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Optimal control for cooperative systems involving fractional Laplace operators

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Abstract

In this work, the elliptic 2×2 cooperative systems involving fractional Laplace operators are studied. Due to the nonlocality of the fractional Laplace operator, we reformulate the problem into a local problem by an extension problem. Then, the existence and uniqueness of the weak solution for these systems are proved. Hence, the existence and optimality conditions are deduced.

MSC: Optimal control; Weak solution; Lax–Milgram lemma; Fractional Laplace operator; Nonlocal operator; Sobolev spaces; Cooperative systems

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1 Introduction

Nonlocal operators have been a useful area of investigation in different branches of mathematics such as operator theory and harmonic analysis. Also, they have gained vital attention because of their strong connection with real-world problems since they form a fundamental part of the modeling and simulation of complex phenomena that span vastly different length scales.

Nonlocal operators appear in several applications such as image processing, boundary-control problems, electromagnetic fluids, materials science, porous-media flow, turbulence, optimization, nonlocal continuum field theories, and others. Consequently, the domain of definition Ω may be in its general form.

In this paper, we discuss the elliptic 2×2 cooperative system containing one of the nonlocal operators namely the fractional Laplace operator $(-\Delta)^s$.

Let $\Omega \subset \mathbb{R}^N, N \geq 2s$, be an open, bounded and connected domain with Lipschitz boundary $\partial\Omega$. Then, we shall study the following system:

$$\begin{cases} (-\Delta)^s y_1 = ay_1 + by_2 + f_1 & \text{in } \Omega, \\ (-\Delta)^s y_2 = cy_1 + dy_2 + f_2 & \text{in } \Omega, \\ y_1 = y_2 = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

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where $y = \{y_1, y_2\}$ are the states of the system, $f = \{f_1, f_2\}$ are the external sources. The fractional Laplace operator $(-\Delta)^s$ is defined by Fourier transform as follows:

$$(-\Delta)^s y(x) = c_{N,s} \quad \text{P.V.} \quad \int_{\mathbb{R}^N} \frac{y(x) - y(t)}{|x - t|^{N+2s}} \, dt, \quad c_{N,s} > 0. \tag{1.2}$$

From (1.2), it becomes clear that the fractional laplace operator $(-\Delta)^s$ is a nonlocal operator.

Definition 1.1 For given numbers a, b, c and d the system

$$\begin{cases} (-\Delta)^s y_1 = a y_1 + b y_2 + f_1 & \text{in } \Omega, \\ (-\Delta)^s y_2 = c y_1 + d y_2 + f_2 & \text{in } \Omega, \end{cases} \tag{1.3}$$

is called cooperative if $b, c > 0$; otherwise, the system (1.3) is said to be noncooperative.

Optimal control for partial differential equations (PDEs) has been widely studied in many fields such as biology, ecology, economics, engineering, and finance [5–10, 18, 22, 24, 25, 30, 34, 37]. These results have been expanded in [12, 14, 15, 29, 31–33] to cooperative and noncooperative systems. The fractional optimal control problems are the generalization of standard optimal control problems. Hence, it allows treatment of more general applications in physics, chemistry, and engineering [13, 17, 20, 21, 23, 39]. Several papers discuss time-fractional optimal control. In [27, 28], the distributed optimal control problem for a time-fractional diffusion equation is discussed. Moreover, the optimality conditions are derived. In [19], the distributed control for a time-fractional differential system involving a Schrödinger operator is studied, and the optimality conditions are derived. Furthermore, space-fractional optimal control is introduced. In [1, 2, 11], the nonlocal system is reformulated to a local system by an extension problem. Hence, the optimality conditions are achieved.

Henceforth, to treat with the nonlocality of the fractional Laplace operator, in [4], Caffarelli and Silvestre proved that the fractional Laplace operator can be characterized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem as follows:

Let $D^+ \subset \mathbb{R}^{N+1}$ be a semi-infinite cylinder as

$$D^+ = \left\{ (x, t) : x \in \mathbb{R}^N, t \in (0, \infty) \right\} \subset \mathbb{R}^{N+1}, \tag{1.4}$$

where t is a new extended variable. Therefore, the nonlocal problem (1.1) is reformulated locally as follows:

$$\begin{cases} \nabla \cdot (t^{1-2s} \nabla Y_1) = 0 & \text{in } D^+, \\ \nabla \cdot (t^{1-2s} \nabla Y_2) = 0 & \text{in } D^+, \\ Y_1(x, 0) = Y_2(x, 0) = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \\ \frac{1}{k_s} \frac{\partial Y_1}{\partial \nu} = (f_1 + a \operatorname{Tr}_\Omega Y_1 + b \operatorname{Tr}_\Omega Y_2) & \text{on } \Omega \times \{0\}, \\ \frac{1}{k_s} \frac{\partial Y_2}{\partial \nu} = (f_2 + c \operatorname{Tr}_\Omega Y_1 + d \operatorname{Tr}_\Omega Y_2) & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.5}$$

where $\frac{\partial Y}{\partial \nu} = -\lim_{t \to 0^+} t^{1-2s} \frac{\partial Y(x,t)}{\partial t}$, ν is the unit outer normal to D^+ at $\Omega \times \{0\}$, $\lim_{t \to \infty} Y(x, t) = 0$ and $k_s = 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)} > 0$.

In this paper, we generalize some previous results obtained for the classical cooperative systems. Indeed, we consider the elliptic 2×2 cooperative system involving one of the nonlocal operators called the fractional Laplace operator. The nonlocality of the fractional Laplace operator creates some difficulties. To overcome these, we transform the nonlocal system into a local system via an extension problem. Hence, via the Lax–Milgram lemma, we are able to prove the existence and uniqueness of the weak solution for the local system. Moreover, for both local and nonlocal systems, the optimality conditions are derived via the Lions technique. The results obtained tend to the classical results if $s \to 1$. This article is organized as follows. In Sect. 2, we introduce some functional spaces to represent the fractional cooperative systems and their extension, and also furnish the existence results. In Sect. 3, the weak solution and the optimality condition are established for the scalar case. In Sect. 4, we can generalize our results to a 2×2 cooperative system involving the fractional Laplace operator. Section 5 is devoted to a summary and discussion.

2 Preliminaries

In our work, the optimal control of the cooperative system depends on the variational formulation. Hence, we introduce the Sobolev spaces, which are the solution spaces for our problem. This section includes three subsections. Section 2.1 provides a short overview of classical fractional Sobolev spaces. In Sect. 2.2, we recollect the idea of the weighted Sobolev spaces and their embedding properties. The characterization of the principal eigenvalue problem is presented in Sect. 2.3.

2.1 Fractional Sobolev spaces

For $0 < s < 1$, define the fractional-order Sobolev space [3, 35]

$$H^s(\Omega) = \left\{ y \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|y(x_1) - y(x_2)|^2}{|x_1 - x_2|^{N+2s}} \, dx_1 \, dx_2 < \infty \right\}, \tag{2.1}$$

which is a Hilbert space endowed with the norm

$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega |y|^2 \, dx + \int_\Omega \int_\Omega \frac{|y(x_1) - y(x_2)|^2}{|x_1 - x_2|^{N+2s}} \, dx_1 \, dx_2 \right)^{\frac{1}{2}}. \tag{2.2}$$

Also, define the space $H_0^s(\Omega)$ as:

$$H_0^s(\Omega) = \left\{ y \in H^s(\Omega) : y = 0 \text{ on } \partial\Omega \right\}, \tag{2.3}$$

which can be endowed with norm

$$\|u\|_{H_0^s(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|y(x_1) - y(x_2)|^2}{|x_1 - x_2|^{N+2s}} \, dx_1 \, dx_2 \right)^{\frac{1}{2}}. \tag{2.4}$$

Moreover, the Lions–Magenes space is given by [26]

$$H_{00}^{\frac{1}{2}}(\Omega) = \left\{ u \in H^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{u^2(x)}{d(x, \partial\Omega)} \, dx < \infty \right\}, \tag{2.5}$$

where $d(x, \partial\Omega)$ is the distance from x to $\partial\Omega$. Combining (2.1), (2.3) and (2.5) we have for any $s \in (0, 1)$, the following fractional Sobolev space:

$$\mathcal{H}^s(\Omega) = \begin{cases} H^s(\Omega); & s \in (0, \frac{1}{2}), \\ H_{00}^{1/2}(\Omega); & s = \frac{1}{2}, \\ H_0^s(\Omega); & s \in (\frac{1}{2}, 1). \end{cases} \tag{2.6}$$

Moreover, we denote by $\mathcal{H}^{-s}(\Omega)$ the dual space of $\mathcal{H}^s(\Omega)$ such that

$$(-\Delta)^s : \mathcal{H}^s(\Omega) \to \mathcal{H}^{-s}(\Omega). \tag{2.7}$$

Also, we have the following embedding

$$\mathcal{H}^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathcal{H}^{-s}(\Omega). \tag{2.8}$$

By a Cartesian product, we have the following chain of Sobolev spaces

$$\left(\mathcal{H}^s(\Omega)\right)^2 \hookrightarrow \left(L^2(\Omega)\right)^2 \hookrightarrow \left(\mathcal{H}^{-s}(\Omega)\right)^2. \tag{2.9}$$

2.2 Weighted Sobolev spaces

To set the weak solution for problem (1.5), it is useful to establish the following weighted space [16]

$$X^s(\mathcal{D}^+) = \left\{ Y \in H_{\mathrm{loc}}^1(\mathcal{D}^+) : \int_{\mathcal{D}^+} t^{1-2s} |\nabla Y(x, t)|^2 \, dx \, dt < +\infty \right\}, \tag{2.10}$$

equipped with the norm

$$\|Y\|_{X^s(\mathcal{D}^+)} := \left(\int_{\mathcal{D}^+} t^{1-2s} |\nabla Y(x, t)|^2 \, dx \, dt \right)^{\frac{1}{2}}. \tag{2.11}$$

Hence, the space of all functions in $X^s(\mathcal{D}^+)$, whose trace over $\mathbb{R}^N$ vanishes outside of Ω, is given by

$$X_\Omega^s(\mathcal{D}^+) = \left\{ Y \in X^s(\mathcal{D}^+) : Y|_{\mathbb{R}^N \times \{0\}} = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}, \tag{2.12}$$

which furnishes a precise meaning of the solutions to problem (1.5) in a bounded domain Ω. It is clear that $\mathcal{H}^s(\Omega) = \{Y|_{\Omega \times \{0\}} : Y \in X_\Omega^s(\mathcal{D}^+)\}$. In addition, we have the following compact embedding.

Lemma 2.1 *Let* $1 \le p < 2_s^\# = \frac{2N}{N-2s}$. *Then,* $\mathrm{Tr}_\Omega(X_\Omega^s(\mathcal{D}^+))$ *is compactly embedded in* $L^p(\Omega)$.

Remark 2.1　For a function $Y \in X_\Omega^s(\mathcal{D}^+)$, the operator $\mathrm{Tr}_\Omega : X_\Omega^s(\mathcal{D}^+) \to \mathcal{H}^s(\Omega)$ is called the trace operator and satisfies

$$\|\mathrm{Tr}_\Omega Y\|_{\mathcal{H}^s(\Omega)} \le \delta \|Y\|_{X_\Omega^s(\mathcal{D}^+)}, \quad \delta > 0. \tag{2.13}$$

Furthermore, $\mathrm{Tr}_\Omega Y = Y(x, 0) = y(x)$ is the trace of Y onto $\Omega \times \{0\}$.

2.3 Eigenvalue problem

In this subsection, we state some results given in [36] concerning the eigenvalue problem for the following fractional elliptic equation

$$
\begin{cases}
(-\Delta)^s y = \lambda y & \text{in } \Omega, \\
y = 0 & \text{in } \mathbb{R}^N \backslash \Omega.
\end{cases}
\tag{2.14}
$$

Theorem 2.1 ([36]) *The first eigenvalue of problem* (2.14) *is positive and can be characterized as follows*:

$$
\lambda = \min_{Y \in X_\Omega^s(D^+)} \int_{D^+} t^{1-2s} \nabla Y \cdot \nabla Y \, dx \, dt, \quad \big\| Y(x,0) \big\|_{L^2(\Omega)} = 1,
\tag{2.15}
$$

or equivalently,

$$
\lambda = \min_{Y \in X_\Omega^s(D^+)} \frac{\int_{D^+} t^{1-2s} \nabla Y \cdot \nabla Y \, dx \, dt}{\int_\Omega |Y(x,0)|^2 \, dx}, \quad Y(x,0) \neq 0.
\tag{2.16}
$$

3 Scalar case

For $a > 0$, consider the following system:

$$
\begin{cases}
(-\Delta)^s y(x) = a y(x) + f(x) & \text{in } \Omega, \\
y(x) = 0 & \text{on } \mathbb{R}^N \backslash \Omega.
\end{cases}
\tag{3.1}
$$

Using the extension problem, the nonlocal problem (3.1) is reformulated in a local way as follows:

$$
\begin{cases}
\nabla(t^{1-2s} \nabla Y(x,t)) = 0 & \text{in } D^+, \\
Y(x,0) = 0 & x \in \mathbb{R}^N \backslash \Omega, \\
-\frac{1}{k_s} \lim_{t \to 0^+} t^{1-2s} \frac{\partial Y}{\partial t}(x,t) = a \operatorname{Tr}_\Omega Y(x,t) + f(x) & x \in \Omega.
\end{cases}
\tag{3.2}
$$

3.1 Weak solution

Multiplying the first equation in (3.2) by a test function $\phi(x,t) \in X_\Omega^s(D^+)$ and integrating over D^+ we obtain

$$
\int_{D^+} \nabla \cdot \left(t^{1-2s} \nabla Y \right) \phi(x,t) \, dx \, dt = 0.
\tag{3.3}
$$

Applying Green's formula, we have

$$
\int_{D^+} \left(t^{1-2s} \nabla Y \right) \nabla \phi(x,t) \, dx \, dt = \int_{\Omega \times \{0\}} \frac{\partial Y}{\partial \nu} \operatorname{Tr}_\Omega \phi(x,t) \, dx
\tag{3.4}
$$

$$
= k_s \int_{\Omega \times \{0\}} \left(a \operatorname{Tr}_\Omega Y(x,t) + f(x) \right) \operatorname{Tr}_\Omega \phi(x,t) \, dx.
\tag{3.5}
$$

Take the bilinear form $a(Y, \phi)$ as follows:

$$
a(Y, \phi) = \frac{1}{k_s} \int_{D^+} t^{1-2s} \nabla Y \nabla \phi(x,t) \, dx \, dt - a \int_{\Omega \times \{0\}} \left(\operatorname{Tr}_\Omega Y(x,t) \right) \operatorname{Tr}_\Omega \phi(x,t) \, dx.
\tag{3.6}
$$

Also, take the linear form $F(\phi)$ as follows:

$$F(\phi) = \int_{\Omega \times \{0\}} f(x) \operatorname{Tr}_\Omega \phi(x,t)\,dx. \tag{3.7}$$

Lemma 3.1 *If $\lambda > a\,k_s$, the bilinear form $a(Y,\phi)$ defined in (3.6) is coercive.*

Proof Replacing $\phi(x,t)$ by $Y(x,t)$ in (3.6) we obtain

$$a(Y,Y) = \frac{1}{k_s} \int_{D^+} t^{1-2s} |\nabla Y|^2 \, dx\,dt - a \int_{\Omega \times \{0\}} \left| \operatorname{Tr}_\Omega Y(x,t) \right|^2 dx. \tag{3.8}$$

Hence, using (2.16) we obtain

$$a(Y,Y) \geq \frac{1}{k_s} \left(1 - \frac{a\,k_s}{\lambda} \right) \int_{D^+} t^{1-2s} |\nabla Y|^2 \, dx\,dt. \tag{3.9}$$

Then, for $\lambda > a\,k_s$ the coerciveness condition is satisfied, i.e.,

$$a(Y,Y) \geq C_1 \|Y\|^2, \quad C_1 > 0. \tag{3.10}$$

$\square$

Remark 3.1 ([38]) If $Y(x,t)$ is a solution of the extended problem (3.2), then the trace function $y(x) = \operatorname{Tr}_\Omega Y(x,t) = Y(x,0)$ will be called a weak solution to problem (3.1).

3.2 The optimality condition

Consider $L^2(\Omega)$ as the space of controls. For a control $u \in L^2(\Omega)$, the state $Y(u)$ solves the systems

$$\begin{cases} \nabla(t^{1-2s} \nabla Y(u)) = 0 & \text{in } D^+, \\ Y(u;x,0) = 0 & x \in \mathbb{R}^N \backslash \Omega, \\ \frac{1}{k_s} \frac{\partial Y}{\partial \nu}(u) = a \operatorname{Tr}_\Omega Y(u) + u & x \in \Omega. \end{cases} \tag{3.11}$$

For a given $z_d \in L^2(\Omega)$ and $v \in L^2(\Omega)$, the cost-functional subject to the systems (3.11) is given by

$$\min \quad J(v) = \frac{1}{2} \left\| \operatorname{Tr}_\Omega Y(v) - z_d \right\|_{L^2(\Omega)}^2 + (\mathbf{N} v, v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{U}_{ad},$$

where $\mathbf{N} \in \mathcal{L}(L^2(\Omega))^{1}$ is a positive-definite Hermitian operator satisfying

$$(\mathbf{N} v, v) \geq \alpha \|v\|_{L^2(\Omega)}^2, \quad \alpha > 0. \tag{3.12}$$

---

[1] $\mathcal{L}(X)$ is the space of all bounded and linear operators from X into itself.

Let v belong to a subset $\mathcal{U}_{ad}$ of $L^2(\Omega)$ (the set of admissable controls); we assume $\mathcal{U}_{ad}$ is a closed nonempty subset of $L^2(\Omega)$. Then, the optimal control problem is now

$$
\begin{cases}
\text{Finding } u \in \mathcal{U}_{ad}, \\
\text{such that } J(u) \le J(v), \quad \forall v \in \mathcal{U}_{ad}.
\end{cases}
\tag{3.13}
$$

Theorem 3.1 *If the cost functional is given by* (3.12) *and the condition* (3.13) *is satisfied, then there exists a unique optimal control* $u \in \mathcal{U}_{ad}$. *Moreover, this control is characterized by the following equations*:

$$
\begin{cases}
\nabla \cdot (t^{1-2s} \nabla P) = 0 & in\ D^+, \\
P(x,0) = 0 & x \in \mathbb{R}^N \backslash \Omega, \\
(\frac{\partial P}{\partial v}, \mathrm{Tr}_\Omega Y) = (\mathrm{Tr}_\Omega P, \frac{\partial Y}{\partial v}) & on\ \Omega \times \{0\},
\end{cases}
\tag{3.14}
$$

together with

$$
(\mathrm{Tr}_\Omega P + \mathbf{N} u, v - u) \ge 0, \quad \forall v \in \mathcal{U}_{ad},
\tag{3.15}
$$

where $P \in X_\Omega^s(D^+)$ *is the adjoint state*.

Proof The control $u \in \mathcal{U}_{ad}$ is optimal, if and only if

$$
J'(u) \cdot (v - u) \ge 0, \quad \forall v \in \mathcal{U}_{ad},
\tag{3.16}
$$

and hence, via an explicit computation of $J'(u)$, (3.17) is equivalent to [24]

$$
\big(\mathrm{Tr}_\Omega Y(u) - z_d, \mathrm{T}_\Omega Y(v) - \mathrm{Tr}_\Omega Y(u)\big) + (\mathbf{N} u, v - u) \ge 0, \quad \forall v \in \mathcal{U}_{ad}.
\tag{3.17}
$$

In order to transform (3.18) into a more convenient form, we introduce the adjoint state P defined by $(\mathbf{A} Y, P) = (Y, \mathbf{A}^* P)$, where A^* is the adjoint of A. Now

$$
(\mathbf{A} Y, P) = \big(\nabla \cdot \big(t^{1-2s} \nabla Y\big), P\big) = \int_{D^+} \nabla \cdot \big(t^{1-2s} \nabla Y\big) P(x,t)\, dx\, dt.
\tag{3.18}
$$

By applying Green's formula, (3.19) is transformed to

$$
\begin{aligned}
(\mathbf{A} Y, P) &= \int_{D^+} \nabla \cdot \big(t^{1-2s} \nabla P\big) Y(x,t)\, dx\, dt - \int_{\Omega \times \{0\}} \mathrm{Tr}_\Omega Y \frac{\partial P}{\partial v}\, dx + \int_{\Omega \times \{0\}} \mathrm{Tr}_\Omega P \frac{\partial Y}{\partial v}\, dx \\
&= \big(Y, \mathbf{A}^* P\big),
\end{aligned}
$$

and hence (3.15) is satisfied.

Take the adjoint systems as follows:

$$
\begin{cases}
\nabla \cdot (t^{1-2s} \nabla P) = 0 & in\ D^+, \\
P(x,0) = 0 & x \in \mathbb{R}^N \backslash \Omega, \\
\frac{1}{k_s} \frac{\partial P(u)}{\partial v} - a \, \mathrm{Tr}_\Omega P(u) = \mathrm{Tr}_\Omega Y(u) - z_d & on\ \Omega \times \{0\}.
\end{cases}
\tag{3.19}
$$

Remark 3.2 The variational form of (3.21) is

$$a^*(P, \phi) = \left(\mathrm{Tr}_\Omega Y(u) - z_d, \mathrm{Tr}_\Omega \phi\right), \tag{3.20}$$

where we define

$$a^*(\psi, \phi) = a(\phi, \psi). \tag{3.21}$$

Then, (3.18) is equivalent to

$$\left(\frac{1}{k_s}\frac{\partial P}{\partial v} - a\,\mathrm{Tr}_\Omega P, \mathrm{Tr}_\Omega Y(v) - \mathrm{Tr}_\Omega Y(u)\right) + (\mathbf{N}\,u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

Hence, using the last condition in (3.15), we obtain

$$\frac{1}{k_s}\left(\mathrm{Tr}_\Omega P, \frac{\partial Y(v)}{\partial v} - \frac{\partial Y(u)}{\partial v}\right) - a\left(\mathrm{Tr}_\Omega P, \mathrm{Tr}_\Omega Y(v) - \mathrm{Tr}_\Omega Y(u)\right)$$
$$+ (\mathbf{N}\,u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

By using (3.11), we have

$$\left(\mathrm{Tr}_\Omega P, a\,\mathrm{Tr}_\Omega Y(v) + v - a\,\mathrm{Tr}_\Omega Y(u) - u\right) - a\left(\mathrm{Tr}_\Omega P, \mathrm{Tr}_\Omega Y(v) - \mathrm{Tr}_\Omega Y(u)\right)$$
$$+ (\mathbf{N}\,u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_{ad},$$

and hence, the optimality condition becomes

$$(\mathrm{Tr}_\Omega P, v - u) + (\mathbf{N}\,u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{3.22}$$

Thereby, the proof is completed.   □

4 2 × 2 cooperative system

In this section, we generalize the results obtained in the previous section to a 2×2 co-operative system. This section is divided into two subsections. In Sect. 4.1, we prove the existence and uniqueness of the weak solution by using the Lax–Miligram Lemma. The optimality condition is obtained in Sect. 4.2.

4.1 The weak solution

To obtain a weak solution of the systems (1.5), we first transform (1.5) into a weak form. Indeed, multiplying the first and second equations in (1.5) by a test function $\phi(x, t) = \{\phi_1(x, t), \phi_2(x, t)\} \in (X_\Omega^s(D^+))^2$ and integrating over D^+ we obtain

$$\int_{D^+} \nabla \cdot \left(t^{1-2s}\nabla Y_1\right)\phi_1(x, t)\, dx\, dt = 0, \tag{4.1}$$

$$\int_{D^+} \nabla \cdot \left(t^{1-2s}\nabla Y_2\right)\phi_2(x, t)\, dx\, dt = 0. \tag{4.2}$$

Applying Green's formula, we have

$$\int_{D^+} t^{1-2s} \nabla Y_1 \nabla \phi_1(x,t) \, dx \, dt = -\int_{\Omega \times \{0\}} \text{Tr}_\Omega \, \phi_1(x,t) \lim_{t \to 0^+} t^{1-2s} \frac{\partial Y_1}{\partial t} \, dx, \tag{4.3}$$

$$\int_{D^+} t^{1-2s} \nabla Y_2 \nabla \phi_2(x,t) \, dx \, dt = -\int_{\Omega \times \{0\}} \text{Tr}_\Omega \, \phi_2(x,t) \lim_{t \to 0^+} t^{1-2s} \frac{\partial Y_2}{\partial t} \, dx. \tag{4.4}$$

Then, we obtain

$$\int_{D^+} t^{1-2s} \nabla Y_1 \nabla \phi_1(x,t) \, dx \, dt = \int_{\Omega \times \{0\}} \text{Tr}_\Omega \, \phi_1(x,t) \frac{\partial Y_1}{\partial \nu} \, dx, \tag{4.5}$$

$$\int_{D^+} t^{1-2s} \nabla Y_2 \nabla \phi_2(x,t) \, dx \, dt = \int_{\Omega \times \{0\}} \text{Tr}_\Omega \, \phi_2(x,t) \frac{\partial Y_2}{\partial \nu} \, dx, \tag{4.6}$$

By using the systems (1.5), Eqs. (4.5) and (4.6) are equivalent to

$$\frac{1}{k_s b} \int_{D^+} t^{1-2s} \nabla Y_1 \nabla \phi_1(x,t) \, dx \, dt = \int_{\Omega \times \{0\}} \left(f_1 + \frac{a}{b} y_1 + y_2 \right) \text{Tr}_\Omega \, \phi_1(x,t) \, dx, \tag{4.7}$$

$$\frac{1}{k_s c} \int_{D^+} t^{1-2s} \nabla Y_2 \nabla \phi_2(x,t) \, dx \, dt = \int_{\Omega \times \{0\}} \left(f_1 + y_1 + \frac{d}{c} y_2 \right) \text{Tr}_\Omega \, \phi_2(x,t) \, dx. \tag{4.8}$$

To this end, we can define a bilinear form on $(X_\Omega^s(D^+))^2$ as follows:

$$a(Y,\phi) = \frac{1}{k_s b} \int_{D^+} t^{1-2s} \nabla Y_1 \nabla \phi_1(x,t) \, dx \, dt + \frac{1}{k_s c} \int_{D^+} t^{1-2s} \nabla Y_1 \nabla \phi_1(x,t) \, dx \, dt$$
$$- \int_{\Omega \times \{0\}} \left(\frac{a}{b} y_1 + y_2 \right) \text{Tr}_\Omega \, \phi_1(x,t) \, dx - \int_{\Omega \times \{0\}} \left(y_1 + \frac{c}{d} y_2 \right) \text{Tr}_\Omega \, \phi_2(x,t) \, dx.$$

Also, we can define a linear form as follows:

$$F(\phi) = \int_{\Omega \times \{0\}} \left(f_1(x) \text{Tr}_\Omega \, \phi_1(x,t) + f_2(x) \text{Tr}_\Omega \, \phi_2(x,t) \right) dx, \quad \forall \phi \in \left(X_\Omega^s(D^+) \right)^2. \tag{4.9}$$

Lemma 4.1 *The bilinear form* (4.9) *is coercive and bounded.*

Proof Replacing $\phi = \{\phi_1, \phi_2\}$ by $Y = \{Y_1, Y_2\}$ in (3.9) yields

$$a(Y,Y) = \frac{1}{k_s b} \int_{D^+} t^{1-2s} |\nabla Y_1|^2 \, dx \, dt + \frac{1}{k_s c} \int_{D^+} t^{1-2s} |\nabla Y_2|^2 \, dx \, dt$$
$$- \frac{a}{b} \int_{\Omega \times \{0\}} |\text{Tr}_\Omega \, Y_1|^2 \, dx - \frac{d}{c} \int_{\Omega \times \{0\}} |\text{Tr}_\Omega \, Y_2|^2 \, dx$$
$$- 2 \int_{\Omega \times \{0\}} \text{Tr}_\Omega \, Y_1 \, \text{Tr}_\Omega \, Y_2 \, dx.$$

By using the Cauchy–Schwarz inequality, we have

$$a(Y,Y) \geq \frac{1}{k_s b} \int_{D^+} t^{1-2s} |\nabla Y_1|^2 \, dx \, dt + \frac{1}{k_s c} \int_{D^+} t^{1-2s} |\nabla Y_2|^2 \, dx \, dt$$
$$- \frac{a}{b} \int_{\Omega \times \{0\}} |\text{Tr}_\Omega \, Y_1|^2 \, dx - \frac{d}{c} \int_{\Omega \times \{0\}} |\text{Tr}_\Omega \, Y_2|^2 \, dx$$

$$-2\left(\int_{\Omega\times\{0\}}|\operatorname{Tr}_\Omega Y_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega\times\{0\}}|\operatorname{Tr}_\Omega Y_1|^2\,dx\right)^{\frac{1}{2}},$$

from (2.15), we deduce

$$a(Y,Y) \geq \frac{1}{k_s b}\left(1-\frac{ak_s}{\lambda}\right)\|\nabla Y_1\|^2 + \frac{1}{k_s c}\left(1-\frac{dk_s}{\lambda}\right)\|\nabla Y_2\|^2 - \frac{2}{\lambda}\|Y_1\|\|Y_2\|$$

$$= \frac{1}{\lambda}\left[\left(\frac{1}{k_s b}(\lambda-ak_s)-\frac{\|Y_2\|}{\|Y_1\|}\right)\|\nabla Y_1\|^2 + \left(\frac{1}{k_s c}(\lambda-dk_s)-\frac{\|Y_1\|}{\|Y_2\|}\right)\|\nabla Y_2\|^2\right].$$

Take

$$C_2 = \left(\frac{1}{k_s b}(\lambda-ak_s)-\frac{\|Y_2\|}{\|Y_1\|}\right),$$

and

$$C_3 = \left(\frac{1}{k_s c}(\lambda-dk_s)-\frac{\|Y_1\|}{\|Y_2\|}\right).$$

Then, if $C_2, C_3 \geq 0$, we have

$$a(Y,Y) \geq \min\{C_2, C_3\}\left[\|Y_1\|^2_{X^s_\Omega(D^+)} + \|Y_2\|^2_{X^s_\Omega(D^+)}\right], \tag{4.10}$$

or

$$a(Y,Y) \geq C\|Y\|^2_{(X^s_\Omega(D^+))^2}, \quad C = \min\{C_2, C_3\}. \tag{4.11}$$

Hence, the bilinear form $a(Y,Y)$ is coercive, if and only if the following conditions are satisfied

$$\begin{cases} (\lambda > ak_s, \qquad \lambda > dk_s, \\ (\lambda - ak_s)(\lambda - dk_s) \geq (k_s)^2 bc. \end{cases} \tag{4.12}$$

$\square$

4.2 The optimality conditions

The control-problem formulation is the main target of this work. For the control problem, we construct the adjoint state. Furthermore, we originate the conditions of optimality via the Lions technique [24, 25]. This subsection consists of two parts. Section 4.2.1 contains the derivation of the necessary and sufficient conditions for fractional optimal control. Meanwhile, the equivalence extended optimal control is obtained in Sect. 4.2.2.

4.2.1 Fractional optimal control

Consider $(L^2(\Omega))^2$ as the space of controls. For a control $u = \{u_1, u_2\} \in (L^2(\Omega))^2$, the state $y(u) = \{y_1(u), y_2(u)\}$ solves the following system:

$$\begin{cases} (-\Delta)^s y_1(u) = ay_1(u) + by_2(u) + u_1 & \text{in } \Omega, \\ (-\Delta)^s y_2(u) = cy_1(u) + dy_2(u) + u_2 & \text{in } \Omega, \\ y_1(u) = y_2(u) = 0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{4.13}$$

The observation equations are given by

$$\mathbf{y}_i(u) = y_i(u), \quad i = 1, 2. \tag{4.14}$$

For a given $z_d = \{z_{1d}, z_{2d}\} \in (L^2(\Omega))^2$ and $v = \{v_1, v_2\} \in (L^2(\Omega))^2$, the cost-functional subject to the systems (4.1) is given by

$$J(v) = \frac{1}{2} \|y_1(v) - z_{1d}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y_2(v) - z_{2d}\|_{L^2(\Omega)}^2 + (\mathbf{N} v, v)_{(L^2(\Omega))^2}, \tag{4.15}$$

where $\mathbf{N} \in \mathcal{L}((L^2(\Omega))^2)$ is a positive-definite Hermitian operator satisfying

$$(\mathbf{N} v, v) \geq \alpha \|v\|_{(L^2(\Omega))^2}^2, \quad \alpha > 0. \tag{4.16}$$

Let $\mathcal{U}_{ad}$ be a closed and convex subset of $L^2(\Omega)$. Then, the control problem is given as follows:

$$\begin{cases} \text{Finding } u \in (\mathcal{U}_{ad})^2, \\ \text{such that } J(u) \leq J(v), \quad \forall v \in (\mathcal{U}_{ad})^2. \end{cases} \tag{4.17}$$

Theorem 4.1 *If the cost functional is given by* (4.3), *and the condition* (4.4) *is satisfied, then there exists a unique optimal control* $u \in (\mathcal{U}_{ad})^2$. *Moreover, this control is characterized by the following equations:*

$$\begin{cases} (-\Delta)^s p_1(u) - a p_1(u) - c p_2(u) = y_1(u) - z_{1d} & \text{in } \Omega, \\ (-\Delta)^s p_2(u) - b p_1(u) - d p_2(u) = y_2(u) - z_{2d} & \text{in } \Omega, \\ p_1(u) = p_2(u) = 0 & \text{on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.18}$$

In addition,

$$(p_1, v_1 - u_1) + (p_2, v_2 - u_2) + (\mathbf{N} u, v - u)_{(L^2(\Omega)^2} \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2, \tag{4.19}$$

where $p = \{p_1, p_2\} \in (\mathcal{H}^s(\Omega))^2$ *is the adjoint state.*

Proof Since $\mathbf{N} > 0$, then the cost functional (4.3) is strictly convex. Furthermore, the set $\mathcal{U}_{ad}$ is nonempty, closed, bounded and convex in $L^2(\Omega)$. Therefore, the existence and uniqueness of the optimal control is proved.

The control $u = \{u_1, u_2\} \in (\mathcal{U}_{ad})^2$ is optimal, if and only if

$$J'(u) \cdot (v - u) \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2, \tag{4.20}$$

which is equivalent to [24]

$$\begin{aligned} &\big(y_1(u) - z_{1d}, y_1(v) - y_1(u)\big) + \big(y_2(u) - z_{2d}, y_2(v) - y_2(u)\big) \\ &\quad + (\mathbf{N} u_1, v_1 - u_1) + (\mathbf{N} u_2, v_2 - u_2) \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2. \end{aligned} \tag{4.21}$$

Now, since $(\mathbf{A}\,y, p) = (y, \mathbf{A}^* p)$, where

$$\mathbf{A}\big(y = \{y_1, y_2\}\big) = \big\{(-\Delta)^s y_1 - a y_1 - b y_2, (-\Delta)^s y_2 - c y_1 - d y_2\big\}, \tag{4.22}$$

then

$$
\begin{aligned}
(\mathbf{A}\,y, p) &= \big((-\Delta)^s y_1 - a y_1 - b y_2, p_1\big) + \big((-\Delta)^s y_2 - c y_1 - d y_2, p_2\big) \\
&= \big((-\Delta)^s y_1, p_1\big) - a(y_1, p_1) - b(y_2, p_1) + \big((-\Delta)^s y_2, p_2\big) - c(y_1, p_2) \\
&\quad - b(y_2, p_2) \\
&= \big(y_1, (-\Delta)^s p_1\big) - a(y_1, p_1) - b(y_2, p_1) + \big(y_2, (-\Delta)^s p_2\big) - c(y_1, p_2) \\
&\quad - b(y_2, p_2) \\
&= \big((y_1, (-\Delta)^s p_1 - a p_1 - c p_2\big) + \big((y_2, (-\Delta)^s p_2 - b p_1 - d p_2\big) \\
&= (y, \mathbf{A}^* p),
\end{aligned}
$$

where $\mathbf{A}^*$ is the adjoint operator of $\mathbf{A}$, p is the adjoint state.

Take the adjoint system as follows:

$$
\begin{cases}
(-\Delta)^s p_1(u) - a p_1(u) - c p_2(u) = y_1(u) - z_{1d} & \text{in } \Omega, \\
(-\Delta)^s p_2(u) - b p_1(u) - d p_2(u) = y_2(u) - z_{2d} & \text{in } \Omega, \\
p_1(u) = p_2(u) = 0 & \text{on } \mathbb{R}^N \backslash \Omega.
\end{cases} \tag{4.23}
$$

By using Eqs. (4.1) and (4.12), we deduce

$$
\begin{aligned}
(p_1, v_1 - u_1) &+ (p_2, v_2 - u_2) + (\mathbf{N}\,u_1, v_1 - u_1)_{L^2(\Omega)} \\
&+ (\mathbf{N}\,u_2, v_2 - u_2)_{L^2(\Omega)} \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2. \tag{4.24}
\end{aligned}
$$

Thus, the proof is completed. $\qquad\square$

4.2.2 Extended optimal control

If $y(v) \in (\mathcal{H}^s(\Omega))^2$ is a solution of (4.1) with $v = \{v_1, v_2\} \in (\mathcal{H}^{-s}(\Omega))^2$ and $Y(v) \in (X_\Omega^s(D^+))^2$ solves the following systems:

$$
\begin{cases}
\nabla \cdot (t^{1-2s} \nabla Y_1(u)) = 0 & \text{in } D^+, \\
\nabla \cdot (t^{1-2s} \nabla Y_2(u)) = 0 & \text{in } D^+, \\
Y_1(x, 0) = Y_2(x, 0) = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \\
\frac{1}{k_s} \frac{\partial Y_1(u)}{\partial v} = (u_1 + a y_1(u) + b y_2(u)) & \text{on } \Omega \times \{0\}, \\
\frac{1}{k_s} \frac{\partial Y_2(u)}{\partial v} = (u_2 + c y_1(u) + d y_2(u)) & \text{on } \Omega \times \{0\},
\end{cases} \tag{4.25}
$$

then, we have

$$\text{Tr}_\Omega Y(v) = y(v). \tag{4.26}$$

Hence, the equivalence extended optimal control problem is given by

$$\min \quad \mathbb{J}(v) = \frac{1}{2}\left\|\mathrm{Tr}_{\Omega}\, Y_1(v) - z_{1d}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\mathrm{Tr}_{\Omega}\, Y_2(v) - z_{2d}\right\|^2_{L^2(\mathcal{B})}$$
$$+ (\mathbf{N}\,v, v)_{(L^2(\Omega))^2}, \quad \forall v \in (\mathcal{U}_{ad})^2.$$

Theorem 4.2 *If the cost functional is given by* (4.16) *and the condition* (4.4) *is satisfied, then there exists a unique optimal control* $u = \{u_1, u_2\} \in (\mathcal{U}_{ad})^2$. *Moreover, this control is characterized by the following equations*:

$$\begin{cases} \nabla \cdot (t^{1-2s}\nabla P_1) = 0 & \text{in } D^+, \\ \nabla \cdot (t^{1-2s}\nabla P_2) = 0 & \text{in } D^+, \\ P_1(x,0) = P_2(x,0) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ (\frac{\partial P_1}{\partial \nu}, \mathrm{Tr}_\Omega\, Y_1) = (\mathrm{Tr}_\Omega\, P_1, \frac{\partial Y_1}{\partial \nu}) & \text{on } \Omega \times \{0\}, \\ (\frac{\partial P_2}{\partial \nu}, \mathrm{Tr}_\Omega\, Y_2) = (\mathrm{Tr}_\Omega\, P_2, \frac{\partial Y_2}{\partial \nu}) & \text{on } \Omega \times \{0\}. \end{cases} \tag{4.27}$$

In addition,

$$(\mathrm{Tr}_\Omega\, P_1, v_1 - u_1) + (\mathrm{Tr}_\Omega\, P_2, v_2 - u_2) + (\mathbf{N}\,u, v - u) \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2, \tag{4.28}$$

where $P = \{P_1, P_2\} \in (X^s_\Omega(D^+))^2$ *is the adjoint state.*

Proof The control $u \in (\mathcal{U}_{ad})^2$ is optimal, if and only if

$$\mathbb{J}'(u) \cdot (v - u) \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2, \tag{4.29}$$

which is again equivalent to [24]

$$\left(\mathrm{Tr}_\Omega\, Y_1(v) - z_{1d}, \mathrm{Tr}_\Omega\, Y_1(v) - \mathrm{Tr}_\Omega\, Y_1(u)\right) + \left(\mathrm{Tr}_\Omega\, Y_2(v) - z_{2d}, \mathrm{Tr}_\Omega\, Y_2(v) - \mathrm{Tr}_\Omega\, Y_2(u)\right)$$
$$+ (\mathbf{N}\,u_1, v_1 - u_1) + (\mathbf{N}\,u_2, v_2 - u_2) \geq 0, \quad \forall v \in (\mathcal{U}_{ad})^2. \tag{4.30}$$

Now, since $(\mathbf{A}\,Y, P) = (Y, \mathbf{A}^* P)$, then

$$(\mathbf{A}\,Y, P) = \left(\nabla \cdot \left(t^{1-2s}\nabla Y_1\right), P_1\right) + \left(\nabla \cdot \left(t^{1-2s}\nabla Y_2\right), P_2\right)$$
$$= \int_{D^+} \nabla \cdot \left(t^{1-2s}\nabla Y_1\right) P_1 \, dx\, dt + \int_{D^+} \nabla \cdot \left(t^{1-2s}\nabla Y_2\right) P_2 \, dx\, dt$$
$$= \int_{D^+} \nabla \cdot \left(t^{1-2s}\nabla P_1\right) Y_1(x,t) \, dx\, dt - \int_\Omega \mathrm{Tr}_\Omega\, Y_1 \frac{\partial P_1}{\partial \nu} + \int_\Omega \mathrm{Tr}_\Omega\, P_1 \frac{\partial Y_1}{\partial \nu}$$
$$+ \int_{D^+} \nabla \cdot \left(t^{1-2s}\nabla P_2\right) Y_2(x,t) \, dx\, dt - \int_\Omega \mathrm{Tr}_\Omega\, Y_2 \frac{\partial P_2}{\partial \nu} + \int_\Omega \mathrm{Tr}_\Omega\, P_2 \frac{\partial Y_2}{\partial \nu}$$
$$= \left(Y, \mathbf{A}^* P\right),$$

and hence, (4.17) is satisfied.

Take the adjoint systems as follows:

$$\begin{cases} \nabla \cdot (t^{1-2s}\nabla P_1) = 0 & \text{in } \mathcal{D}^+, \\ \nabla \cdot (t^{1-2s}\nabla P_2) = 0 & \text{in } \mathcal{D}^+, \\ P_1(x,0) = P_2(x,0) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ \frac{1}{k_s}\frac{\partial P_1}{\partial \nu} - a\operatorname{Tr}_\Omega P_1 - c\operatorname{Tr}_\Omega P_2 = \operatorname{Tr}_\Omega Y_1(\nu) - z_{1d} & \text{on } \Omega \times \{0\} \\ \frac{1}{k_s}\frac{\partial P_2}{\partial \nu} - b\operatorname{Tr}_\Omega P_1 - d\operatorname{Tr}_\Omega P_2 = \operatorname{Tr}_\Omega Y_2(\nu) - z_{2d} & \text{on } \Omega \times \{0\}. \end{cases} \quad (4.31)$$

Then, (4.20) is equivalent to

$$\left(\frac{1}{k_s}\frac{\partial P_1}{\partial \nu} - a\operatorname{Tr}_\Omega P_1 - c\operatorname{Tr}_\Omega P_2, \operatorname{Tr}_\Omega Y_1(\nu) - \operatorname{Tr}_\Omega Y_1(u) \right)$$

$$+ \left(\frac{1}{k_s}\frac{\partial P_2}{\partial \nu} - b\operatorname{Tr}_\Omega P_1 - d\operatorname{Tr}_\Omega P_2, \operatorname{Tr}_\Omega Y_2(\nu) - \operatorname{Tr}_\Omega Y_2(u) \right)$$

$$+ (\mathbf{N}u_1, \nu_1 - u_1) + (\mathbf{N}u_2, \nu_2 - u_2) \geq 0, \quad \forall \nu \in (\mathcal{U}_{ad})^2.$$

Hence, using the last two conditions in (4.17), the optimality condition becomes

$$(\operatorname{Tr}_\Omega P_1, \nu_1 - u_1) + (\operatorname{Tr}_\Omega P_2, \nu_2 - u_2) + (\mathbf{N}u, \nu - u) \geq 0, \quad \forall \nu \in (\mathcal{U}_{ad})^2, \quad (4.32)$$

which completes the proof. □

5 Summary and conclusion

In the present work, we investigate the optimal control problem for 2×2 cooperative systems involving the fractional Laplace operator, wherein these systems are subject to the zero Dirichlet condition. Due to the difficulty arising from the nonlocality of the fractional Laplace operator, we follow the Caffarelli and Silvestre technique to extend our problem to local cooperative systems. With the aid of the Lax–Milgram lemma, the existence and uniqueness of the solution to the extended problem are proved. Moreover, the conditions of optimality are proved by the Lions technique for both fractional and extended optimal control. If $s \to 1$, the obtained results are similar to the classical results.

Availability of data and materials
The data that support the findings of this study are available from the authors upon request.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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