Duality theorems for nondifferentiable semi-infinite interval-valued optimization problems with vanishing constraints

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Abstract

In this paper, we study the duality theorems of a nondifferentiable semi-infinite interval-valued optimization problem with vanishing constraints (IOPVC). By constructing the Wolfe and Mond–Weir type dual models, we give the weak duality, strong duality, converse duality, restricted converse duality, and strict converse duality theorems between IOPVC and its corresponding dual models under the assumptions of generalized convexity.

Keywords: Locally Lipschitz function; Nondifferentiable semi-infinite interval-valued optimization problems; Vanishing constraints; Wolfe type dual; Mond–Weir type dual

1 Introduction

In recent years, the mathematical programming problems with vanishing constraints (MPVCs) have been studied extensively by many scholars. Achtziger and Kanzow [1] first proposed an optimization problem with vanishing constraints (MPVCs) and gave the strong stationary point theorem and VC-stationary point theorem of MPVCs under ACQ and improved ACQ assumptions. Under the inspiration of [1], Hoheisel and Kanzow [2] gave the M-stationary point theorem of MPVCs by using the MPVCs-GCQ. Guu et al. [3] studied the strong KKT type optimality conditions for nonsmooth multiobjective semi-infinite programming problems with vanishing constraints by the generalized S-stationary and M-stationary point conditions. Tung [4] studied the necessary and sufficient KKT type optimality conditions for continuously differentiable multiobjective semi-infinite MPVCs by using the ACQ and VC-ACQ in [1].

Accordingly, the study of dual problems related to MPVCs has also been used as a tool to solve optimization problems in various fields in the past decades, such as variational problems, fractional programming problems, semi-infinite programming problems, complex minimax problems, and so on. Tung [4] presented Wolfe and Mond–Weir type dual models for differentiable multiobjective semi-infinite programming with vanishing constraints and discussed the weak and strong duality theorems. Mishra and Singh [5] studied the continuously differentiable MPVCs and gave the weak, strong, converse, restricted converse, and strict converse duality theorems between MPVCs and the corresponding Wolfe and

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Mond–Weir type dual models under the assumptions of convexity and strict convexity. Hu and Wang et al. [6] proposed a new Wolfe and Mond–Weir type dual models related to continuously differentiable MPVCs and studied the weak, strong, converse, restricted converse, and strict converse duality theorems between them under the assumptions of convexity and generalized convexity.

With the development of mathematics, there are more and more researchers paying their attention to interval-valued optimization problems. Wu [7] studied the Wolfe type dual problem for continuously differentiable interval-valued optimization problems. Sun and Wang [8] gave the optimality conditions and duality for nondifferentiable interval-valued optimization problems. Tung [9] studied the optimality conditions and duality for convex semi-infinite multiobjective interval-valued optimization problems. Ahmad et al. [10] studied continuously differentiable interval-valued variational problem and gave the sufficient optimality condition and Mond–Weir type duality of the original problem by using the invexity conditions. Kummari and Ahmad [11] discussed the optimality conditions and duality for nonsmooth interval-valued optimization problems with equality and inequality constraints via the L-invex-infine functions. Jayswal et al. [12, 13] gave the optimality conditions and duality for nonsmooth interval-valued optimization problems with inequality constraints by using generalized convexity. Su and Dinh [14] studied the duality for interval-valued pseudoconvex optimization problem with equilibrium constraints by using the notion of contingent epiderivatives. Recently, Ahmad et al. [15] studied the optimality conditions and Mond–Weir type dual problems for differentiable interval-valued optimization problems with vanishing constraints.

Inspired by the literatures mentioned above, in this paper, we study the duality theorems for nondifferentiable semi-infinite interval-valued optimization problem with vanishing constraints(IOPVC) and explore the dual relationships between IOPVC and its corresponding Wolfe and Mond–Weir type dual models. The paper is organized as follows. In Sect. 2, we introduce some known concepts and formulas; In Sect. 3, we study the weak, strong, converse, restricted converse, and strict converse duality theorems between IOPVC and the Wolfe type dual model; In Sect. 4, we study the weak, strong, converse, restricted converse, and strict converse duality theorems between IOPVC and the Mond–Weir type dual model. Some examples are given to illustrate our conclusions.

2 Preliminary

Let X be a finite-dimensional Euclidean space. The notation $\langle \cdot, \cdot \rangle$ denotes the inner product in X. For a point $\bar{x} \in X$, $B(\bar{x}; \delta) := \{x \in X : \|x - \bar{x}\| < \delta\}$ denotes the open ball of radius δ around $\bar{x}$. For a set $C \subset X$, span C, cone C stand for the linear hull and convex cone of C, respectively. Let $C \neq \emptyset$, the contingent cone of set C at the point x is defined by

$$T(C, x) := \{v \in X \mid \exists t_n \to 0, \exists v_n \to v, \forall n \in N, x + t_n v_n \in C\}.$$

Let $\mathcal{D}$ be the set of all closed intervals in R. For any $A = [a_1, a_2] \in \mathcal{D}, B = [b_1, b_2] \in \mathcal{D}$, one has (see Moore [16])

$$A + B = [a_1 + b_1, a_2 + b_2], \qquad -B = [-b_2, -b_1],$$
$$A - B = [a_1 - b_2, a_2 - b_1], \qquad A + k = [a_1 + k, a_2 + k],$$

where k is any real number. A partial ordering for intervals can be formulated as follows:

$$A \leq_{LU} B \quad \Longleftrightarrow \quad a_1 \leq b_1, \quad a_2 \leq b_2,$$

$$A <_{LU} B \quad \Longleftrightarrow \quad A \leq_{LU} B, \quad A \neq B,$$

$A \nless_{LU} B$ is the negation of $A <_{LU} B$,

$$A <_{LU}^s B \quad \Longleftrightarrow \quad a_1 < b_1, \quad a_2 < b_2,$$

$A \nless_{LU}^s B$ is the negation of $A <_{LU}^s B$.

Let $F : X \to \mathcal{D}$ be a mapping on X defined as

$$F(x) = \left[F^L(x), F^U(x) \right] \quad (\forall x \in X),$$

where F^L, F^U are the locally Lipschitz functions on X with $F^L(x) \leq F^U(x)$.

Now, we consider the following semi-infinite interval-valued optimization problem with vanishing constraints(IOPVC):

$$LU\text{-}\min \quad F(x),$$
$$\text{s.t.} \quad g_j(x) \leq 0, \quad j \in J,$$
$$h_k(x) = 0, \quad k = 1, \ldots, n,$$
$$H_i(x) \geq 0, \quad i = 1, \ldots, l,$$
$$G_i(x)H_i(x) \leq 0, \quad i = 1, \ldots, l,$$

where $g_j : X \to R \cup \{+\infty\}$, $h_k : X \to R$, $H_i : X \to R$, $G_i : X \to R$ are the locally Lipschitz functions on X and the index set J is arbitrary(possibly infinite). Let $I_n := \{1, \ldots, n\}$, $I_l := \{1, \ldots, l\}$. The feasible set of problem (IOPVC) is

$$E := \left\{ x \in X : g_j(x) \leq 0(j \in J), h_k(x) = 0(k \in I_n), H_i(x) \geq 0, G_i(x)H_i(x) \leq 0(i \in I_l) \right\}.$$

Let $R_+^{|J|}$ denote the collection of all the functions $\lambda : J \to R$ taking values $\lambda_j > 0$ only at finitely many points of J and equal to zero at the other points. For any $\bar{x} \in E$, $I_g(\bar{x}) := \{j \in J : g_j(\bar{x}) = 0\}$ signifies the index set of all active constraints at $\bar{x}$, and $\kappa(\bar{x}) := \{\lambda_j \in R_+^{|J|} : \lambda_j g_j(\bar{x}) = 0, \forall j \in J\}$ signifies the active constraint multipliers at $\bar{x}$.

We give the following definitions of optimal solutions of (IOPVC).

Definition 2.1 ([9]) Let $\bar{x} \in E$,
   (i) $\bar{x}$ is said to be a locally LU optimal solution of (IOPVC) if there exists an open ball $B(\bar{x}; \delta)$ such that there is no $x \in E \cap B(\bar{x}; \delta)$ satisfying

$$F(x) <_{LU} F(\bar{x}).$$

   (ii) $\bar{x}$ is said to be a locally weakly LU optimal solution of (IOPVC) if there exists an open ball $B(\bar{x}; \delta)$ such that there is no $x \in E \cap B(\bar{x}; \delta)$ satisfying

$$F(x) <_{LU}^s F(\bar{x}).$$

Let $f : X \to R$ be a locally Lipschitz function around $\bar{x}$. The Clarke directional derivative of f around $\bar{x}$ in the direction $v \in X$ and the Clarke subdifferential of f at $\bar{x}$ are, respectively, given by (see Clarke [17])

$$f_c'(\bar{x}; v) := \limsup_{x \to \bar{x}, t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

$$\partial_c f(\bar{x}) := \left\{ \xi \in X : \langle \xi, v \rangle \le f_c'(\bar{x}; v), \forall v \in X \right\}.$$

Definition 2.2 ([3]) Let $f : X \to R$ be a locally Lipschitz function around $\bar{x}$. Then
   (i) f is said to be ∂_c-pseudoconvex at $\bar{x}$ if, for each $x \in X$ and any $\xi \in \partial_c f(\bar{x})$,

$$f(x) - f(\bar{x}) < 0 \quad \Rightarrow \quad \langle \xi, x - \bar{x} \rangle < 0;$$

   (ii) f is said to be strictly ∂_c-pseudoconvex at $\bar{x}$ if, for each $x \in X$, $x \ne \bar{x}$ and any $\xi \in \partial_c f(\bar{x})$,

$$f(x) - f(\bar{x}) \le 0 \quad \Rightarrow \quad \langle \xi, x - \bar{x} \rangle < 0;$$

   (iii) f is said to be ∂_c-quasiconvex at $\bar{x}$ if, for each $x \in X$ and any $\xi \in \partial_c f(\bar{x})$,

$$f(x) - f(\bar{x}) \le 0 \quad \Rightarrow \quad \langle \xi, x - \bar{x} \rangle \le 0.$$

The following sets of indicators, which will be used in the sequel, are given. Let $x \in E$,

$$I_+(x) := \left\{ i \in I_l \mid H_i(x) > 0 \right\}, \qquad I_0(x) := \left\{ i \in I_l \mid H_i(x) = 0 \right\},$$

$$I_{+0}(x) := \left\{ i \in I_l \mid H_i(x) > 0, G_i(x) = 0 \right\},$$

$$I_{+-}(x) := \left\{ i \in I_l \mid H_i(x) > 0, G_i(x) < 0 \right\},$$

$$I_{0+}(x) := \left\{ i \in I_l \mid H_i(x) = 0, G_i(x) > 0 \right\},$$

$$I_{00}(x) := \left\{ i \in I_l \mid H_i(x) = 0, G_i(x) = 0 \right\},$$

$$I_{0-}(x) := \left\{ i \in I_l \mid H_i(x) = 0, G_i(x) < 0 \right\}.$$

Referring to Definition 4 in [4], we give the following definition.

Definition 2.3 Let $\bar{x} \in E$ be a feasible point of (IOPVC).
   (i) The Abadie constraint qualification (ACQ) is said to hold at $\bar{x}$ iff $T(E, \bar{x}) = L(\bar{x})$, where $L(\bar{x})$ is the linearized cone of (IOPVC) at $\bar{x}$, and

$$L(\bar{x}) := \left\{ v \in X \mid \langle \xi_j^g, v \rangle \le 0, \forall \xi_j^g \in \partial_c g_j(\bar{x}), j \in I_g(\bar{x}); \langle \xi_k^h, v \rangle = 0, \forall \xi_k^h \in \partial_c h_k(\bar{x}), \right.$$
$$k \in I_n; \langle \xi_i^H, v \rangle = 0, \forall \xi_i^H \in \partial_c H_i(\bar{x}), i \in I_{0+}; \langle \xi_i^H, v \rangle \ge 0, \forall \xi_i^H \in \partial_c H_i(\bar{x}),$$
$$\left. i \in I_{00} \cup I_{0-}; \langle \xi_i^G, v \rangle \le 0, \forall \xi_i^G \in \partial_c G_i(\bar{x}), i \in I_{+0} \right\}. \tag{2.1}$$

(ii) The VC-ACQ is said to hold at $\bar{x}$ iff $L_{VC}(\bar{x}) \subseteq T(E, \bar{x})$, where $L_{VC}(\bar{x})$ is the corresponding VC-linearized cone of (IOPVC) at $\bar{x}$, and

$$
\begin{aligned}
L_{VC}(\bar{x}) := \big\{ v \in X \mid & \langle \xi_j^g, v \rangle \le 0, \forall \xi_j^g \in \partial_c g_j(\bar{x}), j \in I_g(\bar{x}); \\
& \langle \xi_k^h, v \rangle = 0, \forall \xi_k^h \in \partial_c h_k(\bar{x}), k \in I_n; \\
& \langle \xi_i^H, v \rangle = 0, \forall \xi_i^H \in \partial_c H_i(\bar{x}), i \in I_{0+}; \\
& \langle \xi_i^H, v \rangle \ge 0, \forall \xi_i^H \in \partial_c H_i(\bar{x}), i \in I_{00} \cup I_{0-}; \\
& \langle \xi_i^G, v \rangle \le 0, \forall \xi_i^G \in \partial_c G_i(\bar{x}), i \in I_{+0} \cup I_{00} \big\}.
\end{aligned} \tag{2.2}
$$

Remark 2.1 If the functions g_j, h_k, H_i, G_i are continuously differentiable, then the linearized cone and VC-linearized cone given in Definition 2.3 are the same as the linearized cones given in [4].

Now, we give the following theorem, the proof of which is similar to Proposition 1(ii) in [4].

Theorem 2.1 *Let $\bar{x} \in E$ be a locally weakly LU optimal solution of (IOPVC) such that (VC-ACQ) holds at $\bar{x}$ and*

$$
\begin{aligned}
\triangle := \operatorname{cone}\Bigg(& \bigcup_{j \in I_g(\bar{x})} \partial_c g_j(\bar{x}) \cup \bigcup_{i \in I_{00} \cup I_{0-}} -\partial_c H_i(\bar{x}) \cup \bigcup_{i \in I_{+0} \cup I_{00}} \partial_c G_i(\bar{x}) \Bigg) \\
& + \operatorname{span}\Bigg(\bigcup_{k \in I_n} \partial_c h_k(\bar{x}) \cup \bigcup_{i \in I_{0+}} \partial_c H_i(\bar{x}) \Bigg)
\end{aligned}
$$

is closed. Then there exist Lagrange multipliers $\alpha^L, \alpha^U \in R_+$, $\lambda^g \in \kappa(\bar{x})$, $\lambda^h \in R^n$, $\lambda^H, \lambda^G \in R^l$ such that

$$
0 \in \alpha^L \partial_c F^L(\bar{x}) + \alpha^U \partial_c F^U(\bar{x}) + \sum_{j \in J} \lambda_j^g \partial_c g_j(\bar{x}) + \sum_{k=1}^n \lambda_k^h \partial_c h_k(\bar{x})
$$

$$
- \sum_{i=1}^l \lambda_i^H \partial_c H_i(\bar{x}) + \sum_{i=1}^l \lambda_i^G \partial_c G_i(\bar{x}) \tag{2.3}
$$

and

$$
\begin{cases}
\alpha^L + \alpha^U = 1, \qquad h_k(\bar{x}) = 0 \quad (k \in I_n), \\
\lambda_j^g \ge 0, \quad g_j(\bar{x}) \le 0, \qquad \lambda_j^g g_j(\bar{x}) = 0 \quad (j \in J), \\
\lambda_i^H = 0 \quad (i \in I_+(\bar{x})), \qquad \lambda_i^H \ge 0 \quad (i \in I_{00}(\bar{x}) \cup I_{0-}(\bar{x})), \\
\lambda_i^H \in R \quad (i \in I_{0+}(\bar{x})), \\
\lambda_i^G = 0 \quad (i \in I_{0+}(\bar{x}) \cup I_{0-}(\bar{x}) \cup I_{+-}(\bar{x})), \qquad \lambda_i^G \ge 0 \quad (i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x})).
\end{cases} \tag{2.4}
$$

Definition 2.4 ([4]) The point x is said to be a VC-stationary point of (IOPVC) if there exist Lagrange multipliers $\alpha^L, \alpha^U \in R_+$, $\lambda^g \in \kappa(x)$, $\lambda^h \in R^n$, $\lambda^H, \lambda^G \in R^l$ such that (2.3) and (2.4) hold.

Now, let x be a VC-stationary point of (IOPVC) with corresponding multipliers $\lambda^g \in R_+^{|J|}$, $\lambda^h \in R^n$, $\lambda^H, \lambda^G \in R^l$, we give the following index sets:

$$I_g^+(x) := \left\{ j \in I_g(x) \mid \lambda_j^g > 0 \right\},$$

$$I_h^+(x) := \left\{ k \in I_n(x) \mid \lambda_k^h > 0 \right\}, \qquad I_h^-(x) := \left\{ k \in I_n(x) \mid \lambda_k^h < 0 \right\},$$

$$I_{00}^+(x) := \left\{ i \in I_{00}(x) \mid \lambda_i^H > 0 \right\}, \qquad I_{0-}^+(x) := \left\{ i \in I_{0-}(x) \mid \lambda_i^H > 0 \right\},$$

$$I_{0+}^+(x) := \left\{ i \in I_{0+}(x) \mid \lambda_i^H > 0 \right\}, \qquad I_{0+}^-(x) := \left\{ i \in I_{0+}(x) \mid \lambda_i^H < 0 \right\},$$

$$I_+^+(x) := \left\{ i \in I_+(x) \mid \lambda_i^H > 0 \right\}, \qquad \tilde{I}_+^+(x) := \left\{ i \in I_+(x) \mid \lambda_i^G > 0 \right\}.$$

3 Wolfe type duality

In this section, we refer to [6] to give the following Wolfe type dual models. First of all, let $\lambda^g \in R_+^{|J|}$, $\lambda^h \in R^n$, $\lambda^H, \lambda^G \in R^l$,

$$\Phi\left(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\right) = F(\cdot) + \sum_{j \in J} \lambda_j^g g_j(\cdot) + \sum_{k=1}^n \lambda_k^h h_k(\cdot)$$

$$- \sum_{i=1}^l \lambda_i^H H_i(\cdot) + \sum_{i=1}^l \lambda_i^G G_i(\cdot)$$

is an interval-valued function, and

$$\Delta(\cdot) := \alpha^L \partial_c F^L(\cdot) + \alpha^U \partial_c F^U(\cdot) + \sum_{j \in J} \lambda_j^g \partial_c g_j(\cdot) + \sum_{k=1}^n \lambda_k^h \partial_c h_k(\cdot)$$

$$- \sum_{i=1}^l \lambda_i^H \partial_c H_i(\cdot) + \sum_{i=1}^l \lambda_i^G \partial_c G_i(\cdot).$$

Now, we give the Wolfe type dual model of (IOPVC). For $x \in E$,

$$\left(D_W(x)\right) \quad LU\text{-}\max \quad \Phi\left(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\right)$$

$$\begin{cases} s.t. \quad 0 \in \Delta(y) \\ \alpha^L, \alpha^U \in R_+, \quad \alpha^L + \alpha^U = 1, \\ \lambda_j^g \geq 0, \quad \forall j \in J, \\ \lambda_i^G = \nu_i H_i(x), \nu_i \geq 0, \quad \forall i \in I_l, \\ \lambda_i^H = \varrho_i - \nu_i G_i(x), \quad \varrho_i \geq 0, \forall i \in I_l. \end{cases} \tag{3.1}$$

$$E_W(x) := \left\{ \left(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu\right) : \right.$$

$$0 \in \Delta(y), y \in X, \alpha^L, \alpha^U \in R_+, \alpha^L + \alpha^U = 1,$$

$$\lambda_j^g \geq 0, \forall j \in J, \lambda_i^G = \nu_i H_i(x), \nu_i \geq 0, \forall i \in I_l$$

$$\left. \lambda_i^H = \varrho_i - \nu_i G_i(x), \varrho_i \geq 0, \forall i \in I_l \right\}$$

denotes the feasible set of $(D_W(x))$. $prE_W(x) := \{y \in X : (y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_W(x)\}$ represents the projection of the set $E_W(x)$ on X.

In order to be independent of (IOPVC), we give another Wolfe type dual model:

$$(D_W)LU\text{-}\max \quad \Phi\big(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big)$$

$$\text{s.t. } \big(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G,\varrho,\nu\big) \in E_W := \bigcap_{x\in E} E_W(x),$$

where E_W denotes the set of all feasible points of (D_W) and prE_W denotes the projection of the set E_W on X.

Definition 3.1 ([4]) Let $x \in E$.

(i) $\big(\bar{y},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G,\varrho,\nu\big) \in E_W(x)$ is said to be a locally *LU* optimal solution of $(D_W(x))$ if there exists $B(\bar{y};\delta)$ such that there is no $y \in E_W(x) \cap B(\bar{y};\delta)$ satisfying

$$\Phi\big(\bar{y},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big) <_{LU} \Phi\big(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big).$$

(ii) $\big(\bar{y},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G,\varrho,\nu\big) \in E_W(x)$ is said to be a locally weakly *LU* optimal solution of $(D_W(x))$ if there exists $B(\bar{y};\delta)$ such that there is no $y \in E_W(x) \cap B(\bar{y};\delta)$ satisfying

$$\Phi\big(\bar{y},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big) <^s_{LU} \Phi\big(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big).$$

Theorem 3.1 (Weak duality) *Let $x \in E$, $(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G,\varrho,\nu) \in E_W$ be feasible points for the (IOPVC) and the (D_W), respectively. If $\Phi^L(\cdot,\alpha^L,\lambda^g,\lambda^h,\lambda^H,\lambda^G)$, $\Phi^U(\cdot,\alpha^U,\lambda^g, \lambda^h,\lambda^H,\lambda^G)$ are ∂_c-pseudoconvex at $y \in E \cup prE_W$, then*

$$F(x) \not<^s_{LU} \Phi\big(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big).$$

Proof Suppose $F(x) <^s_{LU} \Phi(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G)$, then

$$F(x) <^s_{LU} F(y) + \sum_{j\in J}\lambda^g_j g_j(y) + \sum_{k=1}^{n}\lambda^h_k h_k(y) - \sum_{i=1}^{l}\lambda^H_i H_i(y) + \sum_{i=1}^{l}\lambda^G_i G_i(y). \tag{3.2}$$

Since $x \in E$ and $(y,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G,\varrho,\nu) \in E_W$, it follows that

$$
\begin{aligned}
g_j(x) &< 0, & \lambda^g_j &\geq 0, & j &\notin I_g(x), \\
g_j(x) &= 0, & \lambda^g_j &\geq 0, & j &\in I_g(x), \\
h_k(x) &= 0, & \lambda^h_k &\in R, & k &\in I_n, \\
-H_i(x) &< 0, & \lambda^H_i &\geq 0, & i &\in I_+(x), \\
-H_i(x) &= 0, & \lambda^H_i &\in R, & i &\in I_0(x), \\
G_i(x) &> 0, & \lambda^G_i &= 0, & i &\in I_{0+}(x), \\
G_i(x) &= 0, & \lambda^G_i &\geq 0, & i &\in I_{00}(x) \cup I_{+0}(x), \\
G_i(x) &< 0, & \lambda^G_i &\geq 0, & i &\in I_{0-}(x) \cup I_{+-}(x).
\end{aligned} \tag{3.3}
$$

The above formulas imply that

$$\sum_{j \in J} \lambda_j^g g_j(x) + \sum_{k=1}^{n} \lambda_k^h h_k(x) - \sum_{i=1}^{l} \lambda_i^H H_i(x) + \sum_{i=1}^{l} \lambda_i^G G_i(x) \le 0. \tag{3.4}$$

Equation (3.4) together with (3.2) proves that

$$\Phi\big(x, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big) <_{LU}^s \Phi\big(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big).$$

By the ∂_c-pseudoconvexity of $\Phi^L(\cdot, \alpha^L, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, $\Phi^U(\cdot, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ at $y \in E \cup prE_W$, it follows that

$$\begin{aligned}
&\left\langle \xi^L + \sum_{j \in J} \lambda_j^g \xi_j^g + \sum_{k=1}^{n} \lambda_k^h \xi_k^h - \sum_{i=1}^{l} \lambda_i^H \xi_i^H + \sum_{i=1}^{l} \lambda_i^G \xi_i^G, x - y \right\rangle < 0, \\
&\left\langle \xi^U + \sum_{j \in J} \lambda_j^g \xi_j^g + \sum_{k=1}^{n} \lambda_k^h \xi_k^h - \sum_{i=1}^{l} \lambda_i^H \xi_i^H + \sum_{i=1}^{l} \lambda_i^G \xi_i^G, x - y \right\rangle < 0,
\end{aligned} \tag{3.5}$$

where $\xi^L \in \partial_c F^L(y)$, $\xi^U \in \partial_c F^U(y)$, $\xi_j^g \in \partial_c g_j(y)$, $j \in J$, $\xi_k^h \in \partial_c h_k(y)$, $k \in I_n$, $\xi_i^H \in \partial_c H_i(y)$, $i \in I_l$, $\xi_i^G \in \partial_c G_i(y)$, $i \in I_l$. Add α^L and α^U to both sides of inequations (3.5)

$$\langle \xi, x - y \rangle < 0, \quad \forall \xi \in \Delta(y),$$

contradicting $0 \in \Delta(y)$, the result is proved. □

Theorem 3.2 (Weak duality) *Let $x \in E$, $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_W$ be feasible points for the (IOPVC) and the (D_W), respectively. If $\Phi^L(\cdot, \alpha^L, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, $\Phi^U(\cdot, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ are strictly ∂_c-pseudoconvex at $y \in E \cup prE_W$, then*

$$F(x) \not<_{LU} \Phi\big(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big).$$

Proof The proof is similar to Theorem 3.1. □

Theorem 3.3 (Strong duality) *Let $\bar{x} \in E$ be a locally weakly LU optimal solution of (IOPVC) such that the (VC-ACQ) holds at $\bar{x}$ and $\triangle$ is closed. Then there exist Lagrange multipliers $\bar{\alpha}^L, \bar{\alpha}^U \in R_+$, $\bar{\lambda}^g \in R_+^{|J|}$, $\bar{\lambda}^h \in R^n$, $\bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{v} \in R^l$ such that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{v})$ is a feasible point of $(D_W(\bar{x}))$, and*

$$F(\bar{x}) = \Phi\big(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G\big).$$

Moreover, if $\Phi^L(\cdot, \alpha^L, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, $\Phi^U(\cdot, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ are ∂_c-pseudoconvex at $y \in E \cup prE_W(\bar{x})$, then $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{v})$ is a locally weakly LU optimal solution of $(D_W(\bar{x}))$.

Proof By Theorem 2.1, it follows that there exist Lagrange multipliers $\bar{\alpha}^L, \bar{\alpha}^U \in R_+$, $\bar{\lambda}^g \in \kappa(\bar{x})$, $\bar{\lambda}^h \in R^n$, $\bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{v} \in R^l$ such that (2.3) and (2.4) are satisfied. Combined with the def-

inition of $(D_W(\bar{x}))$, one has that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{v})$ is a feasible point of $(D_W(\bar{x}))$,

$$\sum_{j \in J} \bar{\lambda}_j^g g_j(\bar{x}) + \sum_{k=1}^n \bar{\lambda}_k^h h_k(\bar{x}) - \sum_{i=1}^l \bar{\lambda}_i^H H_i(\bar{x}) + \sum_{i=1}^l \bar{\lambda}_i^G G_i(\bar{x}) = 0$$

and

$$F(\bar{x}) = \Phi(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G).$$

Then, from Theorem 3.1 one has, for any $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_W(\bar{x})$,

$$\Phi(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G) = F(\bar{x}) \not\prec_{LU}^s \Phi(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G).$$

Therefore, $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{v})$ is a locally weakly LU optimal solution of $(D_W(\bar{x}))$. □

Theorem 3.4 (Converse duality) *Let $x \in E$ be any feasible point of (IOPVC), and let $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_W$ be a feasible point of (D_W) such that*

$$\begin{cases} \lambda_j^g g_j(\bar{y}) \geq 0, & \forall j \in J, \\ \lambda_k^h h_k(\bar{y}) = 0, & \forall k \in I_n, \\ -\lambda_i^H H_i(\bar{y}) \geq 0, & \forall i \in I_l, \\ \lambda_i^G G_i(\bar{y}) \geq 0, & \forall i \in I_m. \end{cases} \tag{3.6}$$

If one of the following conditions holds:

(i) $\Phi^L(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, $\Phi^U(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ *are ∂_c-pseudoconvex at $\bar{y} \in E \cup prE_W$;*

(ii) $F^L(\cdot), F^U(\cdot)$ *are ∂_c-pseudoconvex at $\bar{y} \in E \cup prE_W(x)$, g_j $(j \in I_g^+(x))$, h_k $(k \in I_h^+(x))$, $-h_k$ $(k \in I_h^-(x))$, $-H_i$ $(i \in I_+^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x))$, H_i $(i \in I_{0+}^-(x))$, G_i $(i \in \tilde{I}_+^+(x))$ are ∂_c-quasiconvex at $\bar{y} \in E \cup prE_W$;*

Then $\bar{y}$ is the locally weakly LU optimal solution of (IOPVC).

Proof Suppose to the contrary that $\bar{y}$ is not a locally weakly LU optimal solution of (IOPVC), then there exists $\bar{x} \in E \cap B(\bar{y}; \delta)$ such that

$$F(\bar{x}) <_{LU}^s F(\bar{y}). \tag{3.7}$$

(i) Since $\bar{x} \in E$ and $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v)$ are feasible points for the (IOPVC) and the (D_W), respectively. Therefore, combined with (3.3) and (3.6), one gets

$$\sum_{j \in J} \lambda_j^g g_j(\bar{x}) + \sum_{k=1}^n \lambda_k^h h_k(\bar{x}) - \sum_{i=1}^l \lambda_i^H H_i(\bar{x}) + \sum_{i=1}^l \lambda_i^G G_i(\bar{x})$$

$$\leq 0 \leq \sum_{j \in J} \lambda_j^g g_j(\bar{y}) + \sum_{k=1}^n \lambda_k^h h_k(\bar{y}) - \sum_{i=1}^l \lambda_i^H H_i(\bar{y}) + \sum_{i=1}^l \lambda_i^G G_i(\bar{y}). \tag{3.8}$$

By (3.7) and (3.8), one has

$$\Phi\big(\bar{x},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big) <^s_{LU} \Phi\big(\bar{y},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G\big).$$

And by the ∂_c-pseudoconvexity of $\Phi^L(\cdot,\alpha^L,\alpha^U\lambda^g,\lambda^h,\lambda^H,\lambda^G)$, $\Phi^U(\cdot,\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G)$ at $\bar{y} \in E \cup prE_W$, one has

$$\begin{cases} \langle \xi^L + \sum_{j\in J} \lambda^g_j \xi^g_j + \sum_{k=1}^n \lambda^h_k \xi^h_k - \sum_{i=1}^l \lambda^H_i \xi^H_i + \sum_{i=1}^l \lambda^G_i \xi^G_i, \bar{x} - \bar{y} \rangle < 0, \\ \langle \xi^U + \sum_{j\in J} \lambda^g_j \xi^g_j + \sum_{k=1}^n \lambda^h_k \xi^h_k - \sum_{i=1}^l \lambda^H_i \xi^H_i + \sum_{i=1}^l \lambda^G_i \xi^G_i, \bar{x} - \bar{y} \rangle < 0, \end{cases} \tag{3.9}$$

where $\xi^L \in \partial_c F^L(\bar{y}), \xi^U \in \partial_c F^U(\bar{y}), \xi^g_j \in \partial_c g_j(\bar{y}), j \in J, \xi^h_k \in \partial_c h_k(\bar{y}), k \in I_n, \xi^H_i \in \partial_c H_i(\bar{y}), i \in I_l,$ $\xi^G_i \in \partial_c G_i(\bar{y}), i \in I_l$. Combining (3.9) with α^L and α^U, we get

$$\langle \xi, \bar{x} - \bar{y} \rangle < 0, \quad \forall \xi \in \Delta(\bar{y}),$$

which contradicts $0 \in \Delta(\bar{y})$, hence the result holds.

(ii) Since $\bar{x} \in E$ and $(\bar{y},\alpha^L,\alpha^U,\lambda^g,\lambda^h,\lambda^H,\lambda^G,\varrho,\nu) \in E_W$, by (3.3) and (3.6), one has

$$\lambda^g_j g_j(\bar{x}) \le \lambda^g_j g_j(\bar{y}), \quad \forall j \in J,$$
$$\lambda^h_k h_k(\bar{x}) = \lambda^h_k h_k(\bar{y}), \quad \forall k \in I_n,$$
$$\lambda^H_i H_i(\bar{x}) \le \lambda^H_i H_i(\bar{y}), \quad \forall i \in I_l,$$
$$\lambda^G_i G_i(\bar{x}) \le \lambda^G_i G_i(\bar{y}), \quad \forall i \in I_l$$

and by the definition of the index sets above, we get

$$\begin{cases} g_j(\bar{x}) \le g_j(\bar{y}), & \forall j \in I^+_g(\bar{x}), \\ h_k(\bar{x}) = h_k(\bar{y}), & \forall k \in I^+_h(\bar{x}) \cup I^-_h(\bar{x}), \\ -H_i(\bar{x}) \le -H_i(\bar{y}), & \forall i \in I^+_+(\bar{x}) \cup I^+_{00}(\bar{x}) \cup I^+_{0-}(\bar{x}) \cup I^+_{0+}(\bar{x}), \\ -H_i(\bar{x}) \ge -H_i(\bar{y}), & \forall i \in I^-_{0+}(\bar{x}), \\ G_i(\bar{x}) \le G_i(\bar{y}), & \forall i \in \tilde{I}^+_+(\bar{x}). \end{cases} \tag{3.10}$$

By the ∂_c-quasiconvexity of the functions in assumption(ii) and (3.10), it follows that

$$\langle \xi^g_j, \bar{x} - \bar{y} \rangle \le 0, \quad \lambda^g_j > 0, \quad \forall \xi^g_j \in \partial_c g_j(\bar{y}), j \in I^+_g(\bar{x}),$$
$$\langle \xi^h_k, \bar{x} - \bar{y} \rangle \le 0, \quad \lambda^h_k > 0, \quad \forall \xi^h_k \in \partial_c h_k(\bar{y}), k \in I^+_h(\bar{x}),$$
$$\langle \xi^h_k, \bar{x} - \bar{y} \rangle \ge 0, \quad \lambda^h_k < 0, \quad \forall \xi^h_k \in \partial_c h_k(\bar{y}), k \in I^-_h(\bar{x}),$$
$$\langle -\xi^H_i, \bar{x} - \bar{y} \rangle \le 0, \quad \lambda^H_i \ge 0, \quad \forall \xi^H_i \in \partial_c H_i(\bar{y}), i \in I^+_+(\bar{x}) \cup I^+_{00}(\bar{x}) \cup I^+_{0-}(\bar{x}) \cup I^+_{0+}(\bar{x}),$$
$$\langle -\xi^H_i, \bar{x} - \bar{y} \rangle \ge 0, \quad \lambda^H_i \le 0, \quad \forall \xi^H_i \in \partial_c H_i(\bar{y}), i \in I^-_{0+}(\bar{x}),$$
$$\langle \xi^G_i, \bar{x} - \bar{y} \rangle \le 0, \quad \lambda^G_i \ge 0, \quad \forall \xi^G_i \in \partial_c G_i(\bar{y}), i \in \tilde{I}^+_+(\bar{x})$$

that is,

$$\left\langle \sum_{j \in J} \lambda_j^g \xi_j^g + \sum_{k=1}^n \lambda_k^h \xi_k^h - \sum_{i=1}^l \lambda_i^H \xi_i^H + \sum_{i=1}^l \lambda_i^G \xi_i^G, \bar{x} - \bar{y} \right\rangle \le 0.$$

By the above inequality and $0 \in \Delta(\bar{y})$, there exist $\xi^L \in \partial_c F^L(\bar{y})$ and $\xi^U \in \partial_c F^U(\bar{y})$ such that

$$\left\langle \alpha^L \xi^L + \alpha^U \xi^U, \bar{x} - \bar{y} \right\rangle \ge 0. \tag{3.11}$$

By (3.7) and the ∂_c-pseudoconvexity of $F^L(\cdot)$ and $F^U(\cdot)$, it follows that

$$\left\langle \xi^L, \bar{x} - \bar{y} \right\rangle < 0, \quad \forall \xi^L \in \partial_c F^L(\bar{y}),$$
$$\left\langle \xi^U, \bar{x} - \bar{y} \right\rangle < 0, \quad \forall \xi^U \in \partial_c F^U(\bar{y}),$$

then $\langle \alpha^L \xi^L + \alpha^U \xi^U, \bar{x} - \bar{y} \rangle < 0, \alpha^L, \alpha^U \in R_+, \alpha^L + \alpha^U = 1$, contradicting (3.11), so the result also holds. $\square$

Theorem 3.5 (Restricted converse duality) *Let $\bar{x} \in E$ be a feasible point of (IOPVC), and let $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_W$ be a feasible point of (D_W) such that $F(\bar{x}) = \Phi(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$. If $\Phi^L(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G), \Phi^U(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ are ∂_c-pseudoconvex at $y \in E \cup prE_W$, then $\bar{x}$ is the locally weakly LU optimal solution of (IOPVC).*

Proof Suppose that $\bar{x}$ is not a locally weakly *LU* optimal solution of (IOPVC), then there exists $\tilde{x} \in E \cap B(\bar{x}; \delta)$ such that $F(\tilde{x}) <_{LU}^s F(\bar{x})$. By $F(\bar{x}) = \Phi(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, we get $F(\tilde{x}) <_{LU}^s \Phi(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, contradicting Theorem 3.1. So, $\bar{x}$ is the locally weakly *LU* optimal solution of (IOPVC). $\square$

The following example shows that the conclusion of Theorem 3.5 holds.

Example 3.1 Let $X = R^2$, $n = 0$, $J = l = 1$, consider the following question:

$$(\text{IOPVC1}) \quad \min \quad F(x) = \left[F^L(x), F^U(x) \right] = \left[x_1^2 - x_2^2, x_1^2 \right]$$
$$\text{s.t.} \quad g_1(x) = -x_1 \le 0,$$
$$H_1(x) = x_1 - x_2 \ge 0,$$
$$G_1(x)H_1(x) = x_1(x_1 - x_2) \le 0.$$

The feasible set of problem (IOPVC1) is given by

$$E_1 := \{x \in R \mid x_1 > 0, x_1 - x_2 = 0\} \cup \{x \in R \mid x_1 = 0, x_2 \le 0\}.$$

For any $x \in E_1$, the Wolfe type dual model to (IOPVC1) is given by

$$(D_W(x)) \quad \max \quad \Phi\left(y, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G\right)$$
$$\text{s.t.} \quad \alpha^L(2y_1, -2y_2) + \alpha^U(2y_1, 0) + \lambda_1^g(-1, 0) - \lambda_1^H(1, -1) + \lambda_1^G(1, 0) = (0, 0),$$

$$\lambda_1^H \geq 0, \quad \text{if } 1 \in I_+(x) \cup I_{0-}(x) \cup I_{00}(x);$$

$$\lambda_1^H \in R, \quad \text{if } 1 \in I_{0+}(x),$$

$$\lambda_1^G \geq 0, \quad \text{if } 1 \in I_+(x); \qquad \lambda_1^G = 0 \quad \text{if } 1 \in I_0(x),$$

where $\Phi(y, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G) = [y_1^2 - y_2^2 - \lambda_1^g y_1 - \lambda_1^H(y_1 - y_2) + \lambda_1^G y_1, y_1^2 - \lambda_1^g y_1 - \lambda_1^H(y_1 - y_2) + \lambda_1^G y_1]$. Therefore, we can get the feasible set of problem (D_W), which is not dependent on x,

$$E_2 := \left\{ (y_1, y_2, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G) : 2y_1 - \lambda_1^g - \lambda_1^H + \lambda_1^G = 0, -2\alpha^L y_2 + \lambda_1^H = 0, \right.$$
$$\left. y_1, y_2 \in X, \alpha^L, \alpha^U \in R_+, \alpha^L + \alpha^U = 1, \lambda_1^g \geq 0, \lambda_1^G = 0, \lambda_1^H \geq 0 \right\}.$$

Let $\alpha^L = \alpha^U = \frac{1}{2}$, $\lambda_1^g = \lambda_1^H = \beta$ $(\beta \geq 0)$, one has $y_1 = \beta, y_2 = \beta$.

And by $F(x) = \Phi(y, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G)$, we get

$$F^L(x) = \Phi^L(y, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G) = -\beta^2 \leq 0 \quad \Rightarrow \quad x_1^2 - x_2^2 \leq 0,$$

$$F^U(x) = \Phi^U(y, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G) = 0 \quad \Rightarrow \quad x_2^2 = 0,$$

that is, $x = (0, 0)$. By $\lambda_1^g g_1(x) \leq 0$, $\lambda_1^G G_1(x) = 0$, $-\lambda_1^H H_1(x) \leq 0$, and the ∂_c-pseudoconvexity of $\Phi^L(\cdot, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G)$ and $\Phi^U(\cdot, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G)$ at $y \in E_1 \cup prE_2$, we get $x = (0, 0)$ is the locally weakly *LU* optimal solution of (IOPVC1).

Theorem 3.6 (Strict converse duality) *Let $\bar{x} \in E$ be the locally weakly LU optimal solution of (IOPVC) such that the (VC-ACQ) holds at $\bar{x}$ and $\triangle$ is closed. Assume that the conditions of Theorem* 3.3 *hold and $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_W(\bar{x})$ is the locally weakly LU optimal solution of $(D_W(\bar{x}))$. If $\Phi^L(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$, $\Phi^U(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ are strictly ∂_c-pseudoconvex at $y \in E \cup prE_W(\bar{x})$, then $\bar{x} = \bar{y}$.*

Proof Suppose that $\bar{x} \neq \bar{y}$. By Theorem 3.3, there exist Lagrange multipliers $\bar{\alpha}^L, \bar{\alpha}^U \in R_+$, $\bar{\lambda}^g \in \kappa(\bar{x})$, $\bar{\lambda}^h \in R^n$, $\bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu} \in R^l$ such that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu})$ is the locally *LU* optimal solution of $(D_W(\bar{x}))$, that is,

$$F(\bar{x}) = \Phi(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G) = \Phi(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G).$$

Since $\bar{x} \in E$, $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_W(\bar{x})$, one has

$$g_j(\bar{x}) < 0, \qquad \lambda_j^g \geq 0, \quad j \notin I_g(\bar{x}),$$

$$g_j(\bar{x}) = 0, \qquad \lambda_j^g \geq 0, \quad j \in I_g(\bar{x}),$$

$$h_k(\bar{x}) = 0, \qquad \lambda_k^h \in R, \quad k \in I_n,$$

$$-H_i(\bar{x}) < 0, \qquad \lambda_i^H \geq 0, \quad i \in I_+(\bar{x}),$$

$$-H_i(\bar{x}) = 0, \qquad \lambda_i^H \in R, \quad i \in I_0(\bar{x}),$$

$$G_i(\bar{x}) > 0, \qquad \lambda_i^G = 0, \quad i \in I_{0+}(\bar{x}),$$

$$G_i(\bar{x}) = 0, \qquad \lambda_i^G \geq 0, \quad i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}),$$

$$G_i(\bar{x}) < 0, \qquad \lambda_i^G \geq 0, \quad i \in I_{0-}(\bar{x}) \cup I_{+-}(\bar{x}),$$

and

$$\sum_{j \in J} \lambda_j^g g_j(\bar{x}) + \sum_{k=1}^{n} \lambda_k^h h_k(\bar{x}) - \sum_{i=1}^{l} \lambda_i^H H_i(\bar{x}) + \sum_{i=1}^{l} \lambda_i^G G_i(\bar{x}) \le 0.$$

And then we get

$$\Phi^L\big(\bar{x}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big) \le \Phi^L\big(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big),$$

$$\Phi^U\big(\bar{x}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big) \le \Phi^U\big(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G\big).$$

By the strict ∂_c-pseudoconvexity of $\Phi^L(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ and $\Phi^U(\cdot, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G)$ at $\bar{y} \in E \cup prE_W(\bar{x})$, we get (3.9), where $\xi^L \in \partial_c F^L(\bar{y})$, $\xi^U \in \partial_c F^U(\bar{y})$, $\xi_j^g \in \partial_c g_j(\bar{y})$, $j \in J$, $\xi_k^h \in \partial_c h_k(\bar{y})$, $k \in I_n$, $\xi_i^H \in \partial_c H_i(\bar{y})$, $i \in I_l$, $\xi_i^G \in \partial_c G_i(\bar{y})$, $i \in I_l$.

Combining (3.9) with α^L and α^U, one has

$$\langle \xi, \bar{x} - \bar{y} \rangle < 0, \quad \forall \xi \in \Delta(\bar{y}),$$

contradicting $0 \in \Delta(\bar{y})$.   □

4 Mond–Weir type duality

In this section, we give the Mond–Weir type dual model of (IOPVC) by referring to the new Mond–Weir type dual model($VC - MWD(x)$) in [6]: for $x \in E$,

$$\big(D_{MW}(x)\big) \quad LU - \max \quad F(y)$$

$$\text{s.t.} \begin{cases} 0 \in \Delta(y) \\ \alpha^L, \alpha^U \in R_+, \quad \alpha^L + \alpha^U = 1, \\ \lambda_j^g \ge 0, \qquad \lambda_j^g g_j(y) \ge 0, \quad \forall j \in J, \\ \lambda_k^h \in R, \qquad \lambda_k^h h_k(y) = 0, \quad \forall k \in I_n, \\ \lambda_i^G G_i(y) \ge 0, \qquad \lambda_i^G = v_i H_i(x), \quad v_i \ge 0, \forall i \in I_l, \\ -\lambda_i^H H_i(y) \ge 0, \qquad \lambda_i^H = \varrho_i - v_i G_i(x), \quad \varrho_i \ge 0, \forall i \in I_l. \end{cases} \tag{4.1}$$

Let $E_{MW}(x)$ denote the feasible set of $(D_{MW}(x))$, $prE_{MW}(x) := \{y \in X : (y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_{MW}(x)\}$ represents the projection of the set $E_{MW}(x)$ on X.

In order to be independent of (IOPVC), we give the another Mond–Weir type dual model:

$$(D_{MW}) \quad LU\text{-}\max F(y)$$

$$\text{s.t.}\big(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v\big) \in E_{MW} := \bigcap_{x \in E} E_{MW}(x),$$

where E_{MW} denotes the set of all feasible points of (D_{MW}) and prE_{MW} denotes the projection of the set E_{MW} on X.

Definition 4.1 ([4]) Let $x \in E$,

(i) $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_{MW}(x)$ is said to be a locally *LU* optimal solution of $(D_{MW}(x))$ if there exists $B(\bar{y}; \delta)$ such that there is no $y \in E_{MW}(x) \cap B(\bar{y}; \delta)$ satisfying

$$F(\bar{y}) <_{LU} F(y).$$

(ii) $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_{MW}(x)$ is said to be a locally weakly *LU* optimal solution of $(D_{MW}(x))$ if there exists $B(\bar{y}; \delta)$ such that there is no $y \in E_{MW}(x) \cap B(\bar{y}; \delta)$ satisfying

$$F(\bar{y}) <_{LU}^s F(y).$$

Theorem 4.1 (Weak duality) *Let* $x \in E$, $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_{MW}$ *be feasible points for the* (IOPVC) *and the* (D_{MW}), *respectively. If one of the following conditions holds*:

(i) $F^L(\cdot), F^U(\cdot)$ *are* ∂_c-*pseudoconvex at* $y \in E \cup prE_{MW}$, $\sum_{j \in J} \lambda_j^g g_j(\cdot) + \sum_{k=1}^n \lambda_k^h h_k(\cdot) - \sum_{i=1}^l \lambda_i^H H_i(\cdot) + \sum_{i=1}^l \lambda_i^G G_i(\cdot)$ *is* ∂_c-*quasiconvex at* $y \in E \cup prE_{MW}$;

(ii) $F^L(\cdot), F^U(\cdot)$ *are* ∂_c-*pseudoconvex at* $y \in E \cup prE_{MW}$, g_j $(j \in I_g^+(x))$, h_k $(k \in I_h^+(x))$, $-h_k$ $(k \in I_h^-(x))$, $-H_i$ $(i \in I_+^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x))$, H_i $(i \in I_{0+}^-(x))$, G_i $(i \in \tilde{I}_+^+(x))$ *are* ∂_c-*quasiconvex at* $y \in E \cup prE_{MW}$;

Then

$$F(x) \not<_{LU}^s F(y).$$

Proof Suppose that $F(x) <_{LU}^s F(y)$, there exists

$$\left[F^L(x), F^U(x)\right] <_{LU}^s \left[F^L(y), F^U(y)\right]. \tag{4.2}$$

(i) Since $x \in E$, $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_{MW}$, one has (3.3). By (4.1) and (3.3), we get

$$\sum_{j \in J} \lambda_j^g g_j(x) + \sum_{k=1}^n \lambda_k^h h_k(x) - \sum_{i=1}^l \lambda_i^H H_i(x) + \sum_{i=1}^l \lambda_i^G G_i(x)$$

$$\leq \sum_{j \in J} \lambda_i^g g_i(y) + \sum_{k=1}^n \lambda_k^h h_k(y) - \sum_{i=1}^l \lambda_i^H H_i(y) + \sum_{i=1}^l \lambda_i^G G_i(y),$$

and by the ∂_c-quasiconvexity of the above functions, one has

$$\left\langle \sum_{j \in J} \lambda_j^g \xi_j^g + \sum_{k=1}^n \lambda_k^h \xi_k^h - \sum_{i=1}^l \lambda_i^H \xi_i^H + \sum_{i=1}^l \lambda_i^G \xi_i^G, x - y \right\rangle \leq 0,$$

where $\xi_j^g \in \partial_c g_j(y)$, $\xi_k^h \in \partial_c h_k(y)$, $\xi_i^H \in \partial_c H_i(y)$, $\xi_i^G \in \partial_c G_i(y)$. Using the above inequality and $0 \in \Delta(y)$, there exist $\xi^L \in \partial_c F^L(y)$ and $\xi^U \in \partial_c F^U(y)$ such that

$$\left\langle \alpha^L \xi^L + \alpha^U \xi^U, x - y \right\rangle \geq 0. \tag{4.3}$$

By (4.2) and the ∂_c-pseudoconvexity of $F^L(\cdot)$ and $F^U(\cdot)$, it follows that

$$\langle \xi^L, x - y \rangle < 0, \quad \forall \xi^L \in \partial_c F^L(y),$$

$$\langle \xi^U, x - y \rangle < 0, \quad \forall \xi^U \in \partial_c F^U(y).$$

Then $\langle \alpha^L \xi^L + \alpha^U \xi^U, x - y \rangle < 0$, $\alpha^L, \alpha^U \in R_+$, $\alpha^L + \alpha^U = 1$, which contradicts (4.3).

(ii) By (4.1) and (3.3), one has

$$
\begin{cases}
g_j(x) \leq g_j(y), & \forall j \in I_g^+(x), \\
h_k(x) = h_k(y), & \forall k \in I_h^+(x) \cup I_h^-(x), \\
-H_i(x) \leq -H_i(y), & \forall i \in I_+^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x), \\
-H_i(x) \geq -H_i(y), & \forall i \in I_{0+}^-(x), \\
G_i(x) \leq G_i(y), & \forall i \in \tilde{I}_+^+(x).
\end{cases}
\tag{4.4}
$$

Combining (4.4) with the ∂_c-quasiconvexity of the above functions, we get

$$\langle \xi_j^g, x - y \rangle \leq 0, \qquad \lambda_j^g > 0, \quad \forall \xi_j^g \in \partial_c g_j(y), j \in I_g^+(x),$$

$$\langle \xi_k^h, x - y \rangle \leq 0, \qquad \lambda_k^h > 0, \quad \forall \xi_k^h \in \partial_c h_k(y), k \in I_h^+(x),$$

$$\langle \xi_k^h, x - y \rangle \geq 0, \qquad \lambda_k^h < 0, \quad \forall \xi_k^h \in \partial_c h_k(y), k \in I_h^-(x),$$

$$\langle -\xi_i^H, x - y \rangle \leq 0, \qquad \lambda_i^H \geq 0, \quad \forall \xi_i^H \in \partial_c H_i(y), i \in I_+^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x),$$

$$\langle -\xi_i^H, x - y \rangle \geq 0, \qquad \lambda_i^H \leq 0, \quad \forall \xi_i^H \in \partial_c H_i(y), i \in I_{0+}^-(x),$$

$$\langle \xi_i^G, x - y \rangle \leq 0, \qquad \lambda_i^G \geq 0, \quad \forall \xi_i^G \in \partial_c G_i(y), i \in \tilde{I}_+^+(x)$$

that is,

$$\left\langle \sum_{j \in J} \lambda_j^g \xi_j^g + \sum_{k=1}^n \lambda_k^h \xi_k^h - \sum_{i=1}^l \lambda_i^H \xi_i^H + \sum_{i=1}^l \lambda_i^G \xi_i^G, x - y \right\rangle \leq 0.$$

The rest is proved to be the same as the latter part of (i). $\qquad\square$

Theorem 4.2 (Weak duality) *Let $x \in E$, $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_{MW}$ be feasible points for the (IOPVC) and the (D_{MW}), respectively. If one of the following conditions holds:*

(i) *$F^L(\cdot)$, $F^U(\cdot)$ are strictly ∂_c-pseudoconvex at $y \in E \cup prE_{MW}$,*
   $\sum_{j \in J} \lambda_j^g g_j(\cdot) + \sum_{k=1}^n \lambda_k^h h_k(\cdot) - \sum_{i=1}^l \lambda_i^H H_i(\cdot) + \sum_{i=1}^l \lambda_i^G G_i(\cdot)$ is ∂_c-quasiconvex at
   $y \in E \cup prE_{MW}$;

(ii) *$F^L(\cdot)$, $F^U(\cdot)$ are strictly ∂_c-pseudoconvex at $y \in E \cup prE_{MW}$, g_j $(j \in I_g^+(x))$, h_k*
   $(k \in I_h^+(x))$, $-h_k$ $(k \in I_h^-(x))$, $-H_i$ $(i \in I_+^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x))$, H_i $(i \in I_{0+}^-(x))$, G_i
   $(i \in \tilde{I}_+^+(x))$ are ∂_c-quasiconvex at $y \in E \cup prE_{MW}$;

Then

$$F(x) \not\prec_{LU} F(y).$$

Proof The proof is similar to Theorem 4.1. $\qquad\square$

Theorem 4.3 (Strong duality) *Let $\bar{x} \in E$ be the locally weakly LU optimal solution of (IOPVC), and the condition (VC-ACQ) holds at $\bar{x}$ and $\triangle$ is closed. Then there exist $\bar{\alpha}^L, \bar{\alpha}^U \in R_+$, $\bar{\lambda}^g \in R_+^{|J|}$, $\bar{\lambda}^h \in R^n$, $\bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu} \in R^l$ such that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu})$ is a feasible point of $(D_{MW}(\bar{x}))$. Moreover, if the hypotheses of Theorem 4.1 hold, then $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu})$ is the locally weakly LU optimal solution of $(D_{MW}(\bar{x}))$.*

Proof Since $\bar{x}$ is the locally weakly *LU* optimal solution of (IOPVC) and (VC-ACQ) holds at $\bar{x}$, so by Theorem 2.1, there exist $\bar{\alpha}^L, \bar{\alpha}^U \in R_+$, $\bar{\lambda}^g \in \kappa(\bar{x})$, $\bar{\lambda}^h \in R^n$, $\bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu} \in R^l$ such that (2.3) and (2.4) hold. From this together with the definition of $(D_{MW}(\bar{x}))$, one has $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu})$ is a feasible point of $(D_{MW}(\bar{x}))$. By Theorem 4.1, we know

$$F(\bar{x}) \nprec_{LU}^s F(y), \forall (y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_{MW}(\bar{x}),$$

so $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu})$ is the locally weakly *LU* optimal solution of $(D_{MW}(\bar{x}))$. □

Theorem 4.4 (Converse duality) *Let $x \in E$ be any feasible point of (IOPVC) and $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_{MW}$ be a feasible point of (D_{MW}). If one of the following conditions holds*:

(i) *$F^L(\cdot), F^U(\cdot)$ are ∂_c-pseudoconvex at $\bar{y} \in E \cup prE_{MW}$, $\sum_{j \in J} \lambda_j^g g_j(\cdot) + \sum_{k=1}^n \lambda_k^h h_k(\cdot) - \sum_{i=1}^l \lambda_i^H H_i(\cdot) + \sum_{i=1}^l \lambda_i^G G_i(\cdot)$ is ∂_c-quasiconvex at $\bar{y} \in E \cup prE_{MW}$;*

(ii) *$F^L(\cdot), F^U(\cdot)$ are ∂_c-pseudoconvex at $\bar{y} \in E \cup prE_{MW}$, g_j $(j \in I_g^+(x))$, h_k $(k \in I_h^+(x))$, $-h_k$ $(k \in I_h^-(x))$, $-H_i$ $(i \in I_+^+(x) \cup I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{0+}^+(x))$, H_i $(i \in I_{0+}^-(x))$, G_i $(\tilde{I}_+^+(x))$ are ∂_c-quasiconvex at $\bar{y} \in E \cup prE_{MW}$;*

Then $\bar{y}$ is the locally weakly LU optimal solution of (IOPVC).

Proof Suppose to the contrary that $\bar{y}$ is not the locally weakly *LU* optimal solution of (IOPVC), then one has (3.7).

(i) Since $\bar{x} \in E$ and $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu)$ are feasible points for the (IOPVC) and the (D_{MW}), by (4.1) and (3.3), we know that (3.8) holds. By the ∂_c-quasiconvexity of $\sum_{j \in J} \lambda_j^g g_j(\cdot) + \sum_{k=1}^n \lambda_k^h h_k(\cdot) - \sum_{i=1}^l \lambda_i^H H_i(\cdot) + \sum_{i=1}^l \lambda_i^G G_i(\cdot)$ at $\bar{y} \in E \cup prE_{MW}$, we get

$$\left\langle \sum_{j \in J} \lambda_j^g \xi_j^g + \sum_{k=1}^n \lambda_k^h \xi_k^h - \sum_{i=1}^l \lambda_i^H \xi_i^H + \sum_{i=1}^l \lambda_i^G \xi_i^G, \bar{x} - \bar{y} \right\rangle \leq 0,$$

where $\xi_j^g \in \partial_c g_j(\bar{y}), \xi_k^h \in \partial_c h_k(\bar{y}), \xi_i^H \in \partial_c H_i(\bar{y}), \xi_i^G \in \partial_c G_i(\bar{y})$.

Combining this with $0 \in \Delta(\bar{y})$, one has (3.11). And by (3.7) and the ∂_c-pseudoconvexity of $F^L(\cdot)$ and $F^U(\cdot)$, one has

$$\left\langle \xi^L, \bar{x} - \bar{y} \right\rangle < 0, \quad \forall \xi^L \in \partial_c F^L(\bar{y}),$$

$$\left\langle \xi^U, \bar{x} - \bar{y} \right\rangle < 0, \quad \forall \xi^U \in \partial_c F^U(\bar{y})$$

and $\langle \alpha^L \xi^L + \alpha^U \xi^U, \bar{x} - \bar{y} \rangle < 0$, $\alpha^L, \alpha^U \in R_+$, $\alpha^L + \alpha^U = 1$. This is a contradiction to (3.11), and hence the result is proved.

(ii) The proof of (ii) is similar to the proof of Theorem 3.4(ii), so it is omitted. □

Theorem 4.5 (Restricted converse duality) *Let $\bar{x} \in E$ be a feasible point of (IOPVC), and let $(y, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, v) \in E_{MW}$ be a feasible point of (D_{MW}) such that $F(\bar{x}) = F(y)$. If the hypotheses of Theorem 4.1 hold, then $\bar{x}$ is the locally weakly LU optimal solution of (IOPVC).*

Proof Suppose that $\bar{x}$ is not a locally weakly LU optimal solution of (IOPVC). Then there exist $\tilde{x} \in E \cap B(\bar{x}; \delta)$ such that $F(\tilde{x}) <^s_{LU} F(\bar{x})$. By $F(\bar{x}) = F(y)$, one has $F(\tilde{x}) <^s_{LU} F(y)$, contradicting Theorem 4.1.　　　　　　　　　　　　　　　　　　　　　　　　　　□

Let us use the following example to show that the conclusion of Theorem 4.5 holds.

Example 4.1 Let $X = R^2$, $n = 0$, $J = l = 1$, we give the following problem:

$$(\text{IOPVC2}) \quad \min \quad F(x) = \left[F^L(x), F^U(x)\right] = \left[x_1^2 - x_2^2, x_1^2\right]$$
$$\text{s.t.} \quad g_1(x) = -x_1 \leq 0,$$
$$H_1(x) = x_1 - x_2 \geq 0,$$
$$G_1(x)H_1(x) = x_1(x_1 - x_2) \leq 0.$$

It is easy to know that

$$E_3 = \{x \in R \mid x_1 > 0, x_1 - x_2 = 0\} \cup \{x \in R \mid x_1 = 0, x_2 \leq 0\}$$

is the feasible set of (IOPVC2). For any $x \in E_3$, the Mond–Weir type dual model to (IOPVC2) is given by

$$\left(D_{MW}(x)\right) \quad \max \quad F(y) = \left[F^L(y), F^U(y)\right] = \left[y_1^2 - y_2^2, y_1^2\right]$$
$$\text{s.t.} \quad \alpha^L(2y_1, -2y_2) + \alpha^U(2y_1, 0) + \lambda_1^g(-1, 0)$$
$$- \lambda_1^H(1, -1) + \lambda_1^G(1, 0) = (0, 0),$$
$$\lambda_1^g \geq 0, \qquad \lambda_1^g g_1(y) \geq 0,$$
$$\lambda_1^H \geq 0, \quad \text{if } 1 \in I_+(x) \cup I_{0-}(x) \cup I_{00}(x),$$
$$\lambda_1^H \in R, \quad \text{if } 1 \in I_{0+}(x), \qquad -\lambda_1^H H_1(y) \geq 0,$$
$$\lambda_1^G \geq 0, \quad \text{if } 1 \in I_+(x), \qquad \lambda_1^G = 0, \quad \text{if } 1 \in I_0(x), \qquad \lambda_1^G G_1(y) \geq 0.$$

Therefore, we can get the feasible set of problem (D_{MW}), which is not dependent on x,

$$E_4 := \Big\{\left(y_1, y_2, \alpha^L, \alpha^U, \lambda_1^g, \lambda_1^H, \lambda_1^G\right):$$
$$2y_1 - \lambda_1^g - \lambda_1^H + \lambda_1^G = 0, -2\alpha^L y_2 + \lambda_1^H = 0,$$
$$y_1, y_2 \in X, \alpha^L, \alpha^U \in R_+, \alpha^L + \alpha^U = 1, -\lambda_1^g y_1 \geq 0,$$
$$\lambda_1^G y_1 \geq 0, -\lambda_1^H(y_1 - y_2) \geq 0, \lambda_1^H \geq 0, \lambda_1^G = 0\Big\}.$$

Let $\alpha^L = \alpha^U = \frac{1}{2}$, $\lambda_1^g = \lambda_1^H = \beta$ $(\beta \geq 0)$, one has $y_1 = \beta$, $y_2 = \beta$.

By $F(x) = F(y)$, one has

$$F^L(x) = F^L(y) = 0 \quad \Rightarrow \quad x_1^2 - x_2^2 = 0,$$

$$F^U(x) = F^U(y) = \beta^2 \geq 0 \quad \Rightarrow \quad x_1^2 \geq 0.$$

Then we get $\lambda_1^g g_1(x) \leq 0$, $\lambda_1^G G_1(x) = 0$, $-\lambda_1^H H_1(x) \leq 0$. Finally, by the ∂_c-pseudoconvexity of $F^L(\cdot)$ and $F^U(\cdot)$ at $y \in E_3 \cup prE_4$ and the ∂_c-quasiconvexity of $\lambda_1^g g_1(\cdot) - \lambda_1^H H_1(\cdot) + \lambda_1^G G_1(\cdot)$, we get $x = (0,0)$ is the locally weakly *LU* optimal solution of (IOPVC2).

Theorem 4.6 (Strict converse duality) *Let $\bar{x} \in E$ be a locally weakly LU optimal solution of (IOPVC) such that the (VC-ACQ) holds at $\bar{x}$ and $\triangle$ is closed. Assume that the conditions of Theorem 4.3 hold and $(\bar{y}, \alpha^L, \alpha^U, \lambda^g, \lambda^h, \lambda^H, \lambda^G, \varrho, \nu) \in E_{MW}(\bar{x})$ is the locally weakly LU optimal solution of $(D_{MW}(\bar{x}))$. If one of the following conditions holds:*
   (i) *$F^L(\cdot)$, $F^U(\cdot)$ are strictly ∂_c-pseudoconvex at $\bar{y} \in E \cup prE_{MW}(\bar{x})$,*
      $\sum_{j \in J} \lambda_j^g g_j(\cdot) + \sum_{k=1}^n \lambda_k^h h_k(\cdot) - \sum_{i=1}^l \lambda_i^H H_i(\cdot) + \sum_{i=1}^l \lambda_i^G G_i(\cdot)$ is ∂_c-quasiconvex at $\bar{y} \in E \cup prE_{MW}(\bar{x})$;
   (ii) *$F^L(\cdot)$, $F^U(\cdot)$ are strictly ∂_c-pseudoconvex at $\bar{y} \in E \cup prE_{MW}(\bar{x})$, g_j $(j \in I_g^+(\bar{x}))$, h_k $(k \in I_h^+(\bar{x}))$, $-h_k$ $(k \in I_h^-(\bar{x}))$, $-H_i$ $(i \in I_+^+(x) \cup I_{00}^+(\bar{x}) \cup I_{0-}^+(\bar{x}) \cup I_{0+}^+(\bar{x}))$, H_i $(i \in I_{0+}^-(\bar{x}))$, G_i $(\tilde{I}_+^+(\bar{x}))$ are ∂_c-quasiconvex at $\bar{y} \in E \cup prE_{MW}(\bar{x})$; then $\bar{x} = \bar{y}$.*

Proof Suppose that $\bar{x} \neq \bar{y}$. By Theorem 4.3, there exist Lagrange multipliers $\bar{\alpha}^L, \bar{\alpha}^U \in R_+$, $\bar{\lambda}^g \in \kappa(\bar{x})$, $\bar{\lambda}^h \in R^n$, $\bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu} \in R^l$ such that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^H, \bar{\lambda}^G, \bar{\varrho}, \bar{\nu})$ is the locally weakly *LU* optimal solution of $(D_{MW}(\bar{x}))$, it follows that

$$F(\bar{x}) = F(\bar{y}). \tag{4.5}$$

The remaining parts are similar to (i) and (ii) of Theorem 4.4, so they are omitted.   □

5 Concluding remarks

In this paper, we study the duality theorems of nondifferentiable semi-infinite interval-valued optimization problem with vanishing constraints. The weak duality, strong duality, converse duality, restricted converse duality, and strict converse duality theorems between (IOPVC) and its corresponding Wolfe and Mond–Weir type dual models are given under the conditions of ∂_c-pseudoconvex, strictly ∂_c-pseudoconvex, and ∂_c-quasiconvex.

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Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The authors carried out the results and read and approved the current version of the manuscript.

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