Mursaleen and Edely *Journal of Inequalities and Applications* (2021) 2021:178
https://doi.org/10.1186/s13660-021-02713-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Compact operators on sequence spaces associated with the Copson matrix of order α

M. Mursaleen[1,2*] and Osama H.H. Edely[3]

*Correspondence: mursaleenm@gmail.com
[1]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
[2]Al-Qaryah, Doharra, Street No. 1 (West), Aligarh UP 202002, India
Full list of author information is available at the end of the article

Abstract

In this work, we study characterizations of some matrix classes $(\mathcal{C}^{(\alpha)}(\ell^p), \ell^\infty)$, $(\mathcal{C}^{(\alpha)}(\ell^p), c)$, and $(\mathcal{C}^{(\alpha)}(\ell^p), c^0)$, where $\mathcal{C}^{(\alpha)}(\ell^p)$ is the domain of Copson matrix of order α in the space ℓ^p $(0 < p < 1)$. Further, we apply the Hausdorff measures of noncompactness to characterize compact operators associated with these matrices.

MSC: 26D15; 40C05; 40G05; 47B37

Keywords: Sequence spaces; Cesàro matrix; Copson matrix; Copson matrix of order α; Matrix transformations; Hausdorff measure of noncompactness; Compact operators

1 Introduction

By $l^\diamond = \{\zeta = (\zeta_k) : \text{each } \xi_k \text{ is real}\}$. The sequence space ℓ^p is defined by

$$\ell^p := \left\{\zeta = (\zeta_k) \in l^\diamond : \sum_{k=0}^{\infty} |\zeta_k|^p < \infty, p > 0\right\}.$$

This is a Banach space with the norm

$$\|\zeta\|_{\ell^p} = \left(\sum_{k=0}^{\infty} |\zeta_k|^p\right)^{1/p} < \infty \quad (1 \le p < \infty)$$

and complete p-normed space with the p-norm

$$\|\zeta\|_{\ell^p} = \sum_{k=0}^{\infty} |\zeta_k|^p < \infty \quad (0 < p < 1).$$

Further,

$$c^0 := \{\zeta = (\zeta_k) \in l^\diamond : \zeta_k \to 0 (k \to \infty)\},$$

$$c := \left\{\zeta = (\zeta_k) \in l^\diamond : \lim_{k\to\infty} \zeta_k \text{ exists}\right\},$$

$$\ell^{\infty} := \left\{ \zeta = (\zeta_k) \in l^{\diamond} : \sup_k |\zeta_k| < \infty \right\}$$

are Banach spaces with $\|\zeta\|_{\ell^\infty} = \sup_k |\zeta_k|$.

The Copson matrix $\mathcal{C}^{(1)} = (c_{j,k})_{j,k\in\mathbb{N}_0}$ of order 1 is defined by

$$c_{j,k} = \begin{cases} \frac{1}{k+1} & 0 \le j \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|\mathcal{C}^{(1)}\|_{\ell^p} = p$. The Copson matrix is the transpose of the Cesàro matrix

$$c^t_{j,k} = \begin{cases} \frac{1}{k+1} & 0 \le k \le j, \\ 0 & \text{otherwise.} \end{cases}$$

The Copson matrix of order $\alpha > 0$, $\mathcal{C}^{(\alpha)} = (c^{(\alpha)}_{j,k})$ is defined by

$$c^{(\alpha)}_{j,k} = \begin{cases} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} & 0 \le j \le k \\ 0 & \text{otherwise,} \end{cases}$$

which is the transpose of Cesàro matrix of order α, and the ℓ^p-norm of $\mathcal{C}^{(\alpha)}$ is (see [18, 19])

$$\|\mathcal{C}^{(\alpha)}\|_{\ell^p} = \frac{\Gamma(\alpha+1)\Gamma(1/p)}{\Gamma(\alpha+1/p)}.$$

For $\alpha = 0$, $\mathcal{C}^{(0)} = I$, where I is the identity matrix, and for $\alpha = 1$, it is $\mathcal{C}^{(1)}$.

Recently, these types of sequence spaces have been studied in [18–22]. Most recently, Roopaei [19] studied the following spaces:

$$\mathcal{C}^{(\alpha)}(c^0) = \left\{ \zeta = (\zeta_j) \in l^{\diamond} : \lim_{j\to\infty} \sum_{k=j}^{\infty} \frac{\binom{\alpha+k-j-1}{k-j}}{\binom{\alpha+k}{k}} \zeta_k = 0 \right\},$$

$$\mathcal{C}^{(\alpha)}(c) = \left\{ \zeta = (\zeta_j) \in l^{\diamond} : \lim_{j\to\infty} \sum_{k=j}^{\infty} \frac{\binom{\alpha+k-j-1}{k-j}}{\binom{\alpha+k}{k}} \zeta_k \text{ exists} \right\},$$

and

$$\mathcal{C}^{(\alpha)}(\ell^p) = \left\{ \zeta = (\zeta_j) \in l^{\diamond} : \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{\binom{\alpha+k-j-1}{k-j}}{\binom{\alpha+k}{k}} \zeta_k \right|^p < \infty \right\} \quad (0 < p < 1).$$

In terms of matrix domains, these spaces are defined as follows:

$$\mathcal{C}^{(\alpha)}(c^0) = (c^0)_{\mathcal{C}^{(\alpha)}}, \qquad \mathcal{C}^{(\alpha)}(c) = (c)_{\mathcal{C}^{(\alpha)}}, \quad \text{and} \quad \mathcal{C}^{(\alpha)}(\ell^p) = (\ell^p)_{\mathcal{C}^{(\alpha)}}.$$

Throughout the study, $\eta = (\eta_j)$ will be the $\mathcal{C}^{(\alpha)}$-transform of a sequence $\zeta = (\zeta_j)$; that is,

$$\eta_j = (\mathcal{C}^{(\alpha)}\zeta)_j = \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} \zeta_k \tag{1.1}$$

for all $j \in \mathbb{N}_0$. Also, the relation

$$\zeta_k = \sum_{i=k}^{\infty} (-1)^{i-k} \binom{n+k}{k} \binom{n}{i-k} \eta_i \tag{1.2}$$

holds for all $k \in \mathbb{N}_0$.

The spaces $\mathcal{C}^{(\alpha)}(c^0)$ and $\mathcal{C}^{(\alpha)}(c)$ are Banach spaces with the norm $\|\zeta\|_{\mathcal{C}^{(\alpha)}(c^0)} = \|\zeta\|_{\mathcal{C}^{(\alpha)}(c)} = \|\mathcal{C}^{(\alpha)}\zeta\|_{\ell^\infty}$, and $\mathcal{C}^{(\alpha)}(\ell^p)$ $(0 < p < 1)$ is a complete p-normed space with the p-norm $\|\zeta\|_{\mathcal{C}^{(\alpha)}(\ell^p)} = \|\mathcal{C}^{(\alpha)}\zeta\|_{\ell^p}$. Furthermore, $\mathcal{C}^{(\alpha)}(c^0) \simeq c^0$ and $\mathcal{C}^{(\alpha)}(c) \simeq c$, while $\mathcal{C}^{(\alpha)}(\ell^p) \simeq \ell^p$.

The main theme of this article is to characterize some matrix classes $(\mathcal{C}^{(\alpha)}(\ell^p), E)$, where $E = \ell^\infty, c, c^0$. Furthermore, we apply the techniques of measures of noncompactness to characterize compact operators associated with these matrix classes.

2 Matrix classes

Let $c_{00} := \{\zeta = (\zeta_j) \in l^\diamond : \zeta_j \neq 0 \text{ for finite } j; \text{and } 0 \text{ elsewhere}\}$. For a BK-space $\mathfrak{U} \supset c_{00}$ and $\gamma = (\gamma_k) \in l^\diamond$, we define

$$\|\gamma\|_{\mathfrak{U}}^* = \sup_{\zeta \in S_{\mathfrak{X}}} \left| \sum_{k=0}^{\infty} \gamma_k \zeta_k \right| \tag{2.1}$$

provided $\gamma \in \mathfrak{U}^\beta = \{\gamma = (\gamma_k) \in l^\diamond : \sum_{k=0}^{\infty} \gamma_k \zeta_k \text{ converges for all } \zeta = (\zeta_k) \in \mathfrak{U}\}$.

For FK-, BK-, AK-spaces and the relevant literature, we refer to [1, 2, 11], and [12].

We need the following lemmas.

Lemma 2.1 ([23]) *We have the following*:

(i) $D = (d_{jk}) \in (c_0, c_0) \Leftrightarrow$

$$\sup_{j \in \mathbb{N}_0} \sum_{k=0}^{\infty} |d_{jk}| < \infty \tag{2.2}$$

$$\lim_{j \to \infty} d_{jk} = 0 \quad \textit{for each } k \in \mathbb{N}_0. \tag{2.3}$$

(ii) $D = (d_{jk}) \in (c_0, c) \Leftrightarrow$ (2.2) *holds, and*

$$\exists \alpha_k \in \mathbb{R} \ni \lim_{j \to \infty} d_{jk} = \alpha_k \quad \textit{for each } k \in \mathbb{N}_0. \tag{2.4}$$

(iii) $D = (d_{jk}) \in (c : c_0) \Leftrightarrow$ (2.2), (2.3) *hold, and*

$$\lim_{j \to \infty} \sum_{k=0}^{\infty} d_{jk} = 0. \tag{2.5}$$

(iv) $D = (d_{jk}) \in (c, c) \Leftrightarrow$ (2.2) *and* (2.4) *hold, and*

$$\lim_{j \to \infty} \sum_{k=0}^{\infty} d_{jk} \quad \textit{exists}. \tag{2.6}$$

(v) $D = (d_{jk}) \in (c_0, \ell_\infty) = (c, \ell_\infty) \Leftrightarrow$ (2.2) *holds.*

Lemma 2.2 *We have the following*:

(i) [8, *Theorem* 1(*i*) *with* $p_k = p$ *for all* k] $D = (d_{jk}) \in (\ell_p, \ell_\infty) \Leftrightarrow$

$$\sup_{j,k\in\mathbb{N}_0} |d_{jk}|^p < \infty. \tag{2.7}$$

(ii) [8, *Corollary for Theorem* 1 *with* $p_k = p$ *for all* k] $D = (d_{jk}) \in (\ell_p, c) \Leftrightarrow$ (2.4) *and* (2.7) *hold*.

The following results give the relation between $(\mathfrak{U},\mathfrak{V})$ and $\mathcal{B}(\mathfrak{U},\mathfrak{V})$ [1].

Lemma 2.3 *Let* $\mathfrak{U} \supset c^{00}$ *and* $\mathfrak{V}$ *be BK-spaces. Then*,

(a) $(\mathfrak{U},\mathfrak{V}) \subset \mathcal{B}(\mathfrak{U},\mathfrak{V})$, *i.e., every matrix* $\mathfrak{A} \in (\mathfrak{U},\mathfrak{V})$ *is associated with an operator* $L_{\mathfrak{A}} \in \mathcal{B}(\mathfrak{U},\mathfrak{V})$ *by* $L_{\mathfrak{A}}(\zeta) = \mathfrak{A}\xi$ *for all* $\zeta \in \mathfrak{U}$.

(b) *If* $\mathfrak{U}$ *has AK, then the reverse inclusion also holds.*

Lemma 2.4 *Let* $\mathfrak{U} \supset c^{00}$ *be a BK-space and* $\mathfrak{V} \in \{c^0, c, \ell^\infty\}$. *Then*

$$\|L_{\mathfrak{A}}\| = \|\mathfrak{A}\|_{(\mathfrak{U},\ell^\infty)} = \sup_n \|\mathfrak{A}_n\|^*_{\mathfrak{U}} < \infty \quad \textit{for } \mathfrak{A} \in (\mathfrak{U},\mathfrak{V}).$$

Next, we characterize the matrix classes $(\mathcal{C}^{(\alpha)}(\ell^p), \ell^\infty)$, $(\mathcal{C}^{(\alpha)}(\ell^p), c)$, and $(\mathcal{C}^{(\alpha)}(\ell^p), c^0)$. Hereafter, we write $\mathfrak{A} = (a_{jk})_{j,k\in\mathbb{N}_0}$ for an infinite matrix.

The β-dual of a sequence space $\mathfrak{U}$, i.e., $\mathfrak{U}^\beta = \{a = (a_k) \in l^\diamond : \sum_{k=0}^\infty a_k\zeta_k$ converges for all $\zeta = (\zeta_k) \in \mathfrak{U}\}$ plays an important role in matrix transformations. The β-dual of $\mathcal{C}^{(\alpha)}(\ell^p)$ $(0 < p < 1)$ is

$$\left(\mathcal{C}^{(\alpha)}(\ell^p)\right)^\beta := \left\{ b = (b_k) \in l^\diamond : \sup_j \left| \sum_{i=0}^{j} (-1)^{j-i} \binom{n+i}{i} \binom{n}{j-i} b_i \right|^p < \infty \right\}.$$

Theorem 2.5 $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), \ell^\infty) \Leftrightarrow$

$$\sup_{j,k\in\mathbb{N}_0} \left| \sum_{i=0}^{k} (-1)^{k-i} \binom{\alpha+i}{i} \binom{\alpha}{k-i} a_{ji} \right|^p < \infty. \tag{2.8}$$

Proof Necessity. Suppose $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), \ell_\infty)$ and $\xi = (\xi_k) \in \mathcal{C}^{(\alpha)}(\ell^p)$. Then $\mathfrak{A}\xi$ exists and $\mathfrak{A}\xi \in \ell^\infty$. Then $\mathfrak{A}_j = (a_{jk})_{k\in\mathbb{N}_0} \in (\mathcal{C}^{(\alpha)}(\ell^p))^\beta$ for each $j \in \mathbb{N}_0$, and hence (2.8) holds.

Sufficiency. Let (2.8) hold and that $\zeta = (\zeta_k) \in \mathcal{C}^{(\alpha)}(\ell^p)$. Then $\mathfrak{A}_j = (a_{jk})_{k\in\mathbb{N}_0} \in (\mathcal{C}^{(\alpha)}(\ell^p))^\beta$ for each $j \in \mathbb{N}_0$, which guarantees the existence of $\mathfrak{A}\zeta$. Fix $j \in \mathbb{N}$, then by (1.2), for $r \in \mathbb{N}_0$,

$$\begin{aligned}
\sum_{k=0}^{r} a_{jk}\zeta_k &= \sum_{k=0}^{r} \sum_{i=k}^{\infty} (-1)^{i-k} \binom{\alpha+k}{k} \binom{\alpha}{i-k} a_{jk} y_i \\
&= \sum_{k=0}^{r} \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{\alpha+i}{i} \binom{\alpha}{k-i} a_{ji} \right) y_k \\
&\quad + \sum_{k=r+1}^{\infty} \left(\sum_{i=0}^{r} (-1)^{r-i} \binom{\alpha+i}{i} \binom{\alpha}{r-i} a_{ji} \right) y_k
\end{aligned}$$

for all $j, r \in \mathbb{N}_0$. Now, by letting $r \to \infty$, we have

$$(A\zeta)_j = \sum_{k=0}^{\infty} a_{jk}\zeta_k = \sum_{k=0}^{\infty} b_{jk}y_k = (By)_j \tag{2.9}$$

for all $j \in \mathbb{N}_0$, where

$$b_{jk} = \sum_{i=0}^{k} (-1)^{k-i} \binom{\alpha+i}{i} \binom{\alpha}{k-i} a_{ji} \tag{2.10}$$

for all $j, r \in \mathbb{N}_0$. Therefore, condition (2.7) of Lemma 2.2 is satisfied by the matrix $B = (b_{jk})$. Hence $By = \mathfrak{A}\zeta \in \ell^\infty$, i.e., $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), \ell_\infty)$. □

Theorem 2.6 $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), c) \Leftrightarrow$ (2.8) *holds and there exists* $\beta_k \in \mathbb{R}$ *such that*

$$\lim_{j\to\infty} \sum_{i=0}^{k} (-1)^{k-i} \binom{\alpha+i}{i} \binom{\alpha}{k-i} a_{ji} = \beta_k \tag{2.11}$$

for each $k \in \mathbb{N}_0$.

Proof *Necessity*. Let $\mathfrak{A} = (a_{nk}) \in (\mathcal{C}^{(\alpha)}(\ell^p), c)$. Then $\mathfrak{A}\zeta$ exists and $\mathfrak{A}\zeta \in c$ for all $\zeta = (\zeta_k) \in \mathcal{C}^{(\alpha)}(\ell^p)$. Since $c \subset \ell^\infty$, condition (2.8) follows from Theorem 2.5. Condition (2.11) immediate follows by taking the sequence $\zeta^{(i)} = \{\zeta_k^{(i)}\} \in \mathcal{C}^{(\alpha)}(\ell^p)$ defined by

$$\zeta_k^{(i)} := \begin{cases} (-1)^{k-i}\binom{\alpha+i}{i}\binom{\alpha}{k-i}, & k \geq i, \\ 0, & 0 \leq k \leq i-1, \end{cases}$$

for all $i, k \in \mathbb{N}_0$ that $\mathfrak{A}\zeta^{(k)} = \{\sum_{i=0}^{k} (-1)^{k-i}\binom{\alpha+i}{i}\binom{\alpha}{k-i} a_{ji}\} \in c$ for each $k \in \mathbb{N}_0$.

Sufficiency. Suppose that conditions (2.8) and (2.11) hold, and that $\zeta = (\zeta_k) \in \mathcal{C}^{(\alpha)}(\ell^p)$. Existence of $\mathfrak{A}\zeta$ follows from the fact that $\mathfrak{A}_j = (a_{jk})_{k\in\mathbb{N}_0} \in (\mathcal{C}^{(\alpha)}(\ell^p))^\beta$ for each $j \in \mathbb{N}_0$. Therefore, it follows from (2.9) that conditions (2.8) and (2.11) correspond to (2.7) and (2.4) with b_{jk} instead of d_{jk}, respectively, where b_{jk} is given by (2.10). Thus, $By \in c$, and we get by (2.9) that $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), c)$. □

Corollary 2.7 $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), c^0) \Leftrightarrow$ (2.8) *holds and* (2.11) *also holds with* $\beta_k = 0$ *for all* $k \in \mathbb{N}_0$.

Corollary 2.8 *For* $\mathfrak{A} = (a_{nk})$, *write* $c(j,k) = \sum_{i=0}^{j} a_{ik}$ *for all* $k, n \in \mathbb{N}_0$. *Then, from Theorem* 2.5, *Theorem* 2.6, *and Corollary* 2.7, *we get*:

(i) $\mathfrak{A} = (a_{nk}) \in (\mathcal{C}^{(\alpha)}(\ell^p), bs) \Leftrightarrow$ (2.8) *holds with* a_{jk} *is replaced by* $c(j,k)$.

(ii) $\mathfrak{A} = (a_{nk}) \in (\mathcal{C}^{(\alpha)}(\ell^p), cs) \Leftrightarrow$ (2.8) *and* (2.11) *hold with* a_{jk} *is replaced by* $c(j,k)$.

(iii) $\mathfrak{A} = (a_{nk}) \in (\mathcal{C}^{(\alpha)}(\ell^p), cs_0) \Leftrightarrow$ (2.8) *and* (2.11) *hold with* a_{jk} *is replaced by* $c(j,k)$, *with* $\beta_k = 0$ *for all* $k \in \mathbb{N}_0$, *where* bs, cs, *and* s_0 *are the space of bounded, convergent, and null series, respectively.*

3 Compactness of matrix operators

We apply the techniques of [3–7, 9, 10], and [13–17].

Let $\mathcal{M}_{\mathfrak{U}} := \{\mathfrak{B} \subset \mathfrak{U} : \mathfrak{B} \text{ is bounded}\}$. The Hausdorff measure of noncompactness (HMNC) of $\mathfrak{B} \in \mathcal{M}_{\mathfrak{U}}$ is defined by

$$\chi(\mathfrak{B}) = \inf\{\varepsilon > 0 : \mathfrak{B} \text{ has finite } \varepsilon\text{-net}\}.$$

Let $\mathfrak{U}$ and $\mathfrak{V}$ be Banach spaces and $\mathfrak{D} \in \mathcal{B}(\mathfrak{U}, \mathfrak{V})$. Then the HMNC of $\mathfrak{D}$ is defined by

$$\|\mathfrak{D}\|_{\chi} = \chi\bigl(\mathfrak{D}(S_{\mathfrak{U}})\bigr) = \chi\bigl(\mathfrak{D}(\bar{B}_{\mathfrak{U}})\bigr), \tag{3.1}$$

and we have

$$\mathfrak{D} \quad \text{is compact if and only if} \quad \|\mathfrak{D}\|_{\chi} = 0. \tag{3.2}$$

In what follows, we denote the set of all compact operators from $\mathfrak{U}$ into $\mathfrak{V}$ by $\mathfrak{C}(\mathfrak{U}, \mathfrak{V})$.

Theorem 3.1 *Let $\mathfrak{U}$ be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $\mathfrak{D} \in \mathcal{M}_{\mathfrak{U}}$ and $\mathfrak{P}_n : \mathfrak{U} \to \mathfrak{U}$ $(n \in \mathbb{N})$ be the projector onto the linear span of $\{b_0, b_1, \dots, b_n\}$. Then we have*

$$\begin{aligned} &\frac{1}{\limsup_{n\to\infty} \|I - \mathfrak{P}_n\|} \cdot \limsup_{n\to\infty} \Bigl(\sup_{\zeta\in\mathfrak{D}} \bigl\|(I - \mathfrak{P}_n)(\zeta)\bigr\| \Bigr) \\ &\quad \le \chi(\mathfrak{D}) \le \limsup_{n\to\infty} \Bigl(\sup_{x\in\mathfrak{D}} \bigl\|(I - \mathfrak{P}_n)(\zeta)\bigr\| \Bigr). \end{aligned}$$

Theorem 3.2 *Let $\mathfrak{D} \in \mathcal{M}_{\mathfrak{U}}$, where $\mathfrak{U} = \ell_p$ $(1 \le p < \infty)$ or c^0. If $\mathfrak{P}_n : \mathfrak{U} \to \mathfrak{U}$ $(n \in \mathbb{N})$ is the operator defined by $\mathfrak{P}_n(\zeta) = \zeta^{[n]} = (\zeta_0, \zeta_1, \dots, \zeta_n, 0, 0, \dots)$ for all $\zeta = (\zeta_k)_{k=0}^{\infty} \in \mathfrak{U}$, then*

$$\chi(\mathfrak{D}) = \lim_{n\to\infty} \Bigl(\sup_{\zeta\in\mathfrak{D}} \bigl\|(I - \mathfrak{P}_n)(\zeta)\bigr\| \Bigr).$$

Lemma 3.3 ([13]) *Let $\mathfrak{U} \supset c^{00}$ be a BK-space with AK or $\mathfrak{U} = \ell_{\infty}$. If $\mathfrak{A} \in (\mathfrak{U}, c)$, then*

$$\alpha_k = \lim_{j\to\infty} a_{jk} \quad \textit{exists for every } k \in \mathbb{N}, \tag{3.3}$$

$$\alpha = (\alpha_k) \in \mathfrak{U}^{\beta}, \tag{3.4}$$

$$\sup_j \|\mathfrak{A}_j - \alpha\|_{\mathfrak{U}}^* < \infty, \tag{3.5}$$

$$\lim_{j\to\infty} \mathfrak{A}_j(x) = \sum_{k=0}^{\infty} \alpha_{jk} x_k \quad \textit{for all } x = (x_k) \in \mathfrak{U}. \tag{3.6}$$

Theorem 3.4 ([13]) *Let $\mathfrak{U} \supset c^{00}$ be a BK-space. Then we have*

(a)

$$\|L_{\mathfrak{A}}\|_{\chi} = \limsup_{n\to\infty} \|\mathfrak{A}_n\|_{\mathfrak{U}}^* \quad \textit{for } \mathfrak{A} \in \bigl(\mathfrak{U}, c^0\bigr)$$

and

$$L_{\mathfrak{A}} \in \mathfrak{C}\bigl(\mathfrak{U}, c^0\bigr) \quad \Leftrightarrow \quad \lim_{n\to\infty} \|\mathfrak{A}_n\|_{\mathfrak{U}}^* = 0.$$

(b) *If* $\mathfrak{U}$ *has AK or* $\mathfrak{U} = \ell^\infty$, *then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \|\mathfrak{A}_n - \alpha\|_{\mathfrak{U}}^* \le \|L_{\mathfrak{A}}\|_\chi \le \limsup_{n\to\infty} \|\mathfrak{A}_n - \alpha\|_{\mathfrak{U}}^* \quad \textit{for } \mathfrak{A} \in (\mathfrak{U}, c)$$

and

$$L_{\mathfrak{A}} \in \mathfrak{C}(\mathfrak{U}, c) \quad \Leftrightarrow \quad \lim_{n\to\infty} \|\mathfrak{A}_n - \alpha\|_{\mathfrak{U}}^* = 0,$$

where $\alpha = (\alpha_k) = (\lim_{n\to\infty} a_{nk})$ *for all* $k \in \mathbb{N}$.

(c)

$$0 \le \|L_{\mathfrak{A}}\|_\chi \le \limsup_{n\to\infty} \|\mathfrak{A}_n\|_{\mathfrak{U}}^* \quad \textit{for } \mathfrak{A} \in \bigl(\mathfrak{U}, \ell^\infty\bigr)$$

and

$$L_{\mathfrak{A}} \in \mathfrak{C}\bigl(\mathfrak{U}, \ell^\infty\bigr) \quad \textit{if } \lim_{n\to\infty} \|\mathfrak{A}_n\|_{\mathfrak{U}}^* = 0. \tag{3.7}$$

We now state and prove the following.

Theorem 3.5 *Let* $1 \le p < \infty$. *Then we have*

(a)

$$\|L_{\mathfrak{A}}\|_\chi = \lim_{r\to\infty} \sup_j \Biggl(\sum_{k=r+1}^{\infty} |a_{jk}|^p\Biggr)^{1/p} \quad \textit{for } \mathfrak{A} \in \bigl(\mathcal{C}^{(\alpha)}\bigl(\ell^p\bigr), c^0\bigr). \tag{3.8}$$

(b)

$$\begin{aligned} &\frac{1}{2} \cdot \lim_{r\to\infty} \sup_j \Biggl(\sum_{k=r+1}^{\infty} |a_{jk} - \beta_k|^p\Biggr)^{1/p} \\ &\quad \le \|L_{\mathfrak{A}}\|_\chi \le \lim_{r\to\infty} \sup_j \Biggl(\sum_{k=r+1}^{\infty} |a_{jk} - \beta_k|^p\Biggr)^{1/p} \quad \textit{for } \mathfrak{A} \in \bigl(\mathcal{C}^{(\alpha)}\bigl(\ell^p\bigr), c\bigr), \end{aligned} \tag{3.9}$$

where $\beta = (\beta_k) = (\lim_{j\to\infty} b_{jk})$ *for all* $k \in \mathbb{N}$.

(c)

$$0 \le \|L_{\mathfrak{A}}\|_\chi \le \lim_{r\to\infty} \sup_j \Biggl(\sum_{k=r+1}^{\infty} |a_{jk}|^p\Biggr)^{1/p} \quad \textit{for } \mathfrak{A} \in \bigl(\mathcal{C}^{(\alpha)}\bigl(\ell^p\bigr), \ell^\infty\bigr). \tag{3.10}$$

Proof (a) Note that the limits in (3.8), (3.9), and (3.10) exist by Lemmas 2.4 and 3.3. Let $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), c^0)$. Then $\mathfrak{A}_j = (a_{jk})_{k\in\mathbb{N}_0} \in [\mathcal{C}^{(\alpha)}(\ell^p)]^\beta$ for each $j \in \mathbb{N}_0$, and we have

$$\|\mathfrak{A}\|_{\mathcal{C}^{(\alpha)}(\ell^p)}^* = \|B_j\|_{\ell^p} = \Biggl(\sum_{k=0}^{\infty} |a_{jk}|^p\Biggr)^{1/p}. \tag{3.11}$$

Write $S = S_{\mathcal{C}^{(\alpha)}(\ell^p)}$ for short. Then we have $\mathfrak{A}S \in \mathcal{M}_{c^0}$. From Theorem 3.2, we get

$$\|L_{\mathfrak{A}}\|_\chi = \chi(\mathfrak{A}S) = \lim_{r\to\infty} \sup_{\zeta\in S} \bigl\|(I-\mathfrak{P}_r)(\mathfrak{A}\zeta)\bigr\|_{\ell^p}. \tag{3.12}$$

$$\lim_{r\to\infty} \sup_{y\in S_{\ell^p}} \bigl\|(I-\mathfrak{P}_r)(By)\bigr\|_{\ell_p} = \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} |a_{jk}|^p\right)^{1/p}. \tag{3.13}$$

We get (3.8) by (3.13).

(b) We have $\mathfrak{A}S \in \mathcal{M}_c$. Suppose that $\mathfrak{P}_r : c \to c$ $(r \in \mathbb{N})$ are the projectors defined by (2.3).

Now, since $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), c)$, we have $B \in (\ell^p, c)$ and $\mathfrak{A}\xi = By$. Thus, it follows from Lemma 3.3 that the limits $\beta_k = \lim_{j\to\infty} a_{jk}$ exist for all k, $\beta = (\beta_k) \in \ell^1 = c^\beta$ and $\lim_{j\to\infty} B_j(y) = \sum_{k=0}^{\infty} a_{jk} y_k$. Therefore, we get

$$\begin{aligned}\|(I-\mathfrak{P}_r)(\mathfrak{A}\zeta)\|_{\ell^p} &= \|(I-\mathfrak{P}_r)(By)\|_{\ell^p} \\ &= \sup_j \left(\sum_{k=r+1}^{\infty} |a_{jk}-\beta_k|^p\right)^{1/p}\end{aligned}$$

for all $\zeta = (\zeta_k) \in \mathcal{C}^{(\alpha)}(\ell^p)$. Now, (3.12) and (3.1) imply that

$$\frac{1}{2} \cdot \lim_{r\to\infty} \sup \|B_j - \beta\|_{\ell^p} \le \|L_{\mathfrak{A}}\|_\chi \le \lim_{r\to\infty} \sup \|B_j - \beta\|_{\ell^p}. \tag{3.14}$$

Hence, we get (3.9) from (3.14), since the limit in (3.9) exists.

(c) Define $\mathfrak{P}_r : \ell^\infty \to \ell^\infty$ $(r \in \mathbb{N})$ as in (a) for all $\zeta = (\zeta_k) \in \ell^\infty$. Then

$$\mathfrak{A}S \subset \mathfrak{P}_r(\mathfrak{A}S) + (I-\mathfrak{P}_r)(\mathfrak{A}S); \quad (r \in \mathbb{N}).$$

Therefore

$$\begin{aligned}0 &\le \chi(\mathfrak{A}S) \\ &\le \chi\bigl(\mathfrak{P}_r(\mathfrak{A}S)\bigr) + \chi\bigl((I-\mathfrak{P}_r)(\mathfrak{A}S)\bigr) \\ &= \chi\bigl((I-\mathfrak{P}_r)(\mathfrak{A}S)\bigr) \\ &\le \sup_{\xi\in S} \|(I-\mathfrak{P}_r)(\mathfrak{A}\xi)\|_{\ell^p} \\ &= \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} |a_{jk}|^p\right)^{1/p}.\end{aligned}$$

From this and (3.12), we get (3.10), which concludes the proof. □

Corollary 3.6 *We have the following*:

(a) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), c_0)$,

$$L_{\mathfrak{A}} \in \mathfrak{C}\bigl(\mathcal{C}^{(\alpha)}(\ell^p), c^0\bigr) \quad \Leftrightarrow \quad \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} |a_{jk}|^p\right)^{1/p} = 0.$$

(b) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell_p), c)$,

$$L_{\mathfrak{A}} \in \mathfrak{C}\big(\mathcal{C}^{(\alpha)}(\ell^p), c\big) \quad \Leftrightarrow \quad \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} |a_{jk} - \beta_k|^p \right)^{1/p} = 0,$$

where $\beta = (\beta_k) = (\lim_{j\to\infty} a_{jk})$ *for all* $k \in \mathbb{N}$.

(c) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), \ell^\infty)$, *then*

$$L_{\mathfrak{A}} \in \mathfrak{C}\big(\mathcal{C}^{(\alpha)}(\ell^p), \ell^\infty\big) \quad \textit{if} \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} |b_{jk}|^p \right)^{1/p} = 0. \tag{3.15}$$

Corollary 3.7 *From Theorem 3.4 and Corollary 2.11, we have the following*:

(a) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), cs^0)$,

$$\|L_{\mathfrak{A}}\|_\chi = \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k)\big|^p \right)^{1/p}. \tag{3.16}$$

(b) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), cs)$,

$$\begin{aligned} &\frac{1}{2} \cdot \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k) - \beta_k\big|^p \right)^{1/p} \\ &\quad \le \|L_{\mathfrak{A}}\|_\chi \le \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k) - \beta_k\big|^p \right)^{1/p}, \end{aligned} \tag{3.17}$$

where $\beta = (\beta_k) = (\lim_{j\to\infty} b_{jk})$ *for all* $k \in \mathbb{N}$.

(c) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), bs)$,

$$0 \le \|L_{\mathfrak{A}}\|_\chi \le \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k)\big|^p \right)^{1/p}. \tag{3.18}$$

Corollary 3.8 *From Corollary 3.5 and Corollary 2.11, we have the following*:

(a) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), cs^0)$,

$$L_{\mathfrak{A}} \in \mathfrak{C}\big(\mathcal{C}^{(\alpha)}(\ell^p), cs^0\big) \quad \Leftrightarrow \quad \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k)\big|^p \right)^{1/p} = 0.$$

(b) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), cs)$,

$$L_{\mathfrak{A}} \in \mathfrak{C}\big(\mathcal{C}^{(\alpha)}(\ell^p), cs\big) \quad \Leftrightarrow \quad \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k) - \beta_k\big|^p \right)^{1/p} = 0,$$

where $\beta = (\beta_k) = (\lim_{j\to\infty} c(j,k))$ *for all* $k \in \mathbb{N}$.

(c) *For* $\mathfrak{A} \in (\mathcal{C}^{(\alpha)}(\ell^p), bs)$,

$$L_{\mathfrak{A}} \in \mathfrak{C}\big(\mathcal{C}^{(\alpha)}(\ell^p), bs\big) \quad \Leftrightarrow \quad \textit{if} \lim_{r\to\infty} \sup_j \left(\sum_{k=r+1}^{\infty} \big|c(j,k)\big|^p \right)^{1/p} = 0.$$

Acknowledgements
We are thankful to the learned reviewers whose suggestions led to the improvement of the presentation.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally. All authors read and approved the final manuscript.

Author details
[1]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan. [2]Al-Qaryah, Doharra, Street No. 1 (West), Aligarh UP 202002, India. [3]Department of Mathematics, Tafila Technical University, P.O. Box 179, Tafila 66110, Jordan.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 4 February 2021 Accepted: 15 October 2021 Published online: 29 October 2021

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