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Some aspects of nonsmooth variational inequalities on Hadamard manifolds

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Abstract

This is the first paper dealing with the study of minimum and maximum principle sufficiency properties for nonsmooth variational inequalities by using gap functions in the setting of Hadamard manifolds. We also provide some characterizations of these two sufficiency properties. We conclude the paper with a discussion of the error bounds for nonsmooth variational inequalities in the setting of Hadamard manifolds.

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1 Introduction

The variational inequality problem, introduced by Hartman and Stampacchia [20], plays an important role in current mathematical technology. It has been extended and generalized to study a wide class of problems arising in optimization, nonlinear programming, physics, economics, transportation equilibrium problems, and engineering sciences. For further details, see [1–4, 7, 8, 21–23, 25, 32, 36] and the references therein.

The weak sharp minimum property for the convex optimization problem was introduced and studied by Ferris [13, 17]. It has significant applications in sensitivity analysis, error bounds, and finite convergence of algorithms for solving convex optimization problems, see [12, 17, 18, 27]. In 1992, Ferris and Mangasarian [19] studied the minimum principle sufficiency property for convex programming and proved that this property is equivalent to the weak sharp minimum property of convex programming. Marcotte and Zhu [24] extended the minimum principle sufficiency property for variational inequalities and characterized the weak sharp solutions for variational inequalities using pseudomonotone[+] and continuous mapping on a compact polyhedral set. Extending the results of Marcotte and Zhu [24], Wu and Wu [35] established the maximum principle sufficiency property for variational inequalities and also characterized the weak sharp solutions of variational inequalities. Recently, Wu [34] characterized the minimum principle sufficiency property when the mapping in a variational inequality is constant and pseudomonotone[+].

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The variational inequality problem can be used to solve any differentiable optimization problem over a convex feasible region. However, in many practical problems, the function involved in the optimization problem need not be differentiable but has some kind of directional derivative. In this case, the relevant optimization problem can be considered by using a nonsmooth variational inequality with a bifunction, see [10]. In 2016, Alshahrani et al. [6] defined the minimum and maximum principle sufficiency properties for nonsmooth variational inequalities (NVI) by using a gap function that is similar to that of Wu and Wu [35] and provided some characterizations for these two properties. They also discussed error bounds for nonsmooth variational inequalities. For further related results, see [5, 11, 33] and the reference therein.

On the other hand, in the last two decades, several classical problems have been extended from a linear space setting to Riemannian/Hadamard manifolds setting (nonlinear space) because some nonconvex and constrained optimization problems in Euclidean space become convex and unconstrained ones in Riemannian/Hadamard manifolds, see [31]. In 2003, Németh [25] extended the notion of variational inequalities to Hadamard manifolds and related it to geodesic convex optimization problems. He also proved the existence and uniqueness results for variational inequalities on Hadamard manifolds. Then, Colao et al. [15] developed the equilibrium theory in Hadamard manifolds and also proved some existence results. In 2017, we [9] derived the generalized convexity of a real-valued function defined on a Riemannian manifold in terms of a bifunction h. We also defined the generalized monotonicity of the bifunction h. We also proved the relationship between the generalized convexity of a real-valued function and the generalized monotonicity of h. In 2020, we [8] extended the notion of the nonsmooth variational inequality problem (NVIP) and the Minty nonsmooth variational inequality problem (MNVIP) in the setting of Hadamard manifolds and proved some existence results for the nonsmooth variational inequality problem. We gave some relations among (NVIP), (MNVIP), and optimization problems in the setting of Hadamard manifolds. The gap functions for (NVIP) and (MNVIP) were also studied in the setting of Hadamard manifolds.

Motivated by the above results, in the present paper, we extend the work of Alshahrani et al. [6] in the setting of Hadamard manifolds. Hence, we defined the minimum and maximum principle sufficiency properties for nonsmooth variational inequalities by using a gap function in the Hadamard manifolds setting and provide several characterizations for these two properties. We also study the error bounds for nonsmooth variational inequalities in the setting of Hadamard manifolds.

2 Preliminaries

In the present section, we recall some basic definitions, properties, notions and results from manifolds, which will be needed throughout the paper. For details, see [14, 16, 26, 28–31].

Let M be a finite-dimensional differentiable manifold. For each $x \in M$, let $T_x M$ be the *tangent space* at the point x to M, which is a real vector space of the same dimension as M. The collection of all tangent spaces on M is called a *tangent bundle* and it is denoted by TM. A C^∞ mapping $A : M \rightarrow TM$, which assigns a tangent vector $A(x)$ at x for each $x \in M$, is called a *vector field* on M. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on $T_x M$ with the associated norm $\| \cdot \|_x$, where the subscript x can be omitted if there is no confusion.

A scalar product $\langle \cdot, \cdot \rangle_x$ is called the *Riemannian metric* on $T_x M$. A C^∞ tensor field g of type $(0, 2)$ on M is called the *Riemannian metric* on M if for each $x \in M$, the tensor $g(x)$ is a Riemannian metric on $T_x M$, and the pair (M, g) is called the *Riemannian manifold*.

For any $x, y \in M$, let $\gamma : [0, 1] \to M$ be a piecewise-smooth curve joining x to y. The *arc length* of γ is defined by

$$l(\gamma) := \int_0^1 \| \dot{\gamma}(t) \| \, dt,$$

where $\dot{\gamma}(t)$ denotes the tangent vector at $\gamma(t)$. For any $x, y \in M$, the *Riemannian distance* from x to y is defined by

$$d(x, y) = \inf_\gamma l(\gamma),$$

where the infimum is taken over all piecewise-smooth curves $\gamma : [0, 1] \to M$ joining x to y.

Let ∇ be the Levi–Civita connection on M. A curve $\gamma : [0, 1] \to M$ joining x to y is said to be a *geodesic* if

$$\gamma(0) = x, \qquad \gamma(1) = y \quad \text{and} \quad \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \quad \text{on } [0, 1].$$

A geodesic $\gamma : [0, 1] \to M$ joining x to y is said to be *minimal* if its arc length equals its Riemannian distance between x and y.

A Riemannian manifold M is said to be *complete* if for any $x \in M$, all the geodesics emanating from x are defined for all $t \in \mathbb{R}$.

A simply connected complete Riemannian manifold M of nonpositive sectional curvature is called a *Hadamard manifold* [16].

Let M be a Hadamard manifold. The *exponential mapping* [31] $\exp_x : T_x M \to M$ at x is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_x M$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with the velocity v, that is, $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Moreover, $\exp_x tv = \gamma_v(t, x)$ for each real number t.

Proposition 1 ([16]) *Let M be a Hadamard manifold and $x \in M$. Then, $\exp_x : T_x M \to M$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique minimal geodesic joining x to y*

$$\gamma(t) = \exp_x \big(t \exp_x^{-1} y \big),$$

for all $t \in [0, 1]$.

In particular, the exponential mapping and its inverse are continuous on a Hadamard manifold.

Let M be a Hadamard manifold. We denote by $P_{\gamma(b), \gamma(a)}$ the *parallel transport* from $T_{\gamma(a)} M$ to $T_{\gamma(b)} M$ along the geodesic γ with respect to ∇ and it is defined by

$$P_{\gamma(b), \gamma(a)}(v) = A\big(\gamma(b) \big), \quad \text{for all } a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)} M,$$

where A is the unique vector field such that $\nabla_{\dot{\gamma}(t)} v = 0$ for all t and $A(\gamma(a)) = v$.

Note that

(i) $P_{\gamma(a),\gamma(a)} = \mathrm{Id}$, the identity transformation of $T_{\gamma(a)}M$.

(ii) $P_{\gamma(a),\gamma(b)} \circ P_{\gamma(b),\gamma(c)} = P_{\gamma(a),\gamma(c)}$.

(iii) $P_{\gamma(a),\gamma(b)} \circ P_{\gamma(b),\gamma(a)} = P_{\gamma(a),\gamma(a)}$.

Definition 1 A subset K of a Hadamard manifold M is said to be *geodesic convex* if for any pair of distinct points $x, y \in K$, the geodesic γ joining x to y belongs to K, that is, if for any $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$, then $\gamma(t) = \exp_x(t \exp_x^{-1} y) \in K$ for all $t \in [0,1]$.

3 Formulation of the problems

Throughout the paper, unless otherwise specified, we assume that M is a Hadamard manifold, K is a nonempty geodesic convex subset of M and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ is a bifunction.

Recently, we [8] considered the following nonsmooth variational inequality problems:

Nonsmooth variational inequality problem (in short, NVIP): Find $\bar{x} \in K$ such that

$$h(\bar{x}; \exp_{\bar{x}}^{-1} y) \geq 0, \quad \text{for all } y \in K. \tag{1}$$

Minty-type nonsmooth variational inequality problem (in short, MTNVIP): Find $\bar{x} \in K$ such that

$$h(y; \exp_y^{-1} \bar{x}) \leq 0, \quad \text{for all } y \in K. \tag{2}$$

We denote by S^* and S_* the solution set of NVIP (1) and MTNVIP (2), respectively, and assume that they are nonempty.

Consider the following optimization problem (in short, OP):

$$\min f(x), \quad \text{subject to } x \in K, \tag{3}$$

where $f : K \to \mathbb{R}$ is a real-valued function. Assume that the solution set $\bar{S} = \{x \in K : f(x) \leq f(y) \text{ for all } y \in K\}$ of OP (3) is nonempty.

Definition 2 ([8]) A bifunction $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *geodesic upper sign continuous* if for any $x, y \in K$,

$$h(w; P_{w,y} \exp_y^{-1} x) \leq 0 \quad \Rightarrow \quad h(x; \exp_x^{-1} y) \geq 0,$$

where $w = \exp_y(t \exp_y^{-1} x)$ for all $t \in (0,1)$.

Definition 3 ([8]) A bifunction $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *pseudomonotone* if for all $x, y \in K$,

$$h(x; \exp_x^{-1} y) \geq 0 \quad \Rightarrow \quad h(y; \exp_y^{-1} x) \leq 0.$$

Recently, we [8] established the following equivalence result between the solution set of NVIP (1) and MTNVIP (2) under the pseudomonotonicity and geodesic upper sign continuity assumptions of the bifunction h.

Lemma 1 ([8]) *Let $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be a pseudomonotone and geodesic upper sign continuous bifunction such that h is positively homogeneous in the second argument, that is, for all $\alpha > 0$ and $v \in T_x M$, $h(x; \alpha v) = \alpha h(x; v)$. Then, $\bar{x} \in K$ is a solution of NVIP (1) if and only if it is a solution of MTNVIP (2).*

We consider the following condition, which was first considered by Wu and Wu [35], to develop the weak sharpness of variational inequalities in Hilbert spaces and later it was considered by Alshahrani et al. [6].

Condition A For all $x, y \in K$,

$$\mathbb{U}(x) = \{v \in T_x M : h(x; v) \geq 0\} = \{v \in T_x M : h(y; -P_{y,x} v) \leq 0\} = \mathbb{V}(y).$$

Note that $\mathbb{U}(x)$ is nonempty for all $x \in S^*$ and also $\mathbb{V}(y)$ is nonempty for all $y \in K$, since S_* is assumed to be nonempty. If h is pseudomonotone, then any $\exp_x^{-1} y$ from $\mathbb{U}(x)$ also belongs to $\mathbb{V}(y)$ for all $x, y \in K$.

The following result shows the equivalence between S^* and S_* without pseudomonotonicity and geodesic upper sign continuity of h, but under the assumption of Condition A.

Proposition 2 *Let $\bar{x} \in K$ and assume that $\mathbb{U}(\bar{x}) = \mathbb{V}(y)$ for all $y \in K$. Then,*

$$\bar{x} \in S^* \quad \Leftrightarrow \quad \bar{x} \in S_*.$$

Proof Let $\bar{x} \in S^*$. Then, $h(\bar{x}; \exp_{\bar{x}}^{-1} y) \geq 0$ for all $y \in K$, and therefore, $\exp_{\bar{x}}^{-1} y \in \mathbb{U}(\bar{x})$ for all $y \in K$. By hypothesis, $\mathbb{U}(\bar{x}) = \mathbb{V}(y)$ for all $y \in K$ and thus, $\exp_{\bar{x}}^{-1} y \in \mathbb{V}(y)$ for all $y \in K$. Hence,

$$h\big(y; -P_{y,\bar{x}} \exp_{\bar{x}}^{-1} y\big) \leq 0, \quad \text{for all } y \in K.$$

Since $P_{y,\bar{x}} \exp_{\bar{x}}^{-1} y = - \exp_y^{-1} \bar{x}$, we conclude that $h(y; \exp_y^{-1} \bar{x}) \leq 0$ for all $y \in K$. Hence, $\bar{x} \in S_*$.

Conversely, assume that $\bar{x} \in S_*$. Then, $h(y; \exp_y^{-1} \bar{x}) \leq 0$ for all $y \in K$. Since $P_{y,\bar{x}} \exp_{\bar{x}}^{-1} y = - \exp_y^{-1} \bar{x}$, we have

$$h\big(y; -P_{y,\bar{x}} \exp_{\bar{x}}^{-1} y\big) \leq 0, \quad \text{for all } y \in K.$$

Therefore, $\exp_{\bar{x}}^{-1} y \in \mathbb{V}(y)$ for all $y \in K$. By hypothesis, $\mathbb{U}(\bar{x}) = \mathbb{V}(y)$ for all $y \in K$, therefore we have $\exp_{\bar{x}}^{-1} y \in \mathbb{U}(\bar{x})$, that is,

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} y\big) \geq 0, \quad \text{for all } y \in K,$$

and hence, $\bar{x} \in S^*$. □

Definition 4 ([8]) A function $g : K \to \mathbb{R}$ is said to be a *gap function* for NVIP (1) (respectively, MTNVIP (2)) if it satisfies the following properties:

(a) $g(x) \geq 0$ for all $x \in K$;

(b) $g(\bar{x}) = 0$ if and only if $\bar{x} \in K$ is a solution of NVIP (1) (respectively, MTNVIP (2)).

The *primal gap function* $\varphi(x)$ associated with NVIP (1) is defined by

$$\varphi(x) := \sup_{y \in K} \left\{ -h\big(x; \exp_x^{-1} y\big) \right\}, \quad \text{for all } x \in K,$$

and we set

$$\Gamma(x) := \left\{ y \in K : h\big(x; \exp_x^{-1} y\big) = -\varphi(x) \right\}.$$

In a similar way, the *dual gap function* $\Phi(x)$ associated with MTNVIP (2) is defined by

$$\Phi(x) := \sup_{y \in K} h\big(y; \exp_y^{-1} x\big), \quad \text{for all } x \in K,$$

and we set

$$\Lambda(x) := \left\{ y \in K : h\big(y; \exp_y^{-1} x\big) = \Phi(x) \right\}.$$

Note that both the functions φ and Φ are nonnegative on K and vanish on S^* and S_*, respectively. Therefore, they are also gap functions for NVIP (1) and MTNVIP (2), respectively.

4 Characterizations of solution sets

Definition 5 ([9]) A function $f : K \to \mathbb{R}$ is said to be *geodesic radially upper semicontinuous* (respectively, geodesic radially lower semicontinuous) on K if for every pair of distinct points $x, y \in K$, the function f is upper semicontinuous (respectively, lower semicontinuous) along the geodesic segment $\gamma_{xy}(t)$ for all $t \in [0, 1]$, that is, $t \mapsto f(\gamma_{xy}(t))$ is upper semicontinuous (respectively, lower semicontinuous) on $[0, 1]$. The function f is said to be *geodesic radially continuous* on K if it is both geodesic radially upper semicontinuous as well as geodesic radially lower semicontinuous on K.

Definition 6 ([9]) Let $f : M \to \mathbb{R} \cup \{\pm\infty\}$ be a function on a Hadamard manifold M and x be a point where f is finite.

(a) The *Dini-upper directional derivative* at $x \in M$ in the direction $v \in T_x M$ is defined by

$$f^D(x; v) = \limsup_{t \to 0^+} \frac{f(\exp_x tv) - f(x)}{t}.$$

(b) The *Dini-lower directional derivative* at $x \in M$ in the direction $v \in T_x M$ is defined by

$$f_D(x; v) = \liminf_{t \to 0^+} \frac{f(\exp_x tv) - f(x)}{t}.$$

Definition 7 ([9]) A function $f : K \to \mathbb{R}$ is said to be *geodesic h-pseudoconvex* if for any pair of distinct points $x, y \in K$, we have

$$f(y) < f(x) \quad \Rightarrow \quad h\big(x; \exp_x^{-1} y\big) < 0,$$

equivalently,

$$h\big(x; \exp_x^{-1} y\big) \geq 0 \quad \Rightarrow \quad f(x) \leq f(y).$$

The function f is called *geodesic h-pseudoconcave* if in Definition (7), the inequality $<$ is replaced by $>$.

Definition 8 ([9]) A function $f : K \to \mathbb{R}$ is said to be *geodesic h-pseudolinear* if f is both geodesic h-pseudoconvex as well as geodesic h-pseudoconcave.

Theorem 1 ([9]) *Let $f : K \to \mathbb{R}$ be a function and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous in the second argument such that*

$$f_D(x; v) \leq h(x; v) \leq f^D(x; v), \quad \text{for all } x \in K \text{ and } v \in T_x M. \tag{4}$$

Further, assume that one of the following conditions holds:
   (a) *f is geodesic radially continuous*;
   (b) *h is odd in the second argument.*
If f is geodesic h-pseudolinear, then for any $x, y \in K$, we have

$$h\big(x; \exp_x^{-1} y\big) = 0 \quad \text{if and only if} \quad f(x) = f(y).$$

Theorem 2 *Let $f : K \to \mathbb{R}$ be geodesic h-pseudolinear and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous in the second argument such that the condition (4) holds. Assume that one of the following conditions holds*:
   (a) *f is geodesic radially continuous*;
   (b) *h is odd in the second argument.*
Then, the solution set $\bar{S}$ of OP (3) is geodesic convex.

Proof Let $x, y \in \bar{S}$. Then, $f(x) = f(y)$. By Theorem 1, we have

$$h\big(x; \exp_x^{-1} y\big) = 0 \quad \text{and} \quad h\big(y; \exp_y^{-1} x\big) = 0.$$

For any $s \in [0, 1]$, let $w := \exp_y s \exp_y^{-1} x$. Then,

$$\begin{aligned} h\big(y; \exp_y^{-1} w\big) &= h\big(y; \exp_y^{-1}\big(\exp_y s \exp_y^{-1} x\big)\big) \\ &= s h\big(y; \exp_y^{-1} x\big) = 0. \end{aligned} \tag{5}$$

Again, by Theorem 1 and (5), we obtain

$$f(w) = f(y).$$

Therefore, $w \in \bar{S}$ and hence $\bar{S}$ is geodesic convex. $\qquad\square$

For $\bar{x} \in \bar{S}$, consider the sets

$$\bar{S}_1 = \big\{x \in K : h\big(x; \exp_x^{-1} \bar{x}\big) = 0\big\},$$

$$\bar{S}_2 = \left\{ x \in K : h\left(\bar{x}; \exp_{\bar{x}}^{-1} x\right) = 0 \right\}.$$

Theorem 3 *Let $f : K \to \mathbb{R}$ be geodesic h-pseudolinear and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous in the second argument such that the condition* (4) *holds. Assume that one of the following conditions holds*:

  (a) *f is geodesic radially continuous*;

  (b) *h is odd in the second argument.*

If $\bar{x} \in \bar{S}$, then $\bar{S} = \bar{S}_1 = \bar{S}_2$.

Proof Since $x \in \bar{S}$ if and only if $f(x) = f(\bar{x})$, by Theorem 1, we have $f(x) = f(\bar{x})$ if and only if $h(x; \exp_x^{-1} \bar{x}) = 0$. Therefore, $\bar{S} = \bar{S}_1$. Similarly, $\bar{S} = \bar{S}_2$. □

For any $\bar{x} \in \bar{S}$, consider the set

$$\bar{S}_3 = \left\{ x \in K : h\left(w; P_{w,x} \exp_x^{-1} \bar{x}\right) = 0 \right\},$$

where $w = \exp_{\bar{x}} t \exp_{\bar{x}}^{-1} x$ for all $t \in [0, 1]$.

Theorem 4 *Let $f : K \to \mathbb{R}$ be geodesic h-pseudolinear and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous as well as odd in the second argument such that condition* (4) *holds. If $\bar{x} \in \bar{S}$, then $\bar{S} = \bar{S}_3$.*

Proof Let $x \in \bar{S}$. By Theorem 2, $w = \exp_{\bar{x}} s \exp_{\bar{x}}^{-1} x \in \bar{S}$ for all $s \in (0, 1)$. By Theorem 3, we have

$$0 = h\left(w; \exp_w^{-1} \bar{x}\right) = sh\left(w; P_{w,x} \exp_x^{-1} \bar{x}\right) = h\left(w; P_{w,x} \exp_x^{-1} \bar{x}\right), \quad \text{since } s > 0. \tag{6}$$

As $x \in \bar{S}$, by Theorem 3 and the oddness of h in the second argument, we have

$$h\left(\bar{x}; P_{\bar{x},x} \exp_x^{-1} \bar{x}\right) = 0. \tag{7}$$

By combining (6) and (7), we obtain $\bar{S} \subseteq \bar{S}_3$.

Conversely, assume that $x \in \bar{S}_3$, and taking $t = 1$ in particular, we obtain

$$h\left(x; \exp_x^{-1} \bar{x}\right) = 0.$$

Therefore, by Theorem 3, we have $x \in \bar{S}$, and hence, $\bar{S} = \bar{S}_3$. □

For any $\bar{x} \in \bar{S}$, consider the sets

$$\bar{S}_4 = \left\{ x \in K : h\left(x; \exp_x^{-1} \bar{x}\right) \geq 0 \right\},$$

$$\bar{S}_5 = \left\{ x \in K : h\left(\bar{x}; \exp_{\bar{x}}^{-1} x\right) \leq 0 \right\}.$$

Theorem 5 *Let $f : K \to \mathbb{R}$ be geodesic h-pseudolinear and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous in the second argument such that condition* (4) *holds. Assume that one of the following conditions holds*:

(a) *f is geodesic radially continuous*;
(b) *h is odd in the second argument.*
If $\bar{x} \in \bar{S}$, then $\bar{S} = \bar{S}_4 = \bar{S}_5$.

Proof By Theorem 3, we have $\bar{S} \subseteq \bar{S}_4$. For the converse, assume that $x \in \bar{S}_4$, that is,

$$h\left(x; \exp_x^{-1} \bar{x}\right) \geq 0.$$

Since f is geodesic h-pseudolinear, we have

$$f(x) \leq f(\bar{x}).$$

However, $\bar{x}$ is a solution of OP (3), and we have $f(\bar{x}) \leq f(x)$. Therefore, $f(x) = f(\bar{x})$, which yields $x \in \bar{S}$, and hence, $\bar{S} = \bar{S}_4$.

In a similar way, we obtain $\bar{S} = \bar{S}_5$. □

For any $\bar{x} \in \bar{S}$, consider the set

$$\bar{S}_6 = \left\{x \in K : h\left(w; P_{w,x} \exp_x^{-1} \bar{x}\right) \geq 0\right\},$$

where $w = \exp_{\bar{x}} t \exp_{\bar{x}}^{-1} x$ for all $t \in [0, 1]$.

Theorem 6 *Let $f : K \to \mathbb{R}$ be geodesic h-pseudolinear and $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous as well as odd in the second argument such that the condition (4) holds. If $\bar{x} \in \bar{S}$, then $\bar{S} = \bar{S}_6$.*

Proof Since $\bar{S}_6 \subseteq \bar{S}_4 = \bar{S}$. Let $x \in \bar{S}$, then by Theorem 4, we have

$$h\left(w; P_{w,x} \exp_x^{-1} \bar{x}\right) = 0,$$

where $w = \exp_{\bar{x}} t \exp_{\bar{x}}^{-1} x$ for all $t \in [0, 1]$. Thus, $x \in \bar{S}_6$, and hence, $\bar{S} \subseteq \bar{S}_6$. Therefore, $\bar{S} = \bar{S}_6$. □

For any $\bar{x} \in \bar{S}$, consider the sets

$$\bar{S}_7 = \left\{x \in K : h\left(\bar{x}; \exp_{\bar{x}}^{-1} x\right) = h\left(x; \exp_x^{-1} \bar{x}\right)\right\},$$
$$\bar{S}_8 = \left\{x \in K : h\left(\bar{x}; \exp_{\bar{x}}^{-1} x\right) \leq h\left(x; \exp_x^{-1} \bar{x}\right)\right\}.$$

Theorem 7 *Let $f : K \to \mathbb{R}$ be geodesic h-pseudolinear. Let $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ be positively homogeneous in the second argument such that the condition (4) holds. Assume that one of the following conditions holds*:
(a) *f is geodesic radially continuous*;
(b) *h is odd in the second argument.*
If $\bar{x} \in \bar{S}$, then $\bar{S} = \bar{S}_7 = \bar{S}_8$.

Proof By Theorem 3, we have $\bar{S} \subseteq \bar{S}_7$. Now, we show that $\bar{S}_8 \subseteq \bar{S}$. Let $x \in \bar{S}_8$, then,

$$h\left(\bar{x}; \exp_{\bar{x}}^{-1} x\right) \leq h\left(x; \exp_x^{-1} \bar{x}\right). \tag{8}$$

Assume on the contrary that $x \notin \bar{S}$. Then, $f(x) > f(\bar{x})$. Since f is geodesic h-pseudoconcave, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} x\big) > 0.$$

By (8), we have

$$h\big(x; \exp_{x}^{-1} \bar{x}\big) > 0.$$

By the geodesic h-pseudoconvexity of f, we have $f(x) \leq f(\bar{x})$, which contradicts $f(x) > f(\bar{x})$. Therefore $x \in \bar{S}$, and hence, $\bar{S}_8 \subseteq \bar{S}$. Thus, $\bar{S} \subseteq \bar{S}_7 \subseteq \bar{S}_8 \subseteq \bar{S}$, and hence, $\bar{S} = \bar{S}_7 = \bar{S}_8$.   □

Remark 1 Theorems 2–7 extend the Theorems 7.1–7.6 in [10], respectively, from Euclidean space settings to Hadamard manifolds.

5 Relations among S^*, S_*, $\Gamma(x)$ and $\Lambda(x)$

In the present section, we study the relationships among the solution set of NVIP (1) and MTNVIP (2) and sets $\Gamma(x)$ and $\Lambda(x)$.

The following proposition follows directly from the definitions.

Proposition 3 *Let $\bar{x} \in K$. Then,*
   (a) $\bar{x} \in S^* \Leftrightarrow \varphi(\bar{x}) = 0 \Leftrightarrow \bar{x} \in \Gamma(\bar{x})$;
   (b) $\bar{x} \in S_* \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \bar{x} \in \Lambda(\bar{x})$.

Proposition 4 *Let $\bar{x}, \bar{y} \in K$. Then, the following statements are equivalent*:
   (a) $\bar{x} \in S^*$ *and* $\bar{y} \in S_*$;
   (b) $h(y; \exp_y^{-1} \bar{y}) \leq h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) \leq h(\bar{x}; \exp_{\bar{x}}^{-1} x)$, *for all* $x, y \in K$.

Proof (a) $\Rightarrow$ (b): Let $\bar{x} \in S^*$ and $\bar{y} \in S_*$. Then, for all $x, y \in K$, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} x\big) \geq 0 \quad \text{and} \quad h\big(y; \exp_y^{-1} \bar{y}\big) \leq 0.$$

In particular, taking $x = \bar{y}$, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \geq 0,$$

and for $y = \bar{x}$, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \leq 0.$$

Therefore,

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0.$$

Thus, from the definitions of S^* and S_*, we obtain the required result.

(b) $\Rightarrow$ (a): Letting $y = \bar{y}$ and $x = \bar{x}$ in the condition (b), we obtain

$$0 = h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{y}\big) \leq h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \leq h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{x}\big) = 0.$$

Therefore, (a) is obtained. $\square$

Proposition 5 *The following statements hold*:

   (a) $S_* \subseteq \Gamma(\bar{x})$, *for all* $\bar{x} \in S^*$;
   (b) $S^* \subseteq \Lambda(\bar{y})$, *for all* $\bar{y} \in S_*$.

In particular, if h is geodesic upper sign continuous and positively homogeneous in the second argument, then $S_* \subseteq \Gamma(\bar{x})$ for all $\bar{x} \in S_*$. If h is pseudomonotone on S^*, then $S^* \subseteq \Lambda(\bar{x})$ for all $\bar{x} \in S^*$.

Proof By using the proof of Proposition 4, we obtain $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$ for all $\bar{x} \in S^*$ and $\bar{y} \in S_*$. Thus, $\bar{y} \in \Gamma(\bar{x})$ and $\bar{x} \in \Lambda(\bar{y})$, and hence $S_* \subseteq \Gamma(\bar{x})$ and $S^* \subseteq \Lambda(\bar{y})$.

   For the conclusion part, by using the Lemma 1 and the parts (a) and (b), we obtain the required result. $\square$

Proposition 6

   (a) *For all $\bar{x} \in S^*$ and $\bar{y} \in \Gamma(\bar{x})$,*

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \quad \Leftrightarrow \quad h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0;$$

   (b) *For all $\bar{x} \in S_*$ and $\bar{y} \in \Lambda(\bar{x})$,*

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \quad \Leftrightarrow \quad h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0.$$

Proof It follows from

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0, \quad \text{for all } \bar{x} \in S^* \text{ and } \bar{y} \in \Gamma(\bar{x}),$$

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0, \quad \text{for all } \bar{x} \in S_* \text{ and } \bar{y} \in \Lambda(\bar{x}).$$ $\square$

Condition B For any $x, y, z \in K$, there exists $\alpha, \beta > 0$ such that

$$h\big(x; \exp_x^{-1} y\big) \leq \alpha h\big(x; \exp_x^{-1} z\big) + \beta h\big(x; P_{x,z} \exp_z^{-1} y\big).$$

Example 1 Consider $M = (\mathbb{R}_{++}^n, X^{-2})$ is a Hadamard manifold with null sectional curvature, where $X^{-2} = \mathrm{diag}(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2})$ is called the Dikin metric (see [28]). In particular, for $n = 1$, we have $M = (\mathbb{R}_{++}, X^{-2})$. In particular, we take $h(x; \cdot) = \langle Ax, \cdot \rangle$ for all $x \in M$, where $A : M \to TM$ is a vector field and let $Ax = 1$ for all $x \in M$. Then, $\langle Ax, \exp_x^{-1} y \rangle = \exp_x^{-1} y$. Now, for any two points $x, y \in M$, we have

$$\exp_x^{-1} y = x \ln\left(\frac{y}{x}\right).$$

For any $x, y, z \in M$, we can obtain

$$x \ln\left(\frac{y}{x}\right) = x \ln\left(\frac{z}{x}\right) + \frac{x}{z} z \ln\left(\frac{y}{z}\right),$$

that is,

$$\exp_x^{-1} y = \exp_x^{-1} z + \frac{x}{z} P_{x,z} \exp_z^{-1} y.$$

This implies that

$$h\big(x; \exp_x^{-1} y\big) = h\big(x; \exp_x^{-1} z\big) + \frac{x}{z} h\big(x; P_{x,z} \exp_z^{-1} y\big).$$

Therefore, the condition B holds good with $\alpha = 1$ and $\beta = \frac{x}{z}$.

Proposition 7 *Let $\bar{x}, \bar{y} \in K$ be such that $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, that is, they satisfy Condition* A. *Then, the following assertions hold*:

(a) *If h is subodd in the second argument, that is, for any $x \in K$ and $v \in T_x M$, $h(x; v) \geq -h(x; -v)$, then $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$ if and only if $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$.*

(b) *If Condition* B *holds and either $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$ or $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$, then $\bar{x} \in S^*$ if and only if $\bar{y} \in S^*$.*

Proof (a) Suppose that $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$. Since h is subodd in the second argument, we have

$$0 = h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \geq -h\big(\bar{x}; -\exp_{\bar{x}}^{-1} \bar{y}\big) = -h\big(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big).$$

Thus,

$$h\big(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big) \geq 0.$$

Then, both the vectors $\exp_{\bar{x}}^{-1} \bar{y}$ and $P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}$ belong to $\mathbb{U}(\bar{x})$. Since $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, we have $\exp_{\bar{x}}^{-1} \bar{y} \in \mathbb{V}(\bar{y})$ and $P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x} \in \mathbb{V}(\bar{y})$. Therefore,

$$h\big(\bar{y}; -P_{\bar{y}\bar{x}} \exp_{\bar{x}}^{-1} \bar{y}\big) \leq 0,$$

that is,

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0, \tag{9}$$

and

$$h\big(\bar{y}; -P_{\bar{y},\bar{x}} P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0,$$

that is,

$$h\big(\bar{y}; -\exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0.$$

Since h is subodd in the second argument, we obtain

$$0 \geq h\big(\bar{y}; -\exp_{\bar{y}}^{-1} \bar{x}\big) \geq -h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big).$$

Therefore,

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \geq 0. \tag{10}$$

From equations (9) and (10), we have

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0.$$

(b) If either $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$ or $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$, then it follows from part (a).

Let us suppose that $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$ and $\bar{x} \in S^*$. Then, $h(\bar{x}; \exp_{\bar{x}}^{-1} z) \geq 0$ for all $z \in K$. Therefore, by Condition B, we have $\alpha, \beta > 0$ such that

$$0 \leq h\big(\bar{x}; \exp_{\bar{x}}^{-1} z\big) \leq \alpha h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) + \beta h\big(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} z\big), \quad \text{for all } z \in K.$$

Since $\beta > 0$, we have $h(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} z) \geq 0$ for all $z \in K$. Therefore, $P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} z \in \mathbb{U}(\bar{x})$. Since $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, we have

$$h\big(\bar{y}; -P_{\bar{y},\bar{x}} P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} z\big) \leq 0, \quad \text{for all } z \in K,$$

that is,

$$h\big(\bar{y}; -\exp_{\bar{y}}^{-1} z\big) \leq 0, \quad \text{for all } z \in K.$$

Since h is subodd in the second argument, we have for all $z \in K$,

$$0 \geq h\big(\bar{y}; P_{\bar{y},z} \exp_z^{-1} \bar{y}\big) \geq -h\big(\bar{y}; -P_{\bar{y},z} \exp_z^{-1} \bar{y}\big) = -h\big(\bar{y}; \exp_{\bar{y}}^{-1} z\big),$$

that is,

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} z\big) \geq 0.$$

Therefore, $\bar{y} \in S^*$. In a similar manner, the converse part can be proved.　　□

Theorem 8 *Let $\bar{x}, \bar{y} \in K$ be such that $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ and assume that the Condition B holds. Then, the following assertions hold*:

(a) *$\bar{x} \in S^*$ and $\bar{y} \in \Gamma(\bar{x})$ if and only if $\bar{x} \in \Gamma(\bar{x})$ and $\bar{y} \in S^*$;*

(b) *$\bar{x} \in S^*$ and $\bar{y} \in \Lambda(\bar{x})$ if and only if $\bar{x} \in \Lambda(\bar{x})$ and $\bar{y} \in S^*$.*

Proof (a) Let $\bar{x} \in S^*$ and $\bar{y} \in \Gamma(\bar{x})$. Then, $\bar{x} \in \Gamma(\bar{x})$ and $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = -\varphi(\bar{x}) = 0$, and by Proposition 7(b), we obtain $\bar{y} \in S^*$. For the converse part, take $\bar{x} \in \Gamma(\bar{x})$ and $\bar{y} \in S^*$. Then, for any $\bar{x} \in S^*$, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \geq 0.$$

Therefore, $\exp_{\bar{x}}^{-1} \bar{y} \in \mathbb{U}(\bar{x})$. Since $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, we have

$$h\big(\bar{y}; -P_{\bar{y},\bar{x}} \exp_{\bar{x}}^{-1} \bar{y}\big) \leq 0,$$

and hence,

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \le 0.$$

However, $\bar{y} \in S^*$, we have

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \ge 0.$$

Thus,

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0,$$

and by Proposition 7(a), we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0.$$

Therefore, $\bar{y} \in \Gamma(\bar{x})$.

(b) Since $\bar{x} \in S^*$ and $\bar{y} \in \Lambda(\bar{x})$, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \ge 0 \quad \text{and} \quad h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = \Phi(\bar{x}) \ge 0.$$

Hence, $\exp_{\bar{x}}^{-1} \bar{y} \in \mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, and thus, $h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \le 0$. Therefore,

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0,$$

and hence $\Phi(\bar{x}) = 0$, that is, $\bar{x} \in \Lambda(\bar{x})$. Since $\bar{x} \in S^*$ and by Proposition 7(b), we have $\bar{y} \in S^*$.

For the converse part, since $\bar{x} \in \Lambda(\bar{x})$, we have $\Phi(\bar{x}) = 0$, and hence $\bar{x} \in S_*$. Thus, by Proposition 5(b), we obtain $\bar{y} \in \Lambda(\bar{x})$ and by part (a) of Proposition 5, we have $\bar{x} \in \Gamma(\bar{y})$. Therefore, by part (a), we obtain $\bar{x} \in S^*$.   □

Remark 2 The following statements also hold under the assumptions of Theorem 8:

(a) $\bar{y} \in S^*$ and $\bar{x} \in \Gamma(\bar{y})$ if and only if $\bar{y} \in \Gamma(\bar{y})$ and $\bar{x} \in S^*$;

(b) $\bar{y} \in S^*$ and $\bar{x} \in \Lambda(\bar{y})$ if and only if $\bar{y} \in \Lambda(\bar{y})$ and $\bar{x} \in S^*$.

Remark 3 If h is subodd in the second argument. Then, for any $\bar{x}, \bar{y} \in S^*$ such that $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, we have

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0 \quad \text{and} \quad h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0.$$

Proposition 8 *Let $k < 0$ and h be subodd in the second argument, and $\bar{x}, \bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x}) \cup S^* \cup \Gamma(\bar{y}) \cup \Lambda(\bar{y})$. Consider the following statements*

(a) $h\big(\bar{y}; P_{\bar{y},\bar{x}} v\big) = k h(\bar{x}; v)$, *for all $v \in T_{\bar{x}} M$;*

(b) $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$;

(c) $h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \le 0 \Leftrightarrow h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \ge 0$, *and $h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \ge 0 \Leftrightarrow h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \le 0$;*

(d) $h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0$ *and $h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0$;*

(e) $h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big)$.

Then, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). *Furthermore, if* (d), (e) $\Rightarrow$ (a), *then* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) ($\Leftrightarrow$ (e)).

Proof The following implications are trivial. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e). Now, to show (c) $\Rightarrow$ (d), let us suppose that (c) holds for some $\bar{x}, \bar{y} \in K$. Therefore, if

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \geq 0 \quad \text{and} \quad h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \geq 0, \tag{11}$$

then (d) follows. To complete the proof, we assert that if (c) is satisfied for $\bar{x}, \bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x}) \cup S^* \cup \Gamma(\bar{y}) \cup \Lambda(\bar{y})$, then we can consider (11) without loss of generality. Indeed, if $\bar{x} \in \Gamma(\bar{x}) \cup S^* \cup \Lambda(\bar{y})$, then $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) \geq 0$. On the other hand, if $\bar{x} \in \Lambda(\bar{x}) \cup \Gamma(\bar{y})$, then $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) \leq 0$, and hence by (c), we obtain $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) \geq 0$. Now, If $\bar{y} \in S^* \cup \Gamma(\bar{y}) \cup \Lambda(\bar{x})$, we obtain $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) \geq 0$, and if $\bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{y})$, we obtain $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) \leq 0$, by (c), which implies that $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) \geq 0$. $\qquad\square$

Definition 9 A bifunction $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *pseudomonotone*$_*$ if it is pseudomonotone and for some $k < 0$ and for all $x, y \in K$,

$$\left.\begin{matrix} h(x; \exp_x^{-1} y) = 0, \\ h(y; \exp_y^{-1} x) = 0 \end{matrix}\right\} \quad \Rightarrow \quad h(x; v) = kh(y; -P_{y,x} v), \quad \text{for all } v \in T_x M.$$

Remark 4 Since for $\bar{x} \in S_*$ and $\bar{y} \in \Lambda(\bar{x})$, we have $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$, assertions (d) and (e) in Proposition 8 are equivalent to $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$. Therefore, if we add the assumption pseudomonotone$_*$ of h in the Proposition 8, then (a) to (e) in Proposition 8 are equivalent to each other.

6 Minimum principle sufficiency property for nonsmooth variational inequalities

The NVIP (1) has the *minimum principle sufficiency property* if

$$\Gamma(\bar{x}) = S^*, \quad \text{for all } \bar{x} \in S^*.$$

Theorem 9 *Suppose that the Condition* B *holds. Then, the following statements hold*:

(a) *If* $\bar{x} \in S^*$ *and* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ *for all* $\bar{y} \in \Gamma(\bar{x})$*, then* $\Gamma(\bar{x}) \subseteq S^*$*;*

(b) *If* $\bar{x} \in S_*$ *with* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ *for all* $\bar{y} \in \Gamma(\bar{x})$*, then* $\bar{x} \in \Gamma(\bar{x}) = S^*$*;*

(c) *If* $\bar{x} \in S^* \cup S_*$ *and* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ *for all* $\bar{y} \in S^*$*, then* $\bar{x} \in S^* \subseteq \Gamma(\bar{x})$*.*

Proof (a) Follows directly by using the part (a) of Theorem 8.

(b) For $\bar{x} \in S_*$, let $\bar{y} \in \Gamma(\bar{x})$ with $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$. Then,

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = -\varphi(\bar{x}) \leq 0. \tag{12}$$

Since h is subodd in the second argument, we have

$$0 \geq h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \geq -h\big(\bar{x}; -\exp_{\bar{x}}^{-1} \bar{y}\big) = -h\big(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big),$$

and hence

$$h\big(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big) \geq 0.$$

Therefore, $P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x} \in \mathbb{U}(\bar{x})$. Since $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, we have

$$h\big(\bar{y}; -P_{\bar{y},\bar{x}} P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0,$$

and hence

$$h\big(\bar{y}; - \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0.$$

By the suboddness of h in the second argument, we obtain

$$0 \geq h\big(\bar{y}; - \exp_{\bar{y}}^{-1} \bar{x}\big) \geq -h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big).$$

Therefore,

$$h\big(\bar{y}; \exp_{y}^{-1} \bar{x}\big) \geq 0.$$

Since $\bar{x} \in S_*$, we have

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0$$

and hence

$$h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0.$$

Therefore by Proposition 6(a), we obtain $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$. Now, on using (12), we have $\varphi(\bar{x}) = 0$, which means that $\bar{x} \in S^*$. Hence, by Proposition 7(b), $\bar{y} \in S^*$.

On the other hand, take $\bar{y} \in S^*$, then

$$0 \leq h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0.$$

By Proposition 7(b), we have $\bar{x} \in S^*$, which implies that $\bar{x} \in \Gamma(\bar{x})$. By part (a) of the Proposition 7, we obtain $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$. Therefore, $\bar{y} \in \Gamma(\bar{x})$ and hence $S^* \subseteq \Gamma(\bar{x})$.

(c) Let $\bar{y} \in S^*$. If $\bar{x} \in S^*$, then from part (a) of the Theorem 8, we obtain $\bar{x} \in \Gamma(\bar{x})$, and hence, $S^* \subseteq \Gamma(\bar{x})$. If $\bar{x} \in S_*$, then,

$$0 \leq h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0,$$

that is, $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$. Now, from Proposition 7(b), we obtain $\bar{x} \in S^*$ and by Proposition 7(a), we obtain $h(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}) = 0$, which implies that $\bar{y} \in \Gamma(\bar{x})$, and hence $S^* \subseteq \Gamma(\bar{x})$. $\square$

Remark 5 By Proposition 5 and Theorem 9, the NVIP (1) has the minimum sufficiency property if $S^* \subseteq S_*$ and Condition A holds for all $\bar{x} \in S^*$ and for all $\bar{y} \in \Gamma(\bar{x})$ and Condition B holds.

7 Maximum principle sufficiency property for nonsmooth variational inequalities

The NVIP ([1](#)) has the *maximum principle sufficiency property* if

$$\Lambda(\bar{x}) = S^*, \quad \text{for all } \bar{x} \in S^*.$$

Theorem 10 *Suppose that the Conditions* A *and* B *hold. Then, the following statements hold*:

(a) *If* $\bar{x} \in S^* \cup S_*$ *and* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ *for all* $\bar{y} \in \Lambda(\bar{x})$, *then* $\bar{x} \in \Lambda(\bar{x}) = S^*$;

(b) *If for each* $\bar{x} \in S^*$, *there exists* $\bar{y} \in \Lambda(\bar{x})$ *such that* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, *then* $S^* \subseteq S_*$;

(c) *If for each* $\bar{x} \in S_*$, *there exists* $\bar{y} \in S^*$ *such that* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, *then* $S_* \subseteq S^*$;

(d) *If* $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ *for all* $\bar{x} \in S^* \cup S_*$ *and all* $\bar{y} \in \Lambda(\bar{x})$, $S^* = \Lambda(\bar{x}) = S_*$ *for all* $\bar{x} \in S^* \cup S_*$.

Proof (a) Suppose that $\bar{x} \in S^* \cup S_*$ and $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ for all $\bar{y} \in \Lambda(\bar{x})$. If $\bar{x} \in S^*$, then by part (b) of Theorem [8](#), we obtain $\bar{x} \in \Lambda(\bar{x})$ and hence $\Lambda(\bar{x}) \subseteq S^*$. On the other hand, if $\bar{x} \in S_*$, then Proposition [3](#)(b), $\bar{x} \in \Lambda(\bar{x})$ and by Proposition [5](#)(b), we obtain $S^* \subseteq \Lambda(\bar{x})$, which implies that

$$h\big(z; \exp_z^{-1} \bar{x}\big) = 0, \quad \text{for all } z \in \Lambda(\bar{x}).$$

Therefore, the above equality holds in particular for all $\bar{z} \in S^*$; thus, by Proposition [7](#)(b), $\bar{x} \in S^*$, and again by Proposition [7](#)(b), $\bar{y} \in S^*$. Therefore, $\Lambda(\bar{x}) = S^*$.

(b) It directly follows from Theorem [8](#).

(c) By Proposition [5](#)(b), we have $S^* \subseteq \Lambda(\bar{x})$, for all $\bar{x} \in S_*$. Therefore, if $\bar{x} \in S_*$, then there exists $\bar{y} \in S^* \subseteq \Lambda(\bar{x})$ such that $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$, which implies that $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$ and by Proposition [7](#)(b), we obtain $\bar{x} \in S^*$ since $\bar{y} \in S^*$.

(d) Assume that Condition [A](#) holds for all $\bar{x} \in S^* \cup S_*$ and all $\bar{y} \in \Lambda(\bar{x})$. Then from (a), we obtain

$$\Lambda(\bar{x}) = S^*, \quad \text{for all } \bar{x} \in S^* \cup S_*.$$

Therefore, from (a) and (b), we have $S^* = S_*$. $\qquad\square$

Theorem 11 *Assume that*

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) = 0 \quad and \quad h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) = 0 \quad \Rightarrow \quad h\big(\bar{y}; P_{\bar{y},\bar{x}} v\big) = k h(\bar{x}; v),$$

for some $k < 0$ *and for all* $v \in T_{\bar{x}} M$, $\bar{x} \in S^* \cup S_*$ *and all* $\bar{y} \in \Lambda(\bar{x})$. *Then, the following statements are equivalent*:

(a) $\mathbb{U}(\bar{x}) = \mathbb{V}(\bar{y})$ *for all* $\bar{x} \in S^* \cup S_*$ *and all* $\bar{y} \in \Lambda(\bar{x})$;

(b) $S^* = \Lambda(\bar{x}) = S_*$ *for all* $\bar{x} \in S^* \cup S_*$;

(c) *For all* $\bar{x} \in S^* \cup S_*$ *and all* $\bar{y} \in \Lambda(\bar{x})$,

$$h\big(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\big) \geq 0 \quad \Leftrightarrow \quad h\big(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\big) \leq 0,$$

and

$$h\big(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\big) \geq 0 \quad \Leftrightarrow \quad h\big(\bar{y}; P_{\bar{y},\bar{x}} \exp_{\bar{x}}^{-1} \bar{y}\big) \leq 0;$$

(d) *For all $\bar{x} \in S^* \cup S_*$ and all $\bar{y} \in \Lambda(\bar{x})$,*

$$h\left(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\right) = 0 \quad and \quad h\left(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\right) = 0;$$

(e) $h(\bar{y}; P_{\bar{y},\bar{x}} v) = kh(\bar{x}; v)$ *for all $v \in T_{\bar{x}}M$, for some $k < 0$, all $\bar{x} \in S^* \cup S_*$ and all $\bar{y} \in \Lambda(\bar{x})$.*

Proof By Proposition 8, we obtain the equivalence among (a), (c), (d) and (e). Now, from Theorem 10(d), we have (a) $\Rightarrow$ (b) and lastly from Proposition 4, we obtain (b) $\Rightarrow$ (d). □

Remark 6 In the case $S^* \subseteq S_*$, the expression $S^* \cup S_*$ in Theorem 11 can be reduced to S_* and the equality $h(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}) = 0$ in (d) can also be omitted.

Under geodesic upper sign continuity and pseudomonotonicity$_*$ of h, all the statements of Theorem 11 hold, as shown below.

Theorem 12 *Let h be a geodesic upper sign continuous and pseudomonotone$_*$. Then, for some $k > 0$, (a)–(e) in the Theorem 11 hold.*

Proof If h is geodesic upper sign continuous and pseudomonotone, then $S^* = S_*$. Since (d) can be easily verified for $\bar{x} \in S_*$ and $\bar{y} \in \Lambda(\bar{x})$, then by Theorem 11, (a)–(e) hold.   □

Theorem 13 *For some $k > 0$, the following statements satisfy:*

$$(d) \Rightarrow (a) \Rightarrow (b) \quad \left(respectively, (d)\right) \Rightarrow (e):$$

(a) *The Condition A holds for all $\bar{x} \in S^* \cup S_*$ and all $\bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x})$;*
(b) $S^* = \Gamma(\bar{x}) = \Lambda(\bar{x}) = S_*$ *for all $\bar{x} \in S^* \cup S_*$;*
(c) *For all $\bar{x} \in S^* \cup S_*$ and all $\bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x})$,*

$$h\left(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\right) \geq 0 \quad \Leftrightarrow \quad h\left(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\right) \geq 0,$$

and

$$h\left(\bar{x}; \exp_{\bar{x}}^{-1} \bar{y}\right) \geq 0 \quad \Leftrightarrow \quad h\left(\bar{y}; P_{\bar{y},\bar{x}} \exp_{\bar{x}}^{-1} \bar{y}\right) \geq 0;$$

(d) *For all $\bar{x} \in S^* \cup S_*$ and all $\bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x})$,*

$$h\left(\bar{x}; P_{\bar{x},\bar{y}} \exp_{\bar{y}}^{-1} \bar{x}\right) = 0 \quad and \quad h\left(\bar{y}; \exp_{\bar{y}}^{-1} \bar{x}\right) = 0;$$

(e) $h(\bar{y}; P_{\bar{y},\bar{x}} v) = kh(\bar{x}; v)$ *for all $v \in T_{\bar{x}}M$, for some $k > 0$, all $\bar{x} \in S^* \cup S_*$ and all*
$\bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x})$.
Hence, if (e) $\Rightarrow$ (d), *then* (a)–(e) *are equivalent.*

Proof By Proposition 8, we have (e) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d) and from Proposition 3(b) and Theorem 10(d), we obtain (a) $\Rightarrow$ (b). Now, we assert that (b) $\Rightarrow$ (d). Indeed, let (b) hold. Then, for each $\bar{x} \in S^* \cup S_*$, we have $\bar{x} \in S^* = S_*$ and $\Gamma(\bar{x}) \cup \Lambda(\bar{x}) = \Gamma(\bar{x}) \cap \Lambda(\bar{x})$. Thus, for each $\bar{y} \in \Gamma(\bar{x}) \cup \Lambda(\bar{x})$, (d) holds.   □

8 Error bounds for nonsmooth variational inequalities

Definition 10 A bifunction $h : K \times TM \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *strongly pseudomonotone* if for all $x, y \in K$, there exists $\alpha > 0$ such that

$$h(x; \exp_x^{-1} y) \geq 0 \quad \Rightarrow \quad h(y; \exp_y^{-1} x) \leq -\alpha d^2(x, y),$$

where $d(\cdot, \cdot)$ is the Riemannian distance.

Theorem 14 *Assume that h is odd in the second argument and strongly pseudomonotone with constant $\alpha > 0$. Then,*

$$\frac{1}{\sqrt{\alpha}} \sqrt{\varphi(x)} \geq d(x, S^*),$$

where $d(x, S^*) = \inf_{y \in S^*} d(x, y)$ is the Riemannian distance between the point x and the solution set S^*.

Proof Let $\bar{x} \in S^*$, then

$$h(\bar{x}; \exp_{\bar{x}}^{-1} y) \geq 0, \quad \text{for all } y \in K.$$

Since h is strongly pseudomonotone, there exists $\alpha > 0$ such that

$$h(y; \exp_y^{-1} \bar{x}) \leq -\alpha d^2(y, \bar{x}), \quad \text{for all } y \in K.$$

As h is odd in the second argument and $-\exp_y^{-1} x = P_{y,x} \exp_x^{-1} y$, we have

$$-h(y; P_{y,\bar{x}} \exp_{\bar{x}}^{-1} y) = h(y; \exp_y^{-1} \bar{x}) \leq -\alpha d^2(y, \bar{x}), \quad \text{for all } y \in K,$$

and hence,

$$h(y; P_{y,\bar{x}} \exp_{\bar{x}}^{-1} y) \geq \alpha d^2(y, \bar{x}), \quad \text{for all } y \in K.$$

Since

$$\varphi(x) \geq -h(x; \exp_x^{-1} \bar{x}) = h(x; P_{x,\bar{x}} \exp_{\bar{x}}^{-1} x) \geq \alpha d^2(x, \bar{x}), \quad \text{for all } x \in K,$$

we have

$$\sqrt{\varphi(x)} \geq \sqrt{\alpha} d(x, \bar{x}) \geq \sqrt{\alpha} d(x, S^*).$$

Therefore,

$$\frac{1}{\sqrt{\alpha}} \sqrt{\varphi(x)} \geq d(x, S^*),$$

the desired result. $\square$

The NVIP (1) has *weak sharp solutions* if

$$d(x, S^*) \leq \varphi(x), \quad \text{for all } x \in K.$$

Therefore, if h is odd in the second argument and strongly pseudomonotone with constant $\alpha > 0$, then by Theorem 14, the NVIP (1) has weak sharp solutions.

9 Conclusion

In this paper, we define the notions minimum and maximum principle sufficiency properties for nonsmooth variational inequalities by using gap functions in the setting of Hadamard manifolds. We also characterize these two sufficiency properties in the setting of Hadamard manifolds. We conclude our paper by introducing the idea of the error bounds for nonsmooth variational inequalities in the setting of Hadamard manifolds. The main objective of the paper is to include the existing results in nonlinear space, namely Hadamard manifolds. This is the first paper dealing with these two notions for nonsmooth variational inequalities. Therefore, in the future one can extend the concept of this paper in many other directions.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript. All authors contributed equally to writing this paper.

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