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Applications of new contraction mappings on existence and uniqueness results for implicit ϕ-Hilfer fractional pantograph differential equations

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Abstract

In this paper, we consider initial value problems for two different classes of implicit ϕ-Hilfer fractional pantograph differential equations. We use different approach that is based on $\alpha-\psi$-contraction mappings to demonstrate the existence and uniqueness of solutions for the proposed problems. The mappings are defined in appropriate cones of positive functions. The presented examples demonstrate the efficiency of the used method and the consistency of the proposed results.

Keywords: $\alpha-\psi$-Contraction mapping; ϕ-Hilfer fractional derivative; Fractional differential equation; Pantograph differential equation

1 Introduction

Calculus of arbitrary order integration and differentiation has achieved a remarkable growth over the last few decades due to its applications in a wide range of fields such as engineering, physics, neural networks, control theory, population dynamics, and epidemiology; see for instance [1–4].

In this context, there have appeared many definitions of fractional derivatives including the well-known types of Caputo, Riemann–Liouville, Hadamard, Katugampola derivatives and many others. Consequently, this has led to several problems defined by different fractional operators. However, it has been realized that the most efficient way to deal with such a variety of fractional operators is to accommodate generalized forms of fractional operators that include other operators. In [5], the Hilfer fractional derivative $D_{a^+}^{\upsilon,\iota}$ of order α and type ι was introduced. This definition provides a concatenation betwixt the Riemann–Liouville and Caputo fractional derivatives. The type-parameter ι allows some freedom of action in the initial conditions, which produces more kinds of stationary states. Some models based on this definition can be seen in the papers [6–8]. Meanwhile, for the sake of generalizing the definitions of fractional derivative, the fractional derivatives of a function relative to the other function defined as

$$D_{a^+}^{\upsilon;\phi} f(\varsigma) = \left(\frac{1}{\phi'(\varsigma)} \frac{d}{d\varsigma} \right)^n I_{a^+}^{n-\alpha;\phi} f(\varsigma) \quad \text{and} \quad {}^C D_{a^+}^{\upsilon;\phi} f(\varsigma) = I_{a^+}^{n-\alpha;\phi} \left(\frac{1}{\phi'(\varsigma)} \frac{d}{d\varsigma} \right)^n f(\varsigma)$$

in terms of the Riemann–Liouville and Caputo fractional derivatives were introduced, respectively [9, 10]. In this perspective, the ϕ-Hilfer derivative was defined in [11]. For more relevant applications concerning ϕ-Hilfer derivative, we refer the reader to [12–27].

On the one hand, the contraction mappings are widely utilized to examine of the existence and uniqueness of fixed points. For this purpose, many contraction mappings have been developed and used by interested researchers who applied these mappings in various disciplines. In [28], the authors presented $\alpha - \psi$-contraction mappings. This mapping and its extensions as well as some useful fixed point results can be seen in several papers [29–39].

On the other hand, the pantograph is a mechanical connection set in a manner based on parallelograms so that the movement of one pen, in tracing an image, produces identical movements in a second pen. In the study of the motion of the pantograph head on an electric locomotive, the authors of [40] came across the following delay differential equation:

$$y'(\varsigma) = ay(\varsigma) + by(\varepsilon\varsigma), \quad 0 < \varepsilon < 1, \varepsilon, a, b \in \mathrm{R},$$

which was referred to in the literature as pantograph differential equation. During the last decades, it has been realized that pantograph differential equation has many applications in various disciplines, see the paper [41] and the references cited therein.

Knowing the significant of fractional operators in modeling processes, the consideration of pantograph differential equation in fractional settings has come true. We review some relevant results for the sake of completeness. Nonlinear fractional pantograph differential equation was studied in [42, 43] where the existence of solutions is investigated using fractional calculus and fixed point theorems.

In the recent paper [44], the authors discussed the following:

$$\begin{cases} D_{a^+}^{\upsilon,\iota;\phi} x(\varsigma) = f(\varsigma, x(\varsigma), x(\varepsilon(\varsigma))), & 0 < \varepsilon < 1, \varsigma \in J = (a,b], a > 0, \\ I_{a^+}^{1-\mu;\phi} x(\varsigma)|_{\varsigma=a} = \sum_{i=1}^m c_i x(\vartheta_i), & \vartheta_i \in (a,b], \end{cases} \tag{1}$$

where $D_{a^+}^{\upsilon,\iota;\phi}$ is ϕ-Hilfer fractional derivative of order υ $(0 < \upsilon < 1)$ and type ι $(0 \le \iota \le 1)$, $I_{a^+}^{1-\mu;\phi}$ is ϕ-Riemann–Liouville fractional integral of order $1-\mu$ with respect to the continuous function ϕ such that $\phi'(\cdot) \neq 0$ and $\mu = \upsilon + \iota - \upsilon\iota$. The nonlinearity $f : J \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ is a given continuous function, ϑ_i, $i = 0, 1, \dots, m$, are prefixed points satisfying $a < \vartheta_1 \le \cdots \le \vartheta_m < b$, and c_i is real numbers.

The objective of this paper is to provide different approach to investigate the existence and uniqueness of solutions for Eq. (1). To do this, we use a new technique that is based on the application of $\alpha - \psi$-contraction mappings which are defined in appropriate cones of positive functions. Furthermore, we extend the proposed results to cover the following modified implicit ϕ-Hilfer pantograph fractional differential equation:

$$\begin{cases} D_{0^+}^{\upsilon,\iota;\phi} x(\varsigma) = g(\varsigma, x(\varsigma), x(\varepsilon\varsigma), D_{0^+}^{\upsilon,\iota;\phi} x(\varepsilon\varsigma)), & 0 < \varepsilon < 1, \varsigma \in J = (0,T], \\ I_{0^+}^{1-\mu;\phi} x(0^+) = \sum_{i=1}^m c_i I_{0^+}^{\iota;\phi} x(\vartheta_i), \end{cases} \tag{2}$$

which satisfies the same assumptions with $I_{0^+}^{\iota;\phi}$ is ϕ-Riemann–Liouville fractional integral of order $\iota > 0$ relative to the other function ϕ with $\phi'(\cdot) \neq 0$, $g : J \times \mathrm{R}^3 \to \mathrm{R}$ is continuous, $T > 0$ and $\vartheta_i \in J$ satisfying $0 < \vartheta_1 \leq \vartheta_2 \leq \cdots \leq \vartheta_m < T$ for $i = 1, 2, \ldots, m$. It is to be noted that the structure of the boundary conditions in Eq. (1) and Eq. (2) is visible and allows better interpretations to many physical problems [45]. Reported results in this paper yield new sufficient existence and uniqueness conditions via different approach comparing to the existing results in the literature [43, 44, 46, 47]. These conditions are easily attainable with the feature that they are less restrictive.

The rest of the paper adheres to the following plan: In Sect. 2, we define the norms, notations, and some spaces of functions. Properties of ϕ- Hilfer fractional derivative along with some necessary results on $\alpha - \psi$-contraction mappings are stated for completeness. In Sect. 3, the main results are stated and proved. The examples presented in Sect. 4 demonstrate the visibility and capability of the proposed results. We end the paper with a conclusion in Sect. 5.

2 Requisite preliminaries

Here we state some explanations which are needed throughout this paper. Further, some essential lemmas and theorems are stated as preparations for the main objectives.

Let $0 < a < T < \infty$, and $C[a, T]$ be a Banach space of continuous functions $y : [a, T] \to \mathrm{R}$ with the norm $\|y\| = \max\{|y(\varsigma)| : a \leq \varsigma \leq T\}$. The weighted space $C_{1-\mu;\phi}[a, T]$ of continuous functions y is defined in [11] as follows:

$$C_{1-\mu;\phi}[a, T] = \{y : (a, T] \to \mathrm{R}; [\phi(\varsigma) - \phi(a)]^{1-\mu} y(\varsigma) \in C[a, T]\}, \quad 0 \leq \mu < 1.$$

Obviously, $C_{1-\mu;\phi}[a, T]$ is a Banach space with the norm

$$\|y\|_{C_{1-\mu;\phi}} = \max_{\varsigma \in [a,T]} \left|[\phi(\varsigma) - \phi(a)]^{1-\mu} y(\varsigma)\right|.$$

Definition 2.1 ([11]) Let $\upsilon > 0$, $y \in L_1[a, b]$, and $\phi \in C^1[a, b]$ be an increasing function with $\phi'(\varsigma) \neq 0$, $\varsigma \in [a, b]$. Then the left-sided ϕ-Riemann–Liouville integral is stated by

$$I_{a^+}^{\upsilon,\phi} y(\varsigma) = \frac{1}{\Gamma(\upsilon)} \int_a^\varsigma \phi'(s)\big(\phi(\varsigma) - \phi(s)\big)^{\upsilon-1} y(s)\, ds,$$

with $\Gamma(\upsilon) = \int_0^\infty s^{\upsilon-1} e^{-s}\, ds$.

Definition 2.2 ([9]) Let $\upsilon \in (n - 1, n)$, $(n = [\upsilon] + 1)$, and $y, \phi \in C^n[a, b]$ (ϕ is increasing), and $\phi'(\varsigma) \neq 0$, $\varsigma \in [a, b]$. Then ϕ-Caputo derivative of a function y is defined by

$$D_{a^+}^{\upsilon,\phi} y(\varsigma) = \left(\frac{1}{\phi'(\varsigma)} \frac{d}{dt}\right)^n I_{a^+}^{n-\upsilon,\phi} y(\varsigma)$$

and

$${}^{C}D_{a^+}^{\upsilon,\phi} y(\varsigma) = I_{a^+}^{n-\upsilon,\phi} \left(\frac{1}{\phi'(\varsigma)} \frac{d}{dt}\right)^n y(\varsigma),$$

respectively.

Definition 2.3 ([11]) Let $\upsilon \in (n-1,n)$, $(n \in \mathrm{N})$, and $y,\phi \in C^n[a,T]$ be two functions (ϕ is increasing), and $\phi'(\varsigma) \neq 0$, $\varsigma \in [a,T]$. Then the left-sided ϕ-Hilfer derivative of y defined by

$$\begin{aligned} D_{a^+}^{\upsilon,\iota,\phi} y(\varsigma) &= I_{a^+}^{\iota(n-\upsilon);\phi} \left(\frac{1}{\phi'(\varsigma)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\iota)(n-\upsilon);\phi} y(\varsigma), \quad 0 \le \iota \le 1 \\ &= I_{a^+}^{\iota(n-\upsilon);\phi} D_{a^+}^{\mu;\phi} y(\varsigma), \quad (\mu = \upsilon + n\iota - \upsilon\iota). \end{aligned} \tag{3}$$

Lemma 2.4 ([2]) *For $\upsilon > 0$ and $0 \le \mu < 1$, $I_{a^+}^{\upsilon,\phi}$ is bounded from $C_{1-\mu;\phi}[a,b]$ into $C_{1-\mu;\phi}[a,b]$.*

Now, we introduce the following spaces:

$$C_{1-\mu;\phi}^{\upsilon,\iota}[a,T] = \left\{ y \in C_{1-\mu;\phi}[a,T], D_{a^+}^{\upsilon,\iota;\phi} y \in C_{1-\mu;\phi}[a,T] \right\}, \quad 0 \le \mu < 1,$$

and

$$C_{1-\mu;\phi}^{\mu}[a,T] = \left\{ y \in C_{1-\mu;\phi}[a,T], D_{a^+}^{\mu;\phi} y \in C_{1-\mu;\phi}[a,T] \right\}, \quad 0 \le \mu < 1. \tag{4}$$

Lemma 2.5 ([11]) *If $\mu = \upsilon + \iota - \upsilon\iota$, where $\upsilon \in (0,1)$, $\iota \in [0,1]$, and $y \in C_{1-\mu;\phi}^{\mu}[a,T]$, then the following identities hold:*

$$I_{a^+}^{\mu;\phi} D_{a^+}^{\mu;\phi} y = I_{a^+}^{\upsilon;\phi} D_{a^+}^{\upsilon,\iota;\phi} y$$

and

$$D_{a^+}^{\mu;\phi} I_{a^+}^{\upsilon;\phi} y = D_{a^+}^{\iota(1-\upsilon);\phi} y.$$

Lemma 2.6 ([11]) *Let $\upsilon > 0$, $0 \le \mu < 1$, and $y \in C_{1-\mu}[a,T]$, $\iota \in [0,1]$. Then we get*

$$D_{a^+}^{\upsilon,\iota,\phi} I_{a^+}^{\upsilon,\phi} y(\varsigma) = y(\varsigma).$$

Lemma 2.7 ([2]) *Let $\varsigma > a$. Then, for $\upsilon \ge 0$ and $\mu > 0$, we have*

$$I_{a^+}^{\upsilon,\phi} \left[\phi(\varsigma) - \phi(a)\right]^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\upsilon+\mu)} \left(\phi(\varsigma) - \phi(a)\right)^{\upsilon+\mu-1}, \quad \varsigma > a$$

and

$$D_{a^+}^{\upsilon,\phi} \left[\phi(\varsigma) - \phi(a)\right]^{\upsilon-1} = 0 \quad \text{for } \upsilon \in (0,1).$$

Lemma 2.8 ([11]) *Let $\mu = \upsilon + \iota - \upsilon\iota$, where $\upsilon \in (0,1)$, $\iota \in [0,1]$, $y \in C_{1-\mu;\phi}^{\mu}[a,T]$, and $I_{a^+}^{1-\mu;\phi} y \in C_{1-\mu,\phi}^{1}[a,T]$. Then we have*

$$I_{a^+}^{\mu;\phi} D_{a^+}^{\mu,\phi} y(\varsigma) = y(\varsigma) - \frac{I_{a^+}^{1-\mu;\phi} y(a)}{\Gamma(\mu)} \left(\phi(\varsigma) - \phi(a)\right)^{\mu-1}.$$

Lemma 2.9 ([11]) *Let $\upsilon > 0$, $0 \le \mu < \upsilon$, and $y \in C_{1-\mu,\phi}[a,T]$ $(0 < a < T < \infty)$. If $\mu < \upsilon$, then $I_{a^+}^{\upsilon;\phi} : C_{1-\mu,\phi}[a,T] \to C_{1-\mu,\phi}[a,T]$ is continuous, and the following holds*:

$$I_{a^+}^{\upsilon;\phi} y(a) = \lim_{\varsigma \to a^+} I_{a^+}^{\upsilon;\phi} y(\varsigma) = 0.$$

Definition 2.10 ([28]) Let Ψ be all functions $\psi : [0,\infty) \to [0,\infty)$ with:

P_1: ψ is nondecreasing;

P_2: let ψ^k be the kth iterate of ψ, then $\sum_{k=0}^{\infty} \psi^k(\varrho) < \infty$ for all $\varrho > 0$.

A function $\psi \in \Psi$ is called a (c)-comparison function.

Definition 2.11 ([28]) Let $T : \mathrm{M} \to \mathrm{M}$, T is called an $\alpha - \psi$-contraction if there exist a (c)-comparison $\psi \in \Psi$ and $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ with

$$\alpha(\nu,\omega) d(T\nu, T\omega) \le \psi\big(d(\nu,\omega)\big), \quad \forall \nu,\omega \in \mathrm{M}. \tag{5}$$

Definition 2.12 ([28]) Let $\mathrm{M} \ne \varnothing$, $T : \mathrm{M} \to \mathrm{M}$, and $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$. T is α-admissible if

$$\nu,\omega \in \mathrm{M}, \quad \alpha(\nu,\omega) \ge 1 \quad \Rightarrow \quad \alpha(T\nu, T\omega) \ge 1.$$

In the context, we use the following lemma.

Lemma 2.13 ([28]) *For any $\psi \in \Psi$*,

i) $\psi(\varrho) < \varrho$ *for* $\varrho > 0$;

ii) $\psi(0) = 0$;

iii) *at* $\varrho = 0$, ψ *is continuous.*

We define the following set Σ_ψ for any $\psi \in \Psi$:

$$\Sigma_\psi = \big\{\delta \in (0,\infty) : \delta\psi \in \Psi\big\}.$$

The following proposition is helpful.

Proposition 2.14 ([28]) *Let (M, d) be a metric space and $T : \mathrm{M} \to \mathrm{M}$ be an $\alpha - \psi$ contraction mapping where $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ and $\psi \in \Psi$. Suppose $\delta \in \sum_\psi$ with $\{\xi_i\}_{i=0}^{p} \subset \mathrm{M}$ such that*

$$\begin{aligned} &\xi_0 = \nu_0, \qquad \xi_p = T\nu_0, \qquad \alpha\big(T^n\xi_i, T^n\xi_{i+1}\big) \ge \delta^{-1}, \\ &i = 1,2,\ldots,p-1. \end{aligned} \tag{6}$$

Then $\{T^n\nu_0\}$ is Cauchy in (M, d).

Theorem 2.15 ([28]) *Let $T : \mathrm{M} \to \mathrm{M}$ $((\mathrm{M}, d)$ is a complete space$)$, T is an $\alpha - \psi$-contraction for $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$, and $\psi \in \Psi$. Moreover (6) is satisfied. Then $\{T^n\nu_0\}$ converges to some $\nu^* \in \mathrm{M}$. In addition, if there exists a subsequence $\{T^{\theta(n)}\nu_0\}$ of $\{T^n\nu_0\}$ with*

$$\lim_{n\to\infty} \alpha\big(T^{\theta(n)}\nu_0, \nu^*\big) = l \in (0,\infty),$$

then ν^* is a fixed point of T.

Theorem 2.16 ([28]) *Let $\phi : \mathrm{M} \to \mathrm{M}$ and ϕ be an $\alpha - \psi$ contraction mapping for $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ and $\psi \in \Psi$. Moreover, we have the following:*

i) *$Fix(T) \neq \emptyset$;*

ii) *Let $(\nu, \omega) \in Fix(T) \times Fix(T)$ be an arbitrary pair with $\nu \neq \omega$ and $\alpha(\nu, \omega) < 1$, then there exists $\eta \in \Sigma_\psi$, and for some positive integer q, there exists $\{\zeta_i(\nu,\omega)\}_{i=0}^{q} \subset \mathrm{M}$ such that*

$$\begin{aligned} &\zeta_0(\nu,\omega) = \nu, \qquad \zeta_q(\nu,\omega) = \omega, \qquad \alpha\bigl(\phi^n \zeta_i(\nu,\omega), \phi^n \zeta_{i+1}(\nu,\omega)\bigr) \geq \eta^{-1}, \quad n \in \mathrm{N} \\ &\text{and} \quad i = 1, 2, \ldots, q-1. \end{aligned} \tag{7}$$

Then ϕ has a unique fixed point.

3 Main results

Let $E = C(J, \mathrm{R})$. We define the cone $K \subset E$ by

$$K = \bigl\{x \in E | x(\varsigma) \geq 0, \varsigma \in (a,b]\bigr\}. \tag{8}$$

Let $\mathrm{M} = \{f : J \to \mathrm{R}; f \text{ is continuous}\}$. We endow M with a metric defined by

$$d(x,y) = \|x-y\|_\infty = \sup_{\varsigma \in (a,b]} \bigl\{\bigl|x(\varsigma) - y(\varsigma)\bigr|\bigr\}.$$

Theorem 3.1 ([21]) *Let $\mu = \upsilon + \iota - \upsilon\iota$, where $\upsilon \in (0,1)$ and $\iota \in [0,1]$. A function x is the solution of Eq. (1) if and only if it holds in the following:*

$$\begin{aligned} T_1 x(\varsigma) &:= x(\varsigma) \\ &= \frac{T(\phi(\varsigma) - \phi(a))^{\mu-1}}{\Gamma(\upsilon)} \\ &\quad \times \sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\bigl(\phi(\vartheta_i) - \phi(s)\bigr)^{\upsilon-1} f\bigl(s, x(s), x(\varepsilon s)\bigr)\, ds \\ &\quad + \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\bigl(\phi(\varsigma) - \phi(s)\bigr)^{\upsilon-1} f\bigl(s, x(s), x(\varepsilon s)\bigr)\, ds, \end{aligned} \tag{9}$$

where

$$T = \frac{1}{\Gamma(\mu) - \sum_{i=1}^{m} c_i(\phi(\vartheta_i) - \phi(a))^{\mu-1}}.$$

Theorem 3.2 *Suppose that the following conditions hold:*

(i) $\sum_{i=1}^{m} c_i(\phi(\vartheta_i) - \phi(a))^{\mu-1} \leq 1$;

(ii) The function $f : J \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}^+$ satisfies the following condition:

$$\bigl|f\bigl(s, x(s), x(\varepsilon s)\bigr) - f\bigl(s, y(s), y(\varepsilon s)\bigr)\bigr| \leq \frac{\|x-y\|_\infty \upsilon \Gamma(\upsilon)}{2(|T| N^{\mu-1} \sum_{i=1}^{m} c_i(\phi(\vartheta_i) - \phi(a))^{\upsilon} + N^{\upsilon})},$$

where $N = \sup\{\phi(\varsigma) - \phi(a)\}_{\varsigma \in (a,b]}$.

(*iii*) *For* $s \in J$, *we have*

$$x, y \in K, \quad x \leq y \quad \Rightarrow \quad f\big(s, x(s), x(\varepsilon s)\big) \leq f\big(s, y(s), y(\varepsilon s)\big).$$

(*iv*) $\exists\, x_0 \in M$ *such that* $x_0(s) \leq T_1 x_0(s)$, $s \in J$.
Then Eq. (1) *has a unique solution.*

Proof The results are stated in several steps.
Step 1. First we show $T : K \to K$. By condition (*i*), we have

$$\Gamma(\mu) - \sum_{i=1}^{m} c_i \big(\phi(\vartheta_i) - \phi(a)\big)^{\mu-1} \geq 0.$$

Thus, it is easy to see that $T : K \to K$.
Step 2. We demonstrate that T is an $\alpha - \psi$-contraction.
For this purpose, we can write

$$\begin{aligned}
&\big|T_1 x(\varsigma) - T_1 y(\varsigma)\big| \\
&\quad = \Bigg| \frac{T(\phi(\varsigma) - \phi(a))^{\mu-1}}{\Gamma(\upsilon)} \\
&\qquad \times \sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\big(\phi(\vartheta_i) - \phi(s)\big)^{\upsilon-1} f\big(s, x(s), x(\varepsilon s)\big)\, ds \\
&\qquad + \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\big(\phi(\varsigma) - \phi(s)\big)^{\upsilon-1} f\big(s, x(s), x(\varepsilon s)\big)\, ds \\
&\qquad - \frac{T(\phi(\varsigma) - \phi(a))^{\mu-1}}{\Gamma(\upsilon)} \sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\big(\phi(\vartheta_i) - \phi(s)\big)^{\upsilon-1} f\big(s, y(s), y(\varepsilon s)\big)\, ds \\
&\qquad - \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\big(\phi(\varsigma) - \phi(s)\big)^{\upsilon-1} f\big(s, y(s), y(\varepsilon s)\big)\, ds \Bigg| \\
&\quad \leq \frac{|T|(\phi(\varsigma) - \phi(a))^{\mu-1}}{\Gamma(\upsilon)} \\
&\qquad \times \sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\big(\phi(\vartheta_i) - \phi(s)\big)^{\upsilon-1} \big|f\big(s, x(s), x(\varepsilon s)\big) - f\big(s, y(s), y(\varepsilon s)\big)\big|\, ds \\
&\qquad + \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\big(\phi(\varsigma) - \phi(s)\big)^{\upsilon-1} \big|f\big(s, x(s), x(\varepsilon s)\big) - f\big(s, y(s), y(\varepsilon s)\big)\big|\, ds \\
&\quad \leq \left\{ \frac{\|x - y\|_\infty \upsilon \Gamma(\upsilon)}{2(|T| N^{\mu-1} \sum_{i=1}^{m} c_i (\phi(\vartheta_i) - \phi(a))^{\upsilon} + N^{\upsilon})} \right\} \Bigg[\frac{|T|(\phi(\varsigma) - \phi(a))^{\mu-1}}{\Gamma(\upsilon)} \\
&\qquad \times \sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\big(\phi(\vartheta_i) - \phi(s)\big)^{\upsilon-1}\, ds \\
&\qquad + \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\big(\phi(\varsigma) - \phi(s)\big)^{\upsilon-1}\, ds \Bigg] \\
&\quad = \frac{\|x - y\|_\infty}{2}.
\end{aligned}$$

Define $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x(\varsigma) \le y(\varsigma), \varsigma \in J, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then

$$\|T_1x - T_1y\|_\infty \le \frac{\|x-y\|_\infty}{2} \le \alpha(x,y)\frac{\|x-y\|_\infty}{2}.$$

Setting $\psi(\varsigma) = \frac{\varsigma}{2}$, we obtain

$$d(T_1x, T_1y) \le \alpha(x,y)\psi\bigl(d(x,y)\bigr),$$

hence T_1 is an $\alpha - \psi$-contraction.

Step 3. From (iv), we have $\alpha(x_0, T_1x_0) = 1$, hence T_1 is increasing. By induction, we easily obtain that $\alpha(T_1^n x_0, T_1^{n+1}x_0) = 1$, $n \in \mathrm{N}$.

Step 4. Using Theorem 2.14 and from Step (3), there exists a subsequence $\{T_1^{\theta(n)}x_0\}$ of $\{T_1^n x_0\}$ such that

$$\lim_{n\to\infty} \alpha\bigl(T_1^{\theta(n)}x_0, x^*\bigr) = l \in (0,\infty).$$

Step 5. By Theorem 2.15, x^* is a fixed point of T_1 and hence $x^* \in \mathrm{M}$ is a solution for (1).

Step 6. In order to show the uniqueness of the solution, we let $(u,v) \in Fix(T_1) \times Fix(T_1)$ be an arbitrary pair with $u \neq v$. Without loss generality, we set $u \le v$, then the conclusion is obvious. Therefore, by Theorem 2.16 the proof is complete. □

Next, we investigate the existence and uniqueness of the solutions for Eq. (2).

Theorem 3.3 ([43]) *Let $0 < \upsilon < 1$, $0 \le \iota \le 1$, and $q = \upsilon + \iota - \upsilon\iota$. Suppose $g \in C_{1-\mu;\phi}[J,\mathrm{R}]$ for any $z \in C_{1-\mu;\phi}[J,\mathrm{R}]$. If $z \in C^{\mu}_{1-\mu;\phi}[J,\mathrm{R}]$, then x satisfies problem (2) if and only if x satisfies the mixed-type integral equation*

$$\begin{aligned} T_1x(\varsigma) &:= x(\varsigma) \\ &= \left(\frac{T\Gamma(\rho+\mu)(\phi(\varsigma)-\phi(0))^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)}\right) \\ &\quad \times \left(\sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\bigl(\phi(\vartheta_i)-\phi(s)\bigr)^{\upsilon+\rho-1} g\bigl(\varsigma, x(\varsigma), x(\varepsilon\varsigma), {}^H D_{0^+}^{\upsilon,\iota;\phi} x(\varepsilon\varsigma)\bigr)\,ds\right) \\ &\quad + \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\bigl(\phi(\varsigma)-\phi(s)\bigr)^{\upsilon-1} g\bigl(\varsigma, x(\varsigma), x(\varepsilon\varsigma), {}^H D_{0^+}^{\upsilon,\iota;\phi} x(\varepsilon\varsigma)\bigr)\,ds, \end{aligned} \tag{11}$$

where

$$T = \frac{1}{\Gamma(\mu+\rho) - \sum_{i=1}^m c_i(\phi(\vartheta_i)-\phi(a))^{\rho+\mu-1}}$$

such that $\Gamma(\mu+\rho) \neq \sum_{i=1}^m c_i(\phi(\vartheta_i)-\phi(a))^{\rho+\mu-1}$.

Theorem 3.4 *Suppose that the following conditions hold*:

(*i*) $\sum_{i=1}^{m} c_i(\phi(\vartheta_i)-\phi(a))^{\rho+\mu-1} < \Gamma(\rho+\mu)$;

(*ii*) *The function* $g : J \times \mathrm{R}^3 \to \mathrm{R}^+$ *satisfies the following condition*:

$$\begin{aligned}&\bigl|g\bigl(s,x(s),x(\varepsilon s),{}^{H}D_{0^+}^{\upsilon,\iota;\phi}x(\varepsilon\varsigma)\bigr)-g\bigl(s,y(s),y(\varepsilon s),{}^{H}D_{0^+}^{\upsilon,\iota;\phi}y(\varepsilon\varsigma)\bigr)\bigr|\\&\quad\le\left\{\frac{2\|x-y\|_\infty}{3(\frac{T\Gamma(\rho+\mu)N^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)}\sum_{i=1}^{m}\frac{c_i}{\upsilon+\rho}(\phi(\vartheta_i)-\phi(a))^{\upsilon+\rho}+\frac{1}{\upsilon\Gamma(\upsilon)}N^{\upsilon})}\right\},\end{aligned}$$

where $N=\sup\{\phi(\varsigma)-\phi(a)\}_{\varsigma\in(a,b]}$.

(*iii*) *For* $s \in J$, *we have*

$$\begin{aligned}&x,y\in K,\quad x\le y\\&\quad\Rightarrow\quad g\bigl(s,x(s),x(\varepsilon s),{}^{H}D_{0^+}^{\upsilon,\iota;\phi}x(\varepsilon\varsigma)\bigr)\le g\bigl(s,y(s),y(\varepsilon s),{}^{H}D_{0^+}^{\upsilon,\iota;\phi}y(\varepsilon\varsigma)\bigr).\end{aligned}$$

(*iv*) $\exists\, x_0 \in M$ *such that* $x_0(s) \le T_1 x_0(s)$, $s \in J$.

Then problem (2) *has a unique solution*.

Proof The results are stated in several steps.

Step 1. First we show $T_1 : K \to K$. By condition (*i*), we have

$$\Gamma(\mu+\rho)-\sum_{i=1}^{m}c_i\bigl(\phi(\vartheta_i)-\phi(a)\bigr)^{\mu+\rho-1}>0.$$

Thus, $T_1 : K \to K$.

Step 2. We demonstrate that T_1 is an $\alpha-\psi$-contraction.

For this purpose, we can write

$$\begin{aligned}&\bigl|T_1z(\varsigma)-T_1y(\varsigma)\bigr|\\&\quad=\left|\frac{T\Gamma(\rho+\mu)(\phi(\varsigma)-\phi(0))^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)}\sum_{i=1}^{m}c_i\int_a^{\vartheta_i}\phi'(s)\bigl(\phi(\vartheta_i)-\phi(s)\bigr)^{\upsilon+\rho-1}T_{1z}(s)\,ds\right.\\&\qquad+\frac{1}{\Gamma(\upsilon)}\int_a^{\varsigma}\phi'(s)\bigl(\phi(\varsigma)-\phi(s)\bigr)^{\upsilon-1}T_{1z}(s)\,ds\\&\qquad-\frac{T\Gamma(\rho+\mu)(\phi(\varsigma)-\phi(0))^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)}\sum_{i=1}^{m}c_i\int_a^{\vartheta_i}\phi'(s)\bigl(\phi(\vartheta_i)-\phi(s)\bigr)^{\upsilon+\rho-1}T_{1y}(s)\,ds\\&\qquad\left.-\frac{1}{\Gamma(\upsilon)}\int_a^{\varsigma}\phi'(s)\bigl(\phi(\varsigma)-\phi(s)\bigr)^{\upsilon-1}T_{1y}(s)\,ds\right|\\&\quad\le\frac{T\Gamma(\rho+\mu)(\phi(\varsigma)-\phi(0))^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)}\sum_{i=1}^{m}c_i\int_a^{\vartheta_i}\phi'(s)\bigl(\phi(\vartheta_i)-\phi(s)\bigr)^{\upsilon+\rho-1}\bigl|T_{1z}(s)-T_{1y}(s)\bigr|\,ds\\&\qquad+\frac{1}{\Gamma(\upsilon)}\int_a^{\varsigma}\phi'(s)\bigl(\phi(\varsigma)-\phi(s)\bigr)^{\upsilon-1}\bigl|T_{1z}(s)-T_{1y}(s)\bigr|\,ds\\&\quad\le\left\{\frac{2\|x-y\|_\infty}{3(\frac{T\Gamma(\rho+\mu)N^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)}\sum_{i=1}^{m}\frac{c_i}{\upsilon+\rho}(\phi(\vartheta_i)-\phi(a))^{\upsilon+\rho}+\frac{1}{\upsilon\Gamma(\upsilon)}N^{\upsilon})}\right\}\end{aligned}$$

$$\times \left[\frac{T\Gamma(\rho+\mu)(\phi(\varsigma)-\phi(0))^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)} \right.$$
$$\left. \times \sum_{i=1}^{m} c_i \int_a^{\vartheta_i} \phi'(s)\big(\phi(\vartheta_i)-\phi(s)\big)^{\upsilon+\rho-1}\,ds + \frac{1}{\Gamma(\upsilon)} \int_a^{\varsigma} \phi'(s)\big(\phi(\varsigma)-\phi(s)\big)^{\upsilon-1}\,ds \right]$$
$$\leq \left\{ \frac{2\|x-y\|_\infty}{3(\frac{T\Gamma(\rho+\mu)N^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)} \sum_{i=1}^{m} \frac{c_i}{\upsilon+\rho}(\phi(\vartheta_i)-\phi(a))^{\upsilon+\rho} + \frac{1}{\upsilon\Gamma(\upsilon)}N^{\upsilon})} \right\} \left[\frac{T\Gamma(\rho+\mu)N^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)} \right.$$
$$\left. \times \sum_{i=1}^{m} \frac{c_i}{\upsilon+\rho}\big(\phi(\vartheta_i)-\phi(a)\big)^{\upsilon+\rho} + \frac{1}{\upsilon\Gamma(\upsilon)}N^{\upsilon} \right]$$
$$= \frac{2\|x-y\|_\infty}{3}.$$

Define $\alpha : \mathrm{M} \times \mathrm{M} \to \mathrm{R}$ by the following:

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x(\varsigma) \leq y(\varsigma), \varsigma \in J, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Then

$$\|T_1x - T_1y\|_\infty \leq \frac{2\|x-y\|_\infty}{3} \leq \alpha(x,y)\frac{2\|x-y\|_\infty}{3}.$$

Setting $\psi(\varsigma) = \frac{2\varsigma}{3}$, we obtain

$$d(T_1x, T_1y) \leq \alpha(x,y)\psi\big(d(x,y)\big).$$

Hence T_1 is an $\alpha-\psi$-contraction.

Step 3. By (iv), $\alpha(x_0, T_1x_0) = 1$. Further, as T_1 is increasing, we can easily get $\alpha(T_1^n x_0, T_1^{n+1}x_0) = 1$, $n \in \mathrm{N}$.

Step 4. From step (3), there exists a subsequence $\{T_1^{\theta(n)}x_0\}$ of $\{T_1^n x_0\}$ with

$$\lim_{n\to\infty} \alpha\big(T_1^{\theta(n)}x_0, x^*\big) = l \in (0,\infty).$$

Step 5. By applying Theorem 2.15, we conclude that x^* is a fixed point of T_1, that is, $x^* \in \mathrm{M}$ is a solution to equation (2).

Step 6. To prove the uniqueness, we let $(u,v) \in Fix(T_1) \times Fix(T_1)$ be an arbitrary pair with $u \neq v$. Without loss generality, we set $u \leq v$, then the result is concluded. By Theorem 2.16 the proof is perfect. □

4 Application

Here, we present two examples concerning Theorem 3.2 and Theorem 3.4. The validity of the examples demonstrates the applicability of the proposed results. For our purpose, we translate the pantograph fractional differential equations to an integral equation. Due to the complexity of the obtained integral equation, however, it is difficult to use other fixed point theorems in the study of the existence and uniqueness. Hence, our technique proves to be helpful to overcome this limitation.

Example 4.1 Consider the following terminal value problem:

$$\begin{cases} D_{1^+}^{\frac{1}{2},0;e^\varsigma} y(\varsigma) = \frac{8\sqrt{\pi}}{29e^{-\varsigma+2}}(1 + |y(\varsigma)| + |y(\varepsilon\varsigma)|), & 0 < \varepsilon < 1, \varsigma \in (1,2], \\ I_{1^+}^{\frac{1}{2};e^\varsigma}|_{\varsigma=1} = \sum_{i=1}^m c_i x(\vartheta_i), & \vartheta_i \in (1,2]. \end{cases} \tag{13}$$

Setting $f(\varsigma, u, v) = \frac{8\sqrt{\pi}}{29e^{-\varsigma+2}}(1 + |u| + |v|)$ for each $u, v \in \mathrm{R}$, $\varsigma \in (1,2]$, and

$$C_{1-\mu;\psi}^{\iota(1-\upsilon)}[1,2] = C_{\frac{1}{2};e^\varsigma}^0[1,2] = \{f : (1,2] \times \mathrm{R}^2 \to \mathrm{R}; (e^\varsigma - e)^{\frac{1}{2}} f \in C[1,2]\}.$$

Letting $\vartheta_1 = \frac{5}{4}$, $\vartheta_2 = \frac{3}{2}$, $\vartheta_3 = \frac{7}{4}$, $\vartheta_4 = 2$ and

$$c_1 = \frac{\sqrt{e^{\frac{5}{4}} - e}}{4}, \qquad c_2 = \frac{\sqrt{e^{\frac{3}{2}} - e}}{4}, \qquad c_3 = \frac{\sqrt{e^{\frac{7}{4}} - e}}{4}, \qquad c_4 = \frac{\sqrt{e^2 - e}}{4},$$

we can obtain $\sum_{i=1}^4 c_i(\phi(\vartheta_i) - \phi(1))^{\frac{-1}{2}} \leq 1$. Hence condition (i) from Theorem 3.2 is satisfied. On the other hand,

$$\begin{aligned} \bigl|f(s, x(s), x(\varepsilon s)) - f(s, y(s), y(\varepsilon s))\bigr| &= \frac{8\sqrt{\pi}}{29e^{-s+2}} |(|x(s)| - |y(s)| + |x(\varepsilon s)| - |y(\varepsilon s)|) \\ &\leq \frac{\sqrt{\pi}\,\|x - y\|_\infty}{2(|T|(e^2 - e)^{\frac{-1}{2}} \sum_{i=1}^4 c_i \sqrt{e^{\vartheta_i} - e} + \sqrt{e^2 - e})} \\ &\leq \frac{\|x - y\|_\infty \upsilon \Gamma(\upsilon)}{2(|T| N^{\mu-1} \sum_{i=1}^m c_i(\phi(\vartheta_i) - \phi(a))^\upsilon + N^\upsilon)}, \end{aligned}$$

where $N = \sup\{\phi(\varsigma) - \phi(1)\}_{\varsigma \in (1,2]} = \sup\{e^\varsigma - e\}_{\varsigma \in (1,2]} = e^2 - e$ and $\|x - y\|_\infty = \sup\{|x(\varsigma) - y(\varsigma)|\}_{\varsigma \in (0,2]}$. Therefore condition (ii) of Theorem 3.2 holds. Further, it is easy to show that item (iii) in Theorem 3.2 is true. Now, since $T_1(0) \geq 0$, so by choosing $x_0 = 0$, we can establish condition (iv) from Theorem 3.2. Therefore, all requirements of Theorem 3.2 hold which guarantee the existence and uniqueness of solution for (13).

Example 4.2 Consider the following terminal value problem:

$$\begin{cases} D_{1^+}^{\frac{1}{2},0;e^\varsigma} y(\varsigma) = \frac{2\sqrt{\pi(e^2-e)}}{9(e^{\frac{5}{7}} + e^{\frac{1}{3}} + e^{\frac{4}{7}} + 2e^2 - 5e)}(1 + |y(\varsigma)| + |y(\varepsilon\varsigma)| + |D_{1^+}^{\frac{1}{2},0;e^\varsigma} y(\varepsilon\varsigma)|), \\ I_{1^+}^{\frac{1}{2};e^\varsigma}|_{\varsigma=1} = \sum_{i=1}^m c_i x(\vartheta_i), \quad \vartheta_i \in (1,2], \end{cases} \tag{14}$$

where $0 < \varepsilon < 1$ and $\varsigma \in (1,2]$.

Let us find solution of the problem from the following cone:

$$K = \{x, y \in C(J, \mathrm{R}) : \|D_{1^+}^{\frac{1}{2},0;e^\varsigma} x - D_{1^+}^{\frac{1}{2},0;e^\varsigma} y\|_\infty \leq \|x - y\|_\infty, x \leq y \Rightarrow D_{1^+}^{\frac{1}{2},0;e^\varsigma} x \leq D_{1^+}^{\frac{1}{2},0;e^\varsigma} y\}.$$

Setting $g(\varsigma, u, v, w) = \frac{2\sqrt{\pi(e^2-e)}(1+|u|+|v|+|w|)}{9(e^{\frac{5}{7}} + e^{\frac{1}{3}} + e^{\frac{4}{7}} + 2e^2 - 5e)}$ for each $u, v, w \in \mathrm{R}$, $\varsigma \in (1,2]$, and

$$C_{1-\mu;\psi}^{\iota(1-\upsilon)}[1,2] = C_{\frac{1}{2};e^\varsigma}^0[1,2] = \{g : (1,2] \times \mathrm{R}^3 \to \mathrm{R}; (e^\varsigma - e)^{\frac{1}{2}} g \in C[1,2]\}.$$

Letting $\rho = \frac{1}{2}$, $\vartheta_1 = \frac{5}{4}$, $\vartheta_2 = \frac{3}{2}$, $\vartheta_3 = \frac{7}{4}$, $\vartheta_4 = 2$, and $c_1 = c_2 = c_3 = c_4 = \frac{1}{5}$, we can obtain

$$\sum_{i=1}^{m} c_i\big(\phi(\vartheta_i) - \phi(a)\big)^{\rho+\mu-1} < \Gamma(\rho+\mu).$$

Hence condition (i) from Theorem 3.4 is satisfied. On the other hand,

$$\begin{aligned}
&\big|g\big(s, x(s), x(\varepsilon s), D_{1^+}^{\frac{1}{2},0;e^\varsigma} x(\varepsilon s)\big) - g\big(s, y(s), y(\varepsilon s), D_{1^+}^{\frac{1}{2},0;e^\varsigma} y(\varepsilon s)\big)\big| \\
&\quad = \big|\big|x(s)\big| - \big|y(s)\big| + \big|x(\varepsilon s)\big| - \big|y(\varepsilon s)\big| + \big|D_{1^+}^{\frac{1}{2},0;e^\varsigma} x(\varepsilon\varsigma)\big| - \big|D_{1^+}^{\frac{1}{2},0;e^\varsigma} y(\varepsilon\varsigma)\big|\big| \\
&\quad \le \frac{(2\|x-y\|_\infty + \|D_{1^+}^{\frac{1}{2},0;e^\varsigma} x - D_{1^+}^{\frac{1}{2},0;e^\varsigma} y\|_\infty) 2\sqrt{\pi(e^2-e)}}{9(e^{\frac{5}{7}} + e^{\frac{1}{3}} + e^{\frac{4}{7}} + 2e^2 - 5e)} \\
&\quad \le \frac{2\|x-y\|_\infty \sqrt{\pi(e^2-e)}}{3(e^{\frac{5}{7}} + e^{\frac{1}{3}} + e^{\frac{4}{7}} + 2e^2 - 5e)} \\
&\quad = \frac{2\|x-y\|_\infty}{3(\frac{T\Gamma(\rho+\mu)N^{\mu-1}}{\Gamma(\mu)\Gamma(\rho+\upsilon)} \sum_{i=1}^{m} \frac{c_i}{\upsilon+\rho}(\phi(\vartheta_i) - \phi(a))^{\upsilon+\rho} + \frac{1}{\upsilon\Gamma(\upsilon)} N^\upsilon)}.
\end{aligned}$$

where $N = \sup\{\phi(\varsigma) - \phi(1)\}_{\varsigma\in(1,2]} = \sup\{e^\varsigma - e\}_{\varsigma\in(1,2]} = e^2 - e$ and $\|x-y\|_\infty = \sup\{|x(\varsigma) - y(\varsigma)|\}_{\varsigma\in(0,2]}$. Therefore condition (ii) of Theorem 3.4 holds. Further, it is easy to show that item (iii) in Theorem 3.4 is true. Now since $T_1(0) \ge 0$, so by choosing $x_0 = 0$, we can establish condition (iv) from 3.4. Therefore, all requirements of 3.4 hold which guarantee the existence and uniqueness of solution for (14).

5 Conclusion

The fractional derivatives involving ϕ- Hilfer of a function relative to the other function have widely been used due to their tremendous applications in modeling of various phenomena.

Here, we investigated two initial value problems of implicit ϕ-Hilfer fractional pantograph differential equations. Unlike the methods used in the literature which were based on classical fixed point theorems, we utilized the $\alpha - \psi$-contractions to demonstrate the existence and uniqueness of solutions for the proposed problems. The mappings are defined in appropriate cones of positive functions. In spite of the complex structure of ϕ-Hilfer fractional derivative, which causes some limitations, we proposed different techniques that produced sufficient existence and conditions which are more appropriate than the existing conditions. For the sake of confirmation, we constructed particular examples corresponding to the main theorems that illustrate the applicability of the mentioned assumptions.

Results of this paper provide a new technique that associates the study of theory of contraction mappings with the theory fractional differential equations. We believe that the contents of this paper will be of great significance for enthusiasts in these two theories. In this context, many promising topics could be discussed in the future such as inclusion boundary value problems involving ϕ-Hilfer fractional pantograph differential equations and system of fractional pantograph differential equations by using common fixed point theorems of some contractions.

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