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Note on theoretical and practical solvability of a class of discrete equations generalizing the hyperbolic-cotangent class

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Abstract

There has been some recent interest in investigating the hyperbolic-cotangent types of difference equations and systems of difference equations. Among other things their solvability has been studied. We show that there is a class of theoretically solvable difference equations generalizing the hyperbolic-cotangent one. Our analysis shows a bit unexpected fact, namely that the solvability of the class is based on some algebraic relations, not closely related to some trigonometric ones, which enable us to solve them in an elegant way. Some examples of the difference equations belonging to the class which are practically solvable are presented, as well as some interesting comments on connections of the equations with some iteration processes.

MSC: 39A45

Keywords: Difference equation; Theoretical solvability; Practical solvability; Equations solvable in a closed form

1 Introduction

As usual, by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ we denote the sets of natural numbers, nonnegative integers, integers, real numbers, and complex numbers, respectively, whereas the notation $l = \overline{s,t}$, when $s, t \in \mathbb{Z}$ and $s \le t$, is used instead of writing $s \le l \le t$, $l \in \mathbb{Z}$.

Now we present some motivations for the investigation in this paper and several interesting connections among some classes of difference equations and iteration processes.

1.1 A quick overview of some old results on solvability

After discovering some solvable classes of linear difference equations and presenting a few basic methods for solving them (see, for example, [4, 6, 7, 9, 15, 16]), some researchers started investigating solvability of some classes of nonlinear difference equations and systems of difference equations.

An important paper in this direction, as well as in solvability theory in whole, is [17] by Laplace, where he, among several other ones, investigated the solvability of the difference

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equation

$$x_{n+1} = x_n^2 - 2, \quad n \in \mathbb{N}_0. \tag{1}$$

It should be mentioned that the method for showing the solvability of the equation used by Laplace is based on some simple algebraic relations. He presented the initial value x_0 in the form

$$x_0 = \alpha + \frac{1}{\alpha},$$

and by calculating first few members of x_n found a general solution to the equation in terms of α and n.

It is interesting to note that by employing the change of variables

$$x_n = 2\widehat{x}_n, \quad n \in \mathbb{N}_0,$$

equation (1) becomes

$$\widehat{x}_{n+1} = 2\widehat{x}_n^2 - 1, \quad n \in \mathbb{N}_0,$$

which resembles the double angle identity for the cosine function. This fact suggests its solvability. Namely, one can expect that a sequence of cosines satisfies the equation. Bearing in mind that

$$\cos z = \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}} \right),$$

it becomes clear why the use of the quantity $\alpha + \frac{1}{\alpha}$ enabled Laplace to solve equation (1).

Laplace did not give the explanation, but it was realized by researchers of the time that difference equations which have forms to some trigonometric formulas could be solvable.

Later books, besides solvability of some classes of linear difference equations, also mention some of solvable nonlinear ones (see, e.g., [5, 10, 12, 14, 18–21]), but to a small extent.

1.2 Solvability of some equations and iteration processes

Difference equations naturally appear in many areas of science, among other ones, in numerical mathematics for iteration processes serving for calculating some quantities such as roots of some functions (see, e.g., [8, 11]). One of root-finding algorithms is the secant method. Recall that if the initial values x_0, x_1 are real, to find a root of a function f, one can construct the line through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ and find the intersection point of the line with the x-axis, that is, the root of the linear function

$$y = f(x_1) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1),$$

which is equal to

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}.$$

Repeating the procedure for the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and so on, we obtain the iteration process

$$
\begin{aligned}
x_{n+2} &= x_{n+1} - f(x_{n+1}) \frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)} \\
&= \frac{f(x_{n+1})x_n - f(x_n)x_{n+1}}{f(x_{n+1}) - f(x_n)}, \quad n \in \mathbb{N}_0,
\end{aligned}
\tag{2}
$$

for the secant method.

It is an interesting fact that if in (2) we choose the function

$$
f(x) = x^2 - a,
\tag{3}
$$

then after some simple calculations we get

$$
x_{n+2} = \frac{x_{n+1}x_n + a}{x_{n+1} + x_n}, \quad n \in \mathbb{N}_0,
\tag{4}
$$

which is an iteration process for calculating the square root of number a (it is a root of function (3)).

Another interesting fact is that equation (4) is solvable in a closed form. Moreover, the equation belongs to the class of theoretically solvable difference equations

$$
x_{n+k} = \frac{x_{n+l}x_n + a}{x_{n+l} + x_n}, \quad n \in \mathbb{N}_0,
\tag{5}
$$

where $k, l \in \mathbb{N}_0$, $l < k$ (see [26, 35]).

These facts suggest that equation (5) can be also obtained from an iteration process. Indeed, if in the following modification of the secant iteration process (with delayed indices)

$$
\begin{aligned}
x_{n+k} &= x_{n+l} - f(x_{n+l}) \frac{x_{n+l} - x_n}{f(x_{n+l}) - f(x_n)} \\
&= \frac{f(x_{n+l})x_n - f(x_n)x_{n+l}}{f(x_{n+l}) - f(x_n)},
\end{aligned}
\tag{6}
$$

for $n \in \mathbb{N}_0$, we choose function (3), we really get equation (5). Of course, the process is determined if the initial values x_j, $j = \overline{0, k-1}$ are given.

One of the facts that suggest solvability of the difference equations in (5) is that they look like the cotangent sum formula. Note also that when $a \neq 0$ a linear change of variables reduces the equation to the case $a = 1$, which exactly looks like the cotangent sum formula. For some generalizations to systems of cotangent-type difference equations, see, for example, recent papers [30, 33] and the related references therein.

Some other recent results on solvability and invariants of difference equations and systems and their applications can be found, for example, in [3, 13, 22–25, 27–29, 31–33, 35, 37] and the references therein.

1.3 Our aim

Motivated by all the above mentioned, here we consider the following class of difference equations:

$$
x_{n+k} = \frac{x_{n+l}x_n - ab}{x_{n+l} + x_n - a - b}, \quad n \in \mathbb{N}_0,
\tag{7}
$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $a, b \in \mathbb{C}$ and $x_j \in \mathbb{C}$, $j = \overline{0, k-1}$, which naturally generalizes equation (5).

Our aim is to show that equation (7) is theoretically solvable, and that it is a consequence of some pure algebraic relations which are essentially not closely connected to some trigonometric type relations. This shows that the solvability of equation (5) also relies on the same algebraic relation.

2 Main results

This section presents our main results in this paper. We study the solvability of equation (7) by considering several cases separately.

Case $a = b = 0$. If $a = b = 0$, then equation (7) becomes

$$x_{n+k} = \frac{x_{n+l} x_n}{x_{n+l} + x_n}, \quad n \in \mathbb{N}_0. \tag{8}$$

By using the change of variables

$$x_n = \frac{1}{u_n}, \quad n \in \mathbb{N}_0, \tag{9}$$

equation (8) is transformed to the following one:

$$u_{n+k} = u_{n+l} + u_n, \quad n \in \mathbb{N}_0, \tag{10}$$

which is a homogeneous linear difference equation with constant coefficients of kth order.

It is well known that the linear difference equations with constant coefficients are theoretically solvable (see, for example, [4, 5, 10, 12, 18–21]), from which theoretical solvability of equation (8) follows. This fact and the fact that by using the change of variables (9) equation (8) is transformed to equation (10) are well known. For example, in [2, Problem 8.16.9] a special case of equation (8) with $k = 2$ is solved in this way. Moreover, such equations frequently appear (see, for example, recent paper [30]). Therefore, the case is of no special interest. Nevertheless, some special cases of equation (8) will be solved in a closed form later in the paper.

From now on we consider equation (7) under the following assumption:

$$a \neq 0 \quad \text{or} \quad b \neq 0. \tag{11}$$

Case $a + b = 0$. If $a + b = 0$ and (11) holds, then equation (7) becomes

$$x_{n+k} = \frac{x_{n+l} x_n + a^2}{x_{n+l} + x_n}, \quad n \in \mathbb{N}_0, \tag{12}$$

which is a special case of equation (5). As we have already mentioned, the solvability of equation (5) has been thoroughly investigated (see [26, 35]). Hence, we will not consider this case in the paper. Note also that since (11) holds, from $a + b = 0$, we have $a = -b \neq 0$.

Case $a \neq 0$ or $b \neq 0$, $a \neq b$. In this case equation (7) no more has a form of cotangent-sum formula. Since we do not have a typical trigonometric formula hint for suggesting its solvability, another hint should be found.

First, note that the fixed points of the equation satisfy the algebraic equation

$$x^* = \frac{(x^*)^2 - ab}{2x^* - a - b},$$

(13)

from which it easily follows that

$$x_1^* = a \quad \text{and} \quad x_2^* = b.$$

This observation together with the form of equation (7) suggests to consider the following quantities/sequences:

$$x_n - x_1^* \quad \text{and} \quad x_n - x_2^*$$

for $n \in \mathbb{N}_0$.

From (7) and by some simple calculations, it follows that

$$x_{n+k} - b = \frac{x_{n+l}x_n - b(x_{n+l} + x_n) + b^2}{x_{n+l} + x_n - a - b}, \quad n \in \mathbb{N}_0,$$

(14)

and

$$x_{n+k} - a = \frac{x_{n+l}x_n - a(x_{n+l} + x_n) + a^2}{x_{n+l} + x_n - a - b}, \quad n \in \mathbb{N}_0.$$

(15)

From (14) and (15) we easily obtain

$$\frac{x_{n+k} - b}{x_{n+k} - a} = \frac{(x_{n+l} - b)(x_n - b)}{(x_{n+l} - a)(x_n - a)}, \quad n \in \mathbb{N}_0.$$

(16)

By using the change of variables

$$z_n = \frac{x_n - b}{x_n - a}, \quad n \in \mathbb{N}_0,$$

(17)

equation (16) is transformed to

$$z_{n+k} = z_{n+l}z_n, \quad n \in \mathbb{N}_0.$$

(18)

It is also known that equation (18) is theoretically solvable. Namely, its solvability is closely related to the solvability of equation (10). Some books on difference equations wrongly suggest taking the logarithm of both sides of the equation and then application of the change of variables

$$\widehat{y}_n = \ln z_n, \quad n \in \mathbb{N}_0.$$

(19)

This is only justified if all z_n are positive numbers. Fortunately, there are some correct procedures for finding closed form formulas for general solution to equation (18). We have use them, for example, in [34] (see also the related references therein).

Equation (10) is certainly practically solvable if $k \leq 4$. For $k \geq 5$, the characteristic polynomial $\lambda^k - \lambda^l - 1$ associated with the equation can be certainly solved by radicals when $k \leq 4$. However, if $k \geq 5$, then by a known theorem [1], it need not be solvable.

Hence, for $k \leq 4$, closed form formulas for solutions to equation (10), and consequently closed form formulas for solutions to equation (18), can be found. We can use such obtained formulas in the formula

$$x_n = \frac{az_n - b}{z_n - 1}, \quad n \in \mathbb{N}_0, \tag{20}$$

which easily follows from (17).

Hence, we can claim that the following special cases of equation (18) are certainly practically solvable: 1) $k = 2$, $l = 1$; 2) $k = 3$, $l = 1$; 3) $k = 3$, $l = 2$; 4) $k = 4$, $l = 1$; 5) $k = 4$, $l = 2$; 6) $k = 4$, $l = 1$. The equation in these cases has been solved in some of our papers (see, e.g., [34]) and the following result holds.

Theorem 1 *The following statements hold.*

(a) *General solution to equation* (18) *with $k = 2$ and $l = 1$ is given by the formula*

$$z_n = z_1^{f_n} z_0^{f_{n-1}}, \quad n \in \mathbb{N}_0,$$

where f_n is the Fibonacci sequence (see, e.g., [38]).

(b) *General solution to equation* (18) *with $k = 3$ and $l = 1$ is given by the formula*

$$z_n = z_2^{\alpha_{n-3}} z_1^{\alpha_{n-2}} z_0^{\alpha_{n-4}}, \quad n \in \mathbb{N}_0,$$

where the sequence α_n is given by

$$\alpha_n = \sum_{j=1}^{3} \frac{t_j^{n+3}}{P_3'(t_j)}, \quad n \in \mathbb{Z}, \tag{21}$$

where $P_3(t) = t^3 - t - 1$ and t_j, $j = \overline{1,3}$, are its zeros.

(c) *General solution to equation* (18) *with $k = 3$ and $l = 2$ is given by the formula*

$$z_n = z_2^{\beta_{n-2}} z_1^{\beta_{n-4}} z_0^{\beta_{n-3}},$$

where the sequence β_n is given by

$$\beta_n = \sum_{j=1}^{3} \frac{t_j^{n+2}}{Q_3'(t_j)}, \quad n \in \mathbb{Z}, \tag{22}$$

where $Q_3(t) = t^3 - t^2 - 1 = 0$ and t_j, $j = \overline{1,3}$, are its zeros.

(d) *General solution to equation* (18) *with $k = 4$ and $l = 1$ is given by the formula*

$$z_n = z_3^{\gamma_{n-5}} z_2^{\gamma_{n-4}} z_1^{\gamma_{n-3}} z_0^{\gamma_{n-6}},$$

where the sequence γ_n is given by

$$\gamma_n = \sum_{j=1}^{4} \frac{t_j^{n+5}}{R_4'(t_j)}, \quad n \in \mathbb{Z}, \tag{23}$$

where $R_4(t) = t^4 - t - 1 = 0$ and $t_j, j = \overline{1,4}$, are its zeros.

(e) *General solution to equation* (18) *with $k = 4$ and $l = 2$ is given by the formulas*

$$z_{2n} = z_2^{f_n} z_0^{f_{n-1}}, \quad n \in \mathbb{N}_0,$$

$$z_{2n+1} = z_3^{f_n} z_1^{f_{n-1}}, \quad n \in \mathbb{N}_0,$$

where f_n is the Fibonacci sequence.

(f) *General solution to equation* (18) *with $k = 4$ and $l = 3$ is given by the formula*

$$z_n = z_3^{\delta_{n-3}} z_2^{\delta_{n-6}} z_1^{\delta_{n-5}} z_0^{\delta_{n-4}},$$

where the sequence δ_n is given by

$$\delta_n = \sum_{j=1}^{4} \frac{t_j^{n+3}}{S_4'(t_j)}, \quad n \in \mathbb{Z}, \tag{24}$$

where $S_4(t) = t^4 - t^3 - 1 = 0$ and $t_j, j = \overline{1,4}$, are its zeros.

From Theorem 1, and by using (20) as well as relation (17) with $n = \overline{0,3}$, we obtain the following result.

Theorem 2 *Consider equation* (7). *Assume that $a, b \in \mathbb{C}$, $a \neq b$, and $a + b \neq 0$. Then the following statements hold:*

(a) *Assume that $k = 2, l = 1$. Then the general solution to equation* (7) *is given by*

$$x_n = \frac{a\left(\frac{x_1-b}{x_1-a}\right)^{f_n}\left(\frac{x_0-b}{x_0-a}\right)^{f_{n-1}} - b}{\left(\frac{x_1-b}{x_1-a}\right)^{f_n}\left(\frac{x_0-b}{x_0-a}\right)^{f_{n-1}} - 1}, \quad n \in \mathbb{N}_0,$$

where f_n is the Fibonacci sequence.

(b) *Assume that $k = 3, l = 1$. Then the general solution to equation* (7) *is given by*

$$x_n = \frac{a\left(\frac{x_2-b}{x_2-a}\right)^{\alpha_{n-3}}\left(\frac{x_1-b}{x_1-a}\right)^{\alpha_{n-2}}\left(\frac{x_0-b}{x_0-a}\right)^{\alpha_{n-4}} - b}{\left(\frac{x_2-b}{x_2-a}\right)^{\alpha_{n-3}}\left(\frac{x_1-b}{x_1-a}\right)^{\alpha_{n-2}}\left(\frac{x_0-b}{x_0-a}\right)^{f_{n-1}} - 1}, \quad n \in \mathbb{N}_0,$$

where the sequence α_n is given by (21).

(c) *Assume that $k = 3, l = 2$. Then the general solution to equation* (7) *is given by*

$$x_n = \frac{a\left(\frac{x_2-b}{x_2-a}\right)^{\beta_{n-2}}\left(\frac{x_1-b}{x_1-a}\right)^{\beta_{n-4}}\left(\frac{x_0-b}{x_0-a}\right)^{\beta_{n-3}} - b}{\left(\frac{x_2-b}{x_2-a}\right)^{\beta_{n-2}}\left(\frac{x_1-b}{x_1-a}\right)^{\beta_{n-4}}\left(\frac{x_0-b}{x_0-a}\right)^{\beta_{n-3}} - 1}, \quad n \in \mathbb{N}_0,$$

where the sequence β_n is given by (22).

(d) *Assume that $k = 4$, $l = 1$. Then the general solution to equation* (7) *is given by*

$$x_n = \frac{a\left(\frac{x_3-b}{x_3-a}\right)^{\gamma_{n-5}}\left(\frac{x_2-b}{x_2-a}\right)^{\gamma_{n-4}}\left(\frac{x_1-b}{x_1-a}\right)^{\gamma_{n-3}}\left(\frac{x_0-b}{x_0-a}\right)^{\gamma_{n-6}} - b}{\left(\frac{x_3-b}{x_3-a}\right)^{\gamma_{n-5}}\left(\frac{x_2-b}{x_2-a}\right)^{\gamma_{n-4}}\left(\frac{x_1-b}{x_1-a}\right)^{\gamma_{n-3}}\left(\frac{x_0-b}{x_0-a}\right)^{\gamma_{n-6}} - 1}, \quad n \in \mathbb{N}_0,$$

where the sequence γ_n is given by (23).

(e) *Assume that $k = 4$, $l = 2$. Then the general solution to equation* (7) *is given by*

$$x_{2n} = \frac{a\left(\frac{x_2-b}{x_2-a}\right)^{f_n}\left(\frac{x_0-b}{x_0-a}\right)^{f_{n-1}} - b}{\left(\frac{x_2-b}{x_2-a}\right)^{f_n}\left(\frac{x_0-b}{x_0-a}\right)^{f_{n-1}} - 1}, \quad n \in \mathbb{N}_0,$$

$$x_{2n+1} = \frac{a\left(\frac{x_3-b}{x_3-a}\right)^{f_n}\left(\frac{x_1-b}{x_1-a}\right)^{f_{n-1}} - b}{\left(\frac{x_3-b}{x_3-a}\right)^{f_n}\left(\frac{x_1-b}{x_1-a}\right)^{f_{n-1}} - 1}, \quad n \in \mathbb{N}_0,$$

where f_n is the Fibonacci sequence.

(f) *Assume that $k = 4$, $l = 3$. Then the general solution to equation* (7) *is given by*

$$x_n = \frac{a\left(\frac{x_3-b}{x_3-a}\right)^{\delta_{n-3}}\left(\frac{x_2-b}{x_2-a}\right)^{\delta_{n-6}}\left(\frac{x_1-b}{x_1-a}\right)^{\delta_{n-5}}\left(\frac{x_0-b}{x_0-a}\right)^{\delta_{n-4}} - b}{\left(\frac{x_3-b}{x_3-a}\right)^{\delta_{n-3}}\left(\frac{x_2-b}{x_2-a}\right)^{\delta_{n-6}}\left(\frac{x_1-b}{x_1-a}\right)^{\delta_{n-5}}\left(\frac{x_0-b}{x_0-a}\right)^{\delta_{n-4}} - 1}, \quad n \in \mathbb{N}_0,$$

where the sequence δ_n is given by (24).

Remark 1 Equation (7) in the case $k = 4$, $l = 2$ is a difference equation with interlacing indices [36], and its general solution is obtained by the general solution to equation (7) in the case $k = 2$, $l = 1$. Namely, the subsequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are two different solutions of equation (7) in the case $k = 2$, $l = 1$. The first solution has the initial values x_0 and x_2, whereas the second solution has initial values x_1 and x_3. From this we see that these two solutions are not connected to each other.

Case $a = b \neq 0$. Since $a = b \neq 0$, equation (7) becomes

$$x_{n+k} = \frac{x_{n+l}x_n - a^2}{x_{n+l} + x_n - 2a}, \quad n \in \mathbb{N}_0, \tag{25}$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $a \in \mathbb{C} \setminus \{0\}$, and $x_j \in \mathbb{C}$, $j = \overline{0, k-1}$.

In this case equation (13) has only one solution $x^* = a$. Hence, it is not possible to use the method in the previous case.

On the other hand, we have

$$\begin{aligned}
x_{n+k} - a &= \frac{x_{n+l}x_n - a(x_{n+l} + x_n) + a^2}{x_{n+l} + x_n - 2a} \\
&= \frac{(x_{n+l} - a)(x_n - a)}{x_{n+l} - a + x_n - a}
\end{aligned} \tag{26}$$

for $n \in \mathbb{N}_0$.

Equation (26) strikingly suggests a use of the change of variables

$$y_n = x_n - a, \quad n \in \mathbb{N}_0, \tag{27}$$

by which the equation is transformed to equation (8) where x_n is replaced with y_n.

From (9) and (27) we see that the change of variables

$$x_n = a + \frac{1}{u_n}, \quad n \in \mathbb{N}_0,$$ (28)

transforms equation (26) to equation (10).

Equation (10) when $\max\{k, l\} \leq 4$ is solvable in a closed form, and the following result holds.

Theorem 3 *The following statements hold.*

(a) *General solution to equation* (10) *with* $k = 2$ *and* $l = 1$ *is given by the formula*

$$u_n = f_n u_1 + f_{n-1} u_0, \quad n \in \mathbb{N}_0,$$

where f_n *is the Fibonacci sequence.*

(b) *General solution to equation* (10) *with* $k = 3$ *and* $l = 1$ *is given by the formula*

$$u_n = \alpha_{n-3} u_2 + \alpha_{n-2} u_1 + \alpha_{n-4} u_0, \quad n \in \mathbb{N}_0,$$

where the sequence α_n *is given by* (21).

(c) *General solution to equation* (10) *with* $k = 3$ *and* $l = 2$ *is given by the formula*

$$u_n = \beta_{n-2} u_2 + \beta_{n-4} u_1 + \beta_{n-3} u_0,$$

where the sequence β_n *is given by* (22).

(d) *General solution to equation* (10) *with* $k = 4$ *and* $l = 1$ *is given by the formula*

$$u_n = \gamma_{n-5} u_3 + \gamma_{n-4} u_2 + \gamma_{n-3} u_1 + \gamma_{n-6} u_0,$$

where the sequence γ_n *is given by* (23).

(e) *General solution to equation* (10) *with* $k = 4$ *and* $l = 2$ *is given by the formulas*

$$u_{2n} = f_n u_2 + f_{n-1} u_0, \quad n \in \mathbb{N}_0,$$
$$u_{2n+1} = f_n u_3 + f_{n-1} u_1, \quad n \in \mathbb{N}_0,$$

where f_n *is the Fibonacci sequence.*

(f) *General solution to equation* (10) *with* $k = 4$ *and* $l = 3$ *is given by the formula*

$$u_n = \delta_{n-3} u_3 + \delta_{n-6} u_2 + \delta_{n-5} u_1 + \delta_{n-4} u_0,$$

where the sequence δ_n *is given by* (24).

Proof These statements can be easily anticipated and essentially obtained by Theorem 1. Namely, by using the natural connection (19) between positive solutions to product-type difference equations and some solutions to the corresponding linear difference equations, we see that the above statements follow from the corresponding ones in Theorem 1 for

the case of such initial values (the solutions to equation (10) with real-valued initial values correspond to some uniquely defined solutions to equation (18) with positive initial values). But since equation (10) is linear, the above formulas are not only its solutions for such initial values, but are obviously solutions for all complex-valued initial values from which all the statements follow.   □

Remark 2 All the statements in Theorem 3 can be also easily proved by the method of induction. However, the method does not explain the given representations for the solutions to equation (10), unlike the above given proof.

From Theorem 3 and by using relation (28), we obtain the following result.

Theorem 4 *Consider equation* (7). *Assume that* $a = b \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold.*

(a) *General solution to equation* (7) *with* $k = 2$ *and* $l = 1$ *is given by the formula*

$$x_n = a + \frac{1}{\frac{f_n}{x_1 - a} + \frac{f_{n-1}}{x_0 - a}}, \quad n \in \mathbb{N}_0,$$

   where f_n *is the Fibonacci sequence.*

(b) *General solution to equation* (7) *with* $k = 3$ *and* $l = 1$ *is given by the formula*

$$x_n = a + \frac{1}{\frac{\alpha_{n-3}}{x_2 - a} + \frac{\alpha_{n-2}}{x_1 - a} + \frac{\alpha_{n-4}}{x_0 - a}}, \quad n \in \mathbb{N}_0,$$

   where the sequence α_n *is given by* (21).

(c) *General solution to equation* (7) *with* $k = 3$ *and* $l = 2$ *is given by the formula*

$$x_n = a + \frac{1}{\frac{\beta_{n-2}}{x_2 - a} + \frac{\beta_{n-4}}{x_1 - a} + \frac{\beta_{n-3}}{x_0 - a}},$$

   where the sequence β_n *is given by* (22).

(d) *General solution to equation* (7) *with* $k = 4$ *and* $l = 1$ *is given by the formula*

$$x_n = a + \frac{1}{\frac{\gamma_{n-5}}{x_3 - a} + \frac{\gamma_{n-4}}{x_2 - a} + \frac{\gamma_{n-3}}{x_1 - a} + \frac{\gamma_{n-6}}{x_0 - a}},$$

   where the sequence γ_n *is given by* (23).

(e) *General solution to equation* (7) *with* $k = 4$ *and* $l = 2$ *is given by the formula*

$$x_{2n} = a + \frac{1}{\frac{f_n}{x_2 - a} + \frac{f_{n-1}}{x_0 - a}}, \quad n \in \mathbb{N}_0,$$

$$x_{2n+1} = a + \frac{1}{\frac{f_n}{x_3 - a} + \frac{f_{n-1}}{x_1 - a}}, \quad n \in \mathbb{N}_0,$$

   where f_n *is the Fibonacci sequence.*

(f) *General solution to equation* (7) *with k = 4 and l = 3 is given by the formula*

$$x_n = a + \frac{1}{\frac{\delta_{n-3}}{x_3-a} + \frac{\delta_{n-6}}{x_2-a} + \frac{\delta_{n-5}}{x_1-a} + \frac{\delta_{n-4}}{x_0-a}},$$

where the sequence δ_n is given by (24).

Remark 3 Note that the previous consideration holds also in the case $a = 0$. This means that Theorem 4 gives also general solution to equation (7) in the case $a = b = 0$.

Remark 4 For $l = 0$, equation (7) becomes a difference equation with interlacing indices and is reduced to the case $l = 0$ and $k = 1$. In the case $l = 0$ and $k = 1$, a difference equation of first order is obtained which is solved as explained above. We leave the case to the reader as an exercise.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed equally to the writing of this paper. They read and approved the manuscript.

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