(\mathcal{T}\vartheta_n, \mathcal{T}\vartheta_m) = 0$, for $m > n$. We obtain from (qeb2) that

$$q_e(\vartheta_n, \vartheta^*) \leq w(\vartheta_n, \vartheta^*)\big[q_e(\vartheta_n, \mathcal{T}\vartheta_m) + q_e(\mathcal{T}\vartheta_m, \mathcal{T}\vartheta_n) + q_e(\mathcal{T}\vartheta_n, \vartheta^*)\big].$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} q_e\big(\vartheta_n, \vartheta^*\big) \leq \lim_{n\to\infty} w\big(\vartheta_n, \vartheta^*\big)\big[q_e(\vartheta_n, \mathcal{T}\vartheta_m) + q_e\big(\mathcal{T}\vartheta_n, \vartheta^*\big)\big]. \tag{9}$$

WLOG, we can consider that there exists a subsequence $\{\mathcal{T}\vartheta_{n_k}\}$ of $\{\mathcal{T}\vartheta_n\}$ with distinct elements. Otherwise, there exists $\vartheta_0 \in \Xi$ and $n_1 \in \mathbb{N}$ such that $\mathcal{T}\vartheta_n = \vartheta_0$ for $n \geq n_1$. Since $\lim_{n\to\infty} q_e(\vartheta_n, \mathcal{T}\vartheta_n) = 0$, we obtain

$\lim_{n\to\infty} q_e(\vartheta_n, \vartheta_0) = 0$. If $\vartheta_0 \neq \vartheta^*$, then $\vartheta_0 \neq \mathcal{T}\vartheta_0$ due to the uniqueness of the fixed point of $\mathcal{T}$. For $n \geq n_1$, we obtain $\vartheta_0 = \mathcal{T}\vartheta_n \neq \mathcal{T}\vartheta_0$.

Since $\mathcal{T}$ is a right $\mathcal{G}_w$-implicit mapping, we obtain

$$\mathcal{G}\begin{pmatrix} w(\vartheta_n, \vartheta_0)q_e(\mathcal{T}\vartheta_n, \mathcal{T}\vartheta_0), q_e(\vartheta_n, \vartheta_0), q_e(\vartheta_n, \mathcal{T}\vartheta_n), \\ q_e(\vartheta_0, \mathcal{T}\vartheta_0), q_e(\vartheta_n, \mathcal{T}\vartheta_0), q_e(\vartheta_0, \mathcal{T}\vartheta_n) \end{pmatrix} \leq 0,$$

i.e.,

$$\mathcal{G}\begin{pmatrix} w(\vartheta_n, \vartheta_0)q_e(\vartheta_0, \mathcal{T}\vartheta_0), q_e(\vartheta_n, \vartheta_0), q_e(\vartheta_n, \vartheta_0), \\ q_e(\vartheta_0, \mathcal{T}\vartheta_0), q_e(\vartheta_n, \mathcal{T}\vartheta_0), q_e(\vartheta_0, \vartheta_0) \end{pmatrix} \leq 0,$$

i.e.,

$$\mathcal{G}\begin{pmatrix} w(\vartheta_n, \vartheta_0)q_e(\vartheta_0, \mathcal{T}\vartheta_0), q_e(\vartheta_n, \vartheta_0), q_e(\vartheta_n, \vartheta_0), \\ q_e(\vartheta_0, \mathcal{T}\vartheta_0), q_e(\vartheta_n, \mathcal{T}\vartheta_0), 0 \end{pmatrix} \leq 0.$$

It follows from $(\mathcal{G}_1)$ that there exists $\varphi \in \Phi_w$ such that

$$w(\vartheta_n, \vartheta_0)q_e(\vartheta_0, \mathcal{T}\vartheta_0) \leq \varphi\big(q_e(\vartheta_n, \vartheta_0)\big) \quad \text{for all } n \in \mathbb{N}.$$

Hence, we obtain

$$q_e(\vartheta_0, \mathcal{T}\vartheta_0) \leq w(\vartheta_n, \vartheta_0)q_e(\vartheta_0, \mathcal{T}\vartheta_0) \leq \varphi\big(q_e(\vartheta_n, \vartheta_0)\big) < q_e(\vartheta_n, \vartheta_0),$$

which, on applying the limit as $n \to \infty$, gives $q_e(\vartheta_0, \mathcal{T}\vartheta_0) < 0$, a contradiction. Hence, there exist $m, q, n > n_0$ $(m > q > n)$ such that $\mathcal{T}x_m \neq \mathcal{T}x_q \neq \mathcal{T}x_n \neq \mathcal{T}_n$. Then,

$$q_e(\vartheta_n, \mathcal{T}\vartheta_m) \leq w(\vartheta_n, \mathcal{T}x_m)\big\{q_e(\vartheta_n, \mathcal{T}\vartheta_n) + q_e(\mathcal{T}\vartheta_n, \mathcal{T}\vartheta_q) + q_e(\mathcal{T}\vartheta_q, \mathcal{T}\vartheta_m)\big\},$$

which tends to 0 as $n \to \infty$. On replacing the value in (9), we obtain

$$\lim_{n\to\infty} q_e\big(\vartheta_n, \vartheta^*\big) \leq \lim_{n\to\infty} w\big(\vartheta_n, \vartheta^*\big)q_e\big(\mathcal{T}\vartheta_n, \vartheta^*\big). \tag{10}$$

Again, using the right $\mathcal{G}_w$-implicit condition of $\mathcal{T}$, we obtain

$$\mathcal{G}\begin{pmatrix} w(\vartheta_n, \vartheta^*)q_e(\mathcal{T}\vartheta_n, \mathcal{T}\vartheta^*), q_e(\vartheta_n, \vartheta^*), q_e(\vartheta_n, \mathcal{T}\vartheta_n), \\ q_e(\vartheta^*, \mathcal{T}\vartheta^*), q_e(\vartheta_n, \mathcal{T}\vartheta^*), q_e(\vartheta^*, \mathcal{T}\vartheta_n) \end{pmatrix} \leq 0,$$

i.e.,

$$\mathcal{G}\begin{pmatrix} w(\vartheta_n,\vartheta^*)q_e(\mathcal{T}\vartheta_n,\vartheta^*), q_e(\vartheta_n,\vartheta^*), q_e(\vartheta_n,\mathcal{T}\vartheta_n), \\ 0, q_e(\vartheta_n,\vartheta^*), q_e(\vartheta^*,\mathcal{T}\vartheta_n) \end{pmatrix} \leq 0.$$

Taking the limit as $n\to\infty$ and using the continuity of $\mathcal{G}$, we obtain

$$\mathcal{G}\begin{pmatrix} \lim_{n\to\infty} w(\vartheta_n,\vartheta^*)q_e(\mathcal{T}\vartheta_n,\vartheta^*), \lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*), 0, 0, \\ \lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*), \lim_{n\to\infty} q_e(\vartheta^*,\mathcal{T}\vartheta_n) \end{pmatrix} \leq 0.$$

It follows from $(\mathcal{G}_3)$ that there exists $\varphi\in\Phi_w$ such that

$$\lim_{n\to\infty} w(\vartheta_n,\vartheta^*)q_e(\mathcal{T}\vartheta_n,\vartheta^*) \leq \varphi\Big(\lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*)\Big),$$

which, on replacing the values in (10), gives

$$\begin{aligned}\lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*) &\leq \lim_{n\to\infty} w(\vartheta_n,\vartheta^*)q_e(\mathcal{T}\vartheta_n,\vartheta^*) \\ &\leq \varphi\Big(\lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*)\Big) < \lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*),\end{aligned}$$

a contradiction. Therefore, $\lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*) = 0$. □

The limit shadowing property of fpps has been discussed in [16, 23]. We define the weak limit shadowing property (wlsp) in EBQbDS.

Definition 6.3 Let (Ξ, q_e, w) be a complete EBQbDS and $\mathcal{T}:\Xi\to\Xi$ be a mapping. The fpp of $\mathcal{T}$ is said to have wlsp in Ξ if assuming that $\{\vartheta_n\}$ in Ξ satisfies $q_e(\vartheta_n,\mathcal{T}\vartheta_n)\to 0$ as $n\to\infty$ and $q_e(\mathcal{T}\vartheta_n,\mathcal{T}\vartheta_m)\to 0$, it follows that there exists $\vartheta\in\Xi$ such that $q_e(\vartheta_n,\mathcal{T}^n\vartheta)\to 0$ as $n\to\infty$.

Theorem 6.4 *Let (Ξ, q_e, w) be a complete EBQbDS and $\mathcal{T}:\Xi\to\Xi$ be a continuous and right $\mathcal{G}_w$-implicit mapping for $\mathcal{G}\in\mathfrak{G}'$ and $\varphi\in\Phi_w$ with $\{\vartheta_n\}$ in Ξ such that $\lim_{n\to\infty} q_e(\vartheta_n,\mathcal{T}\vartheta_n) = 0$, $\lim_{n,m\to\infty} q_e(\mathcal{T}\vartheta_n,\mathcal{T}\vartheta_m) = 0$ and $\vartheta^*\in \mathrm{Fix}(\mathcal{T})$. Then, $\mathcal{T}$ has the wlsp, provided $\mathcal{G}\in\mathfrak{G}'$ is continuous.*

Proof Let $\{\vartheta_n\}$ in Ξ be such that $\lim_{n\to\infty} q_e(\vartheta_n,\mathcal{T}\vartheta_n) = 0$ and $\lim_{n,m\to\infty} q_e(\mathcal{T}\vartheta_n,\mathcal{T}\vartheta_m) = 0$. Since $\vartheta^*\in\mathrm{Fix}(\mathcal{T})$, $q_e(\vartheta^*,\mathcal{T}\vartheta^*) = 0$, then by virtue of Theorem 6.2, we have $\lim_{n\to\infty} q_e(\vartheta_n,\vartheta^*) = 0$ and therefore we obtain $\lim_{n\to\infty} q_e(\vartheta_n,\mathcal{T}^n\vartheta^*) = 0$. □

7 Application to nonlinear matrix equations

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n\times n$ Hermitian (resp. positive-semidefinite, positive-definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ the set of all $n\times n$ matrices over $\mathbb{C}$. For a matrix $\mathcal{B}\in\mathcal{H}(n)$, we will denote by $s(\mathcal{B})$ any of its singular values and by $s^+(\mathcal{B})$ the sum of all of its singular values, that is, its trace norm $\operatorname{tr}\mathcal{B} = s^+(\mathcal{B})$. For $\mathcal{C},\mathcal{D}\in\mathcal{H}(n)$, $\mathcal{C}\succeq\mathcal{D}$ (resp. $\mathcal{C}\succ\mathcal{D}$) will mean that the matrix $\mathcal{C}-\mathcal{D}$ is positive- semidefinite (resp. positive-definite).

In [21], Ran and Reurings discussed the existence of solutions of the equation

$$\mathcal{U} + \mathcal{B}^* \hbar(\mathcal{U})\mathcal{B} = \mathcal{Q} \tag{11}$$

in $\mathcal{K}(n)$, where $\mathcal{B} \in \mathcal{M}(n)$, $\mathcal{Q}$ is positive-definite and $\hbar$ is a mapping from $\mathcal{K}(n)$ into $\mathcal{M}(n)$. Note that $\mathcal{U}$ is a solution of (11) if and only if it is a fixed point of the mapping $\mathcal{G}(\mathcal{U}) = \mathcal{Q} - \mathcal{B}^*\hbar(\mathcal{U})\mathcal{B}$. In [22], they used the notion of partial ordering and established a modification of the Banach Contraction Principle, which they applied for solving a class of NMEs of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^*\hbar(\mathcal{U})\mathcal{B}_i$ using the Ky Fan norm in $\mathcal{M}(n)$.

In [24], Sawangsup and Sintunavarat studied the NME of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^*\hbar(\mathcal{U})\mathcal{B}_i$ using the spectral norm of a matrix, and applied a generalized contraction condition in metric spaces endowed with a transitive binary relation; they also tested numerically its approximate solutions. The related work in solving NMEs using fixed-point results can be found in [9, 15].

In this section, we establish the existence and uniqueness of the solution of the nonlinear matrix equation

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{A}_i^*\mathcal{G}(\mathcal{X})\mathcal{A}_i, \tag{12}$$

where $\mathcal{Q}$ is a Hermitian positive-definite matrix, $\mathcal{A}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{A}_i$ and $\mathcal{G}$ is an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive-definite matrices such that $\mathcal{G}(O) = O$.

The following lemmas are needed in the subsequent discussion.

Lemma 7.1 ([21]) *If $\mathcal{A} \succeq O$ and $\mathcal{B} \succeq O$ are $n \times n$ matrices, then*

$$0 \le \operatorname{tr}(\mathcal{A}\mathcal{B}) \le \|\mathcal{A}\| \operatorname{tr}(\mathcal{B}).$$

Lemma 7.2 ([21]) *If $\mathcal{A} \in \mathcal{H}(n)$ such that $\mathcal{A} \prec I_n$, then $\|\mathcal{A}\| < 1$.*

Theorem 7.3 *Consider the problem given by* (12). *Assume that there exists a positive real number η such that*:

(H_1) *there exists $\mathcal{Q} \in \mathcal{P}(n)$ such that $\sum_{i=1}^{m} \mathcal{A}_i^*\mathcal{G}(\mathcal{Q})\mathcal{A}_i \succ 0$;*

(H_2) $\sum_{i=1}^{m} \mathcal{A}_i\mathcal{A}_i^* \prec \eta I_n$;

(H_3) *there exist $0 < \lambda_1, \lambda_2, \lambda_3 < 1$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$ such that for all $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(n)$ such that $\mathcal{X} \preceq \mathcal{Y}$ and*

$$\sum_{i=1}^{m} \mathcal{A}_i^*\mathcal{G}(\mathcal{X})\mathcal{A}_i \neq \sum_{i=1}^{m} \mathcal{A}_i^*\mathcal{G}(\mathcal{Y})\mathcal{A}_i,$$

it is $h = \frac{(s^+(\mathcal{X})+s^+(\mathcal{Y})+3)(\lambda_1+\lambda_2)}{(s^+(\mathcal{X})+s^+(\mathcal{Y})+3)-\lambda_3} < 1$ and

$$\begin{aligned}&\operatorname{tr}\left(\mathcal{G}(\mathcal{X}) - \frac{1}{2}\mathcal{G}(\mathcal{Y})\right)\\ &\quad\le \frac{1}{\eta(s^+(\mathcal{X}) + s^+(\mathcal{Y}) + 3)^{1/2}}\end{aligned}$$

$$\times \left\{ \begin{array}{l} \lambda_1 |\operatorname{tr}(\mathcal{X} - \frac{1}{2}\mathcal{Y})|^2 + \lambda_2 |\operatorname{tr}(\mathcal{X} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X})\mathcal{A}_i)|^2 \\ +\lambda_3 \frac{|\operatorname{tr}(\mathcal{Y} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y})\mathcal{A}_i)|^2 |\operatorname{tr}(\mathcal{X} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y})\mathcal{A}_i)|^2}{1 + |\operatorname{tr}(\mathcal{X} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y})\mathcal{A}_i)|^2 + |\operatorname{tr}(\mathcal{Y} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X})\mathcal{A}_i)|^2} \end{array} \right\}^{1/2}.$$

Then, the matrix equation (12) *has a unique solution. Moreover, the iterations*

$$\mathcal{X}_n = \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}_{n-1})\mathcal{A}_i, \tag{13}$$

where $\mathcal{X}_0 \in \mathcal{P}(n)$ *satisfies* $\mathcal{X}_0 \preceq \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}_0)\mathcal{A}_i$, *converge in the sense of the trace norm* $\|\cdot\|_{\operatorname{tr}}$ *to the solution of the matrix equation* (12).

Proof Define a mapping $\mathcal{T} : \mathcal{P}(n) \to \mathcal{P}(n)$ by

$$\mathcal{T}(\mathcal{X}) = \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X})\mathcal{A}_i, \quad \text{for all } \mathcal{X} \in \mathcal{P}(n).$$

Then, $\mathcal{T}$ is well defined and continuous on $\mathcal{P}(n)$. Then, the fixed point of the mapping $\mathcal{T}$ is a solution of the matrix equation (12).

Now, for $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(n)$, we have

$$\begin{aligned}
&\left\| \mathcal{T}(\mathcal{X}) - \frac{1}{2}\mathcal{T}(\mathcal{Y}) \right\|_{\operatorname{tr}}^2 \\
&= \left[\operatorname{tr}\left(\mathcal{T}(\mathcal{X}) - \frac{1}{2}\mathcal{T}(\mathcal{Y}) \right) \right]^2 = \left[\operatorname{tr}\left(\sum_{i=1}^m \mathcal{A}_i^* \left(\mathcal{G}(\mathcal{X}) - \frac{1}{2}\mathcal{G}(\mathcal{Y}) \right) \mathcal{A}_i \right) \right]^2 \\
&= \left[\sum_{i=1}^m \operatorname{tr}\left(\mathcal{A}_i^* \left(\mathcal{G}(\mathcal{X}) - \frac{1}{2}\mathcal{G}(\mathcal{Y}) \right) \mathcal{A}_i \right) \right]^2 = \left[\sum_{i=1}^m \operatorname{tr}\left(\mathcal{A}_i \mathcal{A}_i^* \left(\mathcal{G}(\mathcal{X}) - \frac{1}{2}\mathcal{G}(\mathcal{Y}) \right) \right) \right]^2 \\
&= \left[\operatorname{tr}\left(\left(\sum_{i=1}^m \mathcal{A}_i \mathcal{A}_i^* \right) \left(\mathcal{G}(\mathcal{X}) - \frac{1}{2}\mathcal{G}(\mathcal{Y}) \right) \right) \right]^2 \le \left\| \sum_{i=1}^m \mathcal{A}_i \mathcal{A}_i^* \right\|^2 \cdot \left\| \left(\mathcal{G}(\mathcal{X}) - \frac{1}{2}\mathcal{G}(\mathcal{Y}) \right) \right\|_{\operatorname{tr}}^2 \\
&\le \frac{\| \sum_{i=1}^m \mathcal{A}_i \mathcal{A}_i^* \|_{\operatorname{tr}}^2}{\eta^2 (\|\mathcal{X}\|_{\operatorname{tr}} + \|\mathcal{Y}\|_{\operatorname{tr}} + 3)} \left\{ \begin{array}{l} \lambda_1 \|\mathcal{X} - \frac{1}{2}\mathcal{Y}\|_{\operatorname{tr}}^2 + \lambda_2 \|\mathcal{X} - \frac{1}{2}\mathcal{T}\mathcal{X}\|_{\operatorname{tr}}^2 \\ +\lambda_3 \frac{\|\mathcal{Y} - \frac{1}{2}\mathcal{T}\mathcal{Y}\|_{\operatorname{tr}}^2 \|\mathcal{X} - \frac{1}{2}\mathcal{T}\mathcal{Y}\|_{\operatorname{tr}}^2}{1 + \|\mathcal{X} - \frac{1}{2}\mathcal{T}\mathcal{Y}\|_{\operatorname{tr}}^2 + \|\mathcal{Y} - \frac{1}{2}\mathcal{T}\mathcal{X}\|_{\operatorname{tr}}^2} \end{array} \right\} \\
&\le \frac{1}{(\|\mathcal{X}\|_{\operatorname{tr}} + \|\mathcal{Y}\|_{\operatorname{tr}} + 3)} \left\{ \begin{array}{l} \lambda_1 \|\mathcal{X} - \frac{1}{2}\mathcal{Y}\|_{\operatorname{tr}}^2 + \lambda_2 \|\mathcal{X} - \frac{1}{2}\mathcal{T}\mathcal{X}\|_{\operatorname{tr}}^2 \\ +\lambda_3 \frac{\|\mathcal{Y} - \frac{1}{2}\mathcal{T}\mathcal{Y}\|_{\operatorname{tr}}^2 \|\mathcal{X} - \frac{1}{2}\mathcal{T}\mathcal{Y}\|_{\operatorname{tr}}^2}{1 + \|\mathcal{X} - \frac{1}{2}\mathcal{T}\mathcal{Y}\|_{\operatorname{tr}}^2 + \|\mathcal{Y} - \frac{1}{2}\mathcal{T}\mathcal{X}\|_{\operatorname{tr}}^2} \end{array} \right\}.
\end{aligned} \tag{14}$$

Let $q_e : \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{R}_+$ be defined by

$$q_e(\mathcal{X}, \mathcal{Y}) = \left\| \mathcal{X} - \frac{1}{2}\mathcal{Y} \right\|_{\operatorname{tr}}^2, \quad \text{for } \mathcal{X} \neq \mathcal{Y}, \text{ and } 0, \text{ for } \mathcal{X} = \mathcal{Y}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(n)$. Then, $(\mathcal{P}(n), q_e)$ is a right-complete EBQbDS with coefficient $w(\mathcal{X}, \mathcal{Y}) = \|\mathcal{X}\|_{\operatorname{tr}} + \|\mathcal{Y}\|_{\operatorname{tr}} + 3$. It follows from (14) that

$$w(\mathcal{X}, \mathcal{Y}) q_e\big(\mathcal{T}(\mathcal{X}), \mathcal{T}(\mathcal{Y})\big) \le \lambda_1 q_e(\mathcal{X}, \mathcal{Y}) + \lambda_2 q_e\big(\mathcal{X}, \mathcal{T}(\mathcal{X})\big)$$

$$+\lambda_3 \frac{q_e(\mathcal{Y},\mathcal{T}(\mathcal{Y}))q_e(\mathcal{X},\mathcal{T}(\mathcal{Y}))}{1+q_e(\mathcal{X},\mathcal{T}(\mathcal{Y}))+q_e(\mathcal{Y},\mathcal{T}(\mathcal{X}))}.$$

Following $\mathcal{G} \in \mathfrak{G}$ as given in Example 3.4, $\mathcal{T}$ is a right $\mathcal{G}_w$-implicit mapping for $\mathcal{G} \in \mathfrak{G}$ and $\varphi \in \Phi_w$,

Since $\sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{Q})\mathcal{A}_i \succ 0$, for some $\mathcal{Q} \in \mathcal{P}(n)$, $\mathcal{Q} \preceq \mathcal{T}(\mathcal{Q})$, all the hypotheses of Theorem 4.2 are satisfied, and therefore there exists $\widehat{\mathcal{X}} \in \mathcal{P}(n)$ such that $\mathcal{T}(\widehat{\mathcal{X}}) = \widehat{\mathcal{X}}$. Hence, the matrix equation (12) has a solution in $\mathcal{P}(n)$. □

Example 7.4 Consider the NME (12) for $m = 3$, $\eta = 0.01$, $\lambda_1 = 0.25$, $\lambda_2 = 0.25$, $\lambda_3 = 0.25$ with $\mathcal{G}(\mathcal{X}) = \mathcal{X}^{1/3}$, i.e.,

$$\mathcal{X} = \mathcal{Q} + \mathcal{A}_1^*\mathcal{X}^{1/3}\mathcal{A}_1 + \mathcal{A}_2^*\mathcal{X}^{1/3}\mathcal{A}_2 + \mathcal{A}_3^*\mathcal{X}^{1/3}\mathcal{A}_3, \tag{15}$$

where

$$\mathcal{A}_1 = \begin{bmatrix} 0.0493 & 0.0664 & 0.1310 \\ 0.0533 & 0.0323 & 0.1340 \\ 0.1373 & 0.0542 & 0.0256 \end{bmatrix}, \qquad \mathcal{A}_2 = \begin{bmatrix} 0.0258 & 0.0333 & 0.0380 \\ 0.0541 & 0.0361 & 0.0423 \\ 0.0564 & 0.0550 & 0.0366 \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} 0.5500 & 0.8600 & 0.2700 \\ 0.4600 & 0.2400 & 0.5200 \\ 0.9600 & 0.3600 & 0.5600 \end{bmatrix}, \qquad \mathcal{Q} = \begin{bmatrix} 11.2301 & 1.1999 & 1.9777 \\ 1.1999 & 10.0864 & 1.6390 \\ 1.9777 & 1.6390 & 11.7297 \end{bmatrix}.$$

The conditions of Theorem 15 can be checked numerically by considering different particular values of the matrices involved. For instance, it can be tested (and verified to be true) for

$$\mathcal{X} = \begin{bmatrix} 2.2300 & 1.1996 & 1.9776 \\ 1.1996 & 1.0792 & 1.6384 \\ 1.9776 & 1.6384 & 2.7296 \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} 9.0001 & 0.0003 & 0.0001 \\ 0.0003 & 9.0072 & 0.0006 \\ 0.0001 & 0.0006 & 9.0001 \end{bmatrix}.$$

Then, $w(\mathcal{X},\mathcal{Y}) = 36.0461 > 1$ and $h = 0.5035 < 1$. To see the behavior of convergence of the sequence $\{\mathcal{X}_n\}$ defined in (13), we take three different initial values:

$$\mathcal{U}_0 = \begin{bmatrix} 0.0232 & 0.0076 & 0.0025 \\ 0.0076 & 0.0390 & 0.0064 \\ 0.0025 & 0.0064 & 0.0223 \end{bmatrix} \quad \text{with } \|\mathcal{U}_0\| = 0.08459,$$

$$\mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with } \|\mathcal{V}_0\| = 1,$$

$$\mathcal{W}_0 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{with } \|\mathcal{W}_0\| = 4.$$

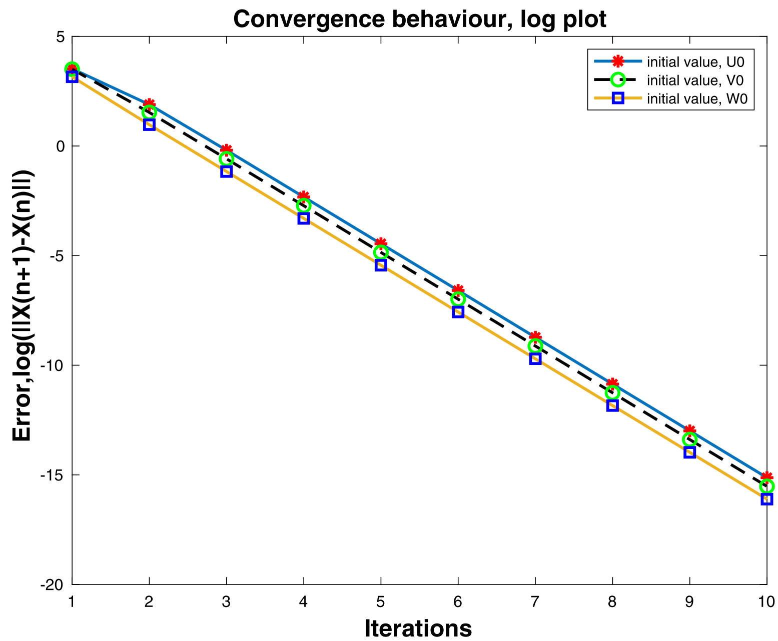

Figure 1 Convergence behavior

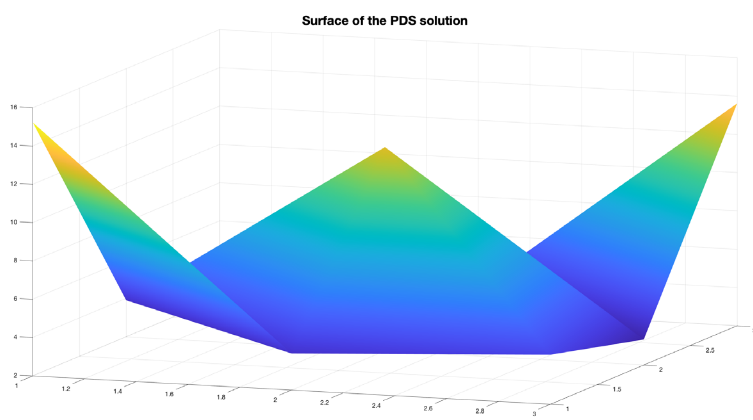

Figure 2 Solution surface plot

After 10 successive iterations, the approximations of the unique PDS of the system (12) are the following:

$$\widehat{\mathcal{U}} \approx \mathcal{U}_{10} = \begin{bmatrix} 15.2273 & 3.9055 & 4.5814 \\ 3.9055 & 12.6350 & 3.3290 \\ 4.5814 & 3.3290 & 13.6115 \end{bmatrix}$$

with Error = 1.3824×10^{-7};

$$\widehat{\mathcal{V}} \approx \mathcal{V}_{10} = \begin{bmatrix} 15.2273 & 3.9055 & 4.5814 \\ 3.9055 & 12.6350 & 3.3290 \\ 4.5814 & 3.3290 & 13.6115 \end{bmatrix}$$

with Error = 9.6297×10^{-8};

$$\widehat{\mathcal{W}} \approx \mathcal{W}_{10} = \begin{bmatrix} 15.2273 & 3.9055 & 4.5814 \\ 3.9055 & 12.6350 & 3.3290 \\ 4.5814 & 3.3290 & 13.6115 \end{bmatrix}$$

with Error = 6.2457×10^{-8}. It can also be verified that the elements of each sequence are order preserving. The convergence behavior and solution graph are shown in Figs. 1 and 2, respectively.

Remark 7.5 It can be noted that the conditions that we have used to guarantee the existence of solutions of the matrix equations, are 'weaker' in the sense of quasinorm than that of the conditions obtained previously in the literature.

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript. All authors contributed equally to the writing of this paper.

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