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On a fractional cantilever beam model in the q-difference inclusion settings via special multi-valued operators

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Abstract

The fundamental goal of the study under consideration is to establish some of the existence criteria needed for a particular fractional inclusion model of cantilever beam in the setting of quantum calculus using new arguments of existence theory. In this way, we investigate a fractional integral equation that corresponds to the aforementioned boundary value problem. In a more concrete sense, we design new multi-valued operators based on this integral equation, which belong to the certain subclasses of functions, called α-admissible and α-ψ-contractive multi-functions, in combination with the AEP-property. Also, we use some inequalities such as Ω-inequality and set-valued version inequalities. Moreover, we add a simulative example for a numerical analysis of our results obtained in this study.

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Keywords: Ω-inequality; α-admissible map; α-ψ-contraction; (AEP)-property; Set-valued version inequality

1 Introduction

Fractional calculus and its corresponding differential equations and BvPs have been widely utilized in the vast fields of science, including biology, chemistry, economy, physics, engineering, etc. [1–3]. Fractional derivatives do not merely represent a generalization of ordinary derivatives but also precisely and accurately describe the complex behavior, in contrast to integer order derivatives, of diverse physical structures. Several investigators have examined differential equation of arbitrary order starting from the existence and uniqueness of solutions to the analytical and computational approaches in search of solutions. A number of monographs and articles are available concerning the developments of theory of fractional differential equations and inclusions [4–31].

On the other hand, the quantum calculus is a field without the concept of limit that corresponds to the traditional infinitesimal one. Regardless of their vast background, both theories are in the domain of mathematical analysis, working on their properties did not emerge two ages later. Quantum difference operators (q-DiffOper) were first exhibited and introduced by Jackson [32] and have been widely analyzed in order to explain complex physical structures with a number of non-differentiable functions. In early nineties, numerous academician [33, 34] came forward with the studies on q-difference equations which lately received great interest and attention [32, 35, 36]. There are some intriguing insights into IVPs and BVPs coupled with q-difference equations in [37–48].

The q-analogue of a second order q-difference inclusion BvP was studied by Ahmad and Ntouyas [49] in 2011, and they explored the existence criteria by utilizing fixed point theory:

$$\begin{cases} {}^{C}\mathfrak{D}_q^2 \mathrm{u}(\varsigma) \in \psi(\varsigma, \mathrm{u}(\varsigma)), \\ \mathrm{u}(0) = \alpha \mathrm{u}(M), \qquad \mathfrak{D}_q \mathrm{u}(0) = \alpha \mathfrak{D}_q \mathrm{u}(M), \end{cases}$$

where $\varsigma \in [0, M]$, $\psi : [0,1] \times \mathbb{R} \to \mathbb{P}(\mathbb{R})$ is a compact-valued map and $\alpha \in \mathbb{R} \setminus \{1\}$.

Ahmad et al. [50] reviewed later the existence criteria of the following q-difference inclusion involving q-antiperiodic boundary conditions:

$$\begin{cases} {}^{C}\mathfrak{D}_q^{\omega} \mathrm{u}(\varsigma) \in \psi(\varsigma, \mathrm{u}(\varsigma), \mathfrak{D}_q \mathrm{u}(\varsigma), \mathfrak{D}_q^2(\varsigma)), \\ \mathrm{u}(0) + \mathrm{u}(1) = 0, \qquad \mathfrak{D}_q \mathrm{u}(0) + \mathfrak{D}_q \mathrm{u}(1) = 0, \qquad \mathfrak{D}_q^2 \mathrm{u}(0) + \mathfrak{D}_q^2 \mathrm{u}(1) = 0, \end{cases}$$

where $\varsigma \in [0,1]$, $q \in (0,1)$, $2 < \omega \leq 3$, ${}^{C}\mathfrak{D}_q^{\beta}$ denotes the q-fractional derivatives in Caputo sense of order ω and $\psi : [0,1] \times \mathbb{R}^3 \to \mathbb{P}(\mathbb{R})$.

The malformations of an elastically balanced beam with fixed and released end points can be represented by means of a mathematical model as the fourth-order BvP

$$\begin{cases} \mathrm{u}^{(4)}(\varsigma) = \psi(\varsigma, \mathrm{u}(\varsigma), \mathrm{u}'(\varsigma), \mathrm{u}''(\varsigma), \mathrm{u}'''(\varsigma)), \\ \mathrm{u}(\varsigma)|_{\varsigma=0} = \mathrm{u}'(\varsigma)|_{\varsigma=0} = \mathrm{u}''(\varsigma)|_{\varsigma=1} = \mathrm{u}'''(\varsigma)|_{\varsigma=1} = 0, \end{cases} \tag{1}$$

where $\psi : [0,1] \times \mathcal{Y} \to \mathbb{R}$ is a continuous function with $\mathcal{Y} = \mathbb{R}^4$. In fact, Li and Gao [51] studied the existence results for lower and upper solution of above fully fourth-order BvP (1) which is named cantilever beam equation in mechanics. In 2019, Li and Chen [52] presented their existence findings to problem (1) by utilizing an approach based on the fixed point theorem due to Leray–Schauder. In 2020, Zhang and Cui [53] utilized the concepts of fixed point index theory and investigated the positivity of solutions of BvP (1) over a cone, considering $\psi : [0,1] \times \mathcal{B} \to [0, +\infty]$, $\mathcal{B} = [0, +\infty] \times (-\infty, +\infty) \times (-\infty, 0) \times (-\infty, +\infty)$.

The following nonlinear Caputo fractional quantum BvP is designed by the above research works and is equipped with the fractional quantum differential conditions:

$$\begin{cases} {}^{C}\mathfrak{D}_q^{\kappa}({}^{C}\mathfrak{D}_q^{\omega} \mathrm{u})(\varsigma) \in \Phi(\varsigma, \mathrm{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega} \mathrm{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1} \mathrm{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2} \mathrm{u}(\varsigma)), \\ \mathrm{u}(\varsigma)|_{\varsigma=0} = {}^{C}\mathfrak{D}_q^{\omega} \mathrm{u}(\varsigma)|_{\varsigma=0} = {}^{C}\mathfrak{D}_q^{\omega+1} \mathrm{u}(\varsigma)|_{\varsigma=1} = {}^{C}\mathfrak{D}_q^{\omega+2} \mathrm{u}(\varsigma)|_{\varsigma=1} = 0, \end{cases} \tag{2}$$

where $q \in (0,1)$, $\varsigma \in \mathbb{S} := [0,1]$, $\kappa \in (2,3]$, $\omega \in (0,1]$, $\omega + 1 \in (1,2]$, $\omega + 2 \in (2,3]$ and $\Phi : \mathbb{S} \times \mathbb{R}^4 \to \mathbb{P}(\mathbb{R})$ is a multi-function with specified properties. Also ${}^{C}\mathfrak{D}_q^{H}$ displays q-Caputo derivative of order $H \in \{\kappa, \omega, \omega + 1, \omega + 2\}$.

Note that the above inclusion q-FBvP (2) is an extension of the standard practical model of the cantilever beam to the fractional q-analogue structure. By assuming $\kappa = 3$, $\omega = 1$, and $q \to 1$ and $\Phi(\cdot) = \{\psi(\cdot)\}$, we obtain the above fourth-order differential inclusion arising in the cantilever beam model (1). One can state some physical interpretations for the q-FBvP model (2) by assuming such assumptions over κ and ω as follows: $u(\varsigma)$ stands for the deformation function, ${}^{C}\mathfrak{D}^{3}_{q\to 1}({}^{C}\mathfrak{D}^{1}_{q\to 1}\mathfrak{u})$ is the load density stiffness, ${}^{C}\mathfrak{D}^{3}_{q\to 1}\mathfrak{u}$ denotes the stiffness of the function $\mathfrak{u}$ under the shear force, ${}^{C}\mathfrak{D}^{2}_{q\to 1}\mathfrak{u}$ denotes the stiffness of the function $\mathfrak{u}$ in the bending moment, and ${}^{C}\mathfrak{D}^{1}_{q\to 1}\mathfrak{u}$ represents the slope [54, 55].

About the novelty of this work, one can state that such a fractional model of cantilever beam based on q-difference operators has not been studied in any research paper so far, and on this new structure, we derive our mathematical and analytical results ensuring the solution's existence by means of some special subclasses of multi-functions. For our applied technique, we here use α-ψ-contractive multi-functions for the confirmation of the existence of a fixed point and also the confirmation of the existence of an end point by making use of another family of multi-functions having the AEP-property.

The rest of the manuscript is structured as follows: Sect. 2 is dedicated to the fundamental ideas of q-analogue of fractional calculus. In the beginning of Sect. 3, we provide a lemma which presents the solution of the cantilever beam q-FBvP (2) in the form of an integral equation, and then, by making use of the α-admissible multi-valued mappings with control function and approximate end point theory, we guarantee the solutions' existence for the cantilever beam q-FBvP (2). Section 4 is assigned to the illustration of the results presented in Sect. 3 with the aid of an example. Finally, Sect. 5 describes the concluded remarks.

2 Preliminaries

We compile and study, in the light of our approaches used in this investigation, some auxiliary and primitive definitions regarding q-calculus.

We suppose that $0 < q < 1$. The q-analogue of the function $(m_1 - m_2)^n$ given for $n \in \mathbb{N}_0$ is defined as $(m_1 - m_2)^{(0)} = 1$ coupled with

$$(m_1 - m_2)^{(n)} = \prod_{k=0}^{n-1} (m_1 - m_2 q^k),$$

so that $m_1, m_2 \in \mathbb{R}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ [56]. Now, $n = \omega$ is a constant which is supposed to belong to $\mathbb{R}$. We represent the following q-analogue of the existing power function $(m_1 - m_2)^n$ in a q-fractional setting:

$$(m_1 - m_2)^{(\omega)} = m_1^{\omega} \prod_{n=0}^{\infty} \frac{1 - (\frac{m_2}{m_1})q^n}{1 - (\frac{m_2}{m_1})q^{\omega+n}} \tag{3}$$

for $m_1 \neq 0$. We consider that, by taking $m_2 = 0$, $m_1^{(\omega)} = m_1^{\omega}$ [56]. For the same $m_1 \in \mathbb{R}$, a q-number $[m_1]_q$ is exhibited as

$$[m_1]_q = \frac{1 - q^{m_1}}{1 - q} = q^{m_1 - 1} + \cdots + q + 1.$$

The q-gamma function is illustrated using such a format

$$\Gamma_q(\varsigma) = \frac{(1-q)^{(\varsigma-1)}}{(1-q)^{\varsigma-1}}, \tag{4}$$

so that $\varsigma \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ [56, 57]. Also, $\Gamma_q(\varsigma+1) = [\varsigma]_q \Gamma_q(\varsigma)$ [57].

Definition 1 ([58, 59] Riemann–Liouville q-integral) For $\omega \geq 0$ and for a given function $\mathfrak{u} \in \mathcal{C}_{\mathbb{R}}([0,+\infty))$, the RL-q-integral of $\mathfrak{u}$ is introduced by

$${}^{R}\mathfrak{I}_q^{\omega}\mathfrak{u}(\varsigma) = \frac{1}{\Gamma_q(\omega)} \int_0^{\varsigma} (\varsigma - qz)^{(\omega-1)}\mathfrak{u}(z)\, \mathrm{d}_q z \quad (\omega > 0),$$

provided that the above value is finite and ${}^{R}\mathfrak{I}_q^{0}\mathfrak{u}(\varsigma) = \mathfrak{u}(\varsigma)$.

For $\omega_1, \omega_2, \omega, \theta > 0$, we have these properties [59]:
(1) ${}^{R}\mathfrak{I}_q^{\omega_1}({}^{R}\mathfrak{I}_q^{\omega_2}\mathfrak{u})(\varsigma) = {}^{R}\mathfrak{I}_q^{\omega_1+\omega_2}\mathfrak{u}(\varsigma)$,
(2) ${}^{R}\mathfrak{I}_q^{\omega}\varsigma^{\theta} = \frac{\Gamma_q(\theta+1)}{\Gamma_q(\theta+\omega+1)}\varsigma^{\theta+\omega}$.
If $\theta = 0$, then ${}^{R}\mathfrak{I}_q^{\omega}1(\varsigma) = \frac{1}{\Gamma_q(\omega+1)}\varsigma^{\omega}$ for any $\varsigma > 0$.

Definition 2 ([58, 59] Caputo q-derivative) Given $n-1 < \omega < n$, i.e., $n = [\omega]+1$ and a function $\mathfrak{u} \in \mathcal{AC}_{\mathbb{R}}^{(n)}([0,+\infty))$, the ωth-Caputo q-derivative for this function is formulated by

$${}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma) = \frac{1}{\Gamma_q(n-\omega)} \int_0^{\varsigma} (\varsigma - qz)^{(n-\omega-1)}\mathfrak{D}_q^{n}\mathfrak{u}(z)\, \mathrm{d}_q z,$$

if the integral exists.

Clearly, ${}^{C}\mathfrak{D}_q^{\omega}c = 0$ for any $c \in \mathbb{R}$ and

$${}^{C}\mathfrak{D}_q^{\omega}\varsigma^{\theta} = \frac{\Gamma_q(\theta+1)}{\Gamma_q(\theta-\omega+1)}\varsigma^{\theta-\omega} \quad (\varsigma > 0).$$

Lemma 3 ([60]) *Assume that $n-1 < \omega < n$ and $\mathfrak{u} \in \mathcal{C}_{\mathbb{R}}^{(n)}([0,+\infty))$. Then*

$$\left({}^{R}\mathfrak{I}_q^{\omega}{}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}\right)(\varsigma) = \mathfrak{u}(\varsigma) - \sum_{k=0}^{n-1} \frac{(\mathfrak{D}_q^{k}\mathfrak{u})(0)}{\Gamma_q(k+1)}\varsigma^{k}.$$

In view of the above lemma, by assuming the constants $\ell_k := \frac{(\mathfrak{D}_q^{k}\mathfrak{u})(0)}{\Gamma_q(k+1)} > 0$, the given fractional homogeneous q-difference equation ${}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma) = 0$ has a general solution which is given by $\mathfrak{u}(\varsigma) = \ell_0 + \ell_1\varsigma + \ell_2\varsigma^2 + \cdots + \ell_{n-1}\varsigma^{n-1}$ such that $\ell_0, \ldots, \ell_{n-1} \in \mathbb{R}$ and $n = [\omega]+1$ [60]. It should be mentioned that, for each continuous function $\mathfrak{u}$, and by Lemma 3, we obtain

$$\left({}^{R}\mathfrak{I}_q^{\omega}{}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}\right)(\varsigma) = \mathfrak{u}(\varsigma) + \ell_0 + \ell_1\varsigma + \ell_2\varsigma^2 + \cdots + \ell_{n-1}\varsigma^{n-1},$$

where $\ell_0, \ldots, \ell_{n-1}$ illustrate constants in $\mathbb{R}$ [60].

Regarding $(\mathfrak{X}_\star, \|\cdot\|)$ as a normed space, the classes $\mathbb{P}_{\mathbb{CL}}(\mathfrak{X}_\star)$ (all closed sets), $\mathbb{P}_{\mathbb{BN}}(\mathfrak{X}_\star)$ (all bounded sets), $\mathbb{P}_{\mathbb{CP}}(\mathfrak{X}_\star)$ (all compact sets), and $\mathbb{P}_{\mathbb{CV}}(\mathfrak{X}_\star)$ (all convex sets) involve the respective form of subsets of $\mathfrak{X}_\star$.

Definition 4 ([61]) An element $\mathfrak{u} \in \mathfrak{X}_\star$ is an end point of the multi-function $\Phi : \mathfrak{X}_\star \to \mathbb{P}(\mathfrak{X}_\star)$ if $\Phi(\mathfrak{u}) = \{\mathfrak{u}\}$.

The multi-function Φ admits an approximate end point property (AEP) if

$$\inf_{\mathfrak{u}_1 \in \mathfrak{X}_\star} \sup_{\mathfrak{u}_2 \in \Phi(\mathfrak{u}_1)} d(\mathfrak{u}_1, \mathfrak{u}_2) = 0.$$

Mohammadi et al. [62] introduced a new notion given as the subclass Ψ of all nondecreasing functions like

$$\psi : [0,\infty) \to [0,\infty)$$

so that $\sum_{n=1}^{\infty} \psi^n(\varsigma) < \infty$ for any $\varsigma > 0$. Now, by making use of such a category, we define a new family of multi-functions.

Definition 5 ([62]) Let $\Phi : \mathfrak{X}_\star \to \mathbb{P}_{\mathbb{CL},\mathbb{BN}}(\mathfrak{X}_\star)$ and $\alpha : \mathfrak{X}_\star^2 \to [0,+\infty)$. Then

(1) Φ is α-admissible if, for each $\mathfrak{u}_1 \in \mathfrak{X}_\star$ and $\mathfrak{u}_2 \in \Phi\mathfrak{u}_1$, inequality $\alpha(\mathfrak{u}_1, \mathfrak{u}_2) \geq 1$ gives $\alpha(\mathfrak{u}_2, \mathfrak{u}_3) \geq 1$ for each $\mathfrak{u}_3 \in \Phi\mathfrak{u}_2$.

(2) Φ is an α-ψ-contractive multi-function if $\forall \mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{X}_\star$,

$$\alpha(\mathfrak{u}_1, \mathfrak{u}_2)\mathbb{H}_d(\Phi\mathfrak{u}_1, \Phi\mathfrak{u}_2) \leq \psi\big(d(\mathfrak{u}_1, \mathfrak{u}_2)\big),$$

where $\mathbb{H}_d$ is the Pompeiu–Hausdorff metric.

Next we recall requisite theorems concerning the investigation of the proposed q-FBvP (2).

Theorem 6 ([62]) *Let us assume a complete metric space $(\mathfrak{X}_\star, d)$, a nonnegative map $\alpha : \mathfrak{X}_\star^2 \to [0,\infty)$, and $\psi \in \Psi$. In addition, assume $\Phi : \mathfrak{X}_\star \to \mathbb{P}_{\mathbb{CL},\mathbb{BN}}(\mathfrak{X}_\star)$ to be an α-ψ-contractive multi-function, and let*

(1) *Φ be α-admissible;*

(2) *$\alpha(\mathfrak{u}_0, \mathfrak{u}_1) \geq 1$ for some $\mathfrak{u}_0 \in \mathfrak{X}_\star$ and $\mathfrak{u}_1 \in \Phi\mathfrak{u}_0$;*

(3) *for each sequence $\{\mathfrak{u}_n\}$ in $\mathfrak{X}_\star$ with $\alpha(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, a subsequence $\{\mathfrak{u}_{n_t}\}$ of $\{\mathfrak{u}_n\}$ exist such that, for all $t \in \mathbb{N}$, $\alpha(\mathfrak{u}_{n_t}, \mathfrak{u}) \geq 1$.*

Then Φ admits a fixed point.

Theorem 7 ([61]) *Let $(\mathfrak{X}_\star, d)$ be a metric space of complete type and consider*

(1) *an upper semi-continuous map $\psi : [0,\infty) \to [0,\infty)$ with $\psi(\varsigma) < \varsigma$ and*

$$\liminf_{\varsigma \to \infty} \big(\varsigma - \psi(\varsigma)\big) > 0, \quad \forall \varsigma > 0;$$

(2) *a multi-function $\Phi : \mathfrak{X}_\star \to \mathbb{P}_{\mathbb{CL},\mathbb{BN}}(\mathfrak{X}_\star)$ such that*

$$\mathbb{H}_d(\Phi\mathfrak{u}_1, \Phi\mathfrak{u}_2) \leq \psi\big(d(\mathfrak{u}_1, \mathfrak{u}_2)\big), \quad \forall \mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{X}_\star.$$

Then a unique end point of Φ exists iff Φ has the AEP-property.

3 Existence results

Regard $\mathfrak{X}_\star = \{\mathrm{u}(\varsigma) : \mathrm{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathrm{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathrm{u}(\varsigma) \in \mathcal{C}(\mathbb{S},\mathbb{R})\}$ as a Banach space of all real-valued continuous functions on $\mathbb{S}$ equipped with a sup-norm

$$\|\mathrm{u}\| = \sup_{\varsigma\in\mathbb{S}}\big|\mathrm{u}(\varsigma)\big| + \sup_{\varsigma\in\mathbb{S}}\big|{}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}(\varsigma)\big| + \sup_{\varsigma\in\mathbb{S}}\big|{}^{C}\mathfrak{D}_q^{\omega+1}\mathrm{u}(\varsigma)\big| + \sup_{\varsigma\in\mathbb{S}}\big|{}^{C}\mathfrak{D}_q^{\omega+2}\mathrm{u}(\varsigma)\big|.$$

In the following proposition, the solution to the proposed fractional cantilever q-problem (2) is presented in the form of an integral equation, which will be helpful in establishing our main findings.

Proposition 8 *Let* $\kappa \in (2,3]$, $\omega \in (0,1]$, *and* $\mathcal{G} \in C(\mathbb{S},\mathbb{R})$. *Then the solution of the linear q-FBvP*

$$\begin{cases} {}^{C}\mathfrak{D}_q^{\kappa}({}^{C}\mathfrak{D}_q^{\omega}\mathrm{u})(\varsigma) = \mathcal{G}(\varsigma), \\ \mathrm{u}(\varsigma)|_{\varsigma=0} = {}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}(\varsigma)|_{\varsigma=0} = {}^{C}\mathfrak{D}_q^{\omega+1}\mathrm{u}(\varsigma)|_{\varsigma=1} = {}^{C}\mathfrak{D}_q^{\omega+2}\mathrm{u}(\varsigma)|_{\varsigma=1} = 0, \end{cases} \tag{5}$$

is presented in the following form:

$$\begin{aligned} \mathrm{u}(\varsigma) = {}& \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma - qz)^{(\kappa+\omega-1)}\mathcal{G}(z)\,\mathrm{d}_q z \\ & - \frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1 (1-qz)^{(\kappa-2)}\mathcal{G}(z)\,\mathrm{d}_q z \\ & + \frac{[\omega+2]_q\varsigma^{\omega+1} - \varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1 (1-qz)^{(\kappa-3)}\mathcal{G}(z)\,\mathrm{d}_q z, \end{aligned} \tag{6}$$

where $[\omega+2]_q = \frac{1-q^{\omega+2}}{1-q}$.

Proof Assume that u^* is a solution of the given q-FBvP (5). Then ${}^{C}\mathfrak{D}_q^{\kappa}({}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}^*)(\varsigma) = \mathcal{G}(\varsigma)$. By taking κth-q-integral of Riemann–Liouville type, we obtain

$${}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}^*(\varsigma) = \frac{1}{\Gamma_q(\kappa)}\int_0^{\varsigma}(\varsigma - qz)^{(\kappa-1)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_0 + \ell_1\varsigma + \ell_2\varsigma^2, \tag{7}$$

where $\ell_m \in \mathbb{R}, m = 0,1,2$. The second condition ${}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}^*(\varsigma)|_{\varsigma=0} = 0$ implies $\ell_0 = 0$. Thus

$${}^{C}\mathfrak{D}_q^{\omega}\mathrm{u}^*(\varsigma) = \frac{1}{\Gamma_q(\kappa)}\int_0^{\varsigma}(\varsigma - qz)^{(\kappa-1)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_1\varsigma + \ell_2\varsigma^2. \tag{8}$$

In the sequel, by using the Riemann–Liouville ωth-q-integral to both sides of (8), we reach

$$\mathrm{u}^*(\varsigma) = \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma - qz)^{(\kappa+\omega-1)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_0^* + \ell_1\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)} + \ell_2\frac{(1+q)\varsigma^{\omega+2}}{\Gamma_q(\omega+3)}. \tag{9}$$

From (9) and the condition $\mathrm{u}^*(\varsigma)|_{\varsigma=0} = 0$, we get $\ell_0^* = 0$. Thus

$$\mathrm{u}^*(\varsigma) = \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma - qz)^{(\kappa+\omega-1)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_1\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)} + \ell_2\frac{(1+q)\varsigma^{\omega+2}}{\Gamma_q(\omega+3)}. \tag{10}$$

As $\omega + 1 \in (1,2]$ and $\omega + 2 \in (2,3]$, we have

$$ {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}^*(\varsigma) = \frac{1}{\Gamma_q(\kappa-1)}\int_0^{\varsigma} (\varsigma - qz)^{(\kappa-2)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_1 + \ell_2(1+q)\varsigma \tag{11} $$

and

$$ {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}^*(\varsigma) = \frac{1}{\Gamma_q(\kappa-2)}\int_0^{\varsigma} (\varsigma - qz)^{(\kappa-3)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_2(1+q). \tag{12} $$

In view of the conditions ${}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}^*(\varsigma)|_{\varsigma=1} = {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}^*(\varsigma)|_{\varsigma=1} = 0$ on the relations (11) and (12), we have

$$ \frac{1}{\Gamma_q(\kappa-1)}\int_0^{1} (1 - qz)^{(\kappa-2)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_1 + \ell_2(1+q) = 0 $$

and

$$ \frac{1}{\Gamma_q(\kappa-2)}\int_0^{1} (1 - qz)^{(\kappa-3)}\mathcal{G}(z)\,\mathrm{d}_q z + \ell_2(1+q) = 0. $$

By solving the above system, we get

$$ \ell_1 = -\frac{1}{\Gamma_q(\kappa-1)}\int_0^{1} (1 - qz)^{(\kappa-2)}\mathcal{G}(z)\,\mathrm{d}_q z + \frac{1}{\Gamma_q(\kappa-2)}\int_0^{1} (1 - qz)^{(\kappa-3)}\mathcal{G}(z)\,\mathrm{d}_q z $$

and

$$ \ell_2 = -\frac{1}{(1+q)\Gamma_q(\kappa-2)}\int_0^{1} (1 - qz)^{(\kappa-3)}\mathcal{G}(z)\,\mathrm{d}_q z. $$

By putting the values ℓ_m $(m = 1,2)$ in (10), we have

$$ \begin{aligned} \mathfrak{u}^*(\varsigma) = {} & \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma} (\varsigma - qz)^{(\kappa+\omega-1)}\mathcal{G}(z)\,\mathrm{d}_q z \\ & - \frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^{1} (1 - qz)^{(\kappa-2)}\mathcal{G}(z)\,\mathrm{d}_q z \\ & + \frac{[\omega+2]_q\varsigma^{\omega+1} - \varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^{1} (1 - qz)^{(\kappa-3)}\mathcal{G}(z)\,\mathrm{d}_q z, \end{aligned} $$

which ensures that $\mathfrak{u}^*$ satisfies (6) and the proof is finished. □

We are now ready to develop our key findings about the existence of solutions for the q-difference inclusion FBvP (2) occurring in the cantilever beam model. The function $\mathfrak{u} \in \mathcal{C}(\mathbb{S}, \mathfrak{X}_\star)$ is referred to as the solution of the fractional cantilever q-FBvP (2) when it settles the given boundary conditions, and a function $\acute{\varphi} \in \mathcal{L}^1(\mathbb{S})$ exists such that, for almost all $\varsigma \in \mathbb{S}$, we have

$$ \acute{\varphi}(\varsigma) \in \Phi\big(\varsigma, \mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big) $$

and

$$\begin{aligned}\mathfrak{u}(\varsigma) &= \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}(z)\,\mathrm{d}_qz \\ &\quad - \frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}(z)\,\mathrm{d}_qz \\ &\quad + \frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}(z)\,\mathrm{d}_qz\end{aligned}$$

for all $\varsigma \in \mathbb{S}$. The selections' set of the multi-function Φ is given by

$$\begin{aligned}(\mathbb{SEL})_{\Phi,\mathfrak{u}} = \big\{&\acute{\varphi} \in \mathcal{L}^1(\mathbb{S}) : \acute{\varphi}(\varsigma) \in \Phi\big(\varsigma, \mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big) \\ &\text{for all } \varsigma \in \mathbb{S}\big\}\end{aligned}$$

for each $\mathfrak{u} \in \mathfrak{X}_\star$. In addition, an operator $\mathfrak{U} : \mathfrak{X}_\star \to \mathbb{P}(\mathfrak{X}_\star)$ is formulated by the following rule:

$$\mathfrak{U}(\mathfrak{u}) = \big\{\nu \in \mathfrak{X}_\star : \text{ there exists } \acute{\varphi} \in (\mathbb{SEL})_{\Phi,\mathfrak{u}} : \nu(\varsigma) = \mathfrak{r}(\varsigma) \text{ for all } \varsigma \in \mathbb{S}\big\}, \tag{13}$$

where

$$\begin{aligned}\mathfrak{r}(\varsigma) &= \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}(z)\,\mathrm{d}_qz \\ &\quad - \frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}(z)\,\mathrm{d}_qz \\ &\quad + \frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}(z)\,\mathrm{d}_qz.\end{aligned}$$

For convenience, we take

$$\begin{aligned}\check{\Upsilon}_1 &= \frac{1}{\Gamma_q(\kappa+\omega+1)} + \frac{1}{\Gamma_q(\omega+2)\Gamma_q(\kappa)} + \frac{[\omega+2]_q+1}{\Gamma_q(\omega+3)\Gamma_q(\kappa-1)}, \\ \check{\Upsilon}_2 &= \frac{1}{\Gamma_q(\kappa+1)} + \frac{1}{\Gamma_q(\kappa)} + \frac{2+q}{(1+q)\Gamma_q(\kappa-1)}, \\ \check{\Upsilon}_3 &= \frac{2}{\Gamma_q(\kappa)} + \frac{2}{\Gamma_q(\kappa-1)}, \\ \check{\Upsilon}_4 &= \frac{2}{\Gamma_q(\kappa-1)},\end{aligned} \tag{14}$$

and

$$\overline{\Pi}_1 = \|\mathfrak{p}\|\check{\Upsilon}_1, \qquad \overline{\Pi}_2 = \|\mathfrak{p}\|\check{\Upsilon}_2, \qquad \overline{\Pi}_3 = \|\mathfrak{p}\|\check{\Upsilon}_3, \qquad \overline{\Pi}_4 = \|\mathfrak{p}\|\check{\Upsilon}_4. \tag{15}$$

Theorem 9 *Consider $\Phi : \mathbb{S} \times \mathfrak{X}_\star^4 \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{X}_\star)$ and assume the following*:

($\mathcal{T}_1$) *The multi-function Φ is bounded and integrable and $\Phi(\cdot, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) : \mathbb{S} \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{X}_\star)$ is measurable for all $\mathfrak{u}_m \in \mathfrak{X}_\star$ (m=1,2,3,4);*

($\mathcal{T}_2$) *There exists* $\mathfrak{p} \in \mathcal{C}(\mathbb{S}, [0,\infty))$ *and* $\psi \in \Psi$ *such that the set-valued version inequality*

$$\begin{aligned} &\mathbb{H}_d\big(\Phi(\varsigma, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4), \Phi(\varsigma, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2, \bar{\mathfrak{u}}_3, \bar{\mathfrak{u}}_4)\big) \\ &\quad \leq \mathfrak{p}(\varsigma)\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\sum_{m=1}^{4} |\mathfrak{u}_m - \bar{\mathfrak{u}}_m|\right), \end{aligned} \tag{16}$$

holds for all $\varsigma \in \mathbb{S}$ *and* $\mathfrak{u}_m, \bar{\mathfrak{u}}_m \in \mathfrak{X}_\star$ $(m = 1,2,3,4)$, *where* $\sup_{\varsigma \in \mathbb{S}} |\mathfrak{p}(\varsigma)| = \|\mathfrak{p}\|$, $\varrho^\star = \frac{1}{\check{\Upsilon}_1 + \check{\Upsilon}_2 + \check{\Upsilon}_3 + \check{\Upsilon}_4}$, *and* $\check{\Upsilon}_m$ $(m = 1,2,3,4)$ *are given by* (14);

($\mathcal{T}_3$) *A function* $\Omega : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ *exists such that, for all* $\mathfrak{u}_m, \bar{\mathfrak{u}}_m \in \mathfrak{X}_\star$ $(m = 1,2,3,4)$, *we have*

$$\Omega\big((\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4), (\bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2, \bar{\mathfrak{u}}_3, \bar{\mathfrak{u}}_4)\big) \geq 0;$$

($\mathcal{T}_4$) *If* $\{\mathfrak{u}_n\}_{n\geq 1}$ *is a sequence in* $\mathfrak{X}_\star$ *with* $\mathfrak{u}_n \to \mathfrak{u}$ *and*

$$\begin{aligned} &\Omega\big(\big(\mathfrak{u}_n(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_n(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_n(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_n(\varsigma)\big), \\ &\quad \big(\mathfrak{u}_{n+1}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_{n+1}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_{n+1}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_{n+1}(\varsigma)\big)\big) \geq 0, \end{aligned}$$

for all $\varsigma \in \mathbb{S}$ *and* $n \geq 1$, *then a subsequence* $\{\mathfrak{u}_{n_t}\}_{t\geq 1}$ *of* $\{\mathfrak{u}_n\}$ *exists such that, for all* $\varsigma \in \mathbb{S}$ *and* $t \geq 1$, *we have*

$$\begin{aligned} &\Omega\big(\big(\mathfrak{u}_{n_t}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_{n_t}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_{n_t}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_{n_t}(\varsigma)\big), \\ &\quad \big(\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)\big) \geq 0; \end{aligned}$$

($\mathcal{T}_5$) *There exist a member* $\mathfrak{u}_0 \in \mathfrak{X}_\star$ *and* $\nu \in \mathfrak{U}(\mathfrak{u}_0)$ *such that, for any* $\varsigma \in \mathbb{S}$,

$$\begin{aligned} &\Omega\big(\big(\mathfrak{u}_0(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_0(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_0(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_0(\varsigma)\big), \\ &\quad \big(\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\nu(\varsigma)\big)\big) \geq 0, \end{aligned}$$

where the multi-function $\mathfrak{U} : \mathfrak{X}_\star \to \mathbb{P}(\mathfrak{X}_\star)$ *is specified by* (13);

($\mathcal{T}_6$) *For every* $\mathfrak{u} \in \mathfrak{X}_\star$ *and* $\nu \in \mathfrak{U}(\mathfrak{u})$ *with*

$$\begin{aligned} &\Omega\big(\big(\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big), \\ &\quad \big(\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\nu(\varsigma)\big)\big) \geq 0, \end{aligned}$$

a member $\mathfrak{r} \in \mathfrak{U}(\mathfrak{u})$ *exists such that the* Ω-*inequality*

$$\begin{aligned} &\Omega\big(\big(\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\nu(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\nu(\varsigma)\big), \\ &\quad \big(\mathfrak{r}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\big)\big) \geq 0, \end{aligned}$$

holds for all $\varsigma \in \mathbb{S}$.

Then the fractional cantilever beam inclusion q-BvP (2) *possesses a solution on* $\mathbb{S}$.

Proof Evidently, the fixed point of the multi-function $\mathfrak{U}:\mathfrak{X}_\star\to\mathbb{P}(\mathfrak{X}_\star)$ given by (13) is identified as the solution for the cantilever beam inclusion q-FBvP (2), so we try to check the assumptions of Theorem 6 on this multi-valued operator. Since the compact-valued set-valued map $\varsigma\mapsto\Phi(\varsigma,\mathrm{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\mathrm{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\mathrm{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\mathrm{u}(\varsigma))$ is measurable as well as closed-valued for any $\mathrm{u}\in\mathfrak{X}_\star$, so the map Φ has a measurable selection and $(\mathbb{SEL})_{\Phi,\mathrm{u}}\neq\emptyset$. At first, we verify that the subset $\mathfrak{U}(\mathrm{u})$ of $\mathfrak{X}_\star$ is closed $\forall\mathrm{u}\in\mathfrak{X}_\star$. For this purpose, consider a sequence $\{\mathrm{u}_n\}_{n\geq1}$ in $\mathfrak{U}(\mathrm{u})$ via $\mathrm{u}_n\to\mathrm{u}$. Now, for each $n\geq1$, we have a member $\acute{\varphi}_n\in(\mathbb{SEL})_{\Phi,\mathrm{u}}$ satisfying

$$\begin{aligned}\mathrm{u}(\varsigma)&=\frac{1}{\Gamma_q(\kappa+\omega)}\int_0^\varsigma(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}_n(z)\,\mathrm{d}_qz\\&\quad-\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}_n(z)\mathrm{d}_qz\\&\quad+\frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}_n(z)\mathrm{d}_qz\end{aligned}$$

for almost all $\varsigma\in\mathbb{S}$. Since Φ admits compact values, we take a subsequence of $\{\acute{\varphi}_n\}_{n\geq1}$ (following the same symbol) that tends to some $\acute{\varphi}\in\mathcal{L}^1(\mathbb{S})$. Thus, $\acute{\varphi}\in(\mathbb{SEL})_{\Phi,\mathrm{u}}$ and

$$\begin{aligned}\mathrm{u}_n(\varsigma)\to\mathrm{u}(\varsigma)&=\frac{1}{\Gamma_q(\kappa+\omega)}\int_0^\varsigma(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}(z)\,\mathrm{d}_qz\\&\quad-\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}(z)\,\mathrm{d}_qz\\&\quad+\frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}(z)\,\mathrm{d}_qz\end{aligned}$$

for all $\varsigma\in\mathbb{S}$. This leads to the conclusion that $\mathrm{u}\in\mathfrak{U}(\mathrm{u})$ and the multi-function $\mathfrak{U}$ is closed-valued. Since Φ is compact-valued, it is easy to ensure the boundedness of $\mathfrak{U}(\mathrm{u})$ for every $\mathrm{u}\in\mathfrak{X}_\star$. Next, we prove that $\mathfrak{U}$ is an α-ψ-contractive multi-function. In view of this intention, we take a function α on $\mathfrak{X}_\star\times\mathfrak{X}_\star$ with nonnegative values which is defined by $\alpha(\mathrm{u},\bar{\mathrm{u}})=1$ if

$$\begin{aligned}&\Omega\big(\big(\mathrm{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\mathrm{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\mathrm{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\mathrm{u}(\varsigma)\big),\\&\quad\big(\bar{\mathrm{u}}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\bar{\mathrm{u}}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathrm{u}}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathrm{u}}(\varsigma)\big)\big)\geq0,\end{aligned}$$

and otherwise, it is defined to be zero for all $\mathrm{u},\bar{\mathrm{u}}\in\mathfrak{X}_\star$. Suppose $\mathrm{u},\bar{\mathrm{u}}\in\mathfrak{X}_\star$ and $\nu_1\in\mathfrak{U}(\bar{\mathrm{u}})$ and select $\acute{\varphi}_1\in(\mathbb{SEL})_{\Phi,\bar{\mathrm{u}}}$ such that

$$\begin{aligned}\nu_1(\varsigma)&=\frac{1}{\Gamma_q(\kappa+\omega)}\int_0^\varsigma(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}_1(z)\,\mathrm{d}_qz\\&\quad-\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}_1(z)\mathrm{d}_qz\\&\quad+\frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}_1(z)\mathrm{d}_qz\end{aligned}$$

for all $\varsigma \in \mathbb{S}$. Utilizing (16), we get

$$\begin{aligned}&\mathbb{H}_d\big(\Phi\big(\varsigma,\mathfrak{u},{}^C\mathfrak{D}_q^{\omega}\mathfrak{u},{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u},{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}\big),\Phi\big(\varsigma,\bar{\mathfrak{u}},{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}},{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}},{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}\big)\big)\\&\quad\le \mathfrak{p}(\varsigma)\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\big(|\mathfrak{u}-\bar{\mathfrak{u}}|+\big|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}\big|+\big|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}\big|\\&\qquad+\big|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}\big|\big)\end{aligned}$$

for all $\mathfrak{u},\bar{\mathfrak{u}} \in \mathfrak{X}_\star$ with

$$\begin{aligned}&\Omega\big(\big(\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big),\\&\quad\big(\bar{\mathfrak{u}}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big)\big)\ge 0\end{aligned}$$

for almost all $\varsigma \in \mathbb{S}$. Thus, a member

$$\mathfrak{r}\in\Phi\big(\varsigma,\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)$$

exists such that

$$\begin{aligned}\big|\acute{\varphi}_1(\varsigma)-\mathfrak{r}\big| &\le \mathfrak{p}(\varsigma)\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\big(\big|\mathfrak{u}(\varsigma)-\bar{\mathfrak{u}}(\varsigma)\big|+\big|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma)\big|\\&\quad+\big|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma)\big|+\big|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big|\big).\end{aligned}$$

Now, consider a map $\aleph:\mathbb{S}\to\mathbb{P}(\mathfrak{X}_\star)$ which is characterized by

$$\begin{aligned}\aleph(\varsigma)=\bigg\{&\mathfrak{r}\in\mathfrak{X}_\star:\big|\acute{\varphi}_1(\varsigma)-\mathfrak{r}\big|\le\mathfrak{p}(\varsigma)\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\big(\big|\mathfrak{u}(\varsigma)-\bar{\mathfrak{u}}(\varsigma)\big|+\big|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma)\big|\\&+\big|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma)\big|+\big|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big|\big)\bigg\}\end{aligned}$$

for any $\varsigma\in\mathbb{S}$. Since $\acute{\varphi}_1$ and

$$\eth=\mathfrak{p}\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\big(|\mathfrak{u}-\bar{\mathfrak{u}}|+\big|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}\big|+\big|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}\big|+\big|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}\big|\big)$$

are measurable, so the multi-map $\aleph(\cdot)\cap\Phi(\cdot,\mathfrak{u}(\cdot),{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\cdot),{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\cdot),{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\cdot))$ is also measurable. Now, choose

$$\acute{\varphi}_2\in\Phi\big(\varsigma,\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)$$

so that, for all $\varsigma\in\mathbb{S}$, we have

$$\begin{aligned}\big|\acute{\varphi}_1(\varsigma)-\acute{\varphi}_2(\varsigma)\big| &\le \mathfrak{p}(\varsigma)\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\big(\big|\mathfrak{u}(\varsigma)-\bar{\mathfrak{u}}(\varsigma)\big|+\big|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma)\big|\\&\quad+\big|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma)\big|+\big|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big|\big).\end{aligned}$$

Consider $\nu_2 \in \mathfrak{U}(\mathfrak{u})$ given by

$$\begin{aligned}\nu_2(\varsigma) &= \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}_2(z)\,\mathrm{d}_qz\\ &\quad-\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}_2(z)\mathrm{d}_qz\\ &\quad+\frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}_2(z)\mathrm{d}_qz\end{aligned}$$

for any $\varsigma \in \mathbb{S}$. Then we obtain the following inequalities:

$$\begin{aligned}\left|\nu_1(\varsigma)-\nu_2(\varsigma)\right| &\le \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma-qz)^{(\kappa+\omega-1)}\left|\acute{\varphi}_1(z)-\acute{\varphi}_2(z)\right|\mathrm{d}_qz\\ &\quad+\frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\left|\acute{\varphi}_1(z)-\acute{\varphi}_2(z)\right|\mathrm{d}_qz\\ &\quad+\frac{|[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}|}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\left|\acute{\varphi}_1(z)-\acute{\varphi}_2(z)\right|\mathrm{d}_qz\\ &\le \frac{1}{\Gamma_q(\kappa+\omega+1)}\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)\\ &\quad+\frac{1}{\Gamma_q(\omega+2)\Gamma_q(\kappa)}\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)\\ &\quad+\frac{[\omega+2]_q+1}{\Gamma_q(\omega+3)\Gamma_q(\kappa-1)}\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)\\ &= \left[\frac{1}{\Gamma_q(\kappa+\omega+1)}+\frac{1}{\Gamma_q(\omega+2)\Gamma_q(\kappa)}+\frac{[\omega+2]_q+1}{\Gamma_q(\omega+3)\Gamma_q(\kappa-1)}\right]\\ &\quad\times\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)\\ &= \varrho^\star\check{\Upsilon}_1\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right).\end{aligned}$$

Also, we have

$$\begin{aligned}&\left|{}^C\mathfrak{D}_q^{\omega}\nu_1(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\nu_2(\varsigma)\right|\\ &\quad\le\left[\frac{1}{\Gamma_q(\kappa+1)}+\frac{1}{\Gamma_q(\kappa)}+\frac{2+q}{(1+q)\Gamma_q(\kappa-1)}\right]\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)\\ &\quad=\varrho^\star\check{\Upsilon}_2\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right),\\ &\left|{}^C\mathfrak{D}_q^{\omega+1}\nu_1(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\nu_2(\varsigma)\right|\\ &\quad\le\left[\frac{2}{\Gamma_q(\kappa)}+\frac{2}{\Gamma_q(\kappa-1)}\right]\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)=\varrho^\star\check{\Upsilon}_3\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right),\end{aligned}$$

and

$$\left|{}^C\mathfrak{D}_q^{\omega+2}\nu_1(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\nu_2(\varsigma)\right|\le\left[\frac{2}{\Gamma_q(\kappa-1)}\right]\|\mathfrak{p}\|\left(\frac{\varrho^\star}{\|\mathfrak{p}\|}\right)\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)=\varrho^\star\check{\Upsilon}_4\psi\left(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\right)$$

for all $\varsigma \in \mathbb{S}$. Hence

$$\begin{aligned}
&\|\nu_1 - \nu_2\| \\
&\quad = \sup_{\varsigma\in\mathbb{S}}\big|\nu_1(\varsigma) - \nu_2(\varsigma)\big| + \sup_{\varsigma\in\mathbb{S}}\big|{}^{C}\mathfrak{D}_q^{\omega}\nu_1(\varsigma) - {}^{C}\mathfrak{D}_q^{\omega}\nu_2(\varsigma)\big| \\
&\qquad + \sup_{\varsigma\in\mathbb{S}}\big|{}^{C}\mathfrak{D}_q^{\omega+1}\nu_1(\varsigma) - {}^{C}\mathfrak{D}_q^{\omega+1}\nu_2(\varsigma)\big| + \sup_{\varsigma\in\mathbb{S}}\big|{}^{C}\mathfrak{D}_q^{\omega+2}\nu_1(\varsigma) - {}^{C}\mathfrak{D}_q^{\omega+2}\nu_2(\varsigma)\big| \\
&\quad \le \varrho^{\star}(\check{\Upsilon}_1 + \check{\Upsilon}_2 + \check{\Upsilon}_3 + \check{\Upsilon}_4)\psi\big(\|\mathfrak{u} - \bar{\mathfrak{u}}\|\big) = \psi\big(\|\mathfrak{u} - \bar{\mathfrak{u}}\|\big).
\end{aligned}$$

Thus,

$$\alpha(\mathfrak{u},\bar{\mathfrak{u}})\mathbb{H}_d\big(\mathfrak{U}(\mathfrak{u}) - \mathfrak{U}(\bar{\mathfrak{u}})\big) \le \psi\big(\|\mathfrak{u} - \bar{\mathfrak{u}}\|\big)$$

for any $\mathfrak{u},\bar{\mathfrak{u}} \in \mathfrak{X}_{\star}$ which indicates that $\mathfrak{U}$ is an α-ψ-contractive multi-function. Now, suppose that $\mathfrak{u} \in \mathfrak{X}_{\star}$ and $\bar{\mathfrak{u}} \in \mathfrak{U}(\mathfrak{u})$ satisfy $\alpha(\mathfrak{u},\bar{\mathfrak{u}}) \ge 1$, and so

$$\begin{aligned}
&\Omega\big(\big(\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big), \\
&\quad \big(\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big)\big) \ge 0.
\end{aligned}$$

Then, from the hypothesis, a member $\mathfrak{r} \in \mathfrak{U}(\bar{\mathfrak{u}})$ exists such that

$$\begin{aligned}
&\Omega\big(\big(\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big), \\
&\quad \big(\mathfrak{r}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\big)\big) \ge 0.
\end{aligned}$$

It implies that $\alpha(\bar{\mathfrak{u}},\mathfrak{r}) \ge 1$, and so we deduce that $\mathfrak{U}$ is α-admissible. Now, take $\mathfrak{u}_0 \in \mathfrak{X}_{\star}$ and $\bar{\mathfrak{u}} \in \mathfrak{U}(\mathfrak{u}_0)$ so that

$$\begin{aligned}
&\Omega\big(\big(\mathfrak{u}_0(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_0(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_0(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_0(\varsigma)\big), \\
&\quad \big(\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\bar{\mathfrak{u}}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\bar{\mathfrak{u}}(\varsigma)\big)\big) \ge 0
\end{aligned}$$

for all $\varsigma \in \mathbb{S}$. Then it follows that $\alpha(\mathfrak{u}_0,\bar{\mathfrak{u}}) \ge 1$. Let us assume that $\{\mathfrak{u}_n\}_{n\ge1}$ is a sequence in $\mathfrak{X}_{\star}$ such that $\mathfrak{u}_n \to \mathfrak{u}$ and $\alpha(\mathfrak{u}_n,\mathfrak{u}_{n+1}) \ge 1$ for all n. Then we obtain

$$\begin{aligned}
&\Omega\big(\big(\mathfrak{u}_n(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_n(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_n(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_n(\varsigma)\big), \\
&\quad \big(\mathfrak{u}_{n+1}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_{n+1}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_{n+1}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_{n+1}(\varsigma)\big)\big) \ge 0.
\end{aligned}$$

Utilization of the assumption $(\mathcal{T}_4)$ leads to the existence of a subsequence $\{\mathfrak{u}_{n_t}\}_{t\ge1}$ of $\{\mathfrak{u}_n\}$ such that

$$\begin{aligned}
&\Omega\big(\big(\mathfrak{u}_{n_t}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}_{n_t}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}_{n_t}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}_{n_t}(\varsigma)\big), \\
&\quad \big(\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^{C}\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)\big) \ge 0
\end{aligned}$$

for all $\varsigma \in \mathbb{S}$. This directly implies that $\alpha(\mathfrak{u}_{n_t},\mathfrak{u}) \ge 1$ for all t. Hence, Theorem 6 is settled and the multi-function $\mathfrak{U}$ possesses a fixed point which is regarded as a solution for the fractional cantilever beam inclusion q-BvP (2). □

Now, we utilize the notion of end points for other subclass of multi-functions to achieve the desired aim.

Theorem 10 *Consider* $\Phi : \mathbb{S} \times \mathfrak{X}_\star^4 \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{X}_\star)$ *and assume*:

($\mathcal{T}_7$) $\exists \psi : [0,\infty) \to [0,\infty)$ *as an increasing and u.s.c. function with* $\liminf_{\varsigma\to\infty}(\varsigma - \psi(\varsigma)) \ge 0$ *and* $\psi(\varsigma) < \varsigma, \forall \varsigma > 0$;

($\mathcal{T}_8$) *the multi-function* $\Phi : \mathbb{S} \times \mathfrak{X}_\star^4 \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{X}_\star)$ *is bounded and integrable such that the map* $\Phi(\cdot, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) : \mathbb{S} \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{X}_\star)$ *is measurable for all* $\mathfrak{u}_m \in \mathfrak{X}_\star$ ($m = 1,2,3,4$);

($\mathcal{T}_9$) $\exists \mathfrak{p} \in \mathcal{C}(\mathbb{S}, [0,\infty))$ *satisfying*

$$\begin{aligned} &\mathbb{H}_d\big(\Phi(\varsigma, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4), \Phi(\varsigma, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2, \bar{\mathfrak{u}}_3, \bar{\mathfrak{u}}_4)\big) \\ &\quad \le \mathfrak{p}(\varsigma)\varrho_\star \psi\big(|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1| + |\mathfrak{u}_2 - \bar{\mathfrak{u}}_2| + |\mathfrak{u}_3 - \bar{\mathfrak{u}}_3| + |\mathfrak{u}_4 - \bar{\mathfrak{u}}_4|\big) \end{aligned} \tag{17}$$

for all $\varsigma \in \mathbb{S}$ *and* $\mathfrak{u}_m, \bar{\mathfrak{u}}_m \in \mathfrak{X}_\star$ ($m = 1,2,3,4$), *where* $\varrho_\star = \frac{1}{\overline{\Pi}_1+\overline{\Pi}_2+\overline{\Pi}_3+\overline{\Pi}_4}$ *and* $\overline{\Pi}_m$ ($m = 1,2,3,4$) *are the constants defined by* (15);

($\mathcal{T}_{10}$) $\mathfrak{U}$, *given by* (13), *has the AEP-property.*

Then the fractional cantilever beam inclusion q-BvP (2) *admits a solution*.

Proof To fulfill Theorem 7, we have to prove that the multi-map $\mathfrak{U} : \mathfrak{X}_\star \to \mathbb{P}(\mathfrak{X}_\star)$, given by (13) has an end point. At first, since

$$\varsigma \mapsto \Phi\big(\varsigma, \mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)$$

is a closed-valued as well as measurable map, so Φ has a measurable selection and $(\mathbb{SEL})_{\Phi,\mathfrak{u}} \neq \emptyset$ for each $\mathfrak{u} \in \mathfrak{X}_\star$. Using the same method as given in Theorem 9, one can deduce that $\mathfrak{U}(\mathfrak{u})$ has closed values. Since the multi-function Φ is compact, so $\mathfrak{U}(\mathfrak{u})$ is bounded for any $\mathfrak{u} \in \mathfrak{X}_\star$. This time, we only try to show that $\mathbb{H}_d(\mathfrak{U}(\mathfrak{u}), \mathfrak{U}(\mathfrak{r})) \le \psi(\|\mathfrak{u} - \mathfrak{r}\|)$. Let us assume that $\mathfrak{u}, \mathfrak{r} \in \mathfrak{X}_\star$ and $\nu_1 \in \mathfrak{U}(\mathfrak{r})$. Select $\acute{\varphi}_1 \in (\mathbb{SEL})_{\Phi,\mathfrak{r}}$ such that

$$\begin{aligned} \nu_1(\varsigma) &= \frac{1}{\Gamma_q(\kappa+\omega)} \int_0^\varsigma (\varsigma - qz)^{(\kappa+\omega-1)} \acute{\varphi}_1(z)\, \mathrm{d}_q z \\ &\quad - \frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)} \int_0^1 (1-qz)^{(\kappa-2)} \acute{\varphi}_1(z)\, \mathrm{d}_q z \\ &\quad + \frac{[\omega+2]_q \varsigma^{\omega+1} - \varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)} \int_0^1 (1-qz)^{(\kappa-3)} \acute{\varphi}_1(z)\, \mathrm{d}_q z \end{aligned}$$

for almost all $\varsigma \in \mathbb{S}$. Since, for any $\varsigma \in \mathbb{S}$, we have

$$\begin{aligned} &\mathbb{H}_d\big(\Phi\big(\varsigma, \mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big), \\ &\Phi\big(\varsigma, \mathfrak{r}(\varsigma), {}^C\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma), {}^C\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma), {}^C\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\big)\big) \\ &\quad \le \mathfrak{p}(\varsigma)\varrho_\star \psi\big(\big|\mathfrak{u}(\varsigma) - \mathfrak{r}(\varsigma)\big| + \big|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma) - {}^C\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma)\big| + \big|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma) - {}^C\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma)\big| \\ &\qquad + \big|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma) - {}^C\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\big|\big), \end{aligned}$$

there exists a member

$$\grave{\lambda} \in \Phi\big(\varsigma, \mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma), {}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)$$

such that

$$\begin{aligned}\left|\acute{\varphi}_1(\varsigma)-\grave{\lambda}\right| \le \mathfrak{p}(\varsigma)\varrho_\star\psi\Big(\left|\mathfrak{u}(\varsigma)-\mathfrak{r}(\varsigma)\right| + \left|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma)\right| \\ + \left|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma)\right| + \left|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\right|\Big)\end{aligned}$$

for any $\varsigma\in\mathbb{S}$. Now, the map $\Xi:\mathbb{S}\to\mathbb{P}(\mathfrak{X}_\star)$ is considered which is given by the following way:

$$\begin{aligned}\Xi(\varsigma) = \Big\{\grave{\lambda}\in\mathfrak{X}_\star : \left|\acute{\varphi}_1(\varsigma)-\grave{\lambda}\right| \le \mathfrak{p}(\varsigma)\varrho_\star\psi\Big(\left|\mathfrak{u}(\varsigma)-\mathfrak{r}(\varsigma)\right| + \left|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma)\right| \\ + \left|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma)\right| + \left|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\right|\Big)\Big\}.\end{aligned}$$

The multi-function $\Xi(\cdot)\cap\Phi(\cdot,\mathfrak{u}(\cdot),{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\cdot),{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\cdot),{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\cdot))$ is measurable because $\acute{\varphi}_1$ and

$$\mathbb{k} = \mathfrak{p}\varrho_\star\psi\Big(|\mathfrak{u}-\mathfrak{r}| + \left|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega}\mathfrak{r}\right| + \left|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{r}\right| + \left|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}-{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{r}\right|\Big)$$

are measurable. Now, a member

$$\acute{\varphi}_2\in\Phi\big(\varsigma,\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma),{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)\big)$$

is selected such that

$$\begin{aligned}\left|\acute{\varphi}_1(\varsigma)-\acute{\varphi}_2(\varsigma)\right| \le \mathfrak{p}(\varsigma)\psi\Big(\left|\mathfrak{u}(\varsigma)-\mathfrak{r}(\varsigma)\right| + \left|{}^C\mathfrak{D}_q^{\omega}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\mathfrak{r}(\varsigma)\right| \\ + \left|{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\mathfrak{r}(\varsigma)\right| \\ + \left|{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{u}(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\mathfrak{r}(\varsigma)\right|\Big)\left[\frac{1}{\overline{\Pi}_1+\overline{\Pi}_2+\overline{\Pi}_3+\overline{\Pi}_4}\right]\end{aligned}$$

for all $\varsigma\in\mathbb{S}$. Choose $\nu_2\in\mathfrak{U}(\mathfrak{u})$ such that, for all $\varsigma\in\mathbb{S}$,

$$\begin{aligned}\nu_2(\varsigma) &= \frac{1}{\Gamma_q(\kappa+\omega)}\int_0^{\varsigma}(\varsigma-qz)^{(\kappa+\omega-1)}\acute{\varphi}_2(z)\,\mathrm{d}_qz \\ &\quad - \frac{\varsigma^{\omega+1}}{\Gamma_q(\omega+2)\Gamma_q(\kappa-1)}\int_0^1(1-qz)^{(\kappa-2)}\acute{\varphi}_2(z)\,\mathrm{d}_qz \\ &\quad + \frac{[\omega+2]_q\varsigma^{\omega+1}-\varsigma^{\omega+2}}{\Gamma_q(\omega+3)\Gamma_q(\kappa-2)}\int_0^1(1-qz)^{(\kappa-3)}\acute{\varphi}_2(z)\,\mathrm{d}_qz.\end{aligned}$$

By employing the same methodology used in the proof of Theorem 9, we reach

$$\begin{aligned}\|\nu_1-\nu_2\| &= \sup_{\varsigma\in\mathbb{S}}\left|\nu_1(\varsigma)-\nu_2(\varsigma)\right| + \sup_{\varsigma\in\mathbb{S}}\left|{}^C\mathfrak{D}_q^{\omega}\nu_1(\varsigma)-{}^C\mathfrak{D}_q^{\omega}\nu_2(\varsigma)\right| \\ &\quad + \sup_{\varsigma\in\mathbb{S}}\left|{}^C\mathfrak{D}_q^{\omega+1}\nu_1(\varsigma)-{}^C\mathfrak{D}_q^{\omega+1}\nu_2(\varsigma)\right| + \sup_{\varsigma\in\mathbb{S}}\left|{}^C\mathfrak{D}_q^{\omega+2}\nu_1(\varsigma)-{}^C\mathfrak{D}_q^{\omega+2}\nu_2(\varsigma)\right| \\ &\le \varrho_\star(\overline{\Pi}_1+\overline{\Pi}_2+\overline{\Pi}_3+\overline{\Pi}_4)\psi\big(\|\mathfrak{u}-\mathfrak{r}\|\big) = \psi\big(\|\mathfrak{u}-\mathfrak{r}\|\big).\end{aligned}$$

Accordingly, we have $\mathbb{H}_d(\mathfrak{U}(\mathfrak{u}),\mathfrak{U}(\mathfrak{r}))\le\psi(\|\mathfrak{u}-\mathfrak{r}\|)$ for any $\mathfrak{u},\mathfrak{r}\in\mathfrak{X}_\star$. From assumption $(\mathcal{T}_{10})$ implying the existence of the AEP-property for the multi-function $\mathfrak{U}$, the application of

Theorem 7 leads to the existence of $\mathfrak{u}^* \in \mathfrak{X}_\star$ satisfying $\mathfrak{U}(\mathfrak{u}^*) = \{\mathfrak{u}^*\}$. This indicates that $\mathfrak{u}^*$ is a solution for the cantilever beam inclusion q-FBvP (2). □

4 An example

Based on the fractional cantilever beam inclusion q-BvP (2), we here present some examples in this framework to confirm the validity of the results.

Example 1 Consider the cantilever beam inclusion q-FBvP

$$\begin{cases} {}^{C}\mathfrak{D}_{0.7}^{2.8}({}^{C}\mathfrak{D}_{0.7}^{0.9}\mathfrak{u})(\varsigma) \\ \quad \in [0, \frac{\varsigma|\arcsin(\mathfrak{u}(\varsigma))|+\varsigma|{}^{C}\mathfrak{D}_{0.7}^{0.9}\mathfrak{u}(\varsigma)|+\varsigma^2|\sin({}^{C}\mathfrak{D}_{0.7}^{1.9}\mathfrak{u}(\varsigma))|+\varsigma^3|\arctan({}^{C}\mathfrak{D}_{0.7}^{2.9}\mathfrak{u}(\varsigma))|}{77} + 12e^{\varsigma}], \\ \mathfrak{u}(\varsigma)|_{\varsigma=0} = {}^{C}\mathfrak{D}_{q}^{0.9}\mathfrak{u}(\varsigma)|_{\varsigma=0} = {}^{C}\mathfrak{D}_{q}^{1.9}\mathfrak{u}(\varsigma)|_{\varsigma=1} = {}^{C}\mathfrak{D}_{q}^{2.9}\mathfrak{u}(\varsigma)|_{\varsigma=1} = 0, \end{cases} \tag{18}$$

where $\varsigma \in \mathbb{S} = [0,1]$, and we have selected $q = 0.7, \kappa = 2.8, \omega = 0.9$. Now, consider the multi-valued map $\Phi : \mathbb{S} \times \mathbb{R}^4 \to \mathbb{P}(\mathbb{R})$ which is determined by

$$\begin{aligned} &\Phi\big(\varsigma, \mathfrak{u}_1(\varsigma), \mathfrak{u}_2(\varsigma), \mathfrak{u}_3(\varsigma), \mathfrak{u}_4(\varsigma)\big) \\ &\quad = \left[0, \frac{\varsigma|\arcsin(\mathfrak{u}_1(\varsigma))| + \varsigma|\mathfrak{u}_2(\varsigma)| + \varsigma^2|\sin(\mathfrak{u}_3(\varsigma))| + \varsigma^3|\arctan(\mathfrak{u}_4(\varsigma))|}{77} + 12e^{\varsigma}\right] \end{aligned}$$

for any $\varsigma \in \mathbb{S}$. Next, consider the function $\mathfrak{p} \in \mathcal{C}(\mathbb{S}, [0,\infty))$ given by $\mathfrak{p}(\varsigma) = \frac{\varsigma}{11}$ for all $\varsigma \in \mathbb{S}$. Then $\|\mathfrak{p}\| = \sup_{\varsigma\in\mathbb{S}} |\frac{\varsigma}{11}| = \frac{1}{11}$. Moreover, we choose $\psi : [0,\infty) \to [0,\infty)$ as an increasing u.s.c map given by $\psi(\varsigma) = \frac{\varsigma}{7}$ for almost all $\varsigma > 0$. It can be easily seen that $\liminf_{\varsigma\to\infty}(\varsigma - \psi(\varsigma)) > 0$ and $\psi(\varsigma) < \varsigma$ for all $\varsigma > 0$. In the light of preceding data, (14) and (15), we obtain

$$\begin{aligned} \check{\Upsilon}_1 &= \frac{1}{\Gamma_{0.7}(2.8 + 0.9 + 1)} + \frac{1}{\Gamma_{0.7}(0.9 + 2)\Gamma_{0.7}(2.8)} \\ &\quad + \frac{[0.9 + 2]_{0.7} + 1}{\Gamma_{0.7}(0.9 + 3)\Gamma_{0.7}(2.8 - 1)} \simeq 1.54029, \\ \check{\Upsilon}_2 &= \frac{1}{\Gamma_{0.7}(2.8 + 1)} + \frac{1}{\Gamma_{0.7}(2.8)} + \frac{2 + 0.7}{(1 + 0.7)\Gamma_{0.7}(2.8 - 1)} \simeq 2.67215, \\ \check{\Upsilon}_3 &= \frac{2}{\Gamma_{0.7}(2.8)} + \frac{2}{\Gamma_{0.7}(2.8 - 1)} \simeq 3.46062, \\ \check{\Upsilon}_4 &= \frac{2}{\Gamma_{0.7}(2.8 - 1)} \simeq 2.11890, \end{aligned}$$

and

$$\overline{\Pi}_1 \simeq 0.14003, \qquad \overline{\Pi}_2 = 0.24292, \qquad \overline{\Pi}_3 = 0.31460, \qquad \overline{\Pi}_4 = 0.19263.$$

For each $\mathfrak{u}_m, \bar{\mathfrak{u}}_m \in \mathbb{R}$ ($m = 1,2,3,4$), we have

$$\begin{aligned} &\mathbb{H}_d\big(\Phi\big(\varsigma, \mathfrak{u}_1(\varsigma), \mathfrak{u}_2(\varsigma), \mathfrak{u}_3(\varsigma), \mathfrak{u}_4(\varsigma)\big), \Phi\big(\varsigma, \bar{\mathfrak{u}}_1(\varsigma), \bar{\mathfrak{u}}_2(\varsigma), \bar{\mathfrak{u}}_3(\varsigma), \bar{\mathfrak{u}}_4(\varsigma)\big)\big) \\ &\quad \le \frac{\varsigma}{11}.\frac{1}{7}\big(\big|\arcsin\big(\mathfrak{u}_1(\varsigma)\big) - \arcsin\big(\bar{\mathfrak{u}}_1(\varsigma)\big)\big| + \big|\mathfrak{u}_2(\varsigma) - \bar{\mathfrak{u}}_2(\varsigma)\big| \\ &\quad\quad + \big|\sin\big(\mathfrak{u}_3(\varsigma)\big) - \sin\big(\bar{\mathfrak{u}}_3(\varsigma)\big)\big| + \big|\arctan\big(\mathfrak{u}_4(\varsigma)\big) - \arctan\big(\bar{\mathfrak{u}}_4(\varsigma)\big)\big|\big) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\varsigma}{11}\cdot\frac{1}{7}\left(\left|\mathfrak{u}_1(\varsigma)-\bar{\mathfrak{u}}_1(\varsigma)\right|+\left|\mathfrak{u}_2(\varsigma)-\bar{\mathfrak{u}}_2(\varsigma)\right|+\left|\mathfrak{u}_3(\varsigma)-\bar{\mathfrak{u}}_3(\varsigma)\right|+\left|\mathfrak{u}_4(\varsigma)-\bar{\mathfrak{u}}_4(\varsigma)\right|\right)\\
&= \frac{\varsigma}{11}\psi\left(\left|\mathfrak{u}_1(\varsigma)-\bar{\mathfrak{u}}_1(\varsigma)\right|+\left|\mathfrak{u}_2(\varsigma)-\bar{\mathfrak{u}}_2(\varsigma)\right|+\left|\mathfrak{u}_3(\varsigma)-\bar{\mathfrak{u}}_3(\varsigma)\right|+\left|\mathfrak{u}_4(\varsigma)-\bar{\mathfrak{u}}_4(\varsigma)\right|\right)\\
&\le \mathfrak{p}(\varsigma)\psi\left(\left|\mathfrak{u}_1(\varsigma)-\bar{\mathfrak{u}}_1(\varsigma)\right|+\left|\mathfrak{u}_2(\varsigma)-\bar{\mathfrak{u}}_2(\varsigma)\right|+\left|\mathfrak{u}_3(\varsigma)-\bar{\mathfrak{u}}_3(\varsigma)\right|+\left|\mathfrak{u}_4(\varsigma)-\bar{\mathfrak{u}}_4(\varsigma)\right|\right)\\
&\quad\times\left[\frac{1}{\overline{\Pi}_1+\overline{\Pi}_2+\overline{\Pi}_3+\overline{\Pi}_4}\right].
\end{aligned}
$$

Now, define the multi-function $\mathfrak{U}:\mathfrak{X}_\star\to\mathbb{P}(\mathfrak{X}_\star)$ by

$$
\mathfrak{U}(\mathfrak{u})=\left\{\nu\in\mathfrak{X}_\star:\exists\acute{\varphi}\in(\mathbb{SEL})_{\Phi,\mathfrak{u}}\text{ s.t. }\nu(\varsigma)=\mathfrak{r}(\varsigma),\forall\varsigma\in\mathbb{S}\right\},
$$

where

$$
\begin{aligned}
\mathfrak{r}(\varsigma) &= \frac{1}{\Gamma_q(2.8+0.9)}\int_0^{\varsigma}(\varsigma-qz)^{(2.8+0.9-1)}\acute{\varphi}(z)\,\mathrm{d}_qz\\
&\quad-\frac{\varsigma^{0.9+1}}{\Gamma_q(0.9+2)\Gamma_q(2.8-1)}\int_0^1(1-qz)^{(2.8-2)}\acute{\varphi}(z)\,\mathrm{d}_qz\\
&\quad+\frac{[0.9+2]_q\varsigma^{0.9+1}-\varsigma^{0.9+2}}{\Gamma_q(0.9+3)\Gamma_q(2.8-2)}\int_0^1(1-qz)^{(2.8-3)}\acute{\varphi}(z)\,\mathrm{d}_qz\\
&= \frac{1}{\Gamma_q(3.7)}\int_0^{\varsigma}(\varsigma-qz)^{2.7}\acute{\varphi}(z)\,\mathrm{d}_qz-\frac{\varsigma^{1.9}}{\Gamma_q(2.9)\Gamma_q(1.8)}\int_0^1(1-qz)^{0.8}\acute{\varphi}(z)\,\mathrm{d}_qz\\
&\quad+\frac{[2.9]_q\varsigma^{1.9}-\varsigma^{2.9}}{\Gamma_q(3.9)\Gamma_q(0.8)}\int_0^1(1-qz)^{-0.2}\acute{\varphi}(z)\,\mathrm{d}_qz,
\end{aligned}
$$

and $[2.9]_{0.7}\simeq 2.14848$. As the multi-function $\mathfrak{U}$ possesses the AEP-property, so utilizing Theorem 10, one can clearly follow that the q-FBvP (18) has a solution.

5 Conclusion

A variety of complex natural phenomena that arise from science and technology are modeled by fractional operators. In the present study, we considered a fractional inclusion model of cantilever beam in the context of quantum calculus. We therefore, specified several operators based on the special classes of α-admissible and α-ψ-contractive multifunctions, relying on the equivalent integral equation. We studied the existence of solutions and, in addition, for such operators, we explored the AEP-property. Lastly, an example was given to examine the results regarding the proposed cantilever beam inclusion q-FBvP. As a future proposal, one can consider some other fractional operators to discuss the existence of solutions and approximating them for different generalized fractional models of the cantilever beam equation.

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