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On the multiplicity of solutions for a kind of fourth-order equation depending on two real parameters

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Abstract

In this paper, by suitable assumptions on nonlinear boundary term, we establish the existence of three distinct weak solutions for a kind of fourth-order boundary value problem depending on two parameters.

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1 Introduction

In this paper, we consider the following fourth-order problem:

$$\begin{cases} u^{(iv)}(x) = \lambda f(x, u(x)), & \text{in } [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = \mu g(u(1)), \end{cases} \tag{1.1}$$

where $\lambda, \mu \in]0, \infty[$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Caratéodory function.

Problem (1.1) describes the static equilibrium of a flexible elastic beam of length 1 when, along its length, a load f is added to cause deformation. Precisely, conditions $u(0) = u'(0) = 0$ mean that the left end of the beam is fixed and conditions $u''(1) = 0$, $u'''(1) = \mu g(u(1))$ mean that the right end of the beam is attached to a bearing device, given by the function g.

Existence and multiplicity of solutions for fourth-order boundary value problems have been discussed by several authors in the last decades; see for example [1, 4, 5, 9–13, 16, 17, 19–21] and the references therein.

In particular, Yang *et al.* [21] used Ricceri's variational principle [18] to establish the existence of at least two classical solutions generated from g for problem (1.1) with $\mu = 1$.

The authors in [9], using a multiplicity result by Cabada and Iannizzotto [8], ensured the existence of at least two nontrivial classical solutions for the problem

$$\begin{cases} u^{(4)}(x) + \lambda f(x, u(x)) = 0, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = \lambda g(u(1)), \end{cases}$$

where the functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and $\lambda \geq 0$ is a real parameter.

Bonanno *et al.* [4], by means of an abstract critical points result of Bonanno [2], studied the existence of at least one nonzero classical solution for problem (1.1).

In [12], by using a smooth version of [7, Theorem 2.1], Heidarkhani *et al.* established the existence of infinitely many generalized solutions for the following perturbed fourth-order problem:

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)) + \mu g(t, u(t)) + p(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = h(u(1)), \end{cases}$$

where $\lambda > 0$, $\mu \geq 0$ are two parameters, f, g are two L^2-Caratéodory functions, and p, h are Lipschitz continuous functions such that $p(0) = h(0) = 0$.

Also in [11], the present authors obtained sufficient conditions to guarantee that problem (1.1) has infinitely many classical solutions.

More recently, Heidarkhani and Gharehgazlouei [13], using an immediate consequence of [3, Theorem 3.3], ensured the existence of at least three generalized solutions for the problem

$$\begin{cases} u^{(iv)}(x) = \lambda f(x, u(x)) + h(u(x)), & \text{in } [0,1], \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = \mu g(u(1)), \end{cases}$$

where $\lambda > 0$, $\mu \geq 0$ are two parameters, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Caratéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $h(0) = 0$.

Motivated by the above works, the aim of the present paper is to offer the existence of three solutions for fourth-order problem (1.1) by using two kinds of critical point theorems obtained in [3, 6].

For completeness, we cite the recent and nice works [14, 15] as general references on the subject treated in this paper.

2 Abstract setting

In order to study problem (1.1), the variational setting is the space

$$X := \{u \in H^2([0,1]) : u(0) = u'(0) = 0\},$$

where $H^2([0,1])$ is the Sobolev space of all function $u:[0,1]\to\mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0,1])$. X is a Hilbert space with the inner product

$$\langle u,v\rangle := \int_0^1 u''(t)v''(t)\,dt$$

and the corresponding norm

$$\|u\| := \left(\int_0^1 \big(u''(t)\big)^2\,dt\right)^{\frac{1}{2}}.$$

We observe that the norm $\|\cdot\|$ on X is equivalent to the usual norm

$$\int_0^1 \left(\left|u(t)\right|^2 + \left|u'(t)\right|^2 + \left|u''(t)\right|^2\right)dt.$$

It is well known that the embedding $X \hookrightarrow C^1([0,1])$ is compact and

$$\|u\|_{C^1([0,1])} := \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty\right\} \le \|u\| \tag{2.1}$$

for all $u\in X$ (see [21]).

We say that $u\in X$ is a *weak solution* of problem (1.1) whenever

$$\int_0^1 u''(x)v''(x)\,dx - \lambda\int_0^1 f\big(x,u(x)\big)v(x)\,dx + \mu g\big(u(1)\big)v(1) = 0$$

for all $v\in X$. By a *classical solution* of problem (1.1) we mean a function $u\in C^1([0,1])$ such that $u^{(iv)}(x)\in C([0,1])$ and the boundary conditions and the equation are satisfied in $[0,1]$.

Our main tools are critical point theorems that we recall here in a convenient form. The first result has been obtained in [6], and it is a more precise version of Theorem 3.2 of [3]. The second one has been established in [3].

Lemma 2.1 ([6, Theorem 3.6]) *Let X be a reflexive real Banach space; $\Phi: X\to\mathbb{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi: X\to\mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0)=\Psi(0)=0.$$

Assume that there exist $r>0$ and $\overline{x}\in X$, with $r<\Phi(\overline{x})$, such that

(a$_1$) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a$_2$) *for each $\lambda\in\Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[$, the functional $\Phi-\lambda\Psi$ is coercive.*

Then, for each $\lambda\in\Lambda_r$, the functional $\Phi-\lambda\Psi$ has at least three distinct critical points in X.

Lemma 2.2 ([3, Theorem 3.3]) *Let X be a reflexive real Banach space; $\Phi: X\to\mathbb{R}$ be a convex, coercive, and continuously Gâteaux differentiable functional whose derivative admits* *a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{x} \in X$, with $2r_1 < \Phi(\overline{x}) < \frac{r_2}{2}$, such that

(b$_1$) $\frac{\sup_{\Phi(x)<r_1} \Psi(x)}{r_1} < \frac{2}{3} \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;
(b$_2$) $\frac{\sup_{\Phi(x)<r_2} \Psi(x)}{r_2} < \frac{1}{3} \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;
(b$_3$) *for each λ in*

$$\Lambda' := \left]\frac{3}{2}\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \min\left\{\frac{r_1}{\sup_{\Phi(x)<r_1} \Psi(x)}, \frac{\frac{r_2}{2}}{\sup_{\Phi(x)<r_2} \Psi(x)}\right\}\right[$$

and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has $\inf_{s\in[0,1]} \Psi(sx_1 + (1-s)x_2) \geq 0$.

Then, for each $\lambda \in \Lambda'$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

We use the following notations:

Corresponding to f, g, we introduce the functions F, G as follows:

$$F(x,\xi) := \int_0^\xi f(x,t)\,dt, \qquad G(\xi) := -\int_0^\xi g(t)\,dt$$

for all $x \in [0,1]$ and $\xi \in \mathbb{R}$. Also, for each θ and η of positive real numbers, define

$$F^\theta = \int_0^1 \sup_{|\xi|\leq\theta} F(x,\xi)\,dx, \qquad G^\theta = \sup_{|\xi|\leq\theta} G(\xi), \qquad G_\eta = \inf_{|\xi|\leq\eta} G(\xi).$$

3 Main results

In this section, we present our main result on the existence of at least three weak solutions for problem (1.1).

In order to introduce our result, we fix $\theta, \eta > 0$ such that

$$\frac{32\pi^4\eta^2}{27\int_{\frac{3}{4}}^1 F(x,\eta)\,dx} < \frac{\theta^2}{2F^\theta},$$

and pick

$$\Lambda := \left]\frac{32\pi^4\eta^2}{27\int_{\frac{3}{4}}^1 F(x,\eta)\,dx}, \frac{\theta^2}{2F^\theta}\right[. \tag{3.1}$$

Set

$$\delta := \min\left\{\frac{\theta^2 - 2\lambda F^\theta}{2G^\theta}, \frac{\frac{32}{27}\pi^4\eta^2 - \lambda\int_{\frac{3}{4}}^1 F(x,\eta)\,dx}{G(\eta)}\right\} \tag{3.2}$$

and

$$\overline{\delta} := \min\left\{\delta, \frac{1}{\max\{0, 4\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{\xi^2}\}}\right\}, \tag{3.3}$$

where we read $r/0 = +\infty$. For instance, $\overline{\delta} = +\infty$ when $\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{\xi^2} \le 0$ and $G(\eta) = G^\theta = 0$.

With the above notations we are able to prove the following multiplicity property.

Theorem 3.1 *Assume that there exist two positive constants θ, η, with $\theta < \frac{8}{3\sqrt{3}}\pi^2\eta$, such that*

(A1) $F(x,t) \ge 0$ *for each* $(x,t) \in [\frac{3}{8}, \frac{3}{4}] \times [0,\eta]$;

(A2) $\frac{F^\theta}{\theta^2} < \frac{27}{64\pi^4}\frac{\int_{\frac{3}{8}}^{1} F(x,\eta)\,dx}{\eta^2}$;

(A3) $\limsup_{|\xi|\to+\infty}\frac{\sup_{x\in[0,1]}F(x,\xi)}{\xi^2} < \frac{F^\theta}{2\theta^2}$.

Then, for every $\lambda \in \Lambda$, where Λ is given by (3.1), *and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{\xi^2} < +\infty,$$

there exists $\overline{\delta} > 0$ given by (3.3) *such that, for each $\mu \in [0, \overline{\delta}[$, problem* (1.1) *admits at least three distinct weak solutions.*

Proof Our aim is to apply Lemma 2.1 to problem (1.1). To this end, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ as follows:

$$\Phi(u) := \frac{1}{2}\|u\|^2,$$
$$\Psi(u) := \int_0^1 F\big(x, u(x)\big)\,dx + \frac{\mu}{\lambda}G\big(u(1)\big).$$

It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 f\big(x, u(x)\big)v(x)\,dx - \frac{\mu}{\lambda}g\big(u(1)\big)v(1)$$

for every $v \in X$. Moreover, in [21], the authors proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator. Furthermore, by standard arguments, Φ is coercive and continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)\,dx$$

for each $v \in X$. Also, in [21] it is proved that Φ' admits a continuous inverse on X^*. Moreover, Φ is sequentially weakly lower semicontinuous. One can show that the weak solutions of problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$.

Fix $\lambda \in \Lambda$ and put $r = \frac{\theta^2}{2}$. Then, for $u \in X$ with $\Phi(u) \leq r$,

$$\begin{aligned}\sup_{\Phi(u)\leq r} \Psi(u) &= \sup_{\Phi(u)\leq r} \left(\int_0^1 F\big(x, u(x)\big)\, dx + \frac{\mu}{\lambda} G\big(u(1)\big) \right) \\ &\leq \int_0^1 \sup_{|t|\leq\theta} F(x,t)\, dx + \frac{\mu}{\lambda} G^\theta = F^\theta + \frac{\mu}{\lambda} G^\theta.\end{aligned}$$

Therefore,

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq \frac{2F^\theta}{\theta^2} + \frac{2\mu}{\lambda}\frac{G^\theta}{\theta^2}.$$

From this, if $G^\theta = 0$, it is clear that

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{1}{\lambda}, \tag{3.4}$$

while if $G^\theta > 0$, it turns out to be true bearing in mind that

$$\mu < \frac{\theta^2 - 2\lambda F^\theta}{2G^\theta}.$$

Denote by w the function of X defined by

$$w(x) := \begin{cases} 0, & x \in [0, \frac{3}{8}], \\ \eta \cos^2(\frac{4\pi x}{3}), & x \in]\frac{3}{8}, \frac{3}{4}[, \\ \eta, & x \in [\frac{3}{4}, 1]. \end{cases} \tag{3.5}$$

It is easy to see that

$$\Phi(w) = \frac{32}{27}\pi^4\eta^2.$$

Therefore, since $\theta < \frac{8}{3\sqrt{3}}\pi^2\eta$, one has $\Phi(w) > r$.

From assumption (A1) we obtain

$$\Psi(w) = \int_0^1 F\big(x, w(x)\big)\, dx + \frac{\mu}{\lambda} G\big(w(1)\big) \geq \int_{\frac{3}{4}}^1 F(x,\eta)\, dx + \frac{\mu}{\lambda} G(\eta).$$

So, we have

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_{\frac{3}{4}}^1 F(x,\eta)\, dx + \frac{\mu}{\lambda} G(\eta)}{\frac{32}{27}\pi^4\eta^2}.$$

Hence, if $G(\eta) \geq 0$, we find

$$\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda}, \tag{3.6}$$

while if $G(\eta) < 0$, the same relation holds since

$$\mu G(\eta) > \frac{32}{27}\pi^4\eta^2 - \lambda \int_{\frac{3}{4}}^{1} F(x,\eta)\,dx.$$

Now, taking into account (3.4) and (3.6) results in

$$\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda} > \frac{\sup_{\Phi(u)\le r}\Psi(u)}{r},$$

and condition (a_1) of Lemma 2.1 is verified.

Now, in order to prove the coercivity of the functional $\Phi - \lambda\Psi$, first we assume that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} > 0.$$

Therefore, fix

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} < \varepsilon < \frac{F^\theta}{2\theta^2},$$

from (A3) there is a function $h_\varepsilon \in L^1([0,1])$ such that

$$F(x,\xi) \le \varepsilon\xi^2 + h_\varepsilon(x)$$

for each $x \in [0,1]$ and $\xi \in \mathbb{R}$. Taking (2.1) into account and since $\lambda < \frac{\theta^2}{2F^\theta}$, it follows that

$$\begin{aligned}
\lambda \int_0^1 F\bigl(x,u(x)\bigr)\,dx &\le \lambda\left(\varepsilon \int_0^1 \bigl(u(x)\bigr)^2\,dx + \int_0^1 h_\varepsilon(x)\,dx\right) \\
&< \frac{\theta^2}{2F^\theta}\left(\varepsilon \int_0^1 \bigl(u(x)\bigr)^2\,dx + \int_0^1 h_\varepsilon(x)\,dx\right) \\
&\le \frac{\theta^2}{2F^\theta}\bigl(\varepsilon\|u\|^2 + \|h_\varepsilon\|_{L^1([0,1])}\bigr)
\end{aligned} \tag{3.7}$$

for each $u \in X$. Moreover, since $\mu < \overline{\delta}$, we obtain

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{\xi^2} < \frac{1}{4\mu}.$$

Thus, there is a positive constant h_μ such that

$$G(\xi) \le \frac{1}{4\mu}\xi^2 + h_\mu$$

for each $\xi \in \mathbb{R}$. Thus, taking again (2.1) into account, it follows that

$$\begin{aligned}
G\bigl(u(1)\bigr) &\le \frac{1}{4\mu}\bigl(u(1)\bigr)^2 + h_\mu \\
&\le \frac{1}{4\mu}\|u\|^2 + h_\mu
\end{aligned} \tag{3.8}$$

for each $u \in X$. Finally, putting together (3.7) and (3.8), we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\theta^2}{2F^\theta}\left(\varepsilon\|u\|^2 + \|h_\varepsilon\|_{L^1([0,1])}\right) - \frac{1}{4}\|u\|^2 - \mu h_\mu \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{\theta^2}{F^\theta}\varepsilon\right)\|u\|^2 - \frac{\theta^2\|h_\varepsilon\|_{L^1([0,1])}}{2F^\theta} - \mu h_\mu.\end{aligned}$$

On the other hand, if

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} \leq 0,$$

there exists a function $h_\varepsilon \in L^1([0,1])$ such that $F(x,\xi) \leq h_\varepsilon(x)$ for each $x \in [0,1]$ and $\xi \in \mathbb{R}$, and arguing as before we obtain

$$\Phi(u) - \lambda\Psi(u) \geq \frac{1}{4}\|u\|^2 - \frac{\theta^2\|h_\varepsilon\|_{L^1([0,1])}}{2F^\theta} - \mu h_\mu.$$

Both cases lead to the coercivity of $\Phi - \lambda\Psi$, and condition (a_2) of Lemma 2.1 is verified.

Since from (3.4) and (3.6)

$$\lambda \in \Lambda \subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[,$$

Lemma 2.1 ensures the existence of at least three critical points for the functional $\Phi - \lambda\Psi$, and the proof is complete. □

Theorem 3.2 *Let θ_1, θ_2, and η be positive constants such that $\frac{3}{4}\sqrt{\frac{3}{2}}\frac{\theta_1}{\pi^2} < \eta < \frac{3}{8}\sqrt{\frac{3}{2}}\frac{\theta_2}{\pi^2}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a mapping for which $f(x,t) \geq 0$ for every $(x,t) \in [0,1] \times [0,\theta_2]$. Assume that*

(B1)

$$\frac{\int_0^1 F(x,\theta_1)\,dx}{\theta_1^2} < \frac{64}{81}\frac{\pi^4\int_{\frac{3}{4}}^1 F(x,\eta)\,dx}{\eta^2};$$

(B2)

$$\frac{\int_0^1 F(x,\theta_2)\,dx}{\theta_2^2} < \frac{32}{81}\frac{\pi^4\int_{\frac{3}{4}}^1 F(x,\eta)\,dx}{\eta^2}.$$

Then, for each

$$\lambda \in \Lambda' := \left]\frac{32}{18}\frac{\pi^4\eta^2}{\int_{\frac{3}{4}}^1 F(x,\eta)\,dx}, \frac{1}{2}\min\left\{\frac{\theta_1^2}{\int_0^1 F(x,\theta_1)\,dx}, \frac{\theta_2^2}{2\int_0^1 F(x,\theta_2)\,dx}\right\}\right[$$

and for every nonpositive continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta^* > 0$, where

$$\delta^* := \min\left\{\frac{\theta_1^2 - 2\lambda\int_0^1 F(x,\theta_1)\,dx}{2G^{\theta_1}}, \frac{\theta_2^2 - 4\lambda\int_0^1 F(x,\theta_2)\,dx}{4G^{\theta_2}}\right\},$$

such that, for all $\mu \in [0, \delta^*[$, *problem* (1.1) *admits at least three distinct weak solutions* u_i *for* $i \in \{1,2,3\}$ *such that* $0 \le \|u_i\|_\infty < \theta_2$ *for every* $i \in \{1,2,3\}$.

Proof Without loss of generality, we can assume $f(x,t) \ge 0$ for every $(x,t) \in [0,1] \times \mathbb{R}$. Fix λ, μ, and g as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. Arguing as in the proof of Theorem 3.1, we observe that the regularity assumptions of Lemma 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1) and (b_2).

To this end, put w as given in (3.5), and let $r_1 = \frac{\theta_1^2}{2}$ and $r_2 = \frac{\theta_2^2}{2}$. It is obvious that $2r_1 < \Phi(w) < \frac{r_2}{2}$. It follows from $\mu < \delta^*$ and $G_\eta = 0$ that

$$\begin{aligned}\frac{\sup_{\Phi(u)<r_1} \Psi(u)}{r_1} &= \frac{\sup_{\Phi(u)<r_1} (\int_0^1 F(x,u(x))\,dx + \frac{\mu}{\lambda} G(u(1)))}{r_1} \\ &\le \frac{2\int_0^1 F(x,\theta_1)\,dx + \frac{2\mu}{\lambda} G^{\theta_1}}{\theta_1^2} \\ &< \frac{1}{\lambda} < \frac{2}{3} \frac{\int_{\frac{3}{4}}^1 F(x,\eta)\,dx + \frac{\mu}{\lambda} G_\eta}{\frac{32}{27}\pi^4\eta^2} \le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{2\sup_{\Phi(u)<r_2} \Psi(u)}{r_2} &\le \frac{4\int_0^1 F(x,\theta_2)\,dx}{\theta_2^2} + \frac{4\mu}{\lambda}\frac{G^{\theta_2}}{\theta_2^2} \\ &< \frac{1}{\lambda} < \frac{2}{3} \frac{\int_{\frac{3}{4}}^1 F(x,\eta)\,dx + \frac{\mu}{\lambda} G_\eta}{\frac{32}{27}\pi^4\eta^2} \le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

This implies that (b_1) and (b_2) of Lemma 2.2 are verified.

Finally, we verify that assumption (b_3) of Lemma 2.2 holds. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so they are weak solutions for problem (1.1). We claim that the weak solutions obtained are nonnegative. Indeed, if u_0 is a weak solution of problem (1.1), then one has

$$\int_0^1 u_0''(x)v''(x)\,dx = \lambda \int_0^1 f\big(x,u_0(x)\big)v(x)\,dx - \mu g\big(u_0(1)\big)v(1)$$

for all $v \in X$. Arguing by a contradiction, assume that the set $A := \{x \in [0,1] : u_0(x) < 0\}$ is nonempty and of positive measure. Put $v_0 := \min\{0, u_0\}$. Clearly, $v_0 \in X$. So, taking into account that u_0 is a weak solution and by choosing $v = v_0$, from our sign assumptions on the data, one has

$$\int_A \big(u_0''(x)\big)^2\,dx = \lambda \int_A f\big(x,u_0(x)\big)u_0(x)\,dx - \mu g\big(u_0(1)\big)u_0(1) \le 0.$$

Hence, $u_0 \equiv 0$ on A which is absurd. Then we deduce $u_1(x) \ge 0$ and $u_2(x) \ge 0$ for each $x \in [0,1]$. Thus, it follows that $su_1 + (1-s)u_2 \ge 0$ for all $s \in [0,1]$, and that

$$\Psi\big(su_1 + (1-s)u_2\big) = \int_0^1 F\big(x, su_1(x) + (1-s)u_2(x)\big)\,dx + \frac{\mu}{\lambda} G\big(su_1(1) + (1-s)u_2(1)\big) \ge 0.$$

So, also (b_3) holds. From Lemma 2.2, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)}, \min\left\{\frac{r_1}{\sup_{\Phi(u)<r_1}\Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{\Phi(u)<r_2}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the solutions of problem (1.1) and the conclusion is achieved. □

4 Conclusion

By using as the main tools two critical point theorems presented recently in the works [3] and [6], we proved two multiplicity properties (Theorems 3.1 and 3.2) that guarantee the existence of an open interval $]\lambda', \lambda''[$ and $\tilde{\delta} > 0$ such that, for each $\lambda \in]\lambda', \lambda''[$ and for each $\mu \in [0, \tilde{\delta}[$, a class of fourth-order boundary value problems depending on parameters λ and μ (problem (1.1)) admits at least three distinct weak solutions.

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