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Asymptotic pointwise error estimates for reconstructing shift-invariant signals with generators in a hybrid-norm space

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Abstract

Sampling and reconstruction of signals in a shift-invariant space are generally studied under the requirement that the generator is in a stronger Wiener amalgam space, and the error estimates are usually given in the sense of $L_{p,1/\omega}$ -norm. Since we often need to reflect the local characteristics of reconstructing error, the asymptotic pointwise error estimates for nonuniform and average sampling in a non-decaying shift-invariant space are discussed under the assumption that the generator is in a hybrid-norm space. Based on Lemma 2.1–Lemma 2.6, we first rewrite the iterative reconstruction algorithms for two kinds of average sampling functionals and prove their convergence. Then, the asymptotic pointwise error estimates are presented for two algorithms under the case that the average samples are corrupted by noise.

Keywords: Hybrid-norm space; Nonuniform average sampling; Non-decaying signals; Iterative algorithm; Asymptotic pointwise error estimate

1 Introduction

The classical Shannon sampling theorem shows that a bandlimited signal which lives in the shift-invariant space generated by the *sinc* function can be recovered from its samples $\{f(n\delta)\}_{n \in \mathbb{Z}}$ when the gap δ is small enough [1]. Since the *sinc* function has infinite support and slow decay, the space of bandlimited functions is often unsuitable for numerical implementations. Retaining some of the simplicity and structure of bandlimited models, sampling in non-bandlimited shift-invariant spaces is more amenable and realistic for many applications [2–13]. Sampling and reconstruction of signals in a shift-invariant space

$$V_{p, \frac{1}{\omega}}(\varphi) := \left\{ \sum_{k \in \mathbb{Z}^d} c_k \varphi(x - k) : c = (c_k)_{k \in \mathbb{Z}^d} \in \ell_{p, \frac{1}{\omega}}(\mathbb{Z}^d) \right\}, \quad 1 \leq p \leq \infty, \quad (1.1)$$

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is generally studied under the condition that the generator φ is in a Winner amalgam space $\tilde{W}_{1,\infty,\omega}(\mathbf{R}^d)$, which is defined as

$$\tilde{W}_{1,\infty,\omega}(\mathbf{R}^d) := \left\{ f : \mathbf{R}^d \rightarrow \mathbf{C} \mid \|f\|_{\tilde{W}_{1,\infty,\omega}} := \sum_{k \in \mathbf{Z}^d} \sup_{x \in [0,1]^d} |f(x+k)| \omega(x+k) < \infty \right\}.$$

Throughout the paper, the weighting function denoted by ω is always assumed to be continuous, symmetric, positive, and submultiplicative,

$$\omega(x+y) \leq C_\omega \omega(x)\omega(y), \quad \forall x, y \in \mathbf{R}^d. \quad (1.2)$$

The decaying weight $1/\omega$ controls the growing rate of the signals living in $V_{p,1/\omega}(\varphi)$.

Recently, regular and ideal sampling [7], nonuniform and average sampling [11] have been restudied under a weaker condition that φ is in the weighted hybrid-norm space $W_{1,q,\omega}(\mathbf{R}^d)$ with $q = \max\{p, p'\}$, p' is the conjugate number of p . Here, for $1 \leq p, q < \infty$, the weighted hybrid-norm space is defined as

$$W_{p,q,\omega}(\mathbf{R}^d) := \{f : \mathbf{R}^d \rightarrow \mathbf{C} \mid \|f\|_{W_{p,q,\omega}} < \infty\}$$

with the norm

$$\|f\|_{W_{p,q,\omega}} := \left(\int_{[0,1]^d} \left(\sum_{k \in \mathbf{Z}^d} |f(x+k)|^p \omega^p(x+k) \right)^{q/p} dx \right)^{1/q}. \quad (1.3)$$

If p or q is infinity, usual adjustments are used. Both [11] and [7] gave the error estimates in the sense of $L_{p,1/\omega}$ -norm, but such estimation can only reflect the mean error information. In some cases, we need to know the local error information. In this paper, we mainly study the asymptotic pointwise error estimates for signals in $V_{p,1/\omega}(\varphi)$ under the assumptions

- (i) $\varphi \in W_{1,\infty,\omega}(\mathbf{R}^d)$.
- (ii) $\lim_{\delta \rightarrow 0} \|\omega_\delta(\varphi)\|_{W_{1,\infty,\omega}} = 0$, where $\omega_\delta(\varphi)$ is the continuous modulus defined by

$$\omega_\delta(\varphi)(x) := \sup_{|y| \leq \delta} |\varphi(x+y) - \varphi(x)|.$$

- (iii) There exist positive constants A and B such that

$$A \leq \sum_{k \in \mathbf{Z}^d} |\widehat{\varphi}(\xi + 2k\pi)|^2 \leq B.$$

In fact, it is easy to verify that assumptions (i)–(ii) are satisfied if $\varphi \in \tilde{W}_{1,\infty,\omega}(\mathbf{R}^d)$ is a continuous function, cf. [4].

The sampling set $\Gamma \subset \mathbf{R}^d$ is assumed to be relatively-separated, that is,

$$B_\Gamma(\delta) := \sup_{x \in \mathbf{R}^d} \sum_{\gamma \in \Gamma} \chi_{B(\gamma,\delta)}(x) < \infty$$

for some $\delta > 0$. Furthermore, $\delta > 0$ is said to be a gap of a relatively-separated subset Γ if

$$A_\Gamma(\delta) := \inf_{x \in \mathbf{R}^d} \sum_{\gamma \in \Gamma} \chi_{B(\gamma,\delta)}(x) \geq 1.$$

Given a relatively-separated sampling set Γ , two kinds of average sampling schemes are considered. The first one is

$$\langle f, \psi_\gamma \rangle = \int_{\mathbb{R}^d} f(x) \psi_\gamma(x) dx, \quad (1.4)$$

where the average sampling functionals $\{\psi_\gamma : \gamma \in \Gamma\}$ satisfy the following:

- (a) $\int_{\mathbb{R}^d} \psi_\gamma(x) dx = 1$ for all $\gamma \in \Gamma$.
- (b) There exists $M > 0$ such that $\int_{\mathbb{R}^d} |\psi_\gamma(x)| dx \leq M$ for all $\gamma \in \Gamma$.
- (c) $\text{supp } \psi_\gamma \subset B(\gamma, a)$ for some $a > 0$.

Note that the first sampling scheme requires that the sampling functions have compact support, we consider the second average sampling scheme which is defined as

$$\langle f, \psi_a(\cdot - \gamma) \rangle = f * \psi_a^*(\gamma), \quad (1.5)$$

where $\psi \in L_1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x) dx = 1$, $\psi_a(\cdot) := \frac{1}{a^d} \psi(\frac{\cdot}{a})$, and $\psi_a^*(\cdot) = \overline{\psi_a(-\cdot)}$.

This paper is organized as follows. In Sect. 2, some necessary lemmas are provided for the subsequent sections. In Sect. 3, the iterative reconstruction algorithms are rewritten and their convergence is proved. The asymptotic pointwise error estimates for both average sampling schemes are presented in Sect. 4.

2 Preliminaries

In this section, we give some lemmas which are important for the subsequent sections.

Lemma 2.1 ([7]) *Let ω be a submultiplicative weighting function. If $\varphi \in W_{1,\infty,\omega}(\mathbb{R}^d)$ and $c \in \ell_{1,\omega}(\mathbb{Z}^d)$, then the function $f = \sum_{k \in \mathbb{Z}^d} c_k \varphi(\cdot - k)$ also belongs to $W_{1,\infty,\omega}(\mathbb{R}^d)$ and*

$$\|f\|_{W_{1,\infty,\omega}} \leq C_\omega \|c\|_{\ell_{1,\omega}} \|\varphi\|_{W_{1,\infty,\omega}}.$$

Lemma 2.2 *Let ω be a submultiplicative weighting function that additionally satisfies the Gelfand–Raikov–Shilov (GRS) condition*

$$\lim_{n \rightarrow \infty} \omega(nk)^{1/n} = 1, \quad \forall k \in \mathbb{Z}^d. \quad (2.1)$$

Suppose that the generator φ satisfies assumptions (i)–(iii), then the dual function $\tilde{\varphi}$ of φ is also in $W_{1,\infty,\omega}(\mathbb{R}^d)$ and satisfies $\lim_{\delta \rightarrow 0} \|\omega_\delta(\tilde{\varphi})\|_{W_{1,\infty,\omega}} = 0$.

Proof By Proposition 6 in [7], $\tilde{\varphi} \in W_{1,\infty,\omega}(\mathbb{R}^d)$. Moreover, there exists a unique $b \in \ell_{1,\omega}(\mathbb{Z}^d)$ such that

$$\tilde{\varphi} = \sum_{k \in \mathbb{Z}^d} b_k \varphi(\cdot - k).$$

Furthermore, it follows from Lemma 2.1 that

$$\begin{aligned} \|\omega_\delta(\tilde{\varphi})\|_{W_{1,\infty,\omega}} &= \left\| \sup_{|y| \leq \delta} |\tilde{\varphi}(x+y) - \tilde{\varphi}(x)| \right\|_{W_{1,\infty,\omega}} \\ &= \left\| \sup_{|y| \leq \delta} \left| \sum_{k \in \mathbb{Z}^d} b_k \varphi(x+y-k) - \sum_{k \in \mathbb{Z}^d} b_k \varphi(x-k) \right| \right\|_{W_{1,\infty,\omega}} \end{aligned}$$

$$\begin{aligned} &\leq \left\| \sum_{k \in \mathbb{Z}^d} |b_k| \omega_\delta(\varphi)(x-k) \right\|_{W_{1,\infty,\omega}} \\ &\leq C_\omega \|b\|_{\ell_{1,\omega}} \|\omega_\delta(\varphi)\|_{W_{1,\infty,\omega}}. \end{aligned} \quad (2.2)$$

Therefore, $\lim_{\delta \rightarrow 0} \|\omega_\delta(\tilde{\varphi})\|_{W_{1,\infty,\omega}} = 0$ is proved. $\square$

Lemma 2.3 ([7]) *Let ω be a submultiplicative weighting function satisfying the Gelfand–Raikov–Shilov (GRS) condition. Suppose that $\varphi \in W_{1,\infty,\omega}(\mathbb{R}^d)$, then the linear operator*

$$Pf(x) := \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}(\cdot - k) \rangle \varphi(x-k) \quad (2.3)$$

is a projector that continuously maps $L_{p,1/\omega}(\mathbb{R}^d)$ into the subspace $V_{p,1/\omega}(\varphi)$.

Lemma 2.4 *Let ω , φ , and $\tilde{\varphi}$ be as in Lemma 2.2. Then the function*

$$K(x, y) := \sum_{k \in \mathbb{Z}^d} \varphi(x-k) \tilde{\varphi}(y-k) \quad (2.4)$$

satisfies

$$\|K\|_w := \max \left\{ \sup_{x \in \mathbb{R}^d} \|K(x, x + \cdot)\|_{L_{1,\omega}}, \sup_{y \in \mathbb{R}^d} \|K(y + \cdot, y)\|_{L_{1,\omega}} \right\} < \infty \quad (2.5)$$

and

$$\lim_{\delta \rightarrow 0} \|\omega_\delta(K)\|_w = 0, \quad (2.6)$$

where the modulus of continuity

$$\omega_\delta(K)(x, y) := \sup_{|x'|, |y'| \leq \delta} |K(x+x', y+y') - K(x, y)|.$$

Proof By direct computation, we have

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \|K(x, x + \cdot)\|_{L_{1,\omega}} \\ &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| \sum_{k \in \mathbb{Z}^d} \varphi(x-k) \tilde{\varphi}(x+y-k) \right| \omega(y) dy \\ &\leq \sup_{x \in \mathbb{R}^d} C_\omega \int_{[0,1]^d} \sum_{\ell \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} \varphi(x-k) \omega(x-k) \right. \\ &\quad \times \left. \tilde{\varphi}(x+y+\ell-k) \omega(x+y+\ell-k) \right| dy \\ &\leq C_\omega \sup_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |\varphi(x-k)| \omega(x-k) \int_{[0,1]^d} \sum_{\ell \in \mathbb{Z}^d} |\tilde{\varphi}(x+y+\ell)| \omega(x+y+\ell) dy \\ &\leq C_\omega \|\varphi\|_{W_{1,\infty,\omega}} \|\tilde{\varphi}\|_{L_{1,\omega}}. \end{aligned}$$

A similar computation yields

$$\sup_{y \in \mathbb{R}^d} \|K(y + \cdot, y)\|_{L_{1,\omega}} \leq C_\omega \|\varphi\|_{L_{1,\omega}} \|\tilde{\varphi}\|_{W_{1,\infty,\omega}}.$$

Note that $W_{1,\infty,\omega}(\mathbb{R}^d) \subseteq W_{1,1,\omega}(\mathbb{R}^d) = L_{1,\omega}(\mathbb{R}^d)$. Then

$$\|K\|_w \leq \max\{C_\omega \|\varphi\|_{W_{1,\infty,\omega}} \|\tilde{\varphi}\|_{L_{1,\omega}}, C_\omega \|\varphi\|_{L_{1,\omega}} \|\tilde{\varphi}\|_{W_{1,\infty,\omega}}\} < \infty. \quad (2.7)$$

Now, we will prove (2.6). In fact,

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} \|\omega_\delta(K)(x, x + \cdot)\|_{L_{1,\omega}} \\ &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{|x'|, |y'| \leq \delta} \left| \sum_{k \in \mathbb{Z}^d} (\varphi(x + x' - k) \tilde{\varphi}(x + y + y' - k) \right. \\ & \quad \left. - \varphi(x - k) \tilde{\varphi}(x + y - k)) \right| \omega(y) dy \\ &\leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}^d} (\omega_\delta(\varphi)(x - k) \sup_{|y'| \leq \delta} |\tilde{\varphi}(x + y + y' - k)| \right. \\ & \quad \left. + |\varphi(x - k)| \omega_\delta(\tilde{\varphi})(x + y - k)) \right) \omega(y) dy \\ &\leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}^d} (\omega_\delta(\varphi)(x - k) (\omega_\delta(\tilde{\varphi})(x + y - k) + |\tilde{\varphi}(x + y - k)|) \right. \\ & \quad \left. + |\varphi(x - k)| \omega_\delta(\tilde{\varphi})(x + y - k)) \right) \omega(y) dy \\ &\leq C_\omega (\|\omega_\delta(\varphi)\|_{W_{1,\infty,\omega}} (\|\omega_\delta(\tilde{\varphi})\|_{L_{1,\omega}} + \|\tilde{\varphi}\|_{L_{1,\omega}}) + \|\varphi\|_{W_{1,\infty,\omega}} \|\omega_\delta(\tilde{\varphi})\|_{L_{1,\omega}}). \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} & \sup_{y \in \mathbb{R}^d} \|\omega_\delta(K)(y + \cdot, y)\|_{L_{1,\omega}} \\ &\leq C_\omega (\|\omega_\delta(\tilde{\varphi})\|_{W_{1,\infty,\omega}} (\|\omega_\delta(\varphi)\|_{L_{1,\omega}} + \|\varphi\|_{L_{1,\omega}}) + \|\tilde{\varphi}\|_{W_{1,\infty,\omega}} \|\omega_\delta(\varphi)\|_{L_{1,\omega}}). \end{aligned}$$

Note that $\|f\|_{L_{1,\omega}} \leq \|f\|_{W_{1,\infty,\omega}}$. Then (2.6) follows from Lemma (2.2) and

$$\begin{aligned} \|\omega_\delta(K)\|_w &\leq \max\left\{\sup_{x \in \mathbb{R}^d} \|\omega_\delta(K)(x, x + \cdot)\|_{L_{1,\omega}}, \sup_{y \in \mathbb{R}^d} \|\omega_\delta(K)(y + \cdot, y)\|_{L_{1,\omega}}\right\} \\ &\leq C_\omega (\|\omega_\delta(\varphi)\|_{W_{1,\infty,\omega}} (\|\omega_\delta(\tilde{\varphi})\|_{W_{1,\infty,\omega}} + \|\tilde{\varphi}\|_{W_{1,\infty,\omega}}) \\ & \quad + \|\varphi\|_{W_{1,\infty,\omega}} \|\omega_\delta(\tilde{\varphi})\|_{W_{1,\infty,\omega}}). \end{aligned} \quad \square$$

Remark 2.1 In fact, $V_{p,1/\omega}(\varphi)$ is the range space of P on $L_{p,1/\omega}(\mathbb{R}^d)$, and

$$Pf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy, \quad f \in L_{p, \frac{1}{\omega}}(\mathbb{R}^d). \quad (2.8)$$

For a relatively-separated subset Γ in $\mathbb{R}^d$, let $U = \{\beta_\gamma\}_{\gamma \in \Gamma}$ be a bounded uniform partition of unity (BUPU) associated with the covering $\{B(\gamma, \delta)\}_{\gamma \in \Gamma}$, which satisfies

- (i) $0 \leq \beta_\gamma(x) \leq 1$;
- (ii) β_γ is supported in $B(\gamma, \delta)$ for all $\gamma \in \Gamma$;
- (iii) $\sum_{\gamma \in \Gamma} \beta_\gamma(x) \equiv 1, \forall x \in \mathbb{R}^d$.

Lemma 2.5 *Let $U = \{\beta_\gamma\}_{\gamma \in \Gamma}$ be a BUPU associated with the covering $\{B(\gamma, \delta)\}_{\gamma \in \Gamma}$. The function $K(x, y)$ defined by (2.4) satisfies $\int_{\mathbb{R}^d} K(x, y) \beta_\gamma(y) dy \in V_{p, 1/\omega}(\varphi)$.*

Proof Let $g(x) := \int_{\mathbb{R}^d} K(x, y) \beta_\gamma(y) dy$. Then

$$\begin{aligned} \|g\|_{L_{1, \frac{1}{\omega}}} &\leq \int_{\mathbb{R}^d} \frac{1}{\omega(x)} \int_{B(\gamma, \delta)} |K(x, y)| \beta_\gamma(y) dy dx \\ &\leq C_\omega \int_{B(\gamma, \delta)} \beta_\gamma(y) \frac{1}{\omega(y)} dy \int_{\mathbb{R}^d} |K(x, y)| \omega(x - y) dx \\ &\leq C_\omega V_\delta \left(\max_{x \in B(\gamma, \delta)} \frac{1}{\omega(x)} \right) \|K\|_w, \end{aligned}$$

where V_δ is the volume of ball $B(\gamma, \delta)$. Moreover,

$$\|g\|_{L_{\infty, \frac{1}{\omega}}} \leq C_\omega \left(\max_{x \in B(\gamma, \delta)} \frac{1}{\omega(x)} \right) \|K\|_w.$$

Therefore, $\|g\|_{L_{p, \frac{1}{\omega}}} \leq \|g\|_{L_{1, \frac{1}{\omega}}}^{\frac{1}{p}} \|g\|_{L_{\infty, \frac{1}{\omega}}}^{1-\frac{1}{p}} < \infty$, which means that $g \in L_{p, 1/\omega}(\mathbb{R}^d)$. Furthermore, we have

$$\begin{aligned} Pg(x) &= \int_{\mathbb{R}^d} K(x, y) \int_{\mathbb{R}^d} K(y, z) \beta_\gamma(z) dz dy \\ &= \int_{\mathbb{R}^d} \beta_\gamma(z) \int_{\mathbb{R}^d} K(x, y) K(y, z) dy dz \\ &= \int_{\mathbb{R}^d} K(x, z) \beta_\gamma(z) dz = g(x). \end{aligned}$$

Then $g \in V_{p, 1/\omega}(\varphi)$. □

Lemma 2.6 *Suppose that $\psi \in L_1(\mathbb{R}^d)$ satisfies $\lim_{a \rightarrow 0} \int_{|t| \geq 1} |\psi_a(t)| \omega(t) dt = 0$, then the function*

$$K_1(x, y) := K(x, y) - K(\cdot, y) * \psi_a^*(x)$$

satisfies

$$\lim_{a \rightarrow 0} \|K_1\|_w = 0 \text{ and } \lim_{a, \delta \rightarrow 0} \|\omega_\delta(K_1)\|_w = 0.$$

Proof Note that

$$|K_1(x, y)| \leq \int_{\mathbb{R}^d} |K(x, y) - K(x + t, y)| |\psi_a(t)| dt$$

$$\begin{aligned}
&= \left(\int_{|t| \leq 1} + \int_{|t| \geq 1} \right) |K(x, y) - K(x + t, y)| |\psi_a(t)| dt \\
&= I_1(x, y) + I_2(x, y).
\end{aligned}$$

We first estimate $I_2(x, y)$.

$$\begin{aligned}
\sup_{x \in \mathbb{R}^d} \|I_2(x + \cdot, x)\|_{L_{1,\omega}} &\leq \sup_{x \in \mathbb{R}^d} \left(\int_{|t| \geq 1} \int_{\mathbb{R}^d} |K(x + z, x)| |\psi_a(t)| \omega(z) dz dt \right) \\
&\quad + \sup_{x \in \mathbb{R}^d} \left(\int_{|t| \geq 1} \int_{\mathbb{R}^d} |K(x + t + z, x)| |\psi_a(t)| \omega(z) dz dt \right) \\
&\leq \|K\|_w \left(\int_{|t| \geq \frac{1}{a}} |\psi(t)| dt + C_\omega \int_{|t| \geq 1} |\psi_a(t)| \omega(t) dt \right).
\end{aligned}$$

Similarly,

$$\sup_{x \in \mathbb{R}^d} \|I_2(x, x + \cdot)\|_{L_{1,\omega}} \leq \|K\|_w \left(\int_{|t| \geq \frac{1}{a}} |\psi(t)| dt + C_\omega \int_{|t| \geq 1} |\psi_a(t)| \omega(t) dt \right).$$

Then we have

$$\|I_2\|_w \leq \|K\|_w \left(\int_{|t| \geq \frac{1}{a}} |\psi(t)| dt + C_\omega \int_{|t| \geq 1} |\psi_a(t)| \omega(t) dt \right) \rightarrow 0, \quad \text{as } a \rightarrow 0.$$

Since $\lim_{\delta \rightarrow 0} \|\omega_\delta(K)\|_w = 0$ for any $\varepsilon > 0$, there exists $0 < \delta' < 1$ such that $\|\omega_\delta(K)\|_w < \varepsilon$ for any $\delta < \delta'$. Then

$$\begin{aligned}
&\sup_{x \in \mathbb{R}^d} \|I_1(x + \cdot, x)\|_{L_{1,\omega}} \\
&\leq \sup_{x \in \mathbb{R}^d} \left(\int_{|t| < \delta'} + \int_{\delta' \leq |t| \leq 1} \right) \\
&\quad \times \left(\int_{\mathbb{R}^d} |K(x + z, x) - K(x + t + z, x)| |\psi_a(t)| \omega(z) dz \right) dt \\
&=: I_{1,1} + I_{1,2}.
\end{aligned} \tag{2.9}$$

Moreover, we obtain

$$\begin{aligned}
I_{1,1} &\leq \sup_{x \in \mathbb{R}^d} \int_{|t| < \delta'} \left(\int_{\mathbb{R}^d} |K(x + z, x) - K(x + t + z, x)| |\psi_a(t)| \omega(z) dz \right) dt \\
&\leq \|\omega_{\delta'}(K)\|_w \left(\int_{|t| < \delta'} |\psi_a(t)| dt \right) < \varepsilon \|\psi\|_1
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
I_{1,2} &\leq \sup_{x \in \mathbb{R}^d} \int_{\delta' \leq |t| \leq 1} \left(\int_{\mathbb{R}^d} |K(x + z, x)| |\psi_a(t)| \omega(z) dz \right) dt \\
&\quad + \sup_{x \in \mathbb{R}^d} \int_{\delta' \leq |t| \leq 1} \left(\int_{\mathbb{R}^d} |K(x + t + z, x)| |\psi_a(t)| \omega(z) dz \right) dt
\end{aligned}$$

$$\leq \|K\|_w \left(1 + \max_{|t| \leq 1} \omega(t)\right) \int_{|t| \geq \frac{\delta'}{a}} |\psi(t)| dt \rightarrow 0, \quad \text{as } a \rightarrow 0.$$

This together with (2.9)–(2.10) proves that $\lim_{a \rightarrow 0} \sup_{x \in \mathbb{R}^d} \|I_1(x + \cdot, x)\|_{L_{1,\omega}} = 0$. Similarly, $\lim_{a \rightarrow 0} \sup_{x \in \mathbb{R}^d} \|I_1(x, x + \cdot)\|_{L_{1,\omega}} = 0$ and $\lim_{a \rightarrow 0} \|K_1\|_w = 0$ is proved.

Now, we will prove $\lim_{a, \delta \rightarrow 0} \|\omega_\delta(K_1)\|_w = 0$. Note that

$$\|\omega_\delta(K_1)\|_w \leq \|\omega_\delta(I_1)\|_w + \|\omega_\delta(I_2)\|_w. \quad (2.11)$$

In the following, we estimate $\lim_{a, \delta \rightarrow 0} \|\omega_\delta(I_1)\|_w = 0$ and $\lim_{a, \delta \rightarrow 0} \|\omega_\delta(I_2)\|_w = 0$, respectively. In fact,

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} \|\omega_\delta(I_1)(x, x + \cdot)\|_{L_{1,\omega}} \\ & \leq 2 \sup_{x \in \mathbb{R}^d} \|I_1(x, x + \cdot)\|_{L_{1,\omega}} + \|\psi\|_1 \left(1 + C_\omega \max_{|t| \leq 1} \omega(t)\right) \|\omega_\delta(K)\|_w \end{aligned}$$

and

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} \|\omega_\delta(I_1)(x + \cdot, x)\|_{L_{1,\omega}} \\ & \leq 2 \sup_{x \in \mathbb{R}^d} \|I_1(x + \cdot, x)\|_{L_{1,\omega}} + \|\psi\|_1 \left(1 + C_\omega \max_{|t| \leq 1} \omega(t)\right) \|\omega_\delta(K)\|_w \end{aligned}$$

Therefore, we have

$$\|\omega_\delta(I_1)\|_w \leq 2\|I_1\|_w + \|\psi\|_1 \left(1 + C_\omega \max_{|t| \leq 1} \omega(t)\right) \|\omega_\delta(K)\|_w. \quad (2.12)$$

Moreover, $\lim_{a, \delta \rightarrow 0} \|\omega_\delta(I_1)\|_w = 0$ follows from $\lim_{a \rightarrow 0} \|I_1\|_w = 0$ and $\lim_{\delta \rightarrow 0} \|\omega_\delta(K)\|_w = 0$. By a similar method, we can obtain

$$\|\omega_\delta(I_2)\|_w \leq 2\|I_2\|_w + \|\omega_\delta(K)\|_w \left(\|\psi\|_1 + C_\omega \int_{|t| \geq 1} |\psi_a(t)| \omega(t) dt\right). \quad (2.13)$$

Therefore, $\lim_{a, \delta \rightarrow 0} \|\omega_\delta(I_2)\|_w = 0$. Finally, $\lim_{a, \delta \rightarrow 0} \|\omega_\delta(K_1)\|_w = 0$ follows from (2.11)–(2.13). $\square$

3 Iterative reconstruction algorithms

For two kinds of average sampling schemes,

$$\begin{aligned} \langle f, \psi_\gamma \rangle &= \int_{\mathbb{R}^d} f(x) \psi_\gamma(x) dx, \\ \langle f, \psi_a(\cdot - \gamma) \rangle &= f * \psi_a^*(\gamma), \end{aligned}$$

define sampling operators for signals in $V_{p,1/\omega}(\varphi)$ as

$$A_\Gamma f := \sum_{\gamma \in \Gamma} \langle Pf, \psi_\gamma \rangle \int_{\mathbb{R}^d} K(x, y) \beta_\gamma(y) dy, \quad (3.1)$$

$$A_{\Gamma,a}f := \sum_{\gamma \in \Gamma} \langle Pf, \psi_a(\cdot - \gamma) \rangle \int_{\mathbb{R}^d} K(x, y) \beta_{\gamma}(y) dy. \quad (3.2)$$

Given an initial sequence $c_0 = (c_0(\gamma))_{\gamma \in \Gamma} \in \ell_{p,1/\omega}(\Gamma)$, the corresponding algorithms are formulated as

$$\begin{cases} f_0 = \sum_{\gamma \in \Gamma} c_0(\gamma) \int_{\mathbb{R}^d} K(\cdot, y) \beta_{\gamma}(y) dy, \\ f_n = f_0 + f_{n-1} - A_{\Gamma} f_{n-1}, \quad n \geq 1, \end{cases} \quad (3.3)$$

and

$$\begin{cases} f_0 = \sum_{\gamma \in \Gamma} c_0(\gamma) \int_{\mathbb{R}^d} K(\cdot, y) \beta_{\gamma}(y) dy, \\ f_n = f_0 + f_{n-1} - A_{\Gamma,a} f_{n-1}, \quad n \geq 1. \end{cases} \quad (3.4)$$

Theorem 3.1 *Let $\Gamma \subset \mathbb{R}^d$ be a relatively-separated subset with gap $\delta > 0$ and $U = \{\beta_{\gamma}\}_{\gamma \in \Gamma}$ be a BUPU associated with the covering $\{B(\gamma, \delta)\}_{\gamma \in \Gamma}$. If δ and a are chosen such that*

$$\begin{aligned} r_1 := MC_{\omega}^2 (\|K\|_w \|\omega_{a+\delta}(K)\|_w + \|K\|_w \|\omega_{2\delta}(K)\|_w \\ + \|\omega_{2\delta}(K)\|_w \|\omega_{a+\delta}(K)\|_w) < 1, \end{aligned} \quad (3.5)$$

then algorithm (3.3) converges to some $f_{\infty} \in V_{p,1/\omega}(\varphi)$. If δ and a are chosen such that

$$r_2 := C_{\omega}^2 (\|K\|_w (\|\omega_{\delta}(K)\|_w + \|K_1\|_w + \|\omega_{\delta}(K_1)\|_w)) < 1, \quad (3.6)$$

then algorithm (3.4) also converges to some $f_{\infty} \in V_{p,1/\omega}(\varphi)$. In particular, if $c_0(\gamma) = \langle f, \psi_{\gamma} \rangle$ or $\langle f, \psi_a(\cdot - \gamma) \rangle$ for signals $f \in V_{p,1/\omega}(\varphi)$, then $f_{\infty} = f$. That is, f can be exactly recovered.

Proof Note that $f_{n+1} - f_n = (P - A_{\Gamma})(f_n - f_{n-1})$, $n \geq 1$, for algorithm (3.3) and $f_{n+1} - f_n = (P - A_{\Gamma,a})(f_n - f_{n-1})$, $n \geq 1$, for algorithm (3.4). It is enough to prove that $\|P - A_{\Gamma}\| < 1$ and $\|P - A_{\Gamma,a}\| < 1$ on $L_{p,1/\omega}(\mathbb{R}^d)$.

For any $f \in L_{p,1/\omega}(\mathbb{R}^d)$, we have

$$\begin{aligned} (P - A_{\Gamma})f(x) &= \int_{\mathbb{R}^d} K(x, z) f(z) dz - \sum_{\gamma \in \Gamma} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} K(t, z) \psi_{\gamma}(t) dt \cdot \int_{\mathbb{R}^d} K(x, y) \beta_{\gamma}(y) dy \right] f(z) dz \\ &=: \int_{\mathbb{R}^d} K_2(x, z) f(z) dz. \end{aligned}$$

Now, we will estimate $\|K_2\|_w$. In fact,

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \|K_2(x, x + \cdot)\|_{L_{1,\omega}} \\ &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| \sum_{\gamma \in \Gamma} \int_{\mathbb{R}^d} K(x, s) K(s, x + z) \beta_{\gamma}(s) ds \int_{\mathbb{R}^d} \psi_{\gamma}(t) dt \int_{\mathbb{R}^d} \frac{\beta_{\gamma}(y)}{\|\beta_{\gamma}\|_1} dy \right. \\ &\quad \left. - \sum_{\gamma \in \Gamma} \|\beta_{\gamma}\|_1 \int_{\mathbb{R}^d} K(t, x + z) \psi_{\gamma}(t) dt \int_{\mathbb{R}^d} K(x, y) \frac{\beta_{\gamma}(y)}{\|\beta_{\gamma}\|_1} dy \right| \omega(z) dz \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{x \in \mathbb{R}^d} \sum_{\gamma \in \Gamma} \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |K(x, s)K(s, x+z) - K(x, y)K(t, x+z)| \right. \\
&\quad \times \left. |\psi_\gamma(t)| \frac{\beta_\gamma(y)}{\|\beta_\gamma\|_1} dt dy \right) \beta_\gamma(s) \omega(z) ds dz \\
&\leq M \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} |K(x, s)| \omega_{a+\delta}(K)(s, x+z) \omega(z) ds dz \\
&\quad + M \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} \omega_{a+\delta}(K)(s, x+z) \omega_{2\delta}(K)(x, s) \omega(z) ds dz \\
&\quad + M \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} K(s, x+z) \omega_{2\delta}(K)(x, s) \omega(z) ds dz \\
&\leq MC_\omega (\|K\|_w \|\omega_{a+\delta}(K)\|_w + \|K\|_w \|\omega_{2\delta}(K)\|_w + \|\omega_{2\delta}(K)\|_w \|\omega_{a+\delta}(K)\|_w) \\
&= \frac{r_1}{C_\omega}.
\end{aligned}$$

Similarly, $\sup_{x \in \mathbb{R}^d} \|K_2(x + \cdot, x)\|_{L_{1,\omega}} \leq r_1/C_\omega$. Therefore,

$$\|P - A_\Gamma\| \leq C_\omega \|K_2\|_w \leq r_1 < 1. \quad (3.7)$$

For any $f \in L_{p,1/\omega}(\mathbb{R}^d)$, let $(P - A_{\Gamma,a})f(x) := \int_{\mathbb{R}^d} K_3(x, y)f(y) dy$. Then

$$\begin{aligned}
K_3(x, y) &= K(x, y) - \sum_{\gamma \in \Gamma} K(\cdot, y) * \psi_a^*(\gamma) \int_{\mathbb{R}^d} K(x, z) \beta_\gamma(z) dz \\
&= \left(\sum_{\gamma \in \Gamma} \int_{\mathbb{R}^d} (K(x, z)K(z, y) - K(\gamma, y)K(x, z)) \beta_\gamma(z) dz \right) \\
&\quad + \left(\sum_{\gamma \in \Gamma} K_1(\gamma, y) \int_{\mathbb{R}^d} K(x, z) \beta_\gamma(z) dz \right) \\
&=: K_{3,1}(x, y) + K_{3,2}(x, y).
\end{aligned}$$

In the following, we estimate $\|K_{3,1}\|_w$ and $\|K_{3,2}\|_w$, respectively. In fact, it is easy to verify that

$$\|K_{3,1}\|_w \leq C_\omega \|K\|_w \|\omega_\delta(K)\|_w$$

and

$$\|K_{3,2}\|_w \leq C_\omega \|K\|_w (\|K_1\|_w + \|\omega_\delta(K_1)\|_w).$$

Therefore, we obtain

$$\begin{aligned}
\|P - A_{\Gamma,a}\| &\leq C_\omega \|K_3\|_w \\
&\leq C_\omega (\|K_{3,1}\|_w + \|K_{3,2}\|_w) \\
&\leq C_\omega^2 \|K\|_w (\|\omega_\delta(K)\|_w + \|K_1\|_w + \|\omega_\delta(K_1)\|_w) \\
&= r_2 < 1.
\end{aligned}$$

Define operators

$$R_{\Gamma} := P + \sum_{n=1}^{\infty} (P - A_{\Gamma})^n \quad (3.8)$$

and

$$R_{\Gamma,a} := P + \sum_{n=1}^{\infty} (P - A_{\Gamma,a})^n. \quad (3.9)$$

Then $R_{\Gamma}A_{\Gamma} = A_{\Gamma}R_{\Gamma} = P$ and $R_{\Gamma,a}A_{\Gamma,a} = A_{\Gamma,a}R_{\Gamma,a} = P$.

If $c_0(\gamma) = \langle f, \psi_{\gamma} \rangle$ or $\langle f, \psi_a(\cdot - \gamma) \rangle$ for $f \in V_{p,1/\omega}(\varphi)$, then

$$f_{\infty} = R_{\Gamma}f_0 = R_{\Gamma}A_{\Gamma}f = Pf = f$$

or

$$f_{\infty} = R_{\Gamma,a}f_0 = R_{\Gamma,a}A_{\Gamma,a}f = Pf = f,$$

which means that $f \in V_{p,1/\omega}(\varphi)$ can be exactly recovered. $\square$

4 Asymptotic pointwise error estimation

In this section, we give the asymptotic pointwise error estimates for algorithms (3.3) and (3.4).

Lemma 4.1 *Let R be R_{Γ} or $R_{\Gamma,a}$, and K_R be its kernel. If δ and a are chosen such that (3.5) or (3.6) is satisfied for algorithm (3.3) or (3.4), respectively, then*

$$\|K_{R_{\Gamma}}\|_w \leq \frac{r_1}{C_{\omega}(1-r_1)} + \|K\|_w \quad (4.1)$$

and

$$\|K_{R_{\Gamma,a}}\|_w \leq \frac{r_2}{C_{\omega}(1-r_2)} + \|K\|_w. \quad (4.2)$$

Proof We only prove (4.1), (4.2) can be proved similarly. Let $K_{\tilde{R}_{\Gamma}}$ be the kernel of $R_{\Gamma} - P$. Then $R_{\Gamma} - P = \sum_{n=1}^{\infty} (P - A_{\Gamma})^n$ means that

$$K_{\tilde{R}_{\Gamma}}(x, y) = K_2(x, y) + \sum_{n=2}^{\infty} \int_{R^{(n-1)d}} K_2(x, z_1) K_2(z_1, z_2) \dots K_2(z_{n-1}, y) dz_1 \dots dz_{n-1}.$$

Moreover, we have

$$\|K_{\tilde{R}_{\Gamma}}\|_w \leq \|K_2\|_w + \sum_{n=2}^{\infty} C_{\omega}^{n-1} \|K_2\|_w^n = \sum_{n=1}^{\infty} C_{\omega}^{n-1} \|K_2\|_w^n \leq \frac{r_1}{C_{\omega}(1-r_1)}.$$

Since $R_{\Gamma} = (R_{\Gamma} - P) + P$, then

$$\|K_{R_{\Gamma}}\|_w \leq \|K\|_w + \|K_{\tilde{R}_{\Gamma}}\|_w \leq \|K\|_w + \frac{r_1}{C_{\omega}(1-r_1)}. \quad \square$$

Theorem 4.1 *Let $1 \leq p \leq \infty$. Suppose that $\Gamma \subset \mathbf{R}^d$ is a relatively-separated subset with gap δ , and $U = \{\beta_\gamma\}_{\gamma \in \Gamma}$ is a BUPU associated with the covering $\{B(\gamma, \delta)\}_{\gamma \in \Gamma}$. Assume that $\{\varepsilon(\gamma)\}_{\gamma \in \Gamma}$ is weighted bounded i.i.d. noise with weighted zero mean and σ^2 variance, that is,*

$$\varepsilon(\gamma)/\omega(\gamma) \in [-B, B], \quad E(\varepsilon(\gamma)/\omega(\gamma)) = 0 \quad \text{and} \quad \text{Var}(\varepsilon(\gamma)/\omega(\gamma)) = \sigma^2$$

for some $B > 0$, and that the initial data c_0 are $(\langle f, \psi_\gamma \rangle + \varepsilon(\gamma))_{\gamma \in \Gamma}$ and $(\langle f, \psi_a(\cdot - \gamma) \rangle + \varepsilon(\gamma))_{\gamma \in \Gamma}$ in algorithms (3.3) and (3.4), respectively. Then, for any $x \in \mathbf{R}^d$,

$$E((f_\infty(x) - f(x))/\omega(x)) = 0$$

and

$$\text{Var}((f_\infty(x) - f(x))/\omega(x)) \leq C_\omega^6 \left(\max_{t \in B(0, \delta)} \omega(t) \right)^2 \|K\|_w^2 \|K_R\|_w^2 \sigma^2.$$

Here, a, δ satisfy (3.5) for algorithm (3.3), (3.6) for algorithm (3.4), respectively.

Proof Note that $f_\infty = Rf_0$ and $f = R(f_0 - h_0)$, where $f_0 = \sum_{\gamma \in \Gamma} c_0(\gamma) \int_{\mathbf{R}^d} K(\cdot, y) \beta_\gamma(y) dy$ and $h_0 = \sum_{\gamma \in \Gamma} \varepsilon(\gamma) \int_{\mathbf{R}^d} K(\cdot, y) \beta_\gamma(y) dy$. Then we have

$$\begin{aligned} (f_\infty(x) - f(x))/\omega(x) &= Rh_0(x) \cdot \frac{1}{\omega(x)} \\ &= \sum_{\gamma \in \Gamma} \frac{\varepsilon(\gamma)}{\omega(\gamma)} R \left(\int_{\mathbf{R}^d} K(\cdot, y) \beta_\gamma(y) dy \right) (x) \frac{\omega(\gamma)}{\omega(x)}. \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{\gamma \in \Gamma} \left| R \left(\int_{\mathbf{R}^d} K(\cdot, y) \beta_\gamma(y) dy \right) (x) \right| \frac{\omega(\gamma)}{\omega(x)} \\ &= \sum_{\gamma \in \Gamma} \left| \int_{\mathbf{R}^d} K_R(x, z) \int_{\mathbf{R}^d} K(z, y) \beta_\gamma(y) dy dz \right| \frac{\omega(\gamma)}{\omega(x)} \\ &\leq C_\omega \int_{\mathbf{R}^{2d}} |K_R(x, z)| |K(z, y)| \beta_\gamma(y) \frac{\omega(\gamma - y) \omega(y)}{\omega(x)} dy dz \\ &\leq C_\omega \left(\max_{t \in B(0, \delta)} \omega(t) \right) \int_{\mathbf{R}^{2d}} |K_R(x, z)| |K(z, y)| \frac{\omega(y)}{\omega(x)} dy dz \\ &\leq C_\omega^3 \left(\max_{t \in B(0, \delta)} \omega(t) \right) \int_{\mathbf{R}^{2d}} |K_R(x, z)| |K(z, y)| \omega(x - z) \omega(z - y) dy dz \\ &\leq C_\omega^3 \left(\max_{t \in B(0, \delta)} \omega(t) \right) \|K\|_w \|K_R\|_w < \infty. \end{aligned} \tag{4.3}$$

Then we obtain

$$E((f_\infty(x) - f(x))/\omega(x)) = \sum_{\gamma \in \Gamma} E(\varepsilon(\gamma)/\omega(\gamma)) R \left(\int_{\mathbf{R}^d} K(\cdot, y) \beta_\gamma(y) dy \right) (x) \frac{\omega(\gamma)}{\omega(x)} = 0.$$

Moreover, for each $x \in \mathbb{R}^d$,

$$\begin{aligned}
 & \text{Var}\left((f_\infty(x) - f(x))/\omega(x)\right) \\
 &= E\left(\sum_{\gamma \in \Gamma} \frac{\varepsilon(\gamma)}{\omega(\gamma)} R\left(\int_{\mathbb{R}^d} K(\cdot, y) \beta_\gamma(y) dy\right)(x) \frac{\omega(\gamma)}{\omega(x)}\right)^2 \\
 &= \sum_{\gamma \in \Gamma} E(\varepsilon(\gamma)/\omega(\gamma))^2 \cdot \left(R\left(\int_{\mathbb{R}^d} K(\cdot, y) \beta_\gamma(y) dy\right)(x) \frac{\omega(\gamma)}{\omega(x)}\right)^2 \\
 &= \sigma^2 \sum_{\gamma \in \Gamma} \left|R\left(\int_{\mathbb{R}^d} K(\cdot, y) \beta_\gamma(y) dy\right)(x) \frac{\omega(\gamma)}{\omega(x)}\right| \\
 &\quad \times \left|R\left(\int_{\mathbb{R}^d} K(\cdot, y) \beta_\gamma(y) dy\right)(x) \frac{\omega(\gamma)}{\omega(x)}\right|. \tag{4.4}
 \end{aligned}$$

Note that

$$\left|R\left(\int_{\mathbb{R}^d} K(\cdot, y) \beta_\gamma(y) dy\right)(x) \frac{\omega(\gamma)}{\omega(x)}\right| \leq C_\omega^3 \left(\max_{t \in B(0, \delta)} \omega(t)\right) \|K\|_w \|K_R\|_w.$$

This together with (4.3) and (4.4) leads to

$$\text{Var}\left((f_\infty(x) - f(x))/\omega(x)\right) \leq C_\omega^6 \left(\max_{t \in B(0, \delta)} \omega(t)\right)^2 \|K\|_w^2 \|K_R\|_w^2 \sigma^2. \quad \square$$

5 Conclusion

In this paper, under a weaker assumption on the generator, we establish the asymptotic pointwise error estimates for reconstructing non-decay shift-invariant signals based on two kinds of average samples. Although we prove the convergence from a theoretical point of view, some numerical experiments are expected to be given for showing the effectiveness of the corresponding iterative reconstruction algorithms, which will be studied in the future work.

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Availability of data and materials

The authors declare that all data and materials in the paper are available and veritable.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

XF carried out the mathematical studies and drafted the manuscript. HZL and YT participated in the design of the study. All authors read and approved the final manuscript.

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