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New fractional identities, associated novel fractional inequalities with applications to means and error estimations for quadrature formulas

Muhammad Uzair Awan¹, Artion Kashuri², Kottakkaran Sooppy Nisar³, Muhammad Zakria Javed¹, Sabah Iftikhar⁴, Poom Kumam^{4,5*} and Parin Chaipunya^{4,6}

*Correspondence:

poom.kum@kmutt.ac.th

⁴Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & KMUTT Fixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Department of Mathematics, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand

⁵Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

Full list of author information is available at the end of the article

Abstract

In this paper, the authors derive some new generalizations of fractional trapezium-like inequalities using the class of harmonic convex functions. Moreover, three new fractional integral identities are given, and on using them as auxiliary results some interesting integral inequalities are found. Finally, in order to show the efficiency of our main results, some applications to special means for different positive real numbers and error estimations for quadrature formulas are obtained.

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1 Introduction and preliminaries

Computational and Fractional Analysis are nowadays more and more at the center of mathematics and of other related sciences, either by themselves because of their rapid development, which is based on very old foundations, or because they cover a great variety of applications in the real world. In recent years, fractional calculus (*FC*) is applied in many phenomena in applied sciences, fluid mechanics, physics and also biology can be described as very effective using the mathematical tools of *FC*. The fractional derivatives have occurred in many applied sciences equations such as reaction and diffusion processes, system identification, velocity signal analysis, relaxation of damping behavior fabrics and creeping of polymer composites [1, 2, 26, 32]. A set $S \in \mathbb{R}$ is said to be convex, if

$$(1 - \tau)b_1 + \tau b_2 \in S, \quad \forall b_1, b_2 \in S, \tau \in [0, 1].$$

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Similarly, a function $\Upsilon : S \rightarrow \mathbb{R}$ is said to be convex, if

$$\Upsilon((1-\tau)b_1 + \tau b_2) \leq (1-\tau)\Upsilon(b_1) + \tau\Upsilon(b_2), \quad b_1, b_2 \in S, \tau \in [0, 1].$$

Recently, İşcan [3] introduced the class of harmonic convex functions as:

A function $\Upsilon : I \subset (0, +\infty) \rightarrow \mathbb{R}$ is said to be harmonic convex, if

$$\Upsilon\left(\frac{b_1 b_2}{\tau b_1 + (1-\tau)b_2}\right) \leq (1-\tau)\Upsilon(b_1) + \tau\Upsilon(b_2), \quad \forall b_1, b_2 \in I, \tau \in [0, 1].$$

The harmonic property has played a significant role in different fields of pure and applied sciences. In [4] the authors discussed the important role of the harmonic mean in Asian options of stock. Interestingly, harmonic means have applications in electric circuit theory. To be more precise, the total resistance of a set of parallel resistors is just half of the total resistor's harmonic means. For example, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are the resistances of two parallel resistors, then the total resistance is computed by the formula:

$$\mathcal{R}_T = \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{1}{2} H(\mathcal{R}_1, \mathcal{R}_2),$$

which is half of the harmonic mean.

Noor [5] showed that the harmonic mean also played a crucial role in developing parallel algorithms for solving nonlinear problems. The author used the harmonic means and harmonic convex functions to suggest some iterative methods for solving linear and nonlinear equations.

The theory of convexity also has a wide range of applications in other areas of pure and applied sciences. It also has a great impact on the development of the theory of inequalities. Several inequalities are consequences of the applications of convex functions. Many generalizations, variants and extensions for the convexity have attracted the attention of many researchers. An interesting result pertaining to convex functions is the trapezium inequality (Hermite–Hadamard inequality) that provides an integral average of a continuous convex function on a compact interval. This result reads as:

Let $\Upsilon : I = [b_1, b_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, then

$$\Upsilon\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \Upsilon(x) dx \leq \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2}.$$

Over the years, a variety of new generalizations of this classical result have been obtained in the literature. For example, İşcan [3] obtained a new refinement of the trapezium inequality using the class of harmonic convex functions. He derived the following version of the trapezium inequality.

Let $\Upsilon : I = [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be an harmonic convex function, then

$$\Upsilon\left(\frac{2b_1 b_2}{b_1 + b_2}\right) \leq \frac{b_1 b_2}{b_2 - b_1} \int_{b_1}^{b_2} \frac{\Upsilon(x)}{x^2} dx \leq \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2}.$$

We now recall some useful definitions. For brevity, the set of integrable functions on the interval $[b_1, b_2]$ is denoted by $L_1[b_1, b_2]$.

Definition 1.1 Let $\Upsilon \in L_1[b_1, b_2]$. The left- and right-sided Riemann–Liouville fractional integrals $J_{b_1^+}^\alpha \Upsilon$ and $J_{b_2^-}^\alpha \Upsilon$ of order $\alpha > 0$ with $b_1 \geq 0$ are defined by

$$J_{b_1^+}^\alpha \Upsilon(x) = \frac{1}{\Gamma(\alpha)} \int_{b_1}^x (x - \tau)^{\alpha-1} \Upsilon(\tau) d\tau, \quad x > b_1$$

and

$$J_{b_2^-}^\alpha \Upsilon(x) = \frac{1}{\Gamma(\alpha)} \int_x^{b_2} (\tau - x)^{\alpha-1} \Upsilon(\tau) d\tau, \quad x < b_2,$$

respectively, and $\Gamma(\alpha)$ is Gamma function. Also, we define $J_{b_1^+}^0 \Upsilon(x) = J_{b_2^-}^0 \Upsilon(x) = \Upsilon(x)$.

Definition 1.2 Let $\Upsilon \in L_1[b_1, b_2]$. The k –Riemann–Liouville fractional integrals ${}_k J_{b_1^+}^\alpha \Upsilon$ and ${}_k J_{b_2^-}^\alpha \Upsilon$ of order $\alpha, k > 0$ with $b_1 \geq 0$ are given as follows:

$${}_k J_{b_1^+}^\alpha \Upsilon(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{b_1}^x (x - \tau)^{\frac{\alpha}{k}-1} \Upsilon(\tau) d\tau, \quad x > b_1$$

and

$${}_k J_{b_2^-}^\alpha \Upsilon(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{b_2} (\tau - x)^{\frac{\alpha}{k}-1} \Upsilon(\tau) d\tau, \quad x < b_2,$$

respectively.

Definition 1.3 A hypergeometric function ${}_2F_1(b_1, b_2, b_3, z)$ has the following integral representation

$${}_2F_1(b_1, b_2, b_3, z) = \frac{1}{\beta(b_2, b_3 - b_2)} \int_0^1 x^{b_2-1} (1-x)^{b_3-b_2-1} (1-zx)^{-b_1} dx, \quad b_3 > b_2 > 0,$$

where $\beta(\cdot)$ is the Beta function and $|z| < 1$.

Sarikaya *et al.* [6] opened up a new direction of research in the field of inequalities involving convex functions. They derived a fractional version of the trapezium inequality. This result reads as:

Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a positive function with $0 \leq b_1 < b_2$ and $\Upsilon \in L_1[b_1, b_2]$. If Υ is a convex function on $[b_1, b_2]$, then

$$\Upsilon\left(\frac{b_1 + b_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b_2 - b_1)^\alpha} [J_{b_1^+}^\alpha \Upsilon(b_2) + J_{b_2^-}^\alpha \Upsilon(b_1)] \leq \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2},$$

with $\alpha > 0$.

Using this idea, İşcan and Wu [7] obtained the fractional trapezium inequality using the class of harmonic convex functions. Their result is stated as follows:

Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a function with $\Upsilon \in L_1[b_1, b_2]$. If Υ is an harmonic convex function, then

$$\begin{aligned}\Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{b_1b_2}{b_2-b_1}\right)^\alpha \left[J_{\frac{1}{b_1}^-}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) + J_{\frac{1}{b_2}^+}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_1}\right)\right] \\ &\leq \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2},\end{aligned}$$

where $\alpha > 0$ and $\Psi(x) := \frac{1}{x}$.

For more details on the trapezium inequality, its generalizations and applications, see [8, 9, 17–25, 27–31].

The aim of this paper is to derive some new generalizations of fractional trapezium-like inequalities using the class of harmonic convex functions. In order to establish some of our main results, we derive three new fractional integral identities. These identities will be used as auxiliary results. Moreover, in order to show the efficiency of our main results, some applications to special means for positive different real numbers and error estimations for quadrature formulas will also be obtained. We also discuss special cases that show that our results represent significant generalizations and under suitable conditions one can obtain many other new and known results.

Before moving to the main results, let us recall some previously known concepts and results that will help us to obtain our main results. Let $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ be a function satisfying the following conditions:

1. $\int_0^1 \frac{\Phi(\tau)}{\tau} d\tau < +\infty$,
2. $\frac{1}{M_1} \leq \frac{\Phi(s)}{\Phi(r)} \leq M_1$ for $\frac{1}{2} \leq \frac{s}{r} \leq 2$,
3. $\frac{\Phi(r)}{r^2} \leq M_2 \frac{\Phi(s)}{s^2}$ for $s \leq r$,
4. $|\frac{\Phi(r)}{r^2} - \frac{\Phi(s)}{s^2}| \leq M_3|r-s|\frac{\Phi(r)}{r^2}$ for $\frac{1}{2} \leq \frac{s}{r} \leq 2$,

where M_1, M_2 and M_3 are independent of $r, s > 0$. Under the assumptions of Φ , the left- and right-sided generalized fractional integrals are

$${}_{{b_1}^+}I_\Phi \Upsilon(x) = \int_{{b_1}}^x \frac{\Phi(x-\tau)}{x-\tau} \Upsilon(\tau) d\tau, \quad x > b_1, \quad (1.1)$$

$${}_{{b_2}^-}I_\Phi \Upsilon(x) = \int_x^{{b_2}} \frac{\Phi(\tau-x)}{\tau-x} \Upsilon(\tau) d\tau, \quad x < b_2. \quad (1.2)$$

Actually, these fractional integrals are the generalization of some well-known fractional integrals like the Riemann–Liouville fractional integrals [10], the k –Riemann–Liouville fractional integrals [11], the Katugampola fractional integrals [12], conformable fractional integrals [13], etc.

1. If we take $\Phi(\tau) = \tau$ in operators (1.1) and (1.2), we have the classical Riemann integrals.
2. If we choose $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in operators (1.1) and (1.2), we obtain the Riemann–Liouville fractional integrals, see [10].
3. If we substitute $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in operators (1.1) and (1.2), we obtain the k –Riemann–Liouville fractional integrals, see [11].
4. If we take $\Phi(\tau) = \tau(x-\tau)^{\alpha-1}$ in operators (1.1) and (1.2), we have conformable fractional integrals that are defined by Khalil *et al.* [14].

5. If we choose $\Phi(\tau) = \frac{\tau}{\alpha} \exp(-\frac{1-\alpha}{\alpha} \tau)$ for $\alpha \in (0, 1]$, in operators (1.1) and (1.2), we get left-sided and right-sided fractional integrals with an exponential kernel that were defined in [15, 16].

2 Main results

In this section, before we discuss our main results, let us denote, respectively

$$\begin{aligned}\Delta(\tau) &:= \int_0^\tau \frac{\Phi(\frac{b_2-b_1}{b_1 b_2} \mu)}{\mu} d\mu \quad \text{and} \quad \delta(\tau) := \int_\tau^1 \frac{\Phi(\frac{b_2-b_1}{b_1 b_2} \mu)}{\mu} d\mu, \\ \eta(\tau) &:= \int_0^\tau \frac{\Phi(\frac{b_2-b_1}{b_1 b_2(m+1)} \mu)}{\mu} d\mu \quad \text{and} \quad \Omega(\tau) := \int_0^\tau \frac{\Phi(\frac{b_2-b_1}{b_1 b_2(\lambda+\mu)} \mu)}{\mu} d\mu.\end{aligned}$$

2.1 Generalized trapezium inequality

We now derive a new generalized fractional trapezium-type integral inequality using the class of harmonic convex functions. For brevity, we denote in the following $\Psi(\tau) := \frac{1}{\tau}$.

Theorem 2.1 *Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be an harmonic convex function, then*

$$\begin{aligned}\Upsilon\left(\frac{2b_1 b_2}{b_1 + b_2}\right) &\leq \frac{1}{2\eta(1)} \left[\left(\frac{1}{b_2}\right)^+ I_\Phi \Upsilon \circ \Psi\left(\frac{mb_1 + b_2}{(m+1)b_1 b_2}\right) + \left(\frac{1}{b_1}\right)^- I_\Phi \Upsilon \circ \Psi\left(\frac{b_1 + mb_2}{(m+1)b_1 b_2}\right) \right] \\ &\leq \frac{[\Upsilon(b_1) + \Upsilon(b_2)]}{2},\end{aligned}$$

where $m \in \mathbb{N}$.

Proof Since Υ is an harmonic convex function, then

$$\Upsilon\left(\frac{2xy}{x+y}\right) \leq \frac{1}{2} [\Upsilon(x) + \Upsilon(y)].$$

This implies

$$2\Upsilon\left(\frac{2b_1 b_2}{b_1 + b_2}\right) \leq \Upsilon\left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2}\right) + \Upsilon\left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2}\right).$$

Multiplying both sides by $\frac{\Phi(\frac{b_2-b_1}{(m+1)b_1 b_2} \tau)}{\tau}$ and integrating with respect to τ on $[0, 1]$, we have

$$\begin{aligned}2\Upsilon\left(\frac{2b_1 b_2}{b_1 + b_2}\right) &\int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1 b_2} \tau)}{\tau} d\tau \\ &\leq \left[\int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1 b_2} \tau)}{\tau} \Upsilon\left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2}\right) d\tau \right. \\ &\quad \left. + \int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1 b_2} \tau)}{\tau} \Upsilon\left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2}\right) d\tau \right].\end{aligned}$$

This implies

$$\begin{aligned}
 & 2\eta(1)\Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) \\
 & \leq \int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1b_2}\tau)}{\tau} \Upsilon\left(\frac{(m+1)b_1b_2}{(1-\tau)b_1+(m+\tau)b_2}\right) d\tau \\
 & \quad + \int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1b_2}\tau)}{\tau} \Upsilon\left(\frac{(m+1)b_1b_2}{(m+\tau)b_1+(1-\tau)b_2}\right) d\tau \\
 & = \int_{\frac{1}{b_2}}^{\frac{mb_1+b_2}{(m+1)b_1b_2}} \frac{\Phi(\frac{mb_1+b_2}{(m+1)b_1b_2}-x)}{(\frac{mb_1+b_2}{(m+1)b_1b_2}-x)} \Upsilon \circ \Psi(x) dx + \int_{\frac{1}{b_1}}^{\frac{b_1+mb_2}{(m+1)b_1b_2}} \frac{\Phi(x-\frac{b_1+mb_2}{(m+1)b_1b_2})}{(x-\frac{b_1+mb_2}{(m+1)b_1b_2})} \Upsilon \circ \Psi(x) dx \\
 & = (\frac{1}{b_2})^+ I_\Phi \Upsilon \circ \Psi\left(\frac{mb_1+b_2}{(m+1)b_1b_2}\right) + (\frac{1}{b_1})^- I_\Phi \Upsilon \circ \Psi\left(\frac{b_1+mb_2}{(m+1)b_1b_2}\right).
 \end{aligned}$$

Now, we prove the second inequality, for this we have

$$\Upsilon\left(\frac{(m+1)b_1b_2}{(1-\tau)b_1+(m+\tau)b_2}\right) \leq \frac{m+\tau}{m+1} \Upsilon(b_1) + \frac{1-\tau}{m+1} \Upsilon(b_2). \quad (2.1)$$

$$\Upsilon\left(\frac{(m+1)b_1b_2}{(m+\tau)b_1+(1-\tau)b_2}\right) \leq \frac{m+\tau}{m+1} \Upsilon(b_2) + \frac{1-\tau}{m+1} \Upsilon(b_1). \quad (2.2)$$

Adding (2.1) and (2.2) and multiplying both sides by $\frac{\Phi(\frac{b_2-b_1}{(m+1)b_1b_2}\tau)}{\tau}$ and integrating with respect to τ on $[0, 1]$, we have

$$\begin{aligned}
 & \int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1b_2}\tau)}{\tau} \Upsilon\left(\frac{(m+1)b_1b_2}{(1-\tau)b_1+(m+\tau)b_2}\right) d\tau \\
 & \quad + \int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1b_2}\tau)}{\tau} \Upsilon\left(\frac{(m+1)b_1b_2}{(m+\tau)b_1+(1-\tau)b_2}\right) d\tau \\
 & \leq [\Upsilon(b_1) + \Upsilon(b_2)] \int_0^1 \frac{\Phi(\frac{b_2-b_1}{(m+1)b_1b_2}\tau)}{\tau} d\tau.
 \end{aligned}$$

Using generalized fractional integrals, we obtain our second inequality. This completes the proof. $\square$

Corollary 2.1 *If we choose $\Phi(\tau) = \tau$ and $m = 1$ in Theorem 2.1, we have*

$$\Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) \leq \frac{b_1b_2}{b_2-b_1} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \leq \frac{[\Upsilon(b_1) + \Upsilon(b_2)]}{2}.$$

Corollary 2.2 *If we choose $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, we obtain*

$$\begin{aligned}
 \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) & \leq \frac{(b_1b_2(m+1))^\alpha \Gamma(\alpha+1)}{2(b_2-b_1)^\alpha} \\
 & \quad \times \left[J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi\left(\frac{mb_1+b_2}{(m+1)b_1b_2}\right) + J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi\left(\frac{b_1+mb_2}{(m+1)b_1b_2}\right) \right]
 \end{aligned}$$

$$\leq \frac{[\Upsilon(b_1) + \Upsilon(b_2)]}{2}.$$

For $m = 1$, we obtain

$$\begin{aligned} \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) &\leq \frac{2^{\alpha-1}(b_1b_2)^\alpha \Gamma(\alpha+1)}{(b_2-b_1)^\alpha} \\ &\quad \times \left[J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi\left(\frac{b_1+b_2}{2b_1b_2}\right) + J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi\left(\frac{b_1+b_2}{2b_1b_2}\right) \right] \\ &\leq \frac{[\Upsilon(b_1) + \Upsilon(b_2)]}{2}. \end{aligned}$$

Corollary 2.3 If we choose $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, we have

$$\begin{aligned} \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) &\leq \frac{((m+1)b_1b_2)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{2(b_2-b_1)^{\frac{\alpha}{k}}} \\ &\quad \times \left[{}^kJ_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi\left(\frac{mb_1+b_2}{(m+1)b_1b_2}\right) + {}^kJ_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi\left(\frac{b_1+mb_2}{(m+1)b_1b_2}\right) \right] \\ &\leq \frac{[\Upsilon(b_1) + \Upsilon(b_2)]}{2}. \end{aligned}$$

For $m = 1$, we obtain

$$\begin{aligned} \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) &\leq \frac{2^{\frac{\alpha}{k}-1}(b_1b_2)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b_2-b_1)^{\frac{\alpha}{k}}} \\ &\quad \times \left[{}^kJ_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi\left(\frac{b_1+b_2}{2b_1b_2}\right) + {}^kJ_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi\left(\frac{b_1+b_2}{2b_1b_2}\right) \right] \\ &\leq \frac{[\Upsilon(b_1) + \Upsilon(b_2)]}{2}. \end{aligned}$$

2.2 Auxiliary results

In this subsection, we derive three new fractional integral identities that will be used in the following.

Lemma 2.2 Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$ and $m \in \mathbb{N}$, then

$$\begin{aligned} &\frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} \\ &\quad - \frac{1}{(m+1)\eta(1)} \left[{}_{(\frac{1}{b_2})^+}I_\Phi \Upsilon \circ \Psi\left(\frac{mb_1+b_2}{(m+1)b_1b_2}\right) + {}_{(\frac{1}{b_1})^-}I_\Phi \Upsilon \circ \Psi\left(\frac{b_1+mb_2}{(m+1)b_1b_2}\right) \right] \\ &= \frac{b_1b_2(b_2-b_1)}{\eta(1)} \left[\int_0^1 \frac{\eta(\tau)}{((m+\tau)b_1 + (1-\tau)b_2)^2} \Upsilon'\left(\frac{(m+1)b_1b_2}{(m+\tau)b_1 + (1-\tau)b_2}\right) d\tau \right. \\ &\quad \left. - \int_0^1 \frac{\eta(\tau)}{((1-\tau)b_1 + (m+\tau)b_2)^2} \Upsilon'\left(\frac{(m+1)b_1b_2}{(1-\tau)b_1 + (m+\tau)b_2}\right) d\tau \right]. \end{aligned}$$

Proof Consider the right-hand side

$$\begin{aligned} I &:= \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\int_0^1 \frac{\eta(\tau)}{((m+\tau)b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2} \right) d\tau \right. \\ &\quad \left. - \int_0^1 \frac{\eta(\tau)}{((1-\tau)b_1 + (m+\tau)b_2)^2} \Upsilon' \left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2} \right) d\tau \right] \\ &= \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} [I_1 - I_2], \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_0^1 \frac{\eta(\tau)}{((m+\tau)b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2} \right) d\tau \\ &= \frac{\eta(1)\Upsilon(b_2)}{(m+1)b_1 b_2 (b_2 - b_1)} \\ &\quad - \frac{1}{b_1 b_2 (b_2 - b_1)(m+1)} \int_{\frac{1}{b_2}}^{\frac{mb_1+b_2}{(m+1)b_1 b_2}} \frac{\Phi(\frac{mb_1+b_2}{(m+1)b_1 b_2} - x)}{\frac{mb_1+b_2}{(m+1)b_1 b_2} - x} \Upsilon \circ \Psi(x) dx \\ &= \frac{\eta(1)\Upsilon(b_2)}{(m+1)b_1 b_2 (b_2 - b_1)} - \frac{1}{b_1 b_2 (b_2 - b_1)(m+1)} \left(\frac{1}{b_2} \right)^+ I_\Phi \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{(m+1)b_1 b_2} \right). \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &:= \int_0^1 \frac{\eta(\tau)}{((1-\tau)b_1 + (m+\tau)b_2)^2} \Upsilon' \left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2} \right) d\tau \\ &= -\frac{\eta(1)\Upsilon(b_1)}{(m+1)b_1 b_2 (b_2 - b_1)} \\ &\quad + \frac{1}{b_1 b_2 (b_2 - b_1)(m+1)} \int_{\frac{b_1+mb_2}{(m+1)b_1 b_2}}^{\frac{1}{b_1}} \frac{\Phi(x - \frac{b_1+mb_2}{(m+1)b_1 b_2})}{x - \frac{b_1+mb_2}{(m+1)b_1 b_2}} \Upsilon \circ \Psi(x) dx \\ &= -\frac{\eta(1)\Upsilon(b_1)}{(m+1)b_1 b_2 (b_2 - b_1)} + \frac{1}{b_1 b_2 (b_2 - b_1)(m+1)} \left(\frac{1}{b_1} \right)^- I_\Phi \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right). \end{aligned}$$

Substituting the values of I_1 and I_2 in I , we obtain our required result. $\square$

Remark 2.1 If we choose $m = 1$ and $\Phi(\tau) = \tau$, we have

$$\begin{aligned} &\frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} - \frac{b_1 b_2}{(b_2 - b_1)} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \\ &= b_1 b_2 (b_2 - b_1) \left[\int_0^1 \frac{\tau}{((1+\tau)b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{2b_1 b_2}{(1+\tau)b_1 + (1-\tau)b_2} \right) d\tau \right. \\ &\quad \left. - \int_0^1 \frac{\tau}{((1-\tau)b_1 + (1+\tau)b_2)^2} \Upsilon' \left(\frac{2b_1 b_2}{(1-\tau)b_1 + (1+\tau)b_2} \right) d\tau \right]. \end{aligned}$$

Corollary 2.4 *If we take $m = 1$ and $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Lemma 2.2, we obtain*

$$\begin{aligned} & \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} \\ & - \frac{2^{\alpha-1}(b_1b_2)^\alpha \Gamma(\alpha+1)}{(b_2-b_1)^\alpha} \left[J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi \left(\frac{b_1+b_2}{2b_1b_2} \right) + J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi \left(\frac{b_1+b_2}{2b_1b_2} \right) \right] \\ & = b_1b_2(b_2-b_1) \left[\int_0^1 \frac{\tau^\alpha}{((1+\tau)b_1+(1-\tau)b_2)^2} \Upsilon' \left(\frac{2b_1b_2}{(1+\tau)b_1+(1-\tau)b_2} \right) d\tau \right. \\ & \quad \left. - \int_0^1 \frac{\tau^\alpha}{((1-\tau)b_1+(1+\tau)b_2)^2} \Upsilon' \left(\frac{2b_1b_2}{(1-\tau)b_1+(1-\tau)b_2} \right) d\tau \right]. \end{aligned}$$

Corollary 2.5 *If we choose $m = 1$ and $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Lemma 2.2, we obtain*

$$\begin{aligned} & \frac{\Upsilon(b_1) + \Upsilon(b_2)}{2} \\ & - \frac{2^{\frac{\alpha}{k}-1}(b_1b_2)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b_2-b_1)^{\frac{\alpha}{k}}} \left[{}^kJ_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi \left(\frac{b_1+b_2}{2b_1b_2} \right) + {}^kJ_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi \left(\frac{b_1+b_2}{2b_1b_2} \right) \right] \\ & = b_1b_2(b_2-b_1) \left[\int_0^1 \frac{\tau^{\frac{\alpha}{k}}}{((1+\tau)b_1+(1-\tau)b_2)^2} \Upsilon' \left(\frac{2b_1b_2}{(1+\tau)b_1+(1-\tau)b_2} \right) d\tau \right. \\ & \quad \left. - \int_0^1 \frac{\tau^{\frac{\alpha}{k}}}{((1-\tau)b_1+(1+\tau)b_2)^2} \Upsilon' \left(\frac{2b_1b_2}{(1-\tau)b_1+(1-\tau)b_2} \right) d\tau \right]. \end{aligned}$$

Lemma 2.3 *Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$ and $\lambda, \mu \in [0, \infty)$ with $\lambda + \mu \neq 0$, then*

$$\begin{aligned} & \frac{\Omega(\lambda)\Upsilon(b_2) + \Omega(\mu)\Upsilon(b_1)}{\lambda + \mu} \\ & - \frac{1}{\lambda + \mu} \left[{}_{(\frac{1}{b_2})^+}I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1b_2(\lambda + \mu)} \right) + {}_{(\frac{1}{b_1})^-}I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1b_2(\lambda + \mu)} \right) \right] \\ & = b_1b_2(b_2-b_1) \left[\int_0^\lambda \frac{\Omega(\tau)}{((\lambda-\tau)b_2+(\mu+\tau)b_1)^2} \Upsilon' \left(\frac{b_1b_2(\lambda+\mu)}{(\lambda-\tau)b_2+(\mu+\tau)b_1} \right) d\tau \right. \\ & \quad \left. - \int_0^\mu \frac{\Omega(\tau)}{((\lambda+\tau)b_2+(\mu-\tau)b_1)^2} \Upsilon' \left(\frac{b_1b_2(\lambda+\mu)}{(\lambda+\tau)b_2+(\mu-\tau)b_1} \right) d\tau \right]. \end{aligned}$$

Proof Consider the right-hand side

$$\begin{aligned} I &:= b_1b_2(b_2-b_1) \left[\int_0^\lambda \frac{\Omega(\tau)}{((\lambda-\tau)b_2+(\mu+\tau)b_1)^2} \Upsilon' \left(\frac{b_1b_2(\lambda+\mu)}{(\lambda-\tau)b_2+(\mu+\tau)b_1} \right) d\tau \right. \\ & \quad \left. - \int_0^\mu \frac{\Omega(\tau)}{((\lambda+\tau)b_2+(\mu-\tau)b_1)^2} \Upsilon' \left(\frac{b_1b_2(\lambda+\mu)}{(\lambda+\tau)b_2+(\mu-\tau)b_1} \right) d\tau \right] \\ &= b_1b_2(b_2-b_1)[I_3 - I_4], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
 I_3 &:= \int_0^\lambda \frac{\Omega(\tau)}{((\lambda - \tau)b_2 + (\mu + \tau)b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau)b_2 + (\mu + \tau)b_1} \right) d\tau \\
 &= \frac{\Omega(\lambda) \Upsilon(b_2)}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} \\
 &\quad - \frac{1}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} \int_{\frac{1}{b_2}}^{\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)}} \frac{\Phi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} - x \right)}{\left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} - x \right)} \Upsilon \circ \Psi(x) dx \\
 &= \frac{\Omega(\lambda) \Upsilon(b_2)}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} - \frac{1}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} \left(\frac{1}{b_2} \right)^+ I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 I_4 &:= \int_0^\mu \frac{\Omega(\tau)}{((\lambda + \tau)b_2 + (\mu - \tau)b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau)b_2 + (\mu - \tau)b_1} \right) d\tau \\
 &= -\frac{\Omega(\mu) \Upsilon(b_1)}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} \\
 &\quad + \frac{1}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} \int_{\frac{1}{b_1}}^{\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)}} \frac{\Phi \left(x - \frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right)}{\left(x - \frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right)} \Upsilon \circ \Psi(x) dx \\
 &= -\frac{\Omega(\mu) \Upsilon(b_1)}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} - \frac{1}{b_1 b_2 (b_2 - b_1) (\lambda + \mu)} \left(\frac{1}{b_1} \right)^- I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right).
 \end{aligned}$$

Substituting the values of I_3 and I_4 in (2.3), we obtain our required result. $\square$

Corollary 2.6 Choosing $\Phi(\tau) = \tau$ in Lemma 2.3, we have

$$\begin{aligned}
 &\frac{\lambda \Upsilon(b_2) + \mu \Upsilon(b_1)}{\lambda + \mu} - \frac{b_1 b_2}{b_2 - b_1} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \\
 &= b_1 b_2 (b_2 - b_1) \left[\int_0^\lambda \frac{\tau}{((\lambda - \tau)b_2 + (\mu + \tau)b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau)b_2 + (\mu + \tau)b_1} \right) d\tau \right. \\
 &\quad \left. - \int_0^\mu \frac{\tau}{((\lambda + \tau)b_2 + (\mu - \tau)b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau)b_2 + (\mu - \tau)b_1} \right) d\tau \right].
 \end{aligned}$$

Corollary 2.7 Taking $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Lemma 2.3, we obtain

$$\begin{aligned}
 &\frac{\lambda^\alpha \Upsilon(b_2) + \mu^\alpha \Upsilon(b_1)}{\lambda + \mu} - \frac{(b_1 b_2)^\alpha (\lambda + \mu)^{\alpha-1} \Gamma(\alpha+1)}{(b_2 - b_1)^\alpha} \left[J_{\left(\frac{1}{b_2} \right)^+}^\alpha \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right. \\
 &\quad \left. + J_{\left(\frac{1}{b_1} \right)^-}^\alpha \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \\
 &= b_1 b_2 (b_2 - b_1) \left[\int_0^\lambda \frac{\tau^\alpha}{((\lambda - \tau)b_2 + (\mu + \tau)b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau)b_2 + (\mu + \tau)b_1} \right) d\tau \right. \\
 &\quad \left. - \int_0^\mu \frac{\tau^\alpha}{((\lambda + \tau)b_2 + (\mu - \tau)b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau)b_2 + (\mu - \tau)b_1} \right) d\tau \right].
 \end{aligned}$$

Corollary 2.8 Choosing $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Lemma 2.3, we obtain

$$\begin{aligned} & \frac{\lambda^{\frac{\alpha}{k}} \Upsilon(b_2) + \mu^{\frac{\alpha}{k}} \Upsilon(b_1)}{\lambda + \mu} - \frac{k(b_1 b_2)^{\frac{\alpha}{k}} (\lambda + \mu)^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + k)}{(b_2 - b_1)^{\frac{\alpha}{k}}} \left[{}^k J_{(\frac{1}{b_2})^+}^{\alpha} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right. \\ & \quad \left. + {}^k J_{(\frac{1}{b_1})^-}^{\alpha} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \\ &= b_1 b_2 (b_2 - b_1) \left[\int_0^{\lambda} \frac{\tau^{\frac{\alpha}{k}}}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau) b_2 + (\mu + \tau) b_1} \right) d\tau \right. \\ & \quad \left. - \int_0^{\mu} \frac{\tau^{\frac{\alpha}{k}}}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau) b_2 + (\mu - \tau) b_1} \right) d\tau \right]. \end{aligned}$$

Lemma 2.4 Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (b_1, b_2) with $b_1 < b_2$, then

$$\begin{aligned} & \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) - \frac{1}{2\Delta(1)} \left[{}^I_{\Phi} \Upsilon \circ \Psi \left(\frac{1}{b_1} \right) + {}^I_{\Phi} \Upsilon \circ \Psi \left(\frac{1}{b_2} \right) \right] \\ &= \frac{b_1 b_2 (b_2 - b_1)}{2\Delta(1)} \sum_{j=1}^4 M_j, \end{aligned}$$

where

$$\begin{aligned} M_1 &:= \int_0^{\frac{1}{2}} \frac{\Delta(\tau)}{(\tau b_1 + (1 - \tau) b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\ M_2 &:= \int_0^{\frac{1}{2}} \frac{(-\Delta(\tau))}{((1 - \tau) b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau, \\ M_3 &:= \int_{\frac{1}{2}}^1 \frac{(-\delta(\tau))}{(\tau b_1 + (1 - \tau) b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\ M_4 &:= \int_{\frac{1}{2}}^1 \frac{\delta(\tau)}{((1 - \tau) b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau. \end{aligned}$$

Proof Integrating by parts M_i for $i = 1, 2, 3, 4$, and changing the variables, we have

$$\begin{aligned} M_1 &= \frac{1}{b_1 b_2 (b_2 - b_1)} \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) \int_0^{\frac{1}{2}} \frac{\Phi(\frac{b_2 - b_1}{b_1 b_2} \mu)}{\mu} d\mu \\ & \quad - \frac{1}{b_1 b_2 (b_2 - b_1)} \int_0^{\frac{1}{2}} \frac{\Phi(\frac{b_2 - b_1}{b_1 b_2} \tau)}{\tau} \Upsilon \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\ M_2 &= \frac{1}{b_1 b_2 (b_2 - b_1)} \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) \int_0^{\frac{1}{2}} \frac{\Phi(\frac{b_2 - b_1}{b_1 b_2} \mu)}{\mu} d\mu \\ & \quad - \frac{1}{b_1 b_2 (b_2 - b_1)} \int_0^{\frac{1}{2}} \frac{\Phi(\frac{b_2 - b_1}{b_1 b_2} \tau)}{\tau} \Upsilon \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau, \end{aligned}$$

$$\begin{aligned}
M_3 &= \frac{1}{b_1 b_2 (b_2 - b_1)} \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) \int_{\frac{1}{2}}^1 \frac{\Phi \left(\frac{b_2 - b_1}{b_1 b_2} \mu \right)}{\mu} d\mu \\
&\quad - \frac{1}{b_1 b_2 (b_2 - b_1)} \int_{\frac{1}{2}}^1 \frac{\Phi \left(\frac{b_2 - b_1}{b_1 b_2} \tau \right)}{\tau} \Upsilon \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\
M_4 &= \frac{1}{b_1 b_2 (b_2 - b_1)} \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) \int_{\frac{1}{2}}^1 \frac{\Phi \left(\frac{b_2 - b_1}{b_1 b_2} \mu \right)}{\mu} d\mu \\
&\quad - \frac{1}{b_1 b_2 (b_2 - b_1)} \int_{\frac{1}{2}}^1 \frac{\Phi \left(\frac{b_2 - b_1}{b_1 b_2} \tau \right)}{\tau} \Upsilon \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau.
\end{aligned}$$

Adding M_1 , M_2 , M_3 and M_4 and multiplying by the factor $\frac{b_1 b_2 (b_2 - b_1)}{2\Delta(1)}$, we obtain our required result. $\square$

Corollary 2.9 Taking $\Phi(\tau) = \tau$ in Lemma 2.4, then

$$\Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) - \frac{b_1 b_2}{b_2 - b_1} \int_{b_1}^{b_2} \frac{\Upsilon(x)}{x^2} dx = \frac{b_1 b_2 (b_2 - b_1)}{2} \sum_{j=1}^4 L_j,$$

where

$$\begin{aligned}
L_1 &:= \int_0^{\frac{1}{2}} \frac{\tau}{(\tau b_1 + (1 - \tau) b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\
L_2 &:= \int_0^{\frac{1}{2}} \frac{-\tau}{((1 - \tau) b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\
L_3 &:= \int_{\frac{1}{2}}^1 \frac{-\tau}{(\tau b_1 + (1 - \tau) b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\
L_4 &:= \int_{\frac{1}{2}}^1 \frac{\tau}{((1 - \tau) b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau.
\end{aligned}$$

Corollary 2.10 Choosing $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Lemma 2.4, then

$$\begin{aligned}
&\Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) - \frac{\Gamma(\alpha + 1)}{2} \left(\frac{b_1 b_2}{b_2 - b_1} \right)^\alpha \left[J_{\frac{1}{b_2}}^\alpha \Upsilon \circ \Psi \left(\frac{1}{b_1} \right) + J_{\frac{1}{b_1}}^\alpha \Upsilon \circ \Psi \left(\frac{1}{b_2} \right) \right] \\
&= \frac{b_1 b_2 (b_2 - b_1)}{2} \sum_{j=5}^8 L_j,
\end{aligned}$$

where

$$\begin{aligned}
L_5 &:= \int_0^{\frac{1}{2}} \frac{\tau^\alpha}{(\tau b_1 + (1 - \tau) b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\
L_6 &:= \int_0^{\frac{1}{2}} \frac{(-\tau)^\alpha}{((1 - \tau) b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau, \\
L_7 &:= \int_{\frac{1}{2}}^1 \frac{(-\tau)^\alpha}{(\tau b_1 + (1 - \tau) b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1 - \tau) b_2} \right) d\tau, \\
L_8 &:= \int_{\frac{1}{2}}^1 \frac{\tau^\alpha}{((1 - \tau) b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1 - \tau) b_1 + \tau b_2} \right) d\tau.
\end{aligned}$$

$$L_8 := \int_{\frac{1}{2}}^1 \frac{\tau^\alpha}{((1-\tau)b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1-\tau)b_1 + \tau b_2} \right) d\tau.$$

Corollary 2.11 Taking $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Lemma 2.4, then

$$\begin{aligned} & \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) - \frac{\Gamma_k(\alpha + k)}{2} \left(\frac{b_1 b_2}{b_2 - b_1} \right)^{\frac{\alpha}{k}} \left[J_{\frac{1}{b_2}}^{\alpha} \Upsilon \circ \Psi \left(\frac{1}{b_1} \right) + J_{\frac{1}{b_1}}^{\alpha} \Upsilon \circ \Psi \left(\frac{1}{b_2} \right) \right] \\ &= \frac{b_1 b_2 (b_2 - b_1)}{2} \sum_{j=9}^{12} L_j, \end{aligned}$$

where

$$\begin{aligned} L_9 &:= \int_0^{\frac{1}{2}} \frac{\tau^{\frac{\alpha}{k}}}{(\tau b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1-\tau)b_2} \right) d\tau, \\ L_{10} &:= \int_0^{\frac{1}{2}} \frac{(-\tau)^{\frac{\alpha}{k}}}{((1-\tau)b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1-\tau)b_1 + \tau b_2} \right) d\tau, \\ L_{11} &:= \int_{\frac{1}{2}}^1 \frac{(-\tau)^{\frac{\alpha}{k}}}{(\tau b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1-\tau)b_2} \right) d\tau, \\ L_{12} &:= \int_{\frac{1}{2}}^1 \frac{\tau^{\frac{\alpha}{k}}}{((1-\tau)b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1-\tau)b_1 + \tau b_2} \right) d\tau. \end{aligned}$$

Corollary 2.12 Choosing $\Phi(\tau) = \frac{\tau}{\alpha} \exp(-A\tau)$ in Lemma 2.4 with $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0, 1]$, then

$$\begin{aligned} & \Upsilon \left(\frac{2b_1 b_2}{b_1 + b_2} \right) - \frac{1-\alpha}{2(1-\exp(-A))} \left(\frac{b_1 b_2}{b_2 - b_1} \right)^{\alpha} \left[I_{\frac{1}{b_2}}^{\alpha} \Upsilon \circ \Psi \left(\frac{1}{b_1} \right) + I_{\frac{1}{b_1}}^{\alpha} \Upsilon \circ \Psi \left(\frac{1}{b_2} \right) \right] \\ &= \frac{b_1 b_2 (b_2 - b_1)}{2(1-\exp(-A))} \sum_{j=13}^{16} L_j, \end{aligned}$$

where

$$\begin{aligned} L_{13} &:= \int_0^{\frac{1}{2}} \frac{[\exp(-A\tau) - 1]}{(\tau b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1-\tau)b_2} \right) d\tau, \\ L_{14} &:= \int_0^{\frac{1}{2}} \frac{[1 - \exp(-A\tau)]}{((1-\tau)b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1-\tau)b_1 + \tau b_2} \right) d\tau, \\ L_{15} &:= \int_{\frac{1}{2}}^1 \frac{[\exp(-A(1-\tau)) - \exp(-A\tau)]}{(\tau b_1 + (1-\tau)b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{\tau b_1 + (1-\tau)b_2} \right) d\tau, \\ L_{16} &:= \int_{\frac{1}{2}}^1 \frac{[\exp(-A\tau) - \exp(-A(1-\tau))]}{((1-\tau)b_1 + \tau b_2)^2} \Upsilon' \left(\frac{b_1 b_2}{(1-\tau)b_1 + \tau b_2} \right) d\tau. \end{aligned}$$

2.3 Further results

Now, utilizing auxiliary results obtained in the previous subsection, we derive some further generalized fractional trapezium-like inequalities using the class of harmonic convex functions.

Theorem 2.5 Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function on (b_1, b_2) with $b_1 < b_2$ and $|\Upsilon'|^q$ be an harmonic convex function with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} & \left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} \right. \\ & \quad \left. - \frac{1}{(m+1)\eta(1)} \left[{}_{(\frac{1}{b_2})^+} I_{\Phi} \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{(m+1)b_1 b_2} \right) + {}_{(\frac{1}{b_1})^-} I_{\Phi} \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right) \right] \right| \\ & \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\pi_1^{\frac{1}{p}} \left(\int_0^1 \eta^q(\tau) \left(\frac{1-\tau}{m+1} |\Upsilon'(b_1)|^q + \frac{m+\tau}{m+1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \pi_2^{\frac{1}{p}} \left(\int_0^1 \eta^q(\tau) \left(\frac{m+\tau}{m+1} |\Upsilon'(b_1)|^q + \frac{1-\tau}{m+1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned} \pi_1 &:= \frac{(mb_1 + b_2)^{1-2p}}{(b_2 - b_1)(1-2p)} \left[1 - \left(\frac{(m+1)b_1}{mb_1 + b_2} \right)^{1-2p} \right], \\ \pi_2 &:= \frac{(b_1 + mb_2)^{1-2p}}{(b_2 - b_1)(1-2p)} \left[\left(\frac{(m+1)b_2}{b_1 + mb_2} \right)^{1-2p} - 1 \right]. \end{aligned}$$

Proof Using Lemma 2.2, the modulus property, Hölder's inequality and the harmonic convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned} & \left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} \right. \\ & \quad \left. - \frac{1}{(m+1)\eta(1)} \left[{}_{(\frac{1}{b_2})^+} I_{\Phi} \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{(m+1)b_1 b_2} \right) + {}_{(\frac{1}{b_1})^-} I_{\Phi} \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right) \right] \right| \\ & \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\int_0^1 \frac{\eta(\tau)}{((m+\tau)b_1 + (1-\tau)b_2)^2} \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2} \right) \right| d\tau \right. \\ & \quad \left. + \int_0^1 \frac{\eta(\tau)}{((1-\tau)b_1 + (m+\tau)b_2)^2} \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2} \right) \right| d\tau \right] \\ & \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\left(\int_0^1 ((m+\tau)b_1 + (1-\tau)b_2)^{-2p} d\tau \right)^{\frac{1}{p}} \right. \\ & \quad \times \left(\int_0^1 \eta^p(\tau) \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2} \right) \right|^q d\tau \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 ((1-\tau)b_1 + (m+\tau)b_2)^{-2p} d\tau \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 \eta^p(\tau) \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2} \right) \right|^q d\tau \right)^{\frac{1}{q}} \Bigg] \\ & \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\left(\int_0^1 ((m+\tau)b_1 + (1-\tau)b_2)^{-2p} d\tau \right)^{\frac{1}{p}} \right. \\ & \quad \times \left(\int_0^1 \eta^q(\tau) \left(\frac{1-\tau}{m+1} |\Upsilon'(b_1)|^q + \frac{m+\tau}{m+1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \Bigg] \end{aligned}$$

$$+ \left(\int_0^1 ((1-\tau)b_1 + (m+\tau)b_2)^{-2p} d\tau \right)^{\frac{1}{p}} \\ \times \left(\int_0^1 \eta^q(\tau) \left(\frac{m+\tau}{m+1} |\Upsilon'(b_1)|^q + \frac{1-\tau}{m+1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \Bigg].$$

After simple calculations, we obtain our required result. $\square$

Corollary 2.13 *Choosing $\Phi(\tau) = \tau$ in Theorem 2.5, we have*

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} - \frac{b_1 b_2}{b_2 - b_1} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \right| \\ \leq b_1 b_2 (b_2 - b_1) \\ \times \left[\pi_1^{\frac{1}{p}} \left(\frac{1}{(m+1)(q+1)(q+2)} |\Upsilon'(b_1)|^q + \frac{m(q+2) + (q+1)}{(m+1)(q+1)(q+2)} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ \left. + \pi_2^{\frac{1}{p}} \left(\frac{1}{(m+1)(q+1)(q+2)} |\Upsilon'(b_2)|^q + \frac{m(q+2) + (q+1)}{(m+1)(q+1)(q+2)} |\Upsilon'(b_1)|^q \right)^{\frac{1}{q}} \right],$$

Corollary 2.14 *Taking $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 2.5, we obtain*

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} - \frac{(m+1)^{\alpha-1} (b_1 b_2)^\alpha \Gamma(\alpha+1)}{(b_2 - b_1)^\alpha} \left[J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{b_1 b_2 (m+1)} \right) \right. \right. \\ \left. \left. + J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right) \right] \right| \\ \leq b_1 b_2 (b_2 - b_1) \left[\pi_1^{\frac{1}{p}} \left(\frac{1}{(m+1)(\alpha q + 1)(\alpha q + 2)} |\Upsilon'(b_1)|^q \right. \right. \\ \left. \left. + \frac{m(\alpha q + 2) + (\alpha q + 1)}{(m+1)(\alpha q + 1)(\alpha q + 2)} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ \left. + \pi_2^{\frac{1}{p}} \left(\frac{1}{(m+1)(\alpha q + 1)(\alpha q + 2)} |\Upsilon'(b_2)|^q \right. \right. \\ \left. \left. + \frac{m(\alpha q + 2) + (\alpha q + 1)}{(m+1)(\alpha q + 1)(\alpha q + 2)} |\Upsilon'(b_1)|^q \right)^{\frac{1}{q}} \right],$$

where π_1 and π_2 are already defined.

Corollary 2.15 *Choosing $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\frac{\alpha}{k})}$ in Theorem 2.5, we obtain*

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} - \frac{(m+1)^{\frac{\alpha}{k}-1} (b_1 b_2)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b_2 - b_1)^{\frac{\alpha}{k}}} \left[J_{(\frac{1}{b_2})^+}^{\frac{\alpha}{k}} \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{b_1 b_2 (m+1)} \right) \right. \right. \\ \left. \left. + J_{(\frac{1}{b_1})^-}^{\frac{\alpha}{k}} \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right) \right] \right| \\ \leq b_1 b_2 (b_2 - b_1) \left[\pi_1^{\frac{1}{p}} \left(\frac{k}{(m+1)(\alpha q + k)(\alpha q + 2k)} |\Upsilon'(b_1)|^q \right. \right. \\ \left. \left. + \frac{km(\alpha q + 2k) + k(\alpha q + k)}{(m+1)(\alpha q + k)(\alpha q + 2k)} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ \left. + \pi_2^{\frac{1}{p}} \left(\frac{k}{(m+1)(\alpha q + k)(\alpha q + 2k)} |\Upsilon'(b_2)|^q \right. \right. \\ \left. \left. + \frac{km(\alpha q + 2k) + k(\alpha q + k)}{(m+1)(\alpha q + k)(\alpha q + 2k)} |\Upsilon'(b_1)|^q \right)^{\frac{1}{q}} \right],$$

$$+ \pi_2^{\frac{1}{p}} \left(\frac{k}{(m+1)(\alpha q + k)(\alpha q + 2k)} |\Upsilon'(b_2)|^q \right. \\ \left. + \frac{mk(\alpha q + 2k) + (k\alpha q + k)}{(m+1)(\alpha q + k)(\alpha q + 2k)} |\Upsilon'(b_1)|^q \right)^{\frac{1}{q}} \Bigg].$$

Theorem 2.6 Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function on (b_1, b_2) with $b_1 < b_2$ and $|\Upsilon'|^q$ be an harmonic convex function with $q \geq 1$, then

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} \right. \\ \left. - \frac{1}{(m+1)\eta(1)} \left[\left(\frac{1}{b_2} \right)^+ I_{\Phi} \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{(m+1)b_1 b_2} \right) + \left(\frac{1}{b_1} \right)^- I_{\Phi} \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right) \right] \right| \\ \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\left(\int_0^1 \eta(\tau) ((m+\tau)b_1 + (1-\tau)b_2)^{-2} d\tau \right)^{1-\frac{1}{q}} \right. \\ \times \left(\int_0^1 \eta(\tau) ((m+\tau)b_1 + (1-\tau)b_2)^{-2} \left(\frac{1-\tau}{m+1} |\Upsilon'(b_1)|^q + \frac{m+\tau}{m+1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \\ \left. + \left(\int_0^1 \eta(\tau) ((1-\tau)b_1 + (m+\tau)b_2)^{-2} d\tau \right)^{1-\frac{1}{q}} \right. \\ \times \left(\int_0^1 \eta(\tau) ((1-\tau)b_1 + (m+\tau)b_2)^{-2} \right. \\ \left. \times \left(\frac{m+\tau}{m+1} |\Upsilon'(b_1)|^q + \frac{1-\tau}{m+1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \Bigg].$$

Proof Using Lemma 2.2, the modulus property, the power mean inequality and the convexity of $|\Upsilon'|^q$, we have

$$\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m+1} \right. \\ \left. - \frac{1}{(m+1)\eta(1)} \left[\left(\frac{1}{b_2} \right)^+ I_{\Phi} \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{(m+1)b_1 b_2} \right) + \left(\frac{1}{b_1} \right)^- I_{\Phi} \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m+1)b_1 b_2} \right) \right] \right| \\ \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\int_0^1 \frac{\eta(\tau)}{((m+\tau)b_1 + (1-\tau)b_2)^2} \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2} \right) \right| d\tau \right. \\ \left. + \int_0^1 \frac{\eta(\tau)}{((1-\tau)b_1 + (m+\tau)b_2)^2} \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2} \right) \right| d\tau \right] \\ \leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\left(\int_0^1 \eta(\tau) ((m+\tau)b_1 + (1-\tau)b_2)^{-2} d\tau \right)^{1-\frac{1}{q}} \right. \\ \times \left(\int_0^1 \eta(\tau) ((m+\tau)b_1 + (1-\tau)b_2)^{-2} \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(m+\tau)b_1 + (1-\tau)b_2} \right) \right|^q d\tau \right)^{\frac{1}{q}} \\ \left. + \left(\int_0^1 \eta(\tau) ((1-\tau)b_1 + (m+\tau)b_2)^{-2} d\tau \right)^{1-\frac{1}{q}} \right. \\ \times \left(\int_0^1 \eta(\tau) ((1-\tau)b_1 + (m+\tau)b_2)^{-2} \left| \Upsilon' \left(\frac{(m+1)b_1 b_2}{(1-\tau)b_1 + (m+\tau)b_2} \right) \right|^q d\tau \right)^{\frac{1}{q}} \Bigg]$$

$$\begin{aligned}
&\leq \frac{b_1 b_2 (b_2 - b_1)}{\eta(1)} \left[\left(\int_0^1 \eta(\tau) ((m + \tau)b_1 + (1 - \tau)b_2)^{-2} d\tau \right)^{1 - \frac{1}{q}} \right. \\
&\quad \times \left(\int_0^1 \eta(\tau) ((m + \tau)b_1 + (1 - \tau)b_2)^{-2} \left(\frac{1 - \tau}{m + 1} |\Upsilon'(b_1)|^q + \frac{m + \tau}{m + 1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \eta(\tau) ((1 - \tau)b_1 + (m + \tau)b_2)^{-2} \right)^{1 - \frac{1}{q}} \\
&\quad \times \left(\int_0^1 \eta(\tau) ((1 - \tau)b_1 + (m + \tau)b_2)^{-2} \right. \\
&\quad \times \left. \left(\frac{m + \tau}{m + 1} |\Upsilon'(b_1)|^q + \frac{1 - \tau}{m + 1} |\Upsilon'(b_2)|^q \right) d\tau \right)^{\frac{1}{q}} \Big].
\end{aligned}$$

After simple calculations, we obtain our required result. $\square$

Corollary 2.16 *If we take $\Phi(\tau) = \tau$ in Theorem 2.6, we have*

$$\begin{aligned}
&\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m + 1} - \frac{b_1 b_2}{b_2 - b_1} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \right| \\
&\leq b_1 b_2 (b_2 - b_1) \left[\pi_3^{1 - \frac{1}{q}} (\pi_4 |\Upsilon'(b_1)|^q + \pi_5 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right. \\
&\quad \left. + \pi_6^{1 - \frac{1}{q}} (\pi_7 |\Upsilon'(b_1)|^q + \pi_8 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right],
\end{aligned}$$

where

$$\begin{aligned}
\pi_3 &:= \frac{(mb_1 + b_2)^{-2}}{2} {}_2F_1\left(2, 2, 3, \frac{b_2 - b_1}{mb_1 + b_2}\right), \\
\pi_4 &:= \frac{(mb_1 + b_2)^{-2}}{6(m + 1)} {}_2F_1\left(2, 2, 4, \frac{b_2 - b_1}{mb_1 + b_2}\right), \\
\pi_5 &:= \frac{m(mb_1 + b_2)^{-2}}{2(m + 1)} {}_2F_1\left(2, 2, 3, \frac{b_2 - b_1}{mb_1 + b_2}\right) + \frac{(mb_1 + b_2)^{-2}}{6(m + 1)} {}_2F_1\left(2, 3, 4, \frac{b_2 - b_1}{mb_1 + b_2}\right), \\
\pi_6 &:= \frac{(b_1 + mb_2)^{-2}}{2} {}_2F_1\left(2, 2, 3, \frac{b_1 - b_2}{b_1 + mb_2}\right), \\
\pi_7 &:= \frac{m(b_1 + mb_2)^{-2}}{2(m + 1)} {}_2F_1\left(2, 2, 3, \frac{b_1 - b_2}{b_1 + mb_2}\right) + \frac{(b_1 + mb_2)^{-2}}{6(m + 1)} {}_2F_1\left(2, 3, 4, \frac{b_1 - b_2}{b_1 + mb_2}\right), \\
\pi_8 &:= \frac{(b_1 + mb_2)^{-2}}{6(m + 1)} {}_2F_1\left(2, 2, 4, \frac{b_1 - b_2}{b_1 + mb_2}\right).
\end{aligned}$$

Corollary 2.17 *If we choose $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 2.6, we obtain*

$$\begin{aligned}
&\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m + 1} - \frac{(m + 1)^{\alpha - 1} (b_1 b_2)^\alpha \Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} \left[J_{\left(\frac{1}{b_2}\right)^+}^\alpha \Upsilon \circ \Psi\left(\frac{mb_1 + b_2}{b_1 b_2 (m + 1)}\right) \right. \right. \\
&\quad \left. \left. + J_{\left(\frac{1}{b_1}\right)^-}^\alpha \Upsilon \circ \Psi\left(\frac{b_1 + mb_2}{(m + 1) b_1 b_2}\right) \right] \right|
\end{aligned}$$

$$\leq b_1 b_2 (b_2 - b_1) \left[\pi_9^{1-\frac{1}{q}} \left(\pi_{10} |\Upsilon'(b_1)|^q + \pi_{11} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ \left. + \pi_{12}^{1-\frac{1}{q}} \left(\pi_{13} |\Upsilon'(b_1)|^q + \pi_{14} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right],$$

where

$$\begin{aligned} \pi_9 &:= \frac{(mb_1 + b_2)^{-2}}{\alpha + 1} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_2 - b_1}{mb_1 + b_2} \right), \\ \pi_{10} &:= \frac{(mb_1 + b_2)^{-2}}{(\alpha + 2)(\alpha + 1)(m + 1)} {}_2F_1 \left(2, \alpha + 1, \alpha + 3, \frac{b_2 - b_1}{mb_1 + b_2} \right), \\ \pi_{11} &:= \frac{m(mb_1 + b_2)^{-2}}{(\alpha + 1)(m + 1)} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_2 - b_1}{mb_1 + b_2} \right) \\ &\quad + \frac{(mb_1 + b_2)^{-2}}{(\alpha + 2)(\alpha + 1)(m + 1)} {}_2F_1 \left(2, \alpha + 2, \alpha + 3, \frac{b_2 - b_1}{mb_1 + b_2} \right), \\ \pi_{12} &:= \frac{(b_1 + mb_2)^{-2}}{\alpha + 1} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_1 - b_2}{b_1 + mb_2} \right), \\ \pi_{13} &:= \frac{m(b_1 + mb_2)^{-2}}{(\alpha + 1)(m + 1)} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_1 - b_2}{b_1 + mb_2} \right) \\ &\quad + \frac{(b_1 + mb_2)^{-2}}{(\alpha + 2)(\alpha + 1)(m + 1)} {}_2F_1 \left(2, \alpha + 2, \alpha + 3, \frac{b_1 - b_2}{b_1 + mb_2} \right), \\ \pi_{14} &:= \frac{(b_1 + mb_2)^{-2}}{(\alpha + 2)(\alpha + 1)(m + 1)} {}_2F_1 \left(2, \alpha + 1, \alpha + 3, \frac{b_1 - b_2}{b_1 + mb_2} \right). \end{aligned}$$

Corollary 2.18 If we take $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.6, we obtain

$$\begin{aligned} &\left| \frac{\Upsilon(b_1) + \Upsilon(b_2)}{m + 1} - \frac{(m + 1)^{\frac{\alpha}{k}-1} (b_1 b_2)^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)}{(b_2 - b_1)^{\frac{\alpha}{k}}} \left[{}_k J_{(\frac{1}{b_2})^+}^{\alpha} \Upsilon \circ \Psi \left(\frac{mb_1 + b_2}{b_1 b_2 (m + 1)} \right) \right. \right. \\ &\quad \left. \left. + {}_k J_{(\frac{1}{b_1})^-}^{\alpha} \Upsilon \circ \Psi \left(\frac{b_1 + mb_2}{(m + 1) b_1 b_2} \right) \right] \right| \\ &\leq b_1 b_2 (b_2 - b_1) \left[\pi_{15}^{1-\frac{1}{q}} \left(\pi_{16} |\Upsilon'(b_1)|^q + \pi_{17} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \pi_{18}^{1-\frac{1}{q}} \left(\pi_{19} |\Upsilon'(b_1)|^q + \pi_{20} |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned} \pi_{15} &:= \frac{k(mb_1 + b_2)^{-2}}{\alpha + k} {}_2F_{1,k} \left(2k, \alpha + k, \alpha + 2k, \frac{b_2 - b_1}{mb_1 + b_2} \right), \\ \pi_{16} &:= \frac{k(mb_1 + b_2)^{-2}}{(\alpha + 2k)(\alpha + k)(m + 1)} {}_2F_{1,k} \left(2k, \alpha + k, \alpha + 3k, \frac{b_2 - b_1}{mb_1 + b_2} \right), \\ \pi_{17} &:= \frac{km(mb_1 + b_2)^{-2}}{(\alpha + k)(m + 1)} {}_2F_{1,k} \left(2k, \alpha + k, \alpha + 2k, \frac{b_2 - b_1}{mb_1 + b_2} \right) \\ &\quad + \frac{k(mb_1 + b_2)^{-2}}{(\alpha + 2k)(\alpha + k)(m + 1)} {}_2F_{1,k} \left(2k, \alpha + 2k, \alpha + 3k, \frac{b_2 - b_1}{mb_1 + b_2} \right), \\ \pi_{18} &:= \frac{k(b_1 + mb_2)^{-2}}{\alpha + k} {}_2F_{1,k} \left(2k, \alpha + k, \alpha + 2k, \frac{b_1 - b_2}{b_1 + mb_2} \right), \end{aligned}$$

$$\begin{aligned}\pi_{19} &:= \frac{km(b_1 + mb_2)^{-2}}{(\alpha + k)(m + 1)} {}_2F_{1,k}\left(2k, \alpha + k, \alpha + 2k, \frac{b_1 - b_2}{b_1 + mb_2}\right) \\ &\quad + \frac{k(b_1 + mb_2)^{-2}}{(\alpha + 2k)(\alpha + k)(m + 1)} {}_2F_{1,k}\left(2k, \alpha + 2k, \alpha + 3k, \frac{b_1 - b_2}{b_1 + mb_2}\right), \\ \pi_{20} &:= \frac{k(b_1 + mb_2)^{-2}}{(\alpha + 2k)(\alpha + k)(m + 1)} {}_2F_{1,k}\left(2k, \alpha + k, \alpha + 3k, \frac{b_1 - b_2}{b_1 + mb_2}\right).\end{aligned}$$

Theorem 2.7 Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function on (b_1, b_2) with $b_1 < b_2$ and $|\Upsilon'|^q$ be an harmonic convex function with $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda, \mu \in [0, \infty)$ with $\lambda + \mu \neq 0$, then

$$\begin{aligned}& \left| \frac{\Omega(\lambda)\Upsilon(b_2) + \Omega(\mu)\Upsilon(b_1)}{\lambda + \mu} \right. \\ & \quad \left. - \frac{1}{\lambda + \mu} \left[\left(\frac{1}{b_2} \right)^+ I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) + \left(\frac{1}{b_1} \right)^- I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^\lambda \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} (\sigma_1 |\Upsilon'(b_2)|^q + \sigma_2 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^\mu \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} (\sigma_3 |\Upsilon'(b_2)|^q + \sigma_4 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right],\end{aligned}$$

where

$$\begin{aligned}\sigma_1 &:= \int_0^\lambda \frac{((\lambda - \tau)b_2 + (\mu + \tau)b_1)^{-2q}(\mu + \tau)}{\lambda + \mu} d\tau \\ &= \frac{\mu(\lambda b_2 + \mu b_1)^{1-2q} - (\lambda + \mu)((\lambda + \mu)b_1)^{1-2q}}{(\lambda + \mu)(b_2 - b_1)(1 - 2q)} - \frac{((\lambda + \mu)b_1)^{2-2q} - (\lambda b_2 + \mu b_1)^{2-2q}}{(\lambda + \mu)(b_2 - b_1)^2(1 - 2q)(2 - 2q)}, \\ \sigma_2 &:= \int_0^\lambda \frac{((\lambda - \tau)b_2 + (\mu + \tau)b_1)^{-2q}(\lambda - \tau)}{\lambda + \mu} d\tau \\ &= \frac{\lambda(\lambda b_2 + \mu b_1)^{1-2q}}{(\lambda + \mu)(b_2 - b_1)(1 - 2q)} + \frac{((\lambda + \mu)b_1)^{2-2q} - (\lambda b_2 + \mu b_1)^{2-2q}}{(\lambda + \mu)(b_2 - b_1)^2(1 - 2q)(2 - 2q)}, \\ \sigma_3 &:= \int_0^\mu \frac{((\lambda + \tau)b_2 + (\mu - \tau)b_1)^{-2q}(\mu - \tau)}{\lambda + \mu} d\tau \\ &= \frac{((\lambda + \mu)b_2)^{2-2q} - (\lambda b_2 + \mu b_1)^{2-2q}}{(\lambda + \mu)(b_2 - b_1)^2(1 - 2q)(2 - 2q)} - \frac{\mu(\lambda b_2 + \mu b_1)^{1-2q}}{(\lambda + \mu)(b_2 - b_1)(1 - 2q)}, \\ \sigma_4 &:= \int_0^\mu \frac{((\lambda + \tau)b_2 + (\mu - \tau)b_1)^{-2q}(\lambda + \tau)}{\lambda + \mu} d\tau \\ &= \frac{(\lambda + \mu)((\lambda + \mu)b_2)^{1-2q} - \lambda(\lambda b_2 + \mu b_1)^{1-2q}}{(\lambda + \mu)(b_2 - b_1)(1 - 2q)} - \frac{((\lambda + \mu)b_2)^{2-2q} - (\lambda b_2 + \mu b_1)^{2-2q}}{(\lambda + \mu)(b_2 - b_1)^2(1 - 2q)(2 - 2q)}.\end{aligned}$$

Proof Using Lemma 2.3, the modulus property, Hölder's inequality and the harmonic convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}& \left| \frac{\Omega(\lambda)\Upsilon(b_2) + \Omega(\mu)\Upsilon(b_1)}{\lambda + \mu} \right. \\ & \quad \left. - \frac{1}{\lambda + \mu} \left[\left(\frac{1}{b_2} \right)^+ I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) + \left(\frac{1}{b_1} \right)^- I_\Phi \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^\lambda \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} (\sigma_1 |\Upsilon'(b_2)|^q + \sigma_2 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^\mu \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} (\sigma_3 |\Upsilon'(b_2)|^q + \sigma_4 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right],\end{aligned}$$

$$\begin{aligned}
&\leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^\lambda \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^\lambda \frac{1}{((\lambda - \tau)b_2 + (\mu + \tau)b_1)^{2p}} \left| \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau)b_2 + (\mu + \tau)b_1} \right) \right|^q d\tau \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^\mu \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\mu \frac{1}{((\lambda + \tau)b_2 + (\mu - \tau)b_1)^{2p}} \left| \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau)b_2 + (\mu - \tau)b_1} \right) \right|^q d\tau \right)^{\frac{1}{q}} \Big] \\
&\leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^\lambda \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} \left(\int_0^\lambda ((\lambda - \tau)b_2 + (\mu + \tau)b_1)^{-2q} \right. \right. \\
&\quad \times \left. \left(\frac{\mu + \tau}{\lambda + \mu} |\Upsilon'(b_2)|^q + \frac{\lambda - \tau}{\lambda + \mu} |\Upsilon'(b_1)|^q \right) d\tau \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^\mu \Omega^p(\tau) d\tau \right)^{\frac{1}{p}} \left(\int_0^\mu ((\lambda + \tau)b_2 + (\mu - \tau)b_1)^{-2q} \right. \\
&\quad \times \left. \left(\frac{\mu - \tau}{\lambda + \mu} |\Upsilon'(b_2)|^q + \frac{\lambda + \tau}{\lambda + \mu} |\Upsilon'(b_1)|^q \right) d\tau \right)^{\frac{1}{q}} \Big].
\end{aligned}$$

After simple calculations, we obtain our required result. $\square$

Corollary 2.19 *Choosing $\Phi(\tau) = \tau$ in Theorem 2.7, we have*

$$\begin{aligned}
&\left| \frac{\lambda \Upsilon(b_2) + \mu \Upsilon(b_1)}{\lambda + \mu} - \frac{b_1 b_2}{b_2 - b_1} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \right| \\
&\leq b_1 b_2 (b_2 - b_1) \left[\left(\frac{\lambda^{p+1}}{p+1} \right)^{\frac{1}{p}} (\sigma_1 |\Upsilon'(b_2)|^q + \sigma_2 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} + \left(\frac{\mu^{p+1}}{p+1} \right)^{\frac{1}{p}} \right. \\
&\quad \times \left. (\sigma_3 |\Upsilon'(b_2)|^q + \sigma_4 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right],
\end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are already defined in Theorem 2.7.

Corollary 2.20 *Taking $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 2.7, we obtain*

$$\begin{aligned}
&\left| \frac{\lambda^\alpha \Upsilon(b_2) + \mu^\alpha \Upsilon(b_1)}{\lambda + \mu} - \frac{(b_1 b_2)^\alpha (\lambda + \mu)^{\alpha-1} \Gamma(\alpha+1)}{(b_2 - b_1)^\alpha} \left[J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right. \right. \\
&\quad \left. \left. + J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \right| \\
&\leq b_1 b_2 (b_2 - b_1) \left[\left(\frac{\lambda^{\alpha p+1}}{\alpha p+1} \right)^{\frac{1}{p}} (\sigma_1 |\Upsilon'(b_2)|^q + \sigma_2 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} + \left(\frac{\mu^{\alpha p+1}}{\alpha p+1} \right)^{\frac{1}{p}} \right. \\
&\quad \times \left. (\sigma_3 |\Upsilon'(b_2)|^q + \sigma_4 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right],
\end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are already defined in Theorem 2.7.

Corollary 2.21 Choosing $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.7, we obtain

$$\begin{aligned} & \left| \frac{\lambda^{\frac{\alpha}{k}} \Upsilon(b_2) + \mu^{\frac{\alpha}{k}} \Upsilon(b_1)}{\lambda + \mu} - \frac{k(b_1 b_2)^{\frac{\alpha}{k}} (\lambda + \mu)^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + k)}{(b_2 - b_1)^{\frac{\alpha}{k}}} \left[k J_{(\frac{1}{b_2})^+}^{\alpha} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right. \right. \\ & \quad \left. \left. + k J_{(\frac{1}{b_1})^-}^{\alpha} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\left(\frac{k \lambda^{\frac{\alpha p + k}{k}}}{\alpha p + k} \right)^{\frac{1}{p}} (\sigma_1 |\Upsilon'(b_2)|^q + \sigma_2 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} + \left(\frac{k \mu^{\frac{\alpha p + k}{k}}}{\alpha p + k} \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times (\sigma_3 |\Upsilon'(b_2)|^q + \sigma_4 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are already defined in Theorem 2.7.

Theorem 2.8 Let $\Upsilon : [b_1, b_2] \rightarrow \mathbb{R}$ be a function on (b_1, b_2) with $b_1 < b_2$ and $|\Upsilon'|^q$ be an harmonic convex function with $q \geq 1$ and $\lambda, \mu \in [0, \infty)$ with $\lambda + \mu \neq 0$, then

$$\begin{aligned} & \left| \frac{\Omega(\lambda) \Upsilon(b_2) + \Omega(\mu) \Upsilon(b_1)}{\lambda + \mu} \right. \\ & \quad \left. - \frac{1}{\lambda + \mu} \left[\left(\frac{1}{b_2} \right)^+ I_{\Phi} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) + \left(\frac{1}{b_1} \right)^- I_{\Phi} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^{\lambda} \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} d\tau \right)^{1 - \frac{1}{q}} \right. \\ & \quad \times \left(\int_0^{\lambda} \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} \left(\frac{\mu + \tau}{\lambda + \mu} |\Upsilon'(b_2)|^q + \frac{\lambda - \tau}{\lambda + \mu} |\Upsilon'(b_1)|^q \right) d\tau \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^{\mu} \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} d\tau \right)^{1 - \frac{1}{q}} \\ & \quad \times \left(\int_0^{\mu} \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} \left(\frac{\mu - \tau}{\lambda + \mu} |\Upsilon'(b_2)|^q + \frac{\lambda + \tau}{\lambda + \mu} |\Upsilon'(b_1)|^q \right) d\tau \right)^{\frac{1}{q}} \Big]. \end{aligned}$$

Proof Using Lemma 2.3, the modulus property, the power mean inequality and the harmonic convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned} & \left| \frac{\Omega(\lambda) \Upsilon(b_2) + \Omega(\mu) \Upsilon(b_1)}{\lambda + \mu} \right. \\ & \quad \left. - \frac{1}{\lambda + \mu} \left[\left(\frac{1}{b_2} \right)^+ I_{\Phi} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) + \left(\frac{1}{b_1} \right)^- I_{\Phi} \Upsilon \circ \Psi \left(\frac{\lambda b_2 + \mu b_1}{b_1 b_2 (\lambda + \mu)} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\int_0^{\lambda} \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} \left| \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau) b_2 + (\mu + \tau) b_1} \right) \right| d\tau \right. \\ & \quad \left. + \int_0^{\mu} \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} \left| \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau) b_2 + (\mu - \tau) b_1} \right) \right| d\tau \right] \\ & \leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^{\lambda} \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} d\tau \right)^{1 - \frac{1}{q}} \right. \\ & \quad \times \left(\int_0^{\lambda} \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} \right. \end{aligned}$$

$$\begin{aligned}
& \times \left| \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda - \tau) b_2 + (\mu + \tau) b_1} \right) \right| d\tau \Big)^{\frac{1}{q}} \\
& + \left(\int_0^\mu \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} d\tau \right)^{1 - \frac{1}{q}} \\
& \times \left(\int_0^\mu \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} \left| \Upsilon' \left(\frac{b_1 b_2 (\lambda + \mu)}{(\lambda + \tau) b_2 + (\mu - \tau) b_1} \right) \right| d\tau \right)^{\frac{1}{q}} \Big] \\
& \leq b_1 b_2 (b_2 - b_1) \left[\left(\int_0^\lambda \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} d\tau \right)^{1 - \frac{1}{q}} \right. \\
& \quad \times \left(\int_0^\lambda \frac{\Omega(\tau)}{((\lambda - \tau) b_2 + (\mu + \tau) b_1)^2} \right. \\
& \quad \times \left. \left(\frac{\mu + \tau}{\lambda + \mu} |\Upsilon'(b_2)|^q + \frac{\lambda - \tau}{\lambda + \mu} |\Upsilon'(b_1)|^q \right) d\tau \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^\mu \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} d\tau \right)^{1 - \frac{1}{q}} \\
& \quad \times \left. \left(\int_0^\mu \frac{\Omega(\tau)}{((\lambda + \tau) b_2 + (\mu - \tau) b_1)^2} \left(\frac{\mu - \tau}{\lambda + \mu} |\Upsilon'(b_2)|^q + \frac{\lambda + \tau}{\lambda + \mu} |\Upsilon'(b_1)|^q \right) d\tau \right)^{\frac{1}{q}} \right].
\end{aligned}$$

This completes the proof. $\square$

Corollary 2.22 *Choosing $\Phi(\tau) = \tau$ and $\lambda = \mu = 1$ in Theorem 2.8, we have*

$$\begin{aligned}
& \left| \frac{\Upsilon(b_2) + \Upsilon(b_1)}{2} - \frac{b_1 b_2}{b_2 - b_1} \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx \right| \\
& \leq b_1 b_2 (b_2 - b_1) \left[\sigma_5^{1 - \frac{1}{q}} (\sigma_6 |\Upsilon'(b_2)|^q + \sigma_7 |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right. \\
& \quad \left. + \sigma_8^{1 - \frac{1}{q}} (\sigma_9 |\Upsilon'(b_2)|^q + \sigma_{10} |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right],
\end{aligned}$$

where

$$\begin{aligned}
\sigma_5 &:= \frac{(b_1 + b_2)^{-2}}{2} {}_2F_1 \left(2, 2, 3, \frac{b_2 - b_1}{b_1 + b_2} \right), \\
\sigma_6 &:= \frac{(b_1 + b_2)^{-2}}{4} {}_2F_1 \left(2, 2, 3, \frac{b_2 - b_1}{b_1 + b_2} \right) + \frac{(b_1 + b_2)^{-2}}{12} {}_2F_1 \left(2, 3, 4, \frac{b_2 - b_1}{b_1 + b_2} \right), \\
\sigma_7 &:= \frac{(b_1 + b_2)^{-2}}{12} {}_2F_1 \left(2, 2, 4, \frac{b_2 - b_1}{b_1 + b_2} \right), \\
\sigma_8 &:= \frac{(b_1 + b_2)^{-2}}{2} {}_2F_1 \left(2, 2, 3, \frac{b_1 - b_2}{b_1 + b_2} \right), \\
\sigma_9 &:= \frac{(b_1 + b_2)^{-2}}{12} {}_2F_1 \left(2, 2, 4, \frac{b_1 - b_2}{b_1 + b_2} \right), \\
\sigma_{10} &:= \frac{(b_1 + b_2)^{-2}}{4} {}_2F_1 \left(2, 2, 3, \frac{b_1 - b_2}{b_1 + b_2} \right) + \frac{(b_1 + b_2)^{-2}}{12} {}_2F_1 \left(2, 3, 4, \frac{b_1 - b_2}{b_1 + b_2} \right).
\end{aligned}$$

Corollary 2.23 Taking $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ and $\lambda = \mu = 1$ in Theorem 2.8, we obtain

$$\begin{aligned} & \left| \frac{\Upsilon(b_2) + \Upsilon(b_1)}{2} - \frac{(b_1 b_2)^\alpha 2^{\alpha-1} \Gamma(\alpha+1)}{(b_2 - b_1)^\alpha} \left[J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi \left(\frac{b_1 + b_2}{2b_1 b_2} \right) \right. \right. \\ & \quad \left. \left. + J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi \left(\frac{b_1 + b_2}{2b_1 b_2} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\sigma_{11}^{1-\frac{1}{q}} (\sigma_{12} |\Upsilon'(b_2)|^q + \sigma_{13} |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + \sigma_{14}^{1-\frac{1}{q}} (\sigma_{15} |\Upsilon'(b_2)|^q + \sigma_{16} |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned} \sigma_{11} &:= \frac{(b_1 + b_2)^{-2}}{\alpha + 1} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_2 - b_1}{b_1 + b_2} \right), \\ \sigma_{12} &:= \frac{(b_1 + b_2)^{-2}}{2(\alpha + 1)} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_2 - b_1}{b_1 + b_2} \right) \\ & \quad + \frac{(b_1 + b_2)^{-2}}{2(\alpha + 1)(\alpha + 2)} {}_2F_1 \left(2, \alpha + 2, \alpha + 3, \frac{b_2 - b_1}{b_1 + b_2} \right), \\ \sigma_{13} &:= \frac{(b_1 + b_2)^{-2}}{2(\alpha + 1)(\alpha + 2)} {}_2F_1 \left(2, \alpha + 1, \alpha + 3, \frac{b_2 - b_1}{b_1 + b_2} \right), \\ \sigma_{14} &:= \frac{(b_1 + b_2)^{-2}}{\alpha + 1} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_1 - b_2}{b_1 + b_2} \right), \\ \sigma_{15} &:= \frac{(b_1 + b_2)^{-2}}{2(\alpha + 1)(\alpha + 2)} {}_2F_1 \left(2, \alpha + 1, \alpha + 3, \frac{b_1 - b_2}{b_1 + b_2} \right), \\ \sigma_{16} &:= \frac{(b_1 + b_2)^{-2}}{2(\alpha + 1)} {}_2F_1 \left(2, \alpha + 1, \alpha + 2, \frac{b_1 - b_2}{b_1 + b_2} \right) \\ & \quad + \frac{(b_1 + b_2)^{-2}}{2(\alpha + 1)(\alpha + 2)} {}_2F_1 \left(2, \alpha + 2, \alpha + 3, \frac{b_1 - b_2}{b_1 + b_2} \right). \end{aligned}$$

Corollary 2.24 Choosing $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\lambda = \mu = 1$ in Theorem 2.8, we obtain

$$\begin{aligned} & \left| \frac{\Upsilon(b_2) + \Upsilon(b_1)}{2} - \frac{(b_1 b_2)^{\frac{\alpha}{k}} 2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + k)}{(b_2 - b_1)^{\frac{\alpha}{k}}} \left[{}^k J_{(\frac{1}{b_2})^+}^\alpha \Upsilon \circ \Psi \left(\frac{b_1 + b_2}{2b_1 b_2} \right) \right. \right. \\ & \quad \left. \left. + {}^k J_{(\frac{1}{b_1})^-}^\alpha \Upsilon \circ \Psi \left(\frac{b_1 + b_2}{2b_1 b_2} \right) \right] \right| \\ & \leq b_1 b_2 (b_2 - b_1) \left[\sigma_{16}^{\frac{1}{p}} (\sigma_{17} |\Upsilon'(b_2)|^q + \sigma_{18} |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + \sigma_{19}^{\frac{1}{p}} (\sigma_{20} |\Upsilon'(b_2)|^q + \sigma_{21} |\Upsilon'(b_1)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\sigma_{16} := \frac{k(b_1 + b_2)^{-2}}{\alpha + k} {}_2F_1 \left(2, \alpha + k, \alpha + 2k, \frac{b_2 - b_1}{b_1 + b_2} \right),$$

$$\begin{aligned}
\sigma_{17} &:= \frac{k(b_1 + b_2)^{-2}}{2(\alpha k)} {}_2F_1\left(2, \alpha + k, \alpha + 2k, \frac{b_2 - b_1}{b_1 + b_2}\right) \\
&\quad + \frac{k(b_1 + b_2)^{-2}}{2(\alpha + k)(\alpha + 2k)} {}_2F_1\left(2, \alpha + 2k, \alpha + 3k, \frac{b_2 - b_1}{b_1 + b_2}\right), \\
\sigma_{18} &:= \frac{k(b_1 + b_2)^{-2}}{2(\alpha + k)(\alpha + 2k)} {}_2F_1\left(2, \alpha + k, \alpha + 3k, \frac{b_2 - b_1}{b_1 + b_2}\right), \\
\sigma_{19} &:= \frac{k(b_1 + b_2)^{-2}}{\alpha + k} {}_2F_1\left(2, \alpha + k, \alpha + 2k, \frac{b_1 - b_2}{b_1 + b_2}\right), \\
\sigma_{20} &:= \frac{k(b_1 + b_2)^{-2}}{2(\alpha + k)(\alpha + 2k)} {}_2F_1\left(2, \alpha + k, \alpha + 3k, \frac{b_1 - b_2}{b_1 + b_2}\right), \\
\sigma_{21} &:= \frac{k(b_1 + b_2)^{-2}}{2(\alpha + k)} {}_2F_1\left(2, \alpha + k, \alpha + 2k, \frac{b_1 - b_2}{b_1 + b_2}\right) \\
&\quad + \frac{k(b_1 + b_2)^{-2}}{2(\alpha + k)(\alpha + 2k)} {}_2F_1\left(2, \alpha + 2k, \alpha + 3k, \frac{b_1 - b_2}{b_1 + b_2}\right).
\end{aligned}$$

Theorem 2.9 Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$. If $|\Upsilon'|^q$ is an harmonic convex function with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
&\left| \Upsilon\left(\frac{2b_1b_2}{b_1 + b_2}\right) - \frac{1}{2\Delta(1)} \left[{}_{\frac{1}{b_2}^+} I_{\Phi} \Upsilon \circ \Psi\left(\frac{1}{b_1}\right) + {}_{\frac{1}{b_1}^-} I_{\Phi} \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) \right] \right| \\
&\leq \frac{b_1b_2(b_2 - b_1)}{2\Delta(1)} \left(\left(\int_0^{\frac{1}{2}} |\Delta(\tau)|^p d\tau \right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^1 |\delta(\tau)|^p d\tau \right)^{\frac{1}{p}} \right) \\
&\quad \times \left((N_1 |\Upsilon'(b_1)|^q + N_2 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} + (N_3 |\Upsilon'(b_1)|^q + N_4 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right),
\end{aligned}$$

where

$$\begin{aligned}
N_1 &:= \int_0^{\frac{1}{2}} \frac{(1 - \tau)}{(\tau b_1 + (1 - \tau)b_2)^{2q}} d\tau \\
&= \frac{2^{2-2q}b_2^{2-2q} - (b_1 + b_2)^{2-2q}}{(b_2 - b_1)^2(2 - 2q)2^{2-2q}} - b_1 \frac{2^{1-2q}b_2^{1-2q} - (b_1 + b_2)^{1-2q}}{(b_2 - b_1)^2(1 - 2q)2^{1-2q}}, \\
N_2 &:= \int_0^{\frac{1}{2}} \frac{\tau}{(\tau b_1 + (1 - \tau)b_2)^{2q}} d\tau \\
&= b_2 \frac{2^{1-2q}b_2^{1-2q} - (b_1 + b_2)^{1-2q}}{(b_2 - b_1)^2(1 - 2q)2^{1-2q}} - \frac{2^{2-2q}b_2^{2-2q} - (b_1 + b_2)^{2-2q}}{(b_2 - b_1)^2(2 - 2q)2^{2-2q}}, \\
N_3 &:= \int_0^{\frac{1}{2}} \frac{\tau}{((1 - \tau)\tau b_1 + \tau b_2)^{2q}} d\tau \\
&= \frac{(b_1 + b_2)^{2-2q} - 2^{2-2q}b_1^{2-2q}}{(b_2 - b_1)^2(2 - 2q)2^{2-2q}} - b_1 \frac{(b_1 + b_2)^{1-2q} - 2^{1-2q}b_1^{1-2q}}{(b_2 - b_1)^2(1 - 2q)2^{1-2q}}, \\
N_4 &:= \int_0^{\frac{1}{2}} \frac{1 - \tau}{((1 - \tau)\tau b_1 + \tau b_2)^{2q}} d\tau \\
&= b_2 \frac{(b_1 + b_2)^{1-2q} - 2^{1-2q}b_1^{1-2q}}{(b_2 - b_1)^2(1 - 2q)2^{1-2q}} - \frac{(b_1 + b_2)^{2-2q} - 2^{2-2q}b_1^{2-2q}}{(b_2 - b_1)^2(2 - 2q)2^{2-2q}},
\end{aligned}$$

$$\begin{aligned}
N_5 &:= \int_{\frac{1}{2}}^1 \frac{1-\tau}{(\tau b_1 + (1-\tau)b_2)^{2q}} d\tau \\
&= \frac{(b_1 + b_2)^{2-2q} - 2^{2-2q} b_1^{2-2q}}{(b_2 - b_1)^2 (2-2q) 2^{2-2q}} - b_1 \frac{(b_1 + b_2)^{1-2q} - 2^{1-2q} b_1^{1-2q}}{(b_2 - b_1)^2 (1-2q) 2^{1-2q}}, \\
N_6 &:= \int_{\frac{1}{2}}^1 \frac{\tau}{(\tau b_1 + (1-\tau)b_2)^{2q}} d\tau \\
&= b_2 \frac{(b_1 + b_2)^{1-2q} - 2^{1-2q} b_1^{1-2q}}{(b_2 - b_1)^2 (1-2q) 2^{1-2q}} - \frac{(b_1 + b_2)^{2-2q} - 2^{2-2q} b_1^{2-2q}}{(b_2 - b_1)^2 (2-2q) 2^{2-2q}}, \\
N_7 &:= \int_{\frac{1}{2}}^1 \frac{\tau}{((1-\tau)b_1 + \tau b_2)^{2q}} d\tau \\
&= \frac{2^{2-2q} b_2^{2-2q} - (b_1 + b_2)^{2-2q}}{(b_2 - b_1)^2 (2-2q) 2^{2-2q}} - b_1 \frac{2^{1-2q} b_2^{1-2q} - (b_1 + b_2)^{1-2q}}{(b_2 - b_1)^2 (1-2q) 2^{1-2q}}, \\
N_8 &:= \int_{\frac{1}{2}}^1 \frac{1-\tau}{((1-\tau)b_1 + \tau b_2)^{2q}} d\tau \\
&= b_2 \frac{2^{1-2q} b_2^{1-2q} - (b_1 + b_2)^{1-2q}}{(b_2 - b_1)^2 (1-2q) 2^{1-2q}} - \frac{2^{2-2q} b_2^{2-2q} - (b_1 + b_2)^{2-2q}}{(b_2 - b_1)^2 (2-2q) 2^{2-2q}}.
\end{aligned}$$

Also, it is easy to verify that $N_1 = N_7$, $N_2 = N_8$, $N_3 = N_5$ and $N_4 = N_6$.

Proof By using Lemma 2.4, the property of modulus, Hölder's inequality and the harmonic convexity of $|\Upsilon'|^q$, we obtain the desired result. We omit here the proof. $\square$

Corollary 2.25 Taking $\Phi(\tau) = \tau$ in Theorem 2.9, then

$$\begin{aligned}
&\left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{b_1b_2}{b_2-b_1} \int_{b_1}^{b_2} \frac{\Upsilon(x)}{x^2} dx \right| \\
&\leq \frac{b_1b_2(b_2-b_1)}{2} \left(\frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \\
&\quad \times \left((N_1 |\Upsilon'(b_1)|^q + N_2 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} + (N_3 |\Upsilon'(b_1)|^q + N_4 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right).
\end{aligned}$$

Corollary 2.26 Choosing $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 2.9, then

$$\begin{aligned}
&\left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{\Gamma(\alpha+1)}{2} \left(\frac{b_1b_2}{b_2-b_1} \right)^\alpha \left[J_{\frac{1}{b_2}}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_1}\right) + J_{\frac{1}{b_1}}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) \right] \right| \\
&\leq \frac{b_1b_2(b_2-b_1)}{2} \left(\frac{(1+2^{\alpha p-1}(\alpha p-1))}{2^{\alpha p}(\alpha p+1)} \right)^{\frac{1}{p}} \\
&\quad \times \left((N_1 |\Upsilon'(b_1)|^q + N_2 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} + (N_3 |\Upsilon'(b_1)|^q + N_4 |\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right).
\end{aligned}$$

Corollary 2.27 Taking $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.9, then

$$\begin{aligned} & \left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{\Gamma_k(\alpha+k)}{2} \left(\frac{b_1b_2}{b_2-b_1}\right)^{\frac{\alpha}{k}} \left[J_{k, \frac{1}{b_2}}^{\alpha} \Upsilon \circ \Psi\left(\frac{1}{b_1}\right) + J_{k, \frac{1}{b_1}}^{\alpha} \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) \right] \right| \\ & \leq \frac{b_1b_2(b_2-b_1)}{2} \left(\frac{(k+2)^{\frac{\alpha p}{k}-1}(\alpha p-k)}{2^{\frac{\alpha p}{k}}(\alpha p+k)} \right)^{\frac{1}{p}} \\ & \quad \times \left((N_1|\Upsilon'(b_1)|^q + N_2|\Upsilon'(b_2)|^q)^{\frac{1}{q}} + (N_3|\Upsilon'(b_1)|^q + N_4|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right). \end{aligned}$$

Theorem 2.10 Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$. If $|\Upsilon'|^q$ is an harmonic convex function with $q \geq 1$, then

$$\begin{aligned} & \left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{1}{2\Delta(1)} \left[I_{\Phi, \frac{1}{b_2}} \Upsilon \circ \Psi\left(\frac{1}{b_1}\right) + I_{\Phi, \frac{1}{b_1}} \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) \right] \right| \\ & \leq \frac{b_1b_2(b_2-b_1)}{2\Delta(1)} \\ & \quad \times \left[\left(\int_0^{\frac{1}{2}} |\Delta(\tau)| d\tau \right)^{1-\frac{1}{q}} \left\{ \left(\int_0^{\frac{1}{2}} \frac{(1-\tau)|\Delta(\tau)|}{(\tau b_1 + (1-\tau)b_2)^2} d\tau |\Upsilon'(b_1)|^q \right. \right. \right. \\ & \quad \left. \left. + \int_0^{\frac{1}{2}} \frac{\tau|\Delta(\tau)|}{(\tau b_1 + (1-\tau)b_2)^2} d\tau |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^{\frac{1}{2}} \frac{\tau|\Delta(\tau)|}{((1-\tau)b_1 + \tau b_2)^2} d\tau |\Upsilon'(b_1)|^q + \int_0^{\frac{1}{2}} \frac{(1-\tau)|\Delta(\tau)|}{((1-\tau)b_1 + \tau b_2)^2} d\tau |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right\} \\ & \quad + \left(\int_{\frac{1}{2}}^1 |\delta(\tau)| d\tau \right)^{1-\frac{1}{q}} \left\{ \left(\int_{\frac{1}{2}}^1 \frac{(1-\tau)|\delta(\tau)|}{(\tau b_1 + (1-\tau)b_2)^2} d\tau |\Upsilon'(b_1)|^q \right. \right. \\ & \quad \left. \left. + \int_{\frac{1}{2}}^1 \frac{\tau|\delta(\tau)|}{(\tau b_1 + (1-\tau)b_2)^2} d\tau |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 \frac{\tau|\delta(\tau)|}{((1-\tau)b_1 + \tau b_2)^2} d\tau |\Upsilon'(b_1)|^q \right. \right. \\ & \quad \left. \left. + \int_{\frac{1}{2}}^1 \frac{(1-\tau)|\delta(\tau)|}{((1-\tau)b_1 + \tau b_2)^2} d\tau |\Upsilon'(b_2)|^q \right)^{\frac{1}{q}} \right\} \Bigg]. \end{aligned}$$

Proof By using Lemma 2.4, the property of modulus, the power mean inequality and the convexity of $|\Upsilon'|^q$ we obtain the desired result. We omit here the proof. $\square$

Corollary 2.28 Taking $\Phi(\tau) = \tau$ in Theorem 2.10, then

$$\begin{aligned} & \left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{b_1b_2}{b_2-b_1} \int_{b_1}^{b_2} \frac{\Upsilon(x)}{x^2} dx \right| \\ & \leq \frac{b_1b_2(b_2-b_1)}{2} \left(\frac{1}{8} \right)^{1-\frac{1}{q}} \\ & \quad \times \left[(M_1|\Upsilon'(b_1)|^q + M_2|\Upsilon'(b_2)|^q)^{\frac{1}{q}} + (M_3|\Upsilon'(b_1)|^q + M_4|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + (M_2|\Upsilon'(b_1)|^q + M_1|\Upsilon'(b_2)|^q)^{\frac{1}{q}} + (M_4|\Upsilon'(b_1)|^q + M_3|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned} M_1 &:= \frac{b_2^{-2}}{8} {}_2F_1\left(2, 2, 3, \frac{b_2 - b_1}{2b_2}\right) - \frac{b_2^{-2}}{24} {}_2F_1\left(2, 3, 4, \frac{b_2 - b_1}{2b_2}\right), \\ M_2 &:= \frac{b_2^{-2}}{24} {}_2F_1\left(2, 3, 4, \frac{b_2 - b_1}{2b_2}\right), \quad M_3 := \frac{b_1^{-2}}{24} {}_2F_1\left(2, 3, 4, \frac{b_1 - b_2}{2b_1}\right), \\ M_4 &:= \frac{b_1^{-2}}{8} {}_2F_1\left(2, 2, 3, \frac{b_1 - b_2}{2b_1}\right) - \frac{b_1^{-2}}{24} {}_2F_1\left(2, 3, 4, \frac{b_1 - b_2}{2b_1}\right). \end{aligned}$$

Corollary 2.29 Choosing $\Phi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ in Theorem 2.10, then

$$\begin{aligned} & \left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{\Gamma(\alpha+1)}{2} \left(\frac{b_1b_2}{b_2-b_1}\right)^\alpha \left[J_{\frac{1}{b_2}^+}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_1}\right) + J_{\frac{1}{b_1}^-}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) \right] \right| \\ & \leq \frac{b_1b_2(b_2-b_1)}{2} \\ & \quad \times \left[\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}} \left\{ (M_5|\Upsilon'(b_1)|^q + M_6|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right. \right. \\ & \quad \left. \left. + (M_7|\Upsilon'(b_1)|^q + M_8|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right\} \right. \\ & \quad \left. + \left(\frac{2^\alpha(\alpha-1)+1}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}} \left\{ (M_6|\Upsilon'(b_1)|^q + M_5|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right. \right. \\ & \quad \left. \left. + (M_8|\Upsilon'(b_1)|^q + M_7|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right\} \right], \end{aligned}$$

where

$$\begin{aligned} M_5 &:= \frac{b_2^{-2}}{2^{\alpha+1}(\alpha+1)} {}_2F_1\left(2, \alpha+1, \alpha+2, \frac{b_2-b_1}{2b_2}\right) \\ & \quad - \frac{b_2^{-2}}{2^{\alpha+2}(\alpha+2)} {}_2F_1\left(2, \alpha+2, \alpha+3, \frac{b_2-b_1}{2b_2}\right), \\ M_6 &:= \frac{b_2^{-2}}{2^{\alpha+2}(\alpha+2)} {}_2F_1\left(2, \alpha+2, \alpha+3, \frac{b_2-b_1}{2b_2}\right), \\ M_7 &:= \frac{b_1^{-2}}{2^{\alpha+2}(\alpha+2)} {}_2F_1\left(2, \alpha+2, \alpha+3, \frac{b_1-b_2}{2b_1}\right), \\ M_8 &:= \frac{b_1^{-2}}{2^{\alpha+1}(\alpha+1)} {}_2F_1\left(2, \alpha+1, \alpha+2, \frac{b_1-b_2}{2b_1}\right) \\ & \quad - \frac{b_1^{-2}}{2^{\alpha+2}(\alpha+2)} {}_2F_1\left(2, \alpha+2, \alpha+3, \frac{b_1-b_2}{2b_1}\right). \end{aligned}$$

Corollary 2.30 Taking $\Phi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.10, then

$$\begin{aligned} & \left| \Upsilon\left(\frac{2b_1b_2}{b_1+b_2}\right) - \frac{\Gamma_k(\alpha+k)}{2} \left(\frac{b_1b_2}{b_2-b_1}\right)^{\frac{\alpha}{k}} \left[J_{\frac{1}{b_2}^+}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_1}\right) + J_{\frac{1}{b_1}^-}^\alpha \Upsilon \circ \Psi\left(\frac{1}{b_2}\right) \right] \right| \\ & \quad \times \left[\left(\frac{k}{2^{\frac{\alpha}{k}+1}(\alpha+k)}\right)^{1-\frac{1}{q}} \left\{ (M_9|\Upsilon'(b_1)|^q + M_{10}|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + (M_{11}|\Upsilon'(b_1)|^q + M_{12}|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \} \\
& + \left(\frac{2^{\frac{\alpha}{k}}(\alpha - k) + k}{2^{\frac{\alpha}{k}+1}(\alpha + k)} \right)^{1-\frac{1}{q}} \{ (M_{10}|\Upsilon'(b_1)|^q + M_9|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \\
& + (M_{12}|\Upsilon'(b_1)|^q + M_{11}|\Upsilon'(b_2)|^q)^{\frac{1}{q}} \} \Big],
\end{aligned}$$

where

$$\begin{aligned}
M_9 &:= \frac{kb_2^{-2}}{2^{\frac{\alpha}{k}+1}(\alpha + k)} {}_2F_1\left(2, \alpha + k, \alpha + 2k, \frac{1}{k}\left(\frac{b_2 - b_1}{2b_2}\right)\right) \\
&\quad - \frac{kb_2^{-2}}{2^{\frac{\alpha}{k}+2}(\alpha + 2k)} {}_2F_1\left(2, \alpha + 2k, \alpha + 3k, \frac{1}{k}\left(\frac{b_2 - b_1}{2b_2}\right)\right), \\
M_{10} &:= \frac{kb_2^{-2}}{2^{\frac{\alpha}{k}+2}(\alpha + 2k)} {}_2F_1\left(2, \alpha + 2k, \alpha + 3k, \frac{1}{k}\left(\frac{b_2 - b_1}{2b_2}\right)\right), \\
M_{11} &:= \frac{kb_1^{-2}}{2^{\frac{\alpha}{k}+2}(\alpha + 2k)} {}_2F_1\left(2, \alpha + 2k, \alpha + 3k, \frac{1}{k}\left(\frac{b_1 - b_2}{2b_1}\right)\right), \\
M_{12} &:= \frac{kb_1^{-2}}{2^{\frac{\alpha}{k}+1}(\alpha + k)} {}_2F_1\left(2, \alpha + k, \alpha + 2k, \frac{1}{k}\left(\frac{b_1 - b_2}{2b_1}\right)\right) \\
&\quad - \frac{kb_1^{-2}}{2^{\frac{\alpha}{k}+2}(\alpha + 2k)} {}_2F_1\left(2, \alpha + 2k, \alpha + 3k, \frac{1}{k}\left(\frac{b_1 - b_2}{2b_1}\right)\right).
\end{aligned}$$

Remark For other suitable choices of function Φ , several new interesting inequalities can be found from our results. We omit here their proofs and the details are left to the interested reader.

3 Applications

3.1 Application to special means

We shall consider the following special means for different positive real numbers b_1 and b_2 , where $b_1 < b_2$:

- Arithmetic mean: $\mathcal{A}(b_1, b_2) = \frac{b_1 + b_2}{2}$;
- Harmonic mean: $\mathcal{H}(b_1, b_2) = \frac{2b_1b_2}{b_1 + b_2}$;
- r -Logarithmic mean: $\mathcal{L}_r(b_1, b_2) = \left(\frac{b_2^{r+1} - b_1^{r+1}}{(r+1)(b_2 - b_1)}\right)^{\frac{1}{r}}$, $r \in \mathbb{R} \setminus \{0, -1\}$.

Using our results, we are in a position to prove the following inequalities regarding the above special means.

Proposition 3.1 *Let $r, b_1, b_2 \in \mathbb{R}$, $0 < b_1 < b_2$ with $r \geq 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
& |A(b_1^{\frac{r}{q}+2}, b_2^{\frac{r}{q}+2}) - \frac{q}{r+q} b_1 b_2 \mathcal{L}_{\frac{r}{q}}(b_1, b_2)| \\
& \leq b_1 b_2 (b_2 - b_1) \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[(\sigma_1^* b_2^{r+q} + \sigma_2^* b_1^{r+q})^{\frac{1}{q}} + (\sigma_3^* b_2^{r+q} + \sigma_4^* b_1^{r+q})^{\frac{1}{q}} \right],
\end{aligned}$$

where

$$\begin{aligned}
 \sigma_1^* &:= \int_0^1 \frac{((1-\tau)b_2 + (1+\tau)b_1)^{-2q}(1+\tau)}{2} d\tau \\
 &= \frac{(b_2 + b_1)^{1-2q} - 2(2b_1)^{1-2q}}{2(b_2 - b_1)(1-2q)} - \frac{(2b_1)^{2-2q} - (b_2 + b_1)^{2-2q}}{2(b_2 - b_1)^2(1-2q)(2-2q)}, \\
 \sigma_2^* &:= \int_0^1 \frac{((1-\tau)b_2 + (1+\tau)b_1)^{-2q}(1-\tau)}{2} d\tau \\
 &= \frac{(b_2 + b_1)^{1-2q}}{2(b_2 - b_1)(1-2q)} + \frac{(2b_1)^{2-2q} - (b_1 + b_2)^{2-2q}}{2(b_2 - b_1)^2(1-2q)(2-2q)}, \\
 \sigma_3^* &:= \int_0^1 \frac{((1+\tau)b_2 + (1-\tau)b_1)^{-2q}(1-\tau)}{2} d\tau \\
 &= \frac{(2b_2)^{2-2q} - (b_2 + b_1)^{2-2q}}{2(b_2 - b_1)^2(1-2q)(2-2q)} - \frac{(b_2 + b_1)^{1-2q}}{2(b_2 - b_1)(1-2q)}, \\
 \sigma_4^* &:= \int_0^1 \frac{((1+\tau)b_2 + (1-\tau)b_1)^{-2q}(1+\tau)}{2} d\tau \\
 &= \frac{(2b_2)^{1-2q} - (b_2 + b_1)^{1-2q}}{2(b_2 - b_1)(1-2q)} - \frac{(2b_2)^{2-2q} - (b_2 + b_1)^{2-2q}}{2(b_2 - b_1)^2(1-2q)(2-2q)}.
 \end{aligned}$$

Proof Taking the harmonic convex function $\Upsilon(\tau) = \frac{q}{r+2q} \tau^{\frac{r}{q}+2}$ for all $\tau > 0$ in Corollary 2.19 and $\lambda = \mu = 1$, we have the desired result. $\square$

Proposition 3.2 Let $r, b_1, b_2 \in \mathbb{R}$, $0 < b_1 < b_2$ with $r \geq 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
 &\left| \mathcal{H}^{\frac{r}{q}+2}(b_1, b_2) - \frac{q}{r+q} b_1 b_2 \mathcal{L}^{\frac{r}{q}}(b_1, b_2) \right| \\
 &\leq \frac{(r+2q)b_1 b_2 (b_2 - b_1)}{q} \left(\frac{1}{2^{p+2}(p+1)} \right)^{\frac{1}{p}} \\
 &\quad \times (\mathcal{A}^{\frac{1}{q}}(N_1 b_1^{r+q}, N_2 b_2^{r+q}) + \mathcal{A}^{\frac{1}{q}}(N_3 b_1^{r+q}, N_4 b_2^{r+q})).
 \end{aligned}$$

Proof Choosing the harmonic convex function $\Upsilon(\tau) = \frac{q}{r+2q} \tau^{\frac{r}{q}+2}$ for all $\tau > 0$ in Corollary 2.25, we obtain the desired result. $\square$

Proposition 3.3 Let $r, b_1, b_2 \in \mathbb{R}$, $0 < b_1 < b_2$ with $r, q \geq 1$, then

$$\begin{aligned}
 &\left| \mathcal{H}^{\frac{r}{q}+2}(b_1, b_2) - \frac{q}{r+q} b_1 b_2 \mathcal{L}^{\frac{r}{q}}(b_1, b_2) \right| \\
 &\leq \frac{(r+2q)b_1 b_2 (b_2 - b_1)}{q} \left(\frac{1}{16} \right)^{1-\frac{1}{q}} \\
 &\quad \times [\mathcal{A}^{\frac{1}{q}}(M_1 b_1^{r+q}, M_2 b_2^{r+q}) + \mathcal{A}^{\frac{1}{q}}(M_3 b_1^{r+q}, M_4 b_2^{r+q}) \\
 &\quad + \mathcal{A}^{\frac{1}{q}}(M_2 b_1^{r+q}, M_1 b_2^{r+q}) + \mathcal{A}^{\frac{1}{q}}(M_4 b_1^{r+q}, M_3 b_2^{r+q})].
 \end{aligned}$$

Proof Taking the harmonic convex function $\Upsilon(\tau) = \frac{q}{r+2q} \tau^{\frac{r}{q}+2}$ for all $\tau > 0$ in Corollary 2.28, we obtain the desired result. $\square$

3.2 Application to error estimations

Finally, let us consider some applications of the integral inequalities obtained above to find new error bounds for the following quadrature formulas. Let $\mathcal{P} : b_1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = b_2$ be a partition of $[b_1, b_2] \subset (0, +\infty)$. We denote, respectively

$$\begin{aligned} \mathcal{T}_1(\mathcal{P}, \Upsilon) &:= \sum_{i=0}^{n-1} \frac{(\Upsilon(x_i) + \Upsilon(x_{i+1}))}{(m+1)x_i x_{i+1}} h_i, & \mathcal{T}_2(\mathcal{P}, \Upsilon) &:= \sum_{i=0}^{n-1} \frac{\Upsilon\left(\frac{2x_i x_{i+1}}{x_i + x_{i+1}}\right)}{x_i x_{i+1}} h_i, \\ \int_{\frac{1}{b_2}}^{\frac{1}{b_1}} \Upsilon \circ \Psi(x) dx &:= \mathcal{T}_1(\mathcal{P}, \Upsilon) + \mathcal{R}_1(\mathcal{P}, \Upsilon), \\ \int_{b_1}^{b_2} \frac{\Upsilon(x)}{x^2} dx &:= \mathcal{T}_2(\mathcal{P}, \Upsilon) + \mathcal{R}_2(\mathcal{P}, \Upsilon), \end{aligned} \quad (3.1)$$

where $m \in \mathbb{N}$, and $\mathcal{R}_1(\mathcal{P}, \Upsilon)$ and $\mathcal{R}_2(\mathcal{P}, \Upsilon)$ are the remainder terms and $h_i = x_{i+1} - x_i$ for $i = 0, 1, 2, \dots, n-1$. Using the above notations, we are in a position to prove the following error estimations.

Proposition 3.4 *Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$ and $m \in \mathbb{N}$. If $|\Upsilon'|^q$ is an harmonic convex function with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} |\mathcal{R}_1(\mathcal{P}, \Upsilon)| &\leq \sum_{i=0}^{n-1} h_i^2 \left[\pi_{i,1}^{\frac{1}{p}} \left(\frac{1}{(m+1)(q+1)(q+2)} |\Upsilon'(x_i)|^q + \frac{m(q+2) + (q+1)}{(m+1)(q+1)(q+2)} |\Upsilon'(x_{i+1})|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \pi_{i,2}^{\frac{1}{p}} \left(\frac{1}{(m+1)(q+1)(q+2)} |\Upsilon'(x_{i+1})|^q + \frac{m(q+2) + (q+1)}{(m+1)(q+1)(q+2)} |\Upsilon'(x_i)|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned} \pi_{i,1} &:= \frac{(mx_i + x_{i+1})^{1-2p}}{h_i(1-2p)} \left[1 - \left(\frac{(m+1)x_i}{mx_i + x_{i+1}} \right)^{1-2p} \right], \\ \pi_{i,2} &:= \frac{(x_i + mx_{i+1})^{1-2p}}{h_i(1-2p)} \left[\left(\frac{(m+1)x_{i+1}}{x_i + mx_{i+1}} \right)^{1-2p} - 1 \right]. \end{aligned}$$

Proof By using Corollary 2.13 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \dots, n-1$) of the partition $\mathcal{P}$ and summing the obtained inequality over i from 0 to $n-1$, we have the desired result. $\square$

Proposition 3.5 *Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$ and $m \in \mathbb{N}$. If $|\Upsilon'|^q$ is an harmonic convex function with $q \geq 1$, then*

$$\begin{aligned} |\mathcal{R}_1(\mathcal{P}, \Upsilon)| &\leq \sum_{i=0}^{n-1} h_i^2 \left[\pi_{i,3}^{1-\frac{1}{q}} \left(\pi_{i,4} |\Upsilon'(x_i)|^q + \pi_{i,5} |\Upsilon'(x_{i+1})|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \pi_{i,6}^{1-\frac{1}{q}} \left(\pi_{i,7} |\Upsilon'(x_i)|^q + \pi_{i,8} |\Upsilon'(x_{i+1})|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned}\pi_{i,3} &:= \frac{(mx_i + x_{i+1})^{-2}}{2} {}_2F_1\left(2, 2, 3, \frac{h_i}{mx_i + x_{i+1}}\right), \\ \pi_{i,4} &:= \frac{(mx_i + x_{i+1})^{-2}}{6(m+1)} {}_2F_1\left(2, 2, 4, \frac{h_i}{mx_i + x_{i+1}}\right), \\ \pi_{i,5} &:= \frac{m(mx_i + x_{i+1})^{-2}}{2(m+1)} {}_2F_1\left(2, 2, 3, \frac{h_i}{mx_i + x_{i+1}}\right) \\ &\quad + \frac{(mx_i + x_{i+1})^{-2}}{6(m+1)} {}_2F_1\left(2, 3, 4, \frac{h_i}{mx_i + x_{i+1}}\right), \\ \pi_{i,6} &:= \frac{(x_i + mx_{i+1})^{-2}}{2} {}_2F_1\left(2, 2, 3, -\frac{h_i}{x_i + mx_{i+1}}\right), \\ \pi_{i,7} &:= \frac{m(x_i + mx_{i+1})^{-2}}{2(m+1)} {}_2F_1\left(2, 2, 3, -\frac{h_i}{x_i + mx_{i+1}}\right) \\ &\quad + \frac{(x_i + mx_{i+1})^{-2}}{6(m+1)} {}_2F_1\left(2, 3, 4, -\frac{h_i}{x_i + mx_{i+1}}\right), \\ \pi_{i,8} &:= \frac{(x_i + mx_{i+1})^{-2}}{6(m+1)} {}_2F_1\left(2, 2, 4, -\frac{h_i}{x_i + mx_{i+1}}\right).\end{aligned}$$

Proof By applying Corollary 2.16 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \dots, n-1$) of the partition $\mathcal{P}$ and summing the obtained inequality over i from 0 to $n-1$, we obtain the desired result. $\square$

Proposition 3.6 Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$. If $|\Upsilon'|^q$ is an harmonic convex function with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}& |\mathcal{R}_2(\mathcal{P}, \Upsilon)| \\ & \leq \frac{1}{2} \left(\frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \sum_{i=0}^{n-1} h_i^2 \left((N_{i,1} |\Upsilon'(x_i)|^q + N_{i,2} |\Upsilon'(x_{i+1})|^q)^{\frac{1}{q}} + (N_{i,3} |\Upsilon'(x_i)|^q + N_{i,4} |\Upsilon'(x_{i+1})|^q)^{\frac{1}{q}} \right),\end{aligned}$$

where

$$\begin{aligned}N_{i,1} &:= \frac{2^{2-2q} x_{i+1}^{2-2q} - (x_i + x_{i+1})^{2-2q}}{h_i^2 (2-2q) 2^{2-2q}} - x_i \frac{2^{1-2q} x_{i+1}^{1-2q} - (x_i + x_{i+1})^{1-2q}}{h_i^2 (1-2q) 2^{1-2q}}, \\ N_{i,2} &:= x_{i+1} \frac{2^{1-2q} x_{i+1}^{1-2q} - (x_i + x_{i+1})^{1-2q}}{h_i^2 (1-2q) 2^{1-2q}} - \frac{2^{2-2q} x_{i+1}^{2-2q} - (x_i + x_{i+1})^{2-2q}}{h_i^2 (2-2q) 2^{2-2q}}, \\ N_{i,3} &:= \frac{(x_i + x_{i+1})^{2-2q} - 2^{2-2q} x_i^{2-2q}}{h_i^2 (2-2q) 2^{2-2q}} - x_i \frac{(x_i + x_{i+1})^{1-2q} - 2^{1-2q} x_i^{1-2q}}{h_i^2 (1-2q) 2^{1-2q}}, \\ N_{i,4} &:= x_{i+1} \frac{(x_i + x_{i+1})^{1-2q} - 2^{1-2q} x_i^{1-2q}}{h_i^2 (1-2q) 2^{1-2q}} - \frac{(x_i + x_{i+1})^{2-2q} - 2^{2-2q} x_i^{2-2q}}{h_i^2 (2-2q) 2^{2-2q}}.\end{aligned}$$

Proof By using Corollary 2.25 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \dots, n-1$) of the partition $\mathcal{P}$ and summing the obtained inequality over i from 0 to $n-1$, we have the desired result. $\square$

Proposition 3.7 *Let $\Upsilon : [b_1, b_2] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on (b_1, b_2) with $b_1 < b_2$. If $|\Upsilon'|^q$ is an harmonic convex function with $q \geq 1$, then*

$$\begin{aligned} & |\mathcal{R}_2(\mathcal{P}, \Upsilon)| \\ & \leq \frac{1}{2} \left(\frac{1}{8} \right)^{1-\frac{1}{q}} \\ & \quad \times \sum_{i=0}^{n-1} h_i^2 \left[(M_{i,1} |\Upsilon'(x_i)|^q + M_{i,2} |\Upsilon'(x_{i+1})|^q)^{\frac{1}{q}} + (M_{i,3} |\Upsilon'(x_i)|^q + M_{i,4} |\Upsilon'(x_{i+1})|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + (M_{i,2} |\Upsilon'(x_i)|^q + M_{i,1} |\Upsilon'(x_{i+1})|^q)^{\frac{1}{q}} + (M_{i,4} |\Upsilon'(x_i)|^q + M_{i,3} |\Upsilon'(x_{i+1})|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$\begin{aligned} M_{i,1} &:= \frac{x_{i+1}^{-2}}{8} {}_2F_1\left(2, 2, 3, \frac{h_i}{2x_{i+1}}\right) - \frac{x_{i+1}^{-2}}{24} {}_2F_1\left(2, 3, 4, \frac{h_i}{2x_{i+1}}\right), \\ M_{i,2} &:= \frac{x_{i+1}^{-2}}{24} {}_2F_1\left(2, 3, 4, \frac{h_i}{2x_{i+1}}\right), \quad M_{i,3} := \frac{x_i^{-2}}{24} {}_2F_1\left(2, 3, 4, -\frac{h_i}{2x_i}\right), \\ M_{i,4} &:= \frac{x_i^{-2}}{8} {}_2F_1\left(2, 2, 3, -\frac{h_i}{2x_i}\right) - \frac{x_i^{-2}}{24} {}_2F_1\left(2, 3, 4, -\frac{h_i}{2x_i}\right). \end{aligned}$$

Proof By applying Corollary 2.28 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \dots, n-1$) of the partition $\mathcal{P}$ and summing the obtained inequality over i from 0 to $n-1$, we obtain the desired result. $\square$

Remark For suitable choices of function Υ , we can obtain new inequalities using special means. Moreover, we can establish new bounds regarding error estimations of the quadrature formulas given above. We omit here their proofs and the details are left to the interested reader.

4 Conclusion

In this paper, we have derived some generalizations of fractional trapezium-like inequalities using the class of harmonic convex functions. Moreover, three new fractional integral identities are given and on applying them as auxiliary results some interesting integral inequalities are found. Our results unified many known ones, and they related some other unrelated results as well. Finally, some applications to special means for different positive real numbers and error estimations for quadrature formulas are obtained. This shows the efficiency of our results. To the best of our knowledge, these results are new in the literature and we believe that they will have a very deep impact in this field of inequalities, and also in pure and applied sciences.

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Author details

¹Department of Mathematics, Government College University, Faisalabad, Pakistan. ²Department of Mathematics, Faculty of Technical Science, University "Ismael Qemali", 9400, Vlorë, Albania. ³Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Al-Kharj, Saudi Arabia. ⁴Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & KMUTT Fixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Department of Mathematics, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand. ⁵Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan. ⁶NCAO Research Center, Fixed Point Theory and Applications Research Group, Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.

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