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Convergence analysis of two-grid methods for second order hyperbolic equation

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Abstract

In this paper, a second-order hyperbolic equation is solved by a two-grid algorithm combined with the expanded mixed finite element method. The error estimate of the expanded mixed finite element method with discrete-time scheme is demonstrated. Moreover, we present a two-grid method and analyze its convergence. It is shown that the algorithm can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = \mathcal{O}(h^{\frac{1}{2}})$. Finally, some numerical experiments are provided to illustrate the efficiency and accuracy of the proposed method.

Keywords: Hyperbolic equation; Expanded mixed finite element method; Two-grid algorithm; Error estimate

1 Introduction

Hyperbolic equations have been one of the principal tools in the nature. They are very significant for many physical problems, such as fluid dynamics and aerodynamics, the theory of elasticity, optics, electromagnetic waves, direct and inverse scattering, and the general theory of relativity [1–3]. In this paper, we consider the following hyperbolic equation:

$$\begin{cases} u_{tt} - \nabla \cdot (K\nabla u) = f(u), & (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = u_1(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times J, \end{cases} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, $J = (0, T]$ and $T > 0$ is some final time. Let u denote the sound pressure, f is the external force, and K is the coefficient. Let u_{tt} and u_t denote $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial u}{\partial t}$, respectively. Moreover, throughout this paper we assume that

(i) For some integer $r \geq 0$, there exists a constant M_0 such that

$$\|u_{tt}\|_{L^2(H^{r+1})} + \|u_t\|_{L^2(H^{r+1})} + \|u\|_{L^2(H^{r+2}\cap W^{r,\infty})} \leq M_0.$$

(ii) The function K is a square-integrable, symmetric, uniformly positive-definite tensor defined on Ω. Additionally, we assume that there exist constants $K_*, K^* > 0$

for every $\boldsymbol{x} \in \bar{\Omega}$, and any vector $\boldsymbol{y} \in \mathbb{R}^2$ such that

$$K_*|\boldsymbol{y}|^2 \le \boldsymbol{y}^T K \boldsymbol{y} \le K^*|\boldsymbol{y}|^2, \quad \forall \boldsymbol{y} \in \mathbb{R}^2.$$

(iii) The function $f = f(u), f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a three times continuously differentiable function with bounded derivatives through the third order. Moreover, there exists a bound M_1 such that

$$|f|, \left|\frac{\partial f}{\partial t}\right|, \left|\frac{\partial f}{\partial u}\right|, \left|\frac{\partial^2 f}{\partial u^2}\right| \le M_1.$$

Many papers have been written about numerical schemes for hyperbolic equations, see [1–8, 13, 21–25] and the references cited therein. Mixed finite element method is such a popular approach and has been widely used in porous media [4, 7, 8]. For the standard mixed methods, problem (1.1) is often rewritten by introducing a new variable $\boldsymbol{p} = -K\nabla u$, or equivalently $K^{-1}\boldsymbol{p} = -\nabla u$. In this paper, we consider a variant of the mixed method, the expanded mixed finite element method (EMFEM), proposed by Arbogast et al. [9]. This approach expands the standard mixed formulation in the sense that three variables are explicitly treated; i.e., the unknown scalar, its gradient, and its flux (the coefficient times the gradient). The EMFEM enables us to compute gradient of pressure directly. In the past two decades, expanded mixed method has been developed and some extensions have been achieved. Chen [10, 11] analyzed the linear/quasilinear elliptic equations by EMFEM. Woodward et al. [12] gave a detailed analysis of the EMFEM for second-order parabolic equation and obtained optimal error estimates for nonlinear problem. Zhou et al. [13] proposed an analysis of EMFEM applied to hyperbolic equations. Recently, Sharma et al. [14] developed and applied the EMFEM for a class of nonlinear and nonlocal parabolic problem for the case of the lowest order RT element.

As we know, the resulting algebraic system of equations is a large system of nonlinear equations using the EMFE approximation for (1.1). Therefore, it is necessary for us to study an effective algorithm for this essential system. Two-grid algorithm was introduced by Xu [15, 16] for the nonsymmetric linear and nonlinear elliptic problems. It is a simple but effective algorithm that has been widely applied to nonlinear problems of various types. For instance, Dawson et al. [17] applied a two-grid method for a class of nonlinear parabolic equations by EMFEM. Wu et al. [18] studied a two-grid EMFEM for solving semilinear reaction-diffusion equations. Chen et al. [19] constructed and analyzed three-steps algorithm by using two-grid method for EMFE solution of parabolic equations. Recently, Hou et al. [20] analyzed the superconvergence property of two-grid EMFEM for semilinear parabolic integro-differential equations. Furthermore, some other research works can also be found, such as [21–26].

To the best of our knowledge, there is no two-grid method convergence analysis for (1.1) in the literature that is combined with EMFEM. In this paper, based on Raviart–Thomas mixed finite spaces, we propose the two-grid method and the corresponding error estimates which partly fill this gap. Our purpose is two fold. First of all, we apply EMFEM and construct fully discrete approximation of $(u, \tilde{\boldsymbol{p}}, \boldsymbol{p})$. Secondly, we use the two-grid algorithm to solve the fully discrete expanded-mixed-method equations. For the purpose of obtaining the optimal approximation, we choose proper relationship between the coarse grid mesh size H and the fine grid mesh size h. It is showed that coarse space can be extremely coarse and asymptotically optimal approximation can be achieved as long as the mesh sizes satisfy $H = \mathcal{O}(h^{\frac{1}{2}})$.

The rest of the paper is organized as follows. In Sect. 2, we introduce some notations and approximation results that are used throughout the paper. In Sect. 3, we construct the EMFEM for (1.1) and derive the error estimates in L^2-norm. In Sect. 4, we propose the two-grid method and analyze the convergence. In Sect. 5, two numerical examples are used to confirm the theoretical results.

Throughout this paper, let C be a generic positive constant independent of any functions and numerical discretization parameters.

2 Notation and approximation results

We denote the standard Lebesgue space defined on Ω by $L^p(\Omega)$ for $p \geq 1$ with the norm $\|\cdot\|_p$. We shall also use the standard Sobolev space $W^{m,p}(\Omega)$ with the norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha \phi\|_{L^p(\Omega)}^p$. To simplify the notation, for $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ and write $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

Let

$$
\begin{aligned}
&W = L^2(\Omega), \\
&\boldsymbol{V} = H(\mathrm{div};\Omega) = \left\{\boldsymbol{v} : \boldsymbol{v} \in \left(L^2(\Omega)\right)^2, \nabla \cdot \boldsymbol{v} \in L^2(\Omega)\right\},
\end{aligned}
$$

with the norm defined by $\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} \equiv (\|\boldsymbol{v}\|^2 + \|\nabla \cdot \boldsymbol{v}\|^2)^{1/2}$.

Let $\mathcal{T}_h$ be a quasi-uniform family of finite element partitions of Ω, where h is the maximal element diameter. We consider finite-dimensional subspaces W_h and $\boldsymbol{V}_h$ of W and $\boldsymbol{V}$, respectively. They are Raviart–Thomas spaces of index k (RT_k) [27] or Brezzi–Douglas–Marini spaces of index k (BDM_k) [28], where k is a fixed nonnegative integer associated with $\mathcal{T}_h$. The following inclusion holds for the RT_k spaces or BDM_k spaces [27, 28]

$$
\nabla \cdot \boldsymbol{v}_h \in W_h, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.1}
$$

The analysis employs standard L^2 projections onto the spaces W_h and $\boldsymbol{V}_h$. Denote by $(\widehat{\phi}, \widehat{\psi}) \in W_h \times \boldsymbol{V}_h$ the L^2 projection of (ϕ, ψ), defined by the conditions

$$
(\phi, w_h) = (\widehat{\phi}, w_h), \quad \forall w_h \in W_h, \tag{2.2}
$$

$$
(\psi, \boldsymbol{v}_h) = (\widehat{\psi}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.3}
$$

Let Π_h be the well-known Fortin projection of $(H^1(\Omega))^2$ into $\boldsymbol{V}_h$ such that

$$
(\nabla \cdot \boldsymbol{z}, w_h) = (\nabla \cdot \Pi_h \boldsymbol{z}, w_h), \quad \forall w_h \in W_h. \tag{2.4}
$$

These projections obey the following identities:

$$
\begin{aligned}
&(\partial\phi/\partial t, w_h) = (\partial\widehat{\phi}/\partial t, w_h), \quad \forall w_h \in W_h, \\
&\left(\partial^2\phi/\partial t^2, w_h\right) = \left(\partial^2\widehat{\phi}/\partial t^2, w_h\right), \quad \forall w_h \in W_h, \\
&(\phi, \nabla \cdot \boldsymbol{v}_h) = (\widehat{\phi}, \nabla \cdot \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.
\end{aligned} \tag{2.5}
$$

From Refs. [27, 28], there are approximation properties: for $\phi \in W^{k+1,q}(\Omega)$ and $\boldsymbol{z} \in (W^{k+1,q}(\Omega))^2$,

$$\|\widehat{\phi}\|_{0,q} \le C\|\phi\|_{0,q}, \quad 2 \le q < \infty, \tag{2.6}$$

$$\|\phi - \widehat{\phi}\|_{0,q} \le C\|\phi\|_{r,q}h^r, \quad 0 \le r \le k+1, \tag{2.7}$$

$$\|\boldsymbol{z} - \Pi_h\boldsymbol{z}\|_{0,q} \le C\|\boldsymbol{z}\|_{r,q}h^r, \quad 1/q < r \le k+1, \tag{2.8}$$

$$\big\|\nabla \cdot (\boldsymbol{z} - \Pi_h\boldsymbol{z})\big\|_{0,q} \le C\|\nabla \cdot \boldsymbol{z}\|_{r,q}h^r, \quad 0 \le r \le k+1. \tag{2.9}$$

3 A priori error estimates of EMFEM

Introduce the auxiliary variables $\widetilde{\boldsymbol{p}} = -\nabla u$, $\boldsymbol{p} = K\widetilde{\boldsymbol{p}}$ to obtain the following first-order system for (1.1):

$$\begin{cases} u_{tt} + \nabla \cdot \boldsymbol{p} = f(u), \quad (\boldsymbol{x},t) \in \Omega \times J, \\ \widetilde{\boldsymbol{p}} + \nabla u = 0, \quad (\boldsymbol{x},t) \in \Omega \times J, \\ \boldsymbol{p} - K\widetilde{\boldsymbol{p}} = 0, \quad (\boldsymbol{x},t) \in \Omega \times J, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad u_t(\boldsymbol{x},0) = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times J. \end{cases} \tag{3.1}$$

The expanded mixed weak formulation of (3.1) is to find $(u,\widetilde{\boldsymbol{p}},\boldsymbol{p}) : J \mapsto W \times \boldsymbol{V} \times \boldsymbol{V}$ such that

$$(u_{tt}, w) + (\nabla \cdot \boldsymbol{p}, w) = \big(f(u), w\big), \quad \forall w \in W, \tag{3.2}$$

$$(\widetilde{\boldsymbol{p}}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, u) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.3}$$

$$(\boldsymbol{p}, \boldsymbol{v}) - (K\widetilde{\boldsymbol{p}}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.4}$$

with $u(0) = u_0$ and $u_t(0) = u_1$.

Let $\{t_n \mid t_n = n\tau; 0 \le n \le N\}$ be a uniform partition of the time interval with the time step $\tau = T/N$. We denote $\varphi^n = \varphi(\cdot, t_n)$. For a sequence of functions $\{\varphi^n\}_{n=0}^N$, we describe some of the notations which will be used in our analysis:

$$\varphi^{n+\frac{1}{2}} = \frac{\varphi^{n+1} + \varphi^n}{2}, \qquad \varphi^{n,\frac{1}{2}} = \frac{\varphi^{n+1} + \varphi^{n-1}}{2},$$

$$\partial_t\varphi^{n+\frac{1}{2}} = \frac{\varphi^{n+1} - \varphi^n}{\tau}, \qquad \partial_t\varphi^n = \frac{\varphi^{n+1} - \varphi^{n-1}}{2\tau},$$

$$\partial_{tt}\varphi^n = \frac{\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}}{\tau^2}.$$

Then, it is easy to verify the following relations:

$$\partial_t\varphi^n = \frac{\partial_t\varphi^{n+\frac{1}{2}} + \partial_t\varphi^{n-\frac{1}{2}}}{2}, \qquad \partial_{tt}\varphi^n = \frac{\partial_t\varphi^{n+\frac{1}{2}} - \partial_t\varphi^{n-\frac{1}{2}}}{\tau}.$$

Now, by the above some notations we can establish the following discrete time EMFE approximation of problem (3.2)–(3.4).

Fully-discrete EMFEM: Given initial value $(u_h^0, \widetilde{\boldsymbol{p}}_h^0, \boldsymbol{p}_h^0) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ such that

$$\big(u_h^0, w_h\big) = (u_0, w_h), \quad \forall w_h \in W_h, \tag{3.5}$$

$$\big(\widetilde{\boldsymbol{p}}_h^0, \boldsymbol{v}_h\big) = (u_0, \nabla \cdot \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.6}$$

$$\big(\boldsymbol{p}_h^0, \boldsymbol{v}_h\big) = (-K\nabla u_0, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.7}$$

and $(u_h^1, \widetilde{\boldsymbol{p}}_h^1, \boldsymbol{p}_h^1) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ such that

$$\left(\frac{u_h^1 - u_h^{-1}}{2\tau}, w_h\right) = (u_1, w_h), \quad \forall w_h \in W_h, \tag{3.8}$$

$$\left(\frac{\widetilde{\boldsymbol{p}}_h^1 - \widetilde{\boldsymbol{p}}_h^{-1}}{2\tau}, w_h\right) = (u_1, \nabla \cdot \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.9}$$

$$\left(\frac{\boldsymbol{p}_h^1 - \boldsymbol{p}_h^{-1}}{2\tau}, w_h\right) = (-K\nabla u_1, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.10}$$

for $n \geq 0$, find $(u_h^{n+1}, \widetilde{\boldsymbol{p}}_h^{n+1}, \boldsymbol{p}_h^{n+1}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ such that

$$\big(\partial_{tt} u_h^n, w_h\big) + \big(\nabla \cdot \boldsymbol{p}_h^{n,\frac{1}{2}}, w_h\big) = \big(f(u_h)^{n,\frac{1}{2}}, w_h\big), \quad \forall w_h \in W_h, \tag{3.11}$$

$$\big(\widetilde{\boldsymbol{p}}_h^{n+1}, \boldsymbol{v}_h\big) - \big(\nabla \cdot \boldsymbol{v}_h, u_h^{n+1}\big) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.12}$$

$$\big(K\widetilde{\boldsymbol{p}}_h^{n+1}, \boldsymbol{v}_h\big) - \big(\boldsymbol{p}_h^{n+1}, \boldsymbol{v}_h\big) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.13}$$

Equations (3.8) and (3.10) arise naturally by defining two fictitious values u_h^{-1} and $\boldsymbol{p}_h^{-1}$ satisfying the conditions

$$u_h^{-1} = u_h^1 - 2\tau u_1,$$
$$\boldsymbol{p}_h^{-1} = -K\nabla u_h^{-1} = -K\nabla\big(u_h^1 - 2\tau u_1\big),$$

and considering (3.11) with $n = 0$.

As in [29], we use the Sobolev embedding inequality

$$\|u_h\|_{0,p} \leq C_0 \|\nabla u_h\|, \quad 1 \leq p < \infty, \tag{3.14}$$

where C_0 is a constant only depending upon the domain and p.

Theorem 3.1 *Let $(u_h^n, \widetilde{\boldsymbol{p}}_h^n, \boldsymbol{p}_h^n) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ be the solution of (3.5)–(3.13). If $u_h^0 = \widehat{u}^0$, $u_h^1 = \widehat{u}^1$, and $\tau < \min\{\frac{1}{2}, \frac{1}{6M_1^2C_0^2}\}$, then for $1 \leq n \leq N$, we have*

$$\sup_n \big\{ \|u^n - u_h^n\| + \|\widetilde{\boldsymbol{p}}^n - \widetilde{\boldsymbol{p}}_h^n\| + \|\boldsymbol{p}^n - \boldsymbol{p}_h^n\| \big\} \leq C\big(h^{k+1} + \tau^2\big), \tag{3.15}$$

where k is associated with the degree of the finite element polynomial.

Proof From (3.2)–(3.4), we see that

$$\big(u_{tt}^{n,\frac{1}{2}}, w_h\big) + \big(\nabla \cdot \boldsymbol{p}^{n,\frac{1}{2}}, w_h\big) = \big(f(u)^{n,\frac{1}{2}}, w_h\big), \quad \forall w_h \in W_h, \tag{3.16}$$

$$\big(\widetilde{\boldsymbol{p}}^{n+1},\boldsymbol{v}_h\big)-\big(\nabla\cdot\boldsymbol{v}_h,u^{n+1}\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{3.17}$$

$$\big(K\widetilde{\boldsymbol{p}}^{n+1},\boldsymbol{v}_h\big)-\big(\boldsymbol{p}^{n+1},\boldsymbol{v}_h\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\tag{3.18}$$

Let $\mu^n=\widehat{u}^n-u_h^n$, $\overline{\chi}^n=\widehat{\widetilde{\boldsymbol{p}}}^n-\widetilde{\boldsymbol{p}}_h^n$, $\chi^n=\Pi_h\boldsymbol{p}^n-\boldsymbol{p}_h^n$, $\xi^n=u^n-\widehat{u}^n$, $\overline{\eta}^n=\widetilde{\boldsymbol{p}}^n-\widehat{\widetilde{\boldsymbol{p}}}^n$, and $\eta=\boldsymbol{p}^n-\Pi_h\boldsymbol{p}^n$. Using (3.11)–(3.13) and (3.16)–(3.18), we get the error equations:

$$\begin{aligned}\big(\partial_{tt}\mu^n,w_h\big)+\big(\nabla\cdot\chi^{n,\frac12},w_h\big)&=\big(\partial_{tt}u^n-u_{tt}^{n,\frac12},w_h\big)-\big(\partial_{tt}\xi^n,w_h\big)\\&\quad+\big(\big(f(u)-f(u_h)\big)^{n,\frac12},w_h\big),\quad\forall w_h\in W_h,\end{aligned}\tag{3.19}$$

$$\big(\overline{\chi}^{n+1},\boldsymbol{v}_h\big)-\big(\mu^{n+1},\nabla\cdot\boldsymbol{v}_h\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{3.20}$$

$$\big(K\overline{\chi}^{n+1},\boldsymbol{v}_h\big)-\big(\chi^{n+1},\boldsymbol{v}_h\big)=\big(\eta^{n+1},\boldsymbol{v}_h\big)-\big(K\overline{\eta}^{n+1},\boldsymbol{v}_h\big),\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\tag{3.21}$$

Subtracting (3.20) from itself, with $n+1$ replaced with $n-1$, we get

$$\big(\overline{\chi}^{n+1}-\overline{\chi}^{n-1},\boldsymbol{v}_h\big)-\big(\mu^{n+1}-\mu^{n-1},\nabla\cdot\boldsymbol{v}_h\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\tag{3.22}$$

It follows from (3.21) that

$$\begin{aligned}&\big(K\big(\overline{\chi}^{n+1}+\overline{\chi}^{n-1}\big),\boldsymbol{v}_h\big)-\big(\chi^{n+1}+\chi^{n-1},\boldsymbol{v}_h\big)\\&\quad=\big(\eta^{n+1}+\eta^{n-1},\boldsymbol{v}_h\big)-\big(K\big(\overline{\eta}^{n+1}+\overline{\eta}^{n-1}\big),\boldsymbol{v}_h\big),\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\end{aligned}\tag{3.23}$$

Let $w_h=\partial_t\mu^n$ in (3.19), $\boldsymbol{v}_h=\frac{\chi^{n,\frac12}}{2\tau}$ in (3.22), and $\boldsymbol{v}_h=\frac{\partial_t\overline{\chi}^n}{2}$ in (3.23), and combine the three resulting equations to get

$$\begin{aligned}&\big(\partial_{tt}\mu^n,\partial_t\mu^n\big)+\big(\partial_t\overline{\chi}^n,K\overline{\chi}^{n,\frac12}\big)\\&\quad=\frac{1}{2\tau}\big(\big\|\partial_t\mu^{n+\frac12}\big\|^2-\big\|\partial_t\mu^{n-\frac12}\big\|^2\big)+\frac{1}{4\tau}\big(\big\|K^{\frac12}\overline{\chi}^{n+1}\big\|^2-\big\|K^{\frac12}\overline{\chi}^{n-1}\big\|^2\big)\\&\quad=\big(\partial_{tt}u^n-u_{tt}^{n,\frac12},\partial_t\mu^n\big)-\big(\partial_{tt}\xi^n,\partial_t\mu^n\big)+\big(\big(f(u)-f(u_h)\big)^{n,\frac12},\partial_t\mu^n\big)\\&\qquad+\big(\eta^{n,\frac12},\partial_t\overline{\chi}^n\big)-\big(K\overline{\eta}^{n,\frac12},\partial_t\overline{\chi}^n\big)\\&\quad:=\sum_{i=1}^5A_i.\end{aligned}\tag{3.24}$$

For A_1, we note that $\partial_{tt}u^n=\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}$, and use Taylor series to expand u^{n+1} and u^{n-1} at u^n as follows:

$$\begin{aligned}u^{n+1}&=u^n+\tau\frac{\partial u^n}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2u^n}{\partial t^2}+\frac{\tau^3}{6}\frac{\partial^3u^n}{\partial t^3}+\frac16\int_{t_n}^{t_{n+1}}(t_{n+1}-t)^3\frac{\partial^4u}{\partial t^4}\,dt,\\u^{n-1}&=u^n-\tau\frac{\partial u^n}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2u^n}{\partial t^2}-\frac{\tau^3}{6}\frac{\partial^3u^n}{\partial t^3}+\frac16\int_{t_{n-1}}^{t_n}(t_{n-1}-t)^3\frac{\partial^4u}{\partial t^4}\,dt.\end{aligned}$$

Then we have

$$u^{n+1}+u^{n-1}=2u^n+\tau^2\frac{\partial^2u^n}{\partial t^2}+\frac16\int_{-\tau}^{\tau}\big(|t|-\tau\big)^3\frac{\partial^4u}{\partial t^4}(t_n+t)\,dt.\tag{3.25}$$

It follows from (3.25) that

$$\partial_{tt}u^n - u_{tt}^n = \frac{1}{6\tau^2}\int_{-\tau}^{\tau}\big(|t|-\tau\big)^3\frac{\partial^4 u}{\partial t^4}(t_n+t)\,dt.$$

Therefore

$$\begin{aligned}\big\|\partial_{tt}u^n - u_{tt}^n\big\|^2 &\le C\tau^2\int_\Omega\left[\int_{t_n-\tau}^{t_n+\tau}\frac{\partial^4 u}{\partial t^4}(t)\right]^2 dx\\ &\le C\tau^3\int_{t_n-\tau}^{t_n+\tau}\left\|\frac{\partial^4 u}{\partial t^4}\right\|^2 dt\\ &\le C\tau^4\left\|\frac{\partial^4 u}{\partial t^4}\right\|^2_{L^\infty(L^2)}.\end{aligned}$$

Using Young's inequality, we have

$$A_1 \le C\tau^4\left\|\frac{\partial^4 u}{\partial t^4}\right\|^2_{L^\infty(L^2)} + \varepsilon_1\big\|\partial_t\mu^n\big\|^2. \tag{3.26}$$

For the second term A_2, using the property of the L^2 projection and Young's inequality, we see that

$$\begin{aligned}A_2 &= \big(\xi_{tt}^n - \xi_{tt}^n - \partial_{tt}\xi^n, \partial_t\mu^n\big)\\ &\le C\left\{h^{2k+2} + \tau^4\left\|\frac{\partial^4 u}{\partial t^4}\right\|^2_{L^\infty(L^2)}\right\} + \varepsilon_2\big\|\partial_t\mu^n\big\|^2.\end{aligned} \tag{3.27}$$

For the third term A_3, by assumption (iii) and (2.7), we obtain

$$\begin{aligned}A_3 &= \big(\big(f(u)-f(u_h)\big)^{n,\frac{1}{2}}, \partial_t\mu^n\big)\\ &\le \big(M_1\big(u^{n,\frac{1}{2}} - \widehat{u}^{n,\frac{1}{2}} + \widehat{u}^{n,\frac{1}{2}} - u_h^{n,\frac{1}{2}}\big), \partial_t\mu^n\big)\\ &\le Ch^{2k+2} + \frac{M_1^2}{2}\big(\big\|\mu^{n+1}\big\|^2 + \big\|\mu^{n-1}\big\|^2\big) + \varepsilon_3\big\|\partial_t\mu^n\big\|^2.\end{aligned} \tag{3.28}$$

For the terms A_4 and A_5, applying Young's inequality, assumption (ii), and (2.8), we have

$$\begin{aligned}A_4 + A_5 &= \big(\eta^{n,\frac{1}{2}}, \partial_t\overline{\chi}^n\big) - \big(K\overline{\eta}^{n,\frac{1}{2}}, \partial_t\overline{\chi}^n\big)\\ &\le C\big\{\big\|\eta^{n+1}\big\|^2 + \big\|\eta^{n-1}\big\|^2 + \big\|\overline{\eta}^{n+1}\big\|^2 + \big\|\overline{\eta}^{n-1}\big\|^2\big\} + (\varepsilon_4+\varepsilon_5)\big\|\partial_t\overline{\chi}^n\big\|^2\\ &\le Ch^{2k+2} + (\varepsilon_4+\varepsilon_5)\big\|\partial_t\overline{\chi}^n\big\|^2.\end{aligned} \tag{3.29}$$

From (3.24), (3.26)–(3.29), choosing $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 1$ and $\varepsilon_4+\varepsilon_5 = \varepsilon > 0$, ε is a small constant, we have

$$\begin{aligned}&\frac{1}{2\tau}\big(\big\|\partial_t\mu^{n+\frac{1}{2}}\big\|^2 - \big\|\partial_t\mu^{n-\frac{1}{2}}\big\|^2\big) + \frac{1}{4\tau}\big(\big\|K^{\frac{1}{2}}\overline{\chi}^{n+1}\big\|^2 - \big\|K^{\frac{1}{2}}\overline{\chi}^{n-1}\big\|^2\big)\\ &\quad\le C\big\{h^{2k+2}+\tau^4\big\} + \frac{M_1^2}{2}\big(\big\|\mu^{n+1}\big\|^2 + \big\|\mu^{n-1}\big\|^2\big) + \big\|\partial_t\mu^n\big\|^2 + \varepsilon\big\|\partial_t\overline{\chi}^n\big\|^2.\end{aligned} \tag{3.30}$$

Multiplying both sides of (3.30) by 4τ and summing from 1 to N, we get

$$\begin{aligned}
&2\left\|\partial_t\mu^{N+\frac{1}{2}}\right\|^2-2\left\|\partial_t\mu^{\frac{1}{2}}\right\|^2+\left\|K^{\frac{1}{2}}\overline{\chi}^{N+1}\right\|^2+\left\|K^{\frac{1}{2}}\overline{\chi}^{N}\right\|^2-\left\|K^{\frac{1}{2}}\overline{\chi}^{1}\right\|^2-\left\|K^{\frac{1}{2}}\overline{\chi}^{0}\right\|^2\\
&\quad\le C\left\{h^{2k+2}+\tau^4\right\}+2M_1^2\tau\sum_{n=1}^{N}\left(\left\|\mu^{n+1}\right\|^2+\left\|\mu^{n-1}\right\|^2\right)+4\tau\sum_{n=1}^{N}\left\|\partial_t\mu^n\right\|^2\\
&\qquad+4\varepsilon\tau\sum_{n=1}^{N}\left\|\partial_t\overline{\chi}^n\right\|^2.
\end{aligned}\tag{3.31}$$

With the proper choice of the initial functions $u_h^0=\widehat{u}^0$ and $u_h^1=\widehat{u}^1$, we have

$$\partial_t\mu^{\frac{1}{2}}=0,\qquad \overline{\chi}^0=\mathbf{0}.$$

Taking $n=0$ in (3.20) and choosing $\boldsymbol{v}_h=\overline{\chi}^1$, we get

$$\overline{\chi}^1=\mathbf{0}.$$

In view of $\|\partial_t\mu^n\|^2\le\|\partial_t\mu^{n+\frac{1}{2}}\|^2+\|\partial_t\mu^{n-\frac{1}{2}}\|^2$, and using assumption (ii), (3.31) can be rewritten as follows:

$$\begin{aligned}
&(2-4\tau)\left\|\partial_t\mu^{N+\frac{1}{2}}\right\|^2+\left\|\overline{\chi}^{N+1}\right\|^2+\left\|\overline{\chi}^{N}\right\|^2\\
&\quad\le C\left\{h^{2k+2}+\tau^4\right\}+2M_1^2\tau\sum_{n=1}^{N}\left(\left\|\mu^{n+1}\right\|^2+\left\|\mu^{n-1}\right\|^2\right)+4\tau\sum_{n=1}^{N-1}\left\|\partial_t\mu^{n+\frac{1}{2}}\right\|^2\\
&\qquad+4\varepsilon\tau\sum_{n=1}^{N}\left\|\partial_t\overline{\chi}^n\right\|^2.
\end{aligned}\tag{3.32}$$

By using the result in (3.14), we see that

$$\left\|\mu^{n+1}\right\|\le C_0\left\|\overline{\chi}^{n+1}\right\|.\tag{3.33}$$

Let $\boldsymbol{v}_h=\chi^{n+1}\in\boldsymbol{V}_h$ in (3.21), we have

$$\left\|\chi^{n+1}\right\|^2\le C\left\|\overline{\chi}^{n+1}\right\|^2.\tag{3.34}$$

Substitute (3.33) and (3.34) into (3.32), we obtain

$$\begin{aligned}
&(2-4\tau)\left\|\partial_t\mu^{N+\frac{1}{2}}\right\|^2+\left(\frac{1}{3C_0^2h^2}-2M_1^2\tau\right)\left\|\mu^{N+1}\right\|^2\\
&\qquad+\frac{1}{3C_1}\left\|\chi^{N+1}\right\|^2+\frac{1}{3}\left\|\overline{\chi}^{N+1}\right\|^2+\left\|\overline{\chi}^{N}\right\|^2\\
&\quad\le C\left\{h^{2k+2}+\tau^4\right\}+C\tau\sum_{n=0}^{N}\left\|\mu^n\right\|^2+4\tau\sum_{n=1}^{N-1}\left\|\partial_t\mu^{n+\frac{1}{2}}\right\|^2+4\varepsilon\tau\sum_{n=1}^{N}\left\|\partial_t\overline{\chi}^n\right\|^2.
\end{aligned}\tag{3.35}$$

Applying the discrete Gronwall's inequality, when $\tau<\min\{\frac{1}{2},\frac{1}{6M_1^2C_0^2}\}$, we see that

$$\left\|\partial_t\mu^{N+\frac{1}{2}}\right\|^2+\left\|\mu^{N+1}\right\|^2+\left\|\chi^{N+1}\right\|^2+\left\|\overline{\chi}^{N+1}\right\|^2\le C\left\{h^{2k+2}+\tau^4\right\}.\tag{3.36}$$

Thus, combining (2.7), (2.8), (3.36) and using the triangle inequality, we can derive (3.15). □

4 A priori error estimates of two-grid method

In this section, we present a two-grid algorithm with expanded mixed element for (3.5)–(3.13). The basic ingredient in our approach is construction of two quasi-uniform triangulation of Ω, $\mathcal{T}_H$, and $\mathcal{T}_h$, with different mesh sizes H and h ($h \ll H < 1$). We introduce the corresponding mixed finite element spaces $W_H \times \boldsymbol{V}_H \times \boldsymbol{V}_H$ and $W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$, which satisfy $W_H \times \boldsymbol{V}_H \times \boldsymbol{V}_H \subset W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$. They will be called the coarse grid and fine grid spaces, respectively. Our two-grid algorithm is as follows.

Algorithm 1 Step 1: On the coarse grid $\mathcal{T}_H$, given the initial value $(u_H^0, \tilde{\boldsymbol{p}}_H^0, \boldsymbol{p}_H^0) \in W_H \times \boldsymbol{V}_H \times \boldsymbol{V}_H$ such that

$$\big(u_H^0, w_H\big) = (u_0, w_H), \quad \forall w_H \in W_H, \tag{4.1}$$

$$\big(\tilde{\boldsymbol{p}}_H^0, \boldsymbol{v}_H\big) = (u_0, \nabla \cdot \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.2}$$

$$\big(\boldsymbol{p}_H^0, \boldsymbol{v}_H\big) = (-K\nabla u_0, \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.3}$$

and $(u_H^1, \tilde{\boldsymbol{p}}_H^1, \boldsymbol{p}_H^1) \in W_H \times \boldsymbol{V}_H \times \boldsymbol{V}_H$ such that

$$\left(\frac{u_H^1 - u_H^{-1}}{2\tau}, w_H\right) = (u_1, w_H), \quad \forall w_H \in W_H, \tag{4.4}$$

$$\left(\frac{\tilde{\boldsymbol{p}}_H^1 - \tilde{\boldsymbol{p}}_H^{-1}}{2\tau}, w_H\right) = (u_1, \nabla \cdot \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.5}$$

$$\left(\frac{\boldsymbol{p}_H^1 - \boldsymbol{p}_H^{-1}}{2\tau}, w_H\right) = (-K\nabla u_1, \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.6}$$

for $n \geq 0$, find $(u_H^{n+1}, \tilde{\boldsymbol{p}}_H^{n+1}, \boldsymbol{p}_H^{n+1}) \in W_H \times \boldsymbol{V}_H \times \boldsymbol{V}_H$, solve the following semi-linear system:

$$\big(\partial_{tt} u_H^n, w_H\big) + \big(\nabla \cdot \boldsymbol{p}_H^{n,\frac{1}{2}}, w_H\big) = \big(f(u_H)^{n,\frac{1}{2}}, w_H\big), \quad \forall w_H \in W_H, \tag{4.7}$$

$$\big(\tilde{\boldsymbol{p}}_H^{n+1}, \boldsymbol{v}_H\big) - \big(\nabla \cdot \boldsymbol{v}_H, u_H^{n+1}\big) = 0, \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.8}$$

$$\big(K\tilde{\boldsymbol{p}}_H^{n+1}, \boldsymbol{v}_H\big) - \big(\boldsymbol{v}_H, \boldsymbol{p}_H^{n+1}\big) = 0, \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{4.9}$$

Step 2: On the fine grid $\mathcal{T}_h$, given the initial value $(\mathcal{U}_h^0, \widetilde{\mathcal{P}}_h^0, \mathcal{P}_h^0) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ such that

$$\big(\mathcal{U}_h^0, w_h\big) = (u_0, w_h), \quad \forall w_h \in W_h, \tag{4.10}$$

$$\big(\widetilde{\mathcal{P}}_h^0, \boldsymbol{v}_h\big) = (u_0, \nabla \cdot \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.11}$$

$$\big(\mathcal{P}_h^0, \boldsymbol{v}_h\big) = (-K\nabla u_0, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.12}$$

and $(\mathcal{U}_h^1, \widetilde{\mathcal{P}}_h^1, \mathcal{P}_h^1) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ such that

$$\left(\frac{\mathcal{U}_h^1 - \mathcal{U}_h^{-1}}{2\tau}, w_h\right) = (u_1, w_h), \quad \forall w_h \in W_h, \tag{4.13}$$

$$\left(\frac{\widetilde{\mathcal{P}}_h^1-\widetilde{\mathcal{P}}_h^{-1}}{2\tau},w_h\right)=(u_1,\nabla\cdot\boldsymbol{v}_h),\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{4.14}$$

$$\left(\frac{\mathcal{P}_h^1-\mathcal{P}_h^{-1}}{2\tau},w_h\right)=(-K\nabla u_1,\boldsymbol{v}_h),\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{4.15}$$

for $n\geq 0$, find $(\mathcal{U}_h^{n+1},\widetilde{\mathcal{P}}_h^{n+1},\mathcal{P}_h^{n+1})\in W_h\times\boldsymbol{V}_h\times\boldsymbol{V}_h$, solve the following linear system:

$$\big(\partial_{tt}\mathcal{U}_h^n,w_h\big)+\big(\nabla\cdot\mathcal{P}_h^{n,\frac12},w_h\big)=\big(\big(f(u_H)+f'(u_H)(\mathcal{U}_h-u_H)\big)^{n,\frac12},w_h\big),\quad\forall w_h\in W_h,\tag{4.16}$$

$$\big(\widetilde{\mathcal{P}}_h^{n+1},\boldsymbol{v}_h\big)-\big(\nabla\cdot\boldsymbol{v}_h,\mathcal{U}_h^{n+1}\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{4.17}$$

$$\big(K\widetilde{\mathcal{P}}_h^{n+1},\boldsymbol{v}_h\big)-\big(\boldsymbol{v}_h,\mathcal{P}_h^{n+1}\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\tag{4.18}$$

We note that, for $n=0$ in (4.16), the values $\mathcal{U}_h^{-1}$ and $\mathcal{P}_h^{-1}$ can be determined by (4.13) and (4.15).

Now we consider the error estimate for the two-grid algorithm. Obviously, Theorem 3.1 holds for the solution of the coarse mesh with $h=H$. In the following, we derive the error estimate of the fine grid.

Theorem 4.1 *Let $(\mathcal{U}_h^n,\widetilde{\mathcal{P}}_h^n,\mathcal{P}_h^n)\in W_h\times\boldsymbol{V}_h\times\boldsymbol{V}_h$ be the solution of (4.10)–(4.18). Assume that (i)–(iii) hold and take $\mathcal{U}_h^0=\widehat{u}^0$, $\mathcal{U}_h^1=\widehat{u}^1$, $\tau<\min\{\frac12,\frac{1}{6C_0{}^2M_1^2}\}$, then for $1\leq n\leq N$, there exists a positive constant C such that*

$$\sup_n\big\{\|u^n-\mathcal{U}_h^n\|+\|\widetilde{\boldsymbol{p}}^n-\widetilde{\mathcal{P}}_h^n\|+\|\boldsymbol{p}^n-\mathcal{P}_h^n\|\big\}\leq C\big(h^{k+1}+H^{2k+2}+\tau^2\big),$$

where k is associated with the degree of the finite element polynomial.

Proof Taking $\beta^n=\widehat{u}^n-\mathcal{U}_h^n$, $\overline{\gamma}^n=\widehat{\widetilde{\boldsymbol{p}}}^n-\widetilde{\mathcal{P}}_h^n$, and $\gamma^n=\Pi_h\boldsymbol{p}^n-\mathcal{P}_h^n$. Let us first note the following error equations from (3.16)–(3.18) and (4.16)–(4.18):

$$\begin{aligned}&\big(\partial_{tt}\beta^n,w_h\big)+\big(\nabla\cdot\gamma^{n,\frac12},w_h\big)\\&\quad=\big(\partial_{tt}u^n-u_{tt}^{n,\frac12},w_h\big)-\big(\partial_{tt}\xi^n,w_h\big)+\big(f(u)^{n,\frac12},w_h\big)\\&\qquad-\big(\big(f(u_H)+f'(u_H)(\mathcal{U}_h-u_H)\big)^{n,\frac12},w_h\big),\quad\forall w_h\in W_h,\end{aligned}\tag{4.19}$$

$$\big(\overline{\gamma}^{n+1},\boldsymbol{v}_h\big)-\big(\beta^{n+1},\nabla\cdot\boldsymbol{v}_h\big)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{4.20}$$

$$\big(K\overline{\gamma}^{n+1},\boldsymbol{v}_h\big)-\big(\gamma^{n+1},\boldsymbol{v}_h\big)=\big(\eta^{n+1},\boldsymbol{v}_h\big)-\big(K\overline{\eta}^{n+1},\boldsymbol{v}_h\big),\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\tag{4.21}$$

We now rewrite terms on the right-hand side of (4.19). A Taylor expansion of f about u_H yields

$$\begin{aligned}&\big(f(u)^{n,\frac12},w_h\big)-\big(\big(f(u_H)+f'(u_H)(\mathcal{U}_h-u_H)\big)^{n,\frac12},w_h\big)\\&\quad=\left(f(u_H)^{n,\frac12}+\big(f'(u_H)(u-u_H)\big)^{n,\frac12}+\frac12\big(f''\big(u^*\big)(u-u_H)^2\big)^{n,\frac12},w_h\right)\\&\qquad-\big(\big(f(u_H)+f'(u_H)(\mathcal{U}_h-u_H)\big)^{n,\frac12},w_h\big)\\&\quad=\big(\big(f'(u_H)(u-\mathcal{U}_h)\big)^{n,\frac12},w_h\big)+\left(\frac12\big(f''\big(u^*\big)(u-u_H)^2\big)^{n,\frac12},w_h\right)\end{aligned}$$

for some function u^*. By (4.20) and (4.21), we have

$$\left(\overline{\gamma}^{n+1}-\overline{\gamma}^{n-1},\boldsymbol{v}_h\right)-\left(\beta^{n+1}-\beta^{n-1},\nabla\cdot\boldsymbol{v}_h\right)=0,\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{4.22}$$

$$\begin{aligned}\left(K\left(\overline{\gamma}^{n+1}+\overline{\gamma}^{n-1}\right),\boldsymbol{v}_h\right)-\left(\gamma^{n+1}+\gamma^{n-1},\boldsymbol{v}_h\right)&=\left(\eta^{n+1}+\eta^{n-1},\boldsymbol{v}_h\right)\\&\quad-\left(K\left(\overline{\eta}^{n+1}+\overline{\eta}^{n-1}\right),\boldsymbol{v}_h\right),\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\end{aligned}\tag{4.23}$$

Choose $w_h=\partial_t\beta^n$, $\boldsymbol{v}_h=\frac{\gamma^{n,\frac{1}{2}}}{2\tau}$, and $\boldsymbol{v}_h=\frac{\partial_t\overline{\gamma}^n}{2}$ as test functions in (4.19), (4.22), and (4.23), respectively. Then, add the resulting equations to obtain

$$\begin{aligned}&\left(\partial_{tt}\beta^n,\partial_t\beta^n\right)+\left(\partial_t\overline{\gamma}^n,K\overline{\gamma}^{n,\frac{1}{2}}\right)\\&=\frac{1}{2\tau}\left(\left\|\partial_t\beta^{n+\frac{1}{2}}\right\|^2-\left\|\partial_t\beta^{n-\frac{1}{2}}\right\|^2\right)+\frac{1}{4\tau}\left(\left\|K^{\frac{1}{2}}\overline{\gamma}^{n+1}\right\|^2-\left\|K^{\frac{1}{2}}\overline{\gamma}^{n-1}\right\|^2\right)\\&=\left(\partial_{tt}u^n-u_{tt}^{n,\frac{1}{2}},\partial_t\beta^n\right)-\left(\partial_{tt}\xi^n,\partial_t\beta^n\right)+\left(\eta^{n,\frac{1}{2}},\partial_t\overline{\gamma}^n\right)-\left(K\overline{\eta}^{n,\frac{1}{2}},\partial_t\overline{\gamma}^n\right)\\&\quad+\left(\left(f'(u_H)(u-\mathcal{U}_h)\right)^{n,\frac{1}{2}},\partial_t\beta^n\right)+\left(\frac{1}{2}\left(f''(u^*)(u-u_H)^2\right)^{n,\frac{1}{2}},\partial_t\beta^n\right)\\&:=\sum_{i=1}^{6}I_i.\end{aligned}\tag{4.24}$$

In the same way as the estimate of (3.24), we get

$$\begin{aligned}&I_1+I_2\le C\left\{h^{2k+2}+\tau^4\left\|\frac{\partial^4u}{\partial t^4}(t)\right\|_{L^\infty(L^2)}^2\right\}+\varepsilon_6\left\|\partial_t\beta^n\right\|^2,\\&I_3+I_4\le Ch^{2k+2}+\varepsilon\left\|\partial_t\overline{\gamma}^n\right\|^2.\end{aligned}\tag{4.25}$$

The last two terms on the right-hand side of (4.24) can be bound using (2.7) and (3.15), we see that

$$\begin{aligned}&I_5\le Ch^{2k+2}+\frac{M_1^2}{2}\left(\left\|\beta^{n+1}\right\|^2+\left\|\beta^{n-1}\right\|^2\right)+\varepsilon_7\left\|\partial_t\mu^n\right\|^2,\\&I_6\le C\left(\left\|(u-u_H)^2\right\|\right)^{n,\frac{1}{2}}\left\|\partial_t\mu^n\right\|\le C\left\{H^{4k+4}+\tau^4\right\}+\varepsilon_8\left\|\partial_t\mu^n\right\|^2.\end{aligned}\tag{4.26}$$

From (4.24)–(4.26), set $\sum_{i=6}^{8}\varepsilon_i=1$, we find that

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|\partial_t\beta^{n+\frac{1}{2}}\right\|^2-\left\|\partial_t\beta^{n-\frac{1}{2}}\right\|^2\right)+\frac{1}{4\tau}\left(\left\|K^{\frac{1}{2}}\overline{\gamma}^{n+1}\right\|^2-\left\|K^{\frac{1}{2}}\overline{\gamma}^{n-1}\right\|^2\right)\\&\quad\le C\left\{h^{2k+2}+H^{4k+4}+\tau^4\right\}+\frac{M_1^2}{2}\left(\left\|\beta^{n+1}\right\|^2+\left\|\beta^{n-1}\right\|^2\right)+\left\|\partial_t\mu^n\right\|^2+\varepsilon\left\|\partial_t\overline{\gamma}^n\right\|^2.\end{aligned}\tag{4.27}$$

Multiplying by 4τ and summing (4.27) from $n=1$ to N, the resulting equation becomes

$$2\left\|\partial_t\beta^{N+\frac{1}{2}}\right\|^2-2\left\|\partial_t\beta^{\frac{1}{2}}\right\|^2+\left\|K^{\frac{1}{2}}\overline{\gamma}^{N+1}\right\|^2+\left\|K^{\frac{1}{2}}\overline{\gamma}^{N}\right\|^2-\left\|K^{\frac{1}{2}}\overline{\gamma}^{1}\right\|^2-\left\|K^{\frac{1}{2}}\overline{\gamma}^{0}\right\|^2$$

$$
\begin{aligned}
&\le C\{h^{2k+2}+H^{4k+4}+\tau^4\}+2M_1^2\tau\sum_{n=1}^{N}\left(\|\beta^{n+1}\|^2+\|\beta^{n-1}\|^2\right)+4\tau\sum_{n=1}^{N}\|\partial_t\beta^n\|^2 \qquad (4.28)\\
&\quad+4\varepsilon\tau\sum_{n=1}^{N}\|\partial_t\overline{\gamma}^n\|^2.
\end{aligned}
$$

In the following, similar to the estimates of (3.33) and (3.34), we have

$$
\begin{aligned}
&\|\beta^{n+1}\|\le C_0\|\overline{\gamma}^{n+1}\|,\\
&\|\gamma^{n+1}\|^2\le C\|\overline{\gamma}^{n+1}\|^2.
\end{aligned} \qquad (4.29)
$$

By the initial conditions $\mathcal{U}_h^0=\widehat{u}^0$ and $\mathcal{U}_h^1=\widehat{u}^1$, we can easily obtain

$$
\partial_t\beta^{\frac{1}{2}}=0,\qquad \overline{\gamma}^0=\overline{\gamma}^1=\mathbf{0}. \qquad (4.30)
$$

Combining (4.28)–(4.30) results in

$$
\begin{aligned}
&(2-4\tau)\|\partial_t\beta^{N+\frac{1}{2}}\|^2+\left(1-2M_1^2\tau\right)\|\beta^{N+1}\|^2+\|\gamma^{N+1}\|^2+\|\overline{\gamma}^{N+1}\|^2+\|\overline{\gamma}^N\|^2\\
&\quad\le C\{h^{2k+2}+H^{4k+4}+\tau^4\}+C\tau\sum_{n=0}^{N}\|\beta^n\|^2\\
&\qquad+4\tau\sum_{n=1}^{N-1}\|\partial_t\beta^{n+\frac{1}{2}}\|^2+4\varepsilon\tau\sum_{n=1}^{N}\|\partial_t\overline{\gamma}^n\|^2.
\end{aligned} \qquad (4.31)
$$

When $\tau<\min\{\frac{1}{2},\frac{1}{6C_0{}^2M_1^2}\}$, the discrete Gronwall's lemma yields

$$
\|\partial_t\beta^{N+\frac{1}{2}}\|^2+\|\beta^{N+1}\|^2+\|\gamma^{N+1}\|^2+\|\overline{\gamma}^{N+1}\|^2\le C\{h^{2k+2}+H^{4k+4}+\tau^4\}. \qquad (4.32)
$$

The final result is obtained by using (2.7), (2.8), (4.32) and the triangle inequality. □

Remark 4.1 From Theorem 4.1, we see that the optimal error estimate is $\mathcal{O}(\tau^2+h^{k+1})$ in L^2-norm by taking $H=\mathcal{O}(h^{1/2})$. This result is consistent with the optimal error result (3.15) obtained for EMFE system (3.5)–(3.13).

5 Numerical experiments

In this section, we give some numerical experiments to support the analysis developed in the paper and to assess the merit of the two-grid method when compared with the EMFEM. In the numerical examples reported below, we choose RT_k ($k=0,1$) as the approximation space. The domain Ω is uniformly divided into two families $\mathcal{T}_H$ and $\mathcal{T}_h$ of triangular elements with $H=h^{1/2}$. In order to obtain the convergence rate of space mesh size, J is also uniformly divided so that τ is a small time step.

Example 1 We consider the following hyperbolic problem:

$$
\begin{cases}
u_{tt}-\nabla\cdot(K\nabla u)-u^5=g, & (\boldsymbol{x},t)\in\Omega\times J,\\
u(\boldsymbol{x},0)=u_0,\qquad u_t(\boldsymbol{x},0)=u_1, & \boldsymbol{x}\in\Omega,\\
u=0,\qquad (\boldsymbol{x},t)\in\partial\Omega\times J,
\end{cases}
$$

Table 1 Errors and computational time of the EMFEM with RT_1 element

h	$\|u-u_h\|$	$\|\widetilde{\boldsymbol{p}}-\widetilde{\boldsymbol{p}}_h\|$	$\|\boldsymbol{p}-\boldsymbol{p}_h\|$	Computing time (s)
2^{-2}	1.8758e–2	9.4517e–2	9.4822e–2	0.54
2^{-4}	1.3067e–3	6.6139e–3	6.6214e–3	5.27
2^{-6}	9.2041e–5	4.6862e–4	4.6851e–4	121.78
2^{-8}	6.3792e–6	3.2754e–5	3.2967e–5	1470.35
Rates	2.0	2.0	2.0	

Table 2 Errors and computational time of the two-grid method with RT_1 element

(H,h)	$\|u-\mathcal{U}_h\|$	$\|\widetilde{\boldsymbol{p}}-\widetilde{\mathcal{P}}_h\|$	$\|\boldsymbol{p}-\mathcal{P}_h\|$	Computing time (s)
$(2^{-1},2^{-2})$	1.8843e–2	9.4908e–2	9.5169e–2	0.71
$(2^{-2},2^{-4})$	1.3341e–3	6.8325e–3	6.9089e–3	1.95
$(2^{-3},2^{-6})$	9.3167e–5	4.8378e–4	4.8919e–4	29.84
$(2^{-4},2^{-8})$	6.5058e–6	3.4161e–5	3.4732e–5	377.51
Rates	2.0	2.0	2.0	

where $\Omega = (0,1)^2$, $J = (0,1]$, $\boldsymbol{x} = (x_1,x_2)^T$, and

$$K = \begin{pmatrix} x_1^2+1 & 0 \\ 0 & x_2^2+1 \end{pmatrix}.$$

The functions g, u_0, and u_1 are computed from the exact solution $u(\boldsymbol{x},t) = e^{-t}\sin(\pi x_1) \times \sin(\pi x_2)$.

We use Raviart–Thomas spaces of index $k = 1$ (RT_1). The error results, convergence rates, and computational time obtained with $\tau = 1.0e$–3 by the EMEFM and the two-grid method are presented in Tables 1 and 2.

From the numerical results in Tables 1 and 2, we can easily observe that the proposed two methods are of second-order accuracy in L^2-norm, which coincides with our theoretical analysis. Moreover, by comparing with the last columns of two tables, it is easy to see that given the same accuracy the two-grid method is much more efficient than the EMFEM.

Example 2 We consider the following hyperbolic problem:

$$\begin{cases} u_{tt} - \nabla\cdot(K\nabla u) - \sin u + u^2 = g, & (\boldsymbol{x},t)\in\Omega\times J, \\ u(\boldsymbol{x},0) = u_0, \quad u_t(\boldsymbol{x},0) = u_1, & \boldsymbol{x}\in\Omega, \\ u = 0, \quad (\boldsymbol{x},t)\in\partial\Omega\times J, \end{cases}$$

where $\Omega = (0,1)^2$, $J = (0,1]$, $\boldsymbol{x} = (x_1,x_2)^T$, and

$$K = \begin{pmatrix} x_1^2+1 & 0 \\ 0 & x_2^2+1 \end{pmatrix}.$$

The functions g, u_0, and u_1 are chosen so that the exact solution $u(\boldsymbol{x},t) = e^{-t}(x_1^4 - x_1^3)(x_2^2 - x_2)$.

We consider the lowest Raviart–Thomas spaces of index $k = 0$ (RT_0). The numerical results and computational time obtained with $\tau = 1.0e$–3 are shown in Tables 3 and 4. It can be seen from Tables 3 and 4 that $\|u-\mathcal{U}_h\|$, $\|\widetilde{\boldsymbol{p}}-\widetilde{\mathcal{P}}_h\|$, and $\|\boldsymbol{p}-\mathcal{P}_h\|$ are convergent at the rate of $\mathcal{O}(h)$, which is in accordance with the theoretical analysis.

Table 3 Errors and computational time of the EMFEM with RT_0 element

h	$\|u-u_h\|$	$\|\widetilde{\boldsymbol{p}}-\widetilde{\boldsymbol{p}}_h\|$	$\|\boldsymbol{p}-\boldsymbol{p}_h\|$	Computing time (s)
2^{-2}	4.2443e–3	1.5198e–2	1.5957e–2	0.37
2^{-4}	1.1176e–3	4.0412e–3	4.2548e–3	4.92
2^{-6}	2.9495e–4	1.0679e–3	1.1212e–3	96.09
2^{-8}	7.6126e–5	2.8173e–4	2.9416e–4	816.48
Rates	1.0	1.0	1.0	

Table 4 Errors and computational time of the two-grid method with RT_0 element

(H,h)	$\|u-\mathcal{U}_h\|$	$\|\widetilde{\boldsymbol{p}}-\widetilde{\mathcal{P}}_h\|$	$\|\boldsymbol{p}-\mathcal{P}_h\|$	Computing time (s)
$(2^{-1},2^{-2})$	4.4721e–3	1.6902e–2	1.7111e–2	0.58
$(2^{-2},2^{-4})$	1.1817e–3	4.4975e–3	4.5216e–3	1.61
$(2^{-3},2^{-6})$	3.0799e–4	1.1758e–3	1.1948e–3	23.76
$(2^{-4},2^{-8})$	7.9547e–5	3.0644e–4	3.1401e–4	227.14
Rates	1.0	1.0	1.0	

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Availability of data and materials
The data used to support the findings of this study are available from the corresponding author upon request.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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