RESEARCH

Open Access

An implicit relation, relational theoretic approach under w -distance and application to nonlinear matrix equations

Reena Jain¹, Hemant Kumar Nashine^{2*} and Manuel De la Sen³

*Correspondence:
hemantkumarnashine@tdtu.edu.vn

²Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, 700000, Vietnam
Full list of author information is available at the end of the article

Abstract

We propose a new class of implicit relations and an implicit type contractive condition based on it in the relational metric spaces under w -distance functional. Further we derive fixed points results based on them. Useful examples illustrate the applicability and effectiveness of the presented results. We apply these results to discuss sufficient conditions ensuring the existence of a unique positive definite solution of the nonlinear matrix equation (NME) of the form $U = Q + \sum_{i=1}^k A_i^* \mathcal{G}(U) A_i$, where Q is an $n \times n$ Hermitian positive definite matrix, $A_1, A_2, \dots, A_m$ are $n \times n$ matrices and $\mathcal{G}$ is a nonlinear self-mapping of the set of all Hermitian matrices which are continuous in the trace norm. In order to demonstrate the obtained conditions, we consider an example together with convergence and error analysis and visualisation of solutions in a surface plot.

MSC: Primary 47H10; secondary 54H25; 34A08

Keywords: Fixed point; w -distance; Implicit relation; Positive definite matrix; Nonlinear matrix equation; Convergence analysis; Binary relation

1 Introductory notes

Several mathematicians have established fixed point findings for contraction type mappings in metric spaces with partial order in recent years. Turinici established some early results in this approach in [21, 22]; it should be noted that their beginning points were amorphous contributions in the area due to Matkowski [10, 11]. Ran and Reurings [16], as well as Nieto and Rodríguez-López [13, 14], have looked at these kinds of findings. Turinici's findings were expanded upon and refined in articles [13, 14]. Fixed point theorems for nonlinear contraction under symmetric closure of an arbitrary relation were recently developed by Samet and Turinici [18]. Alam and Imdad used an amorphous relation to show a relation-theoretic counterpart of the Banach contraction principle, which combines a number of well-known relevant order-theoretic fixed point theorems. According to Mizoguchi and Takahashi [12], Kada et al. [4] developed the idea of ω -distance on a metric space in 1996 and proved a modified Caristi fixed point theorem, Ekeland's ϵ -variational principle and the nonconvex minimisation theorem. Rouzkard et al. [17] used the ω -distance function to find a generalised (ψ, ϕ) -weakly contractive condition

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

for orbital continuous maps. Lakzian et al. [6, 7] established various fixed point theorems for (α, ψ) -contractive mappings and Kannan contraction in the presence of a ω -distance. Senapati and Dey [19] recently published a modified version of the ω -distance function and demonstrated that Alam and Imdad [1]'s conclusion is false. For more details about the ω -distance functional, see [3].

1.1 Motivation

One of the most visually attractive applications of contraction mapping is found in nonlinear matrix equations. The question now is whether the previously described contraction can be improved and generalised. In relational metric spaces with w -distance functional, we investigate a novel class of implicit relations and implicit type contractions. We establish fixed point findings by justifying the applicability and usefulness of the provided results. Contraction and rational type contraction mappings are included in the proposed implicit contraction. We use these findings to examine the necessary conditions for the existence of a unique positive definite solution to the nonlinear matrix equation (NME) of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i$, where $\mathcal{Q}$ is an $n \times n$ Hermitian positive definite matrix, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ are $n \times n$ matrices and $\mathcal{G}$ is a nonlinear self-mapping of the set of all Hermitian matrices which are continuous in the trace norm. In order to demonstrate the obtained conditions, we consider an example together with convergence and error analysis, and visualisation of solutions in a surface plot.

2 Preliminaries

2.1 Relational metric spaces

Throughout this article, the notations $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ have their usual meanings, and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

We call $(\mathcal{W}, \mathcal{R})$ a relational set if (i) $\mathcal{W} \neq \emptyset$ is a set and (ii) $\mathcal{R}$ is a binary relation on $\mathcal{W}$.

In addition, if $(\mathcal{W}, d)$ is a metric space, we call $(\mathcal{W}, d, \mathcal{R})$ a relational metric space (RMS, for short).

The following are some standard terms used in the theory of relational sets (see, e.g. [1, 5, 8, 9, 18]).

Let $(\mathcal{W}, \mathcal{R})$ be a relational set, $(\mathcal{W}, d, \mathcal{R})$ be an RMS, and let $\mathcal{K}$ be a self-mapping on $\mathcal{W}$. Then:

- 1 $v \in \mathcal{W}$ is $\mathcal{R}$ -related to $\vartheta \in \mathcal{W}$ if and only if $(v, \vartheta) \in \mathcal{R}$.
- 2 The set $(\mathcal{W}, \mathcal{R})$ is said to be complete if for all $v, \vartheta \in \mathcal{W}$, $[v, \vartheta] \in \mathcal{R}$, where $[v, \vartheta] \in \mathcal{R}$ means that either $(v, \vartheta) \in \mathcal{R}$ or $(\vartheta, v) \in \mathcal{R}$.
- 3 A sequence (v_n) in $\mathcal{W}$ is said to be $\mathcal{R}$ -preserving if $(v_n, v_{n+1}) \in \mathcal{R}$, $\forall n \in \mathbb{N} \cup \{0\}$.
- 4 $(\mathcal{W}, d, \mathcal{R})$ is said to be $\mathcal{R}$ -complete if every $\mathcal{R}$ -preserving Cauchy sequence converges in $\mathcal{K}$.
- 5 $\mathcal{R}$ is said to be $\mathcal{K}$ -closed if $(v, \vartheta) \in \mathcal{R} \Rightarrow (\mathcal{K}v, \mathcal{K}\vartheta) \in \mathcal{R}$. It is said to be weakly $\mathcal{K}$ -closed if $(v, \vartheta) \in \mathcal{R} \Rightarrow [\mathcal{K}v, \mathcal{K}\vartheta] \in \mathcal{R}$.
- 6 $\mathcal{R}$ is said to be d -self-closed if for every $\mathcal{R}$ -preserving sequence with $v_n \rightarrow v$ there is a subsequence (v_{n_k}) of (v_n) such that $[v_{n_k}, v] \in \mathcal{R}$ for all $k \in \mathbb{N} \cup \{0\}$.
- 7 A subset $\mathfrak{J}$ of $\mathcal{W}$ is called $\mathcal{R}$ -directed if for each $v, \vartheta \in \mathfrak{J}$ there exists $\mu \in \mathcal{W}$ such that $(v, \mu) \in \mathcal{R}$ and $(\vartheta, \mu) \in \mathcal{R}$. It is called $(\mathcal{K}, \mathcal{R})$ -directed if for each $v, \vartheta \in \mathfrak{J}$ there exists $\mu \in \mathcal{W}$ such that $(v, \mathcal{K}\mu) \in \mathcal{R}$ and $(\vartheta, \mathcal{K}\mu) \in \mathcal{R}$.

- 8 $\mathcal{K}$ is said to be $\mathcal{R}$ -continuous at v if for every $\mathcal{R}$ -preserving sequence (v_n) converging to v we get $\mathcal{K}(v_n) \rightarrow \mathcal{K}(v)$ as $n \rightarrow \infty$. Moreover, $\mathcal{K}$ is said to be $\mathcal{R}$ -continuous if it is $\mathcal{R}$ -continuous at every point of $\mathcal{W}$.
- 9 For $v, \vartheta \in \mathcal{W}$, a path of length k (where k is a natural number) in $\mathcal{R}$ from v to ϑ is a finite sequence $\{\mu_0, \mu_1, \mu_2, \dots, \mu_k\} \subset \mathcal{W}$ satisfying the following conditions:
- (i) $\mu_0 = v$ and $\mu_k = \vartheta$,
 - (ii) $(\mu_i, \mu_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$),
- then this finite sequence is called a path of length k joining v to ϑ in $\mathcal{R}$.
- 10 If, for a pair of $v, \vartheta \in \mathcal{W}$, there is a finite sequence $\{\mu_0, \mu_1, \mu_2, \dots, \mu_k\} \subset \mathcal{W}$ satisfying the following conditions:
- (i) $\mathcal{K}\mu_0 = v$ and $\mathcal{K}\mu_k = \vartheta$,
 - (ii) $(\mathcal{K}\mu_i, \mathcal{K}\mu_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$),
- then this finite sequence is called a $\mathcal{K}$ -path of length k joining v to ϑ in $\mathcal{R}$.

Notice that a path of length k involves $k+1$ elements of $\mathcal{W}$, although they are not necessarily distinct.

We fix the following notation for a relational space $(\mathcal{W}, \mathcal{R})$, a self-mapping $\mathcal{K}$ on $\mathcal{W}$ and an $\mathcal{R}$ -directed subset $\mathfrak{D}$ of $\mathcal{W}$:

- (i) $\text{Fix}(\mathcal{K}) :=$ the set of all fixed points of $\mathcal{K}$,
- (ii) $\mathfrak{X}(\mathcal{K}, \mathcal{R}) := \{v \in \mathcal{W} : (v, \mathcal{K}v) \in \mathcal{R}, (\mathcal{K}v, v) \in \mathcal{R}\}$,
- (iii) $\mathfrak{P}(v, \vartheta, \mathcal{R}) :=$ the class of all paths in $\mathcal{R}$ from v to ϑ in $\mathcal{R}$, where $v, \vartheta \in \mathcal{W}$.

2.2 Relational metric spaces with w -distance

The corresponding definitions and lemmas, in the setting of metric spaces endowed with an arbitrary binary relation $\mathcal{R}$, are as follows.

Definition 2.1 ([19]) Let $(\mathcal{W}, d, \mathcal{R})$ be a relational metric space. A function $\omega: \mathcal{W} \times \mathcal{W} \rightarrow [0, +\infty)$ is called a w -distance on $\mathcal{W}$ if it satisfies the following properties:

- (W1) $\omega(\vartheta, \mu) \leq \omega(\vartheta, v) + \omega(v, \mu)$ for any $\vartheta, v, \mu \in \mathcal{W}$;
- (W2') w is $\mathcal{R}$ -LSC in its second variable; i.e. if $\vartheta \in \mathcal{W}$ and $v_n \rightarrow v \in \mathcal{W}$ such that $v_n \mathcal{R} v_{n+1}$, then $\omega(\vartheta, v) \leq \liminf_{n \rightarrow \infty} \omega(\vartheta, v_n)$;
- (W3) for each $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(\mu, \vartheta) \leq \delta$ and $\omega(\mu, v) \leq \delta$ imply $d(\vartheta, v) \leq \epsilon$.

The following lemma is a modified version of Kada et al. [4] due to Senapati and Dey [19].

Lemma 2.2 ([19]) Let $(\mathcal{W}, d, \mathcal{R})$ be a relational metric space, and let ω be a w -distance on $\mathcal{W}$. Suppose that $\{\vartheta_n\}$ and $\{v_n\}$ are $\mathcal{R}$ -preserving sequences in $\mathcal{W}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, +\infty)$ converging to 0, and let $\vartheta, v, \mu \in \mathcal{W}$. Then the following assertions hold:

- (i) If $\omega(\vartheta_n, v) \leq \alpha_n$ and $\omega(\vartheta_n, \mu) \leq \beta_n$ for all $n \in \mathbb{N}$, then $v = \mu$, particularly, if $\omega(\vartheta, v) = \omega(\vartheta, \mu) = 0$, then $v = \mu$;
- (ii) If $\omega(\vartheta_n, v_n) \leq \alpha_n$ and $\omega(\vartheta_n, v) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{v_n\}$ converges to v ;
- (iii) If $\omega(\vartheta_n, \vartheta_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{\vartheta_n\}$ is a Cauchy sequence;
- (iv) If $\omega(v, \vartheta_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{\vartheta_n\}$ is a Cauchy sequence.

Lemma 2.3 ([4, 20]) *Let ω be a w -distance on a metric space $(\mathcal{W}, d)$ and $\{\vartheta_n\}$ be a sequence in $\mathcal{W}$ such that for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $m > n > N_\epsilon$ implies $\omega(\vartheta_n, \vartheta_m) < \epsilon$, i.e. $\lim_{m,n \rightarrow \infty} \omega(\vartheta_n, \vartheta_m) = 0$. Then $\{\vartheta_n\}$ is a Cauchy sequence.*

3 New class of implicit relations

In this section we introduce a modified version of implicit relations and examples discussed in [2, 15].

Let $\Omega := \{\mathcal{H} : \mathbb{R}_+^6 \rightarrow \mathbb{R} \text{ is continuous function}\}$ satisfy the following conditions:

- ($\mathcal{H}_1$) $\mathcal{H}$ is nonincreasing in the fifth and sixth variables;
- ($\mathcal{H}_2$) There exists $\hbar \in [0, 1)$ such that, for all $\zeta, \xi, \mu \geq 0$,
 - ($\mathcal{H}_{2a}$) $\mathcal{H}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \leq 0$ implies that $\zeta \leq \hbar \xi$ and
 - ($\mathcal{H}_{2b}$) $\mathcal{H}(\zeta, \xi, \xi, \zeta, \mu, \zeta + \xi) \leq 0$ implies that $\zeta \leq \hbar \xi$;
- ($\mathcal{H}_3$) $\mathcal{H}(\zeta, \zeta, 0, 0, \zeta, \zeta) > 0$ and $\mathcal{H}(\zeta, 0, 0, \zeta, \zeta, 0) > 0$ for $\zeta > 0$.

Example 3.1 Let $\mathcal{H}(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 + b\frac{q_3^2 + q_4^2}{q_5 + q_6 + 1}$, $a \in (1/2, 1)$, $b \in (0, 1/2)$.

($\mathcal{H}_{2a}$) Then, for $\zeta, \xi, \mu \geq 0$, we have

$$\mathcal{H}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = \zeta^2 - a\xi^2 + b\frac{\zeta^2 + \xi^2}{1 + \zeta + \xi + \mu}.$$

It is evident that $\mathcal{H}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \leq 0$, then $\zeta^2 \leq a\xi^2 - b\frac{\zeta^2 + \xi^2}{1 + \zeta + \xi + \mu}$. Then $\zeta^2 \leq a\xi^2$. Hence, $\zeta \leq \hbar \xi$, $\hbar = \sqrt{a} < 1$.

($\mathcal{H}_{2b}$) Similar to ($\mathcal{H}_{2a}$), if $\mathcal{H}(\zeta, \xi, \xi, \zeta, \mu, \zeta + \xi) \leq 0$, then $\zeta \leq \hbar \xi$.

($\mathcal{H}_3$) For all $\zeta > 0$, $\mathcal{H}(\zeta, \zeta, 0, 0, \zeta, \zeta) = (1 - a)\zeta^2 > 0$.

Example 3.2 Let $\mathcal{H}(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 + b\frac{q_3^2 + q_4^2}{q_5^2 + q_6^2 + 1}$, $a \in (1/2, 1)$, $b \in (0, 1/2)$.

($\mathcal{H}_{2a}$) Then, for $\zeta, \xi, \mu \geq 0$, we have

$$\mathcal{H}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = \zeta^2 - a\xi^2 + b\frac{\zeta^2 + \xi^2}{1 + (\zeta + \xi)^2 + \mu^2}.$$

It is evident that $\mathcal{H}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \leq 0$, then $\zeta^2 \leq a\xi^2 - b\frac{\zeta^2 + \xi^2}{1 + (\zeta + \xi)^2 + \mu^2}$. Then $\zeta^2 \leq a\xi^2$. Hence, $\zeta \leq \hbar \xi$, $\hbar = \sqrt{a} < 1$.

($\mathcal{H}_{2b}$) Similar to ($\mathcal{H}_{2a}$), if $\mathcal{H}(\zeta, \xi, \xi, \zeta, \mu, \zeta + \xi) \leq 0$, then $\zeta \leq \hbar \xi$.

($\mathcal{H}_3$) For all $\zeta > 0$, $\mathcal{H}(\zeta, \zeta, 0, 0, \zeta, \zeta) = (1 - a)\zeta^2 > 0$.

4 $\mathcal{R}_\omega$ -implicit contractive condition under w -distance

We define an $\mathcal{R}_\omega$ -implicit contractive mapping in the metric space under w -distance using the above introduced implicit relation.

Definition 4.1 Let $(\mathcal{W}, d, \mathcal{R})$ be a relational metric space with w -distance w and $\mathcal{K} : \mathcal{W} \rightarrow \mathcal{W}$ be a given mapping. We say that $\mathcal{K}$ is an $\mathcal{R}_\omega$ -implicit contractive mapping if there exists a function $\mathcal{H} \in \Omega$ such that

$$\mathcal{H}(\omega(\mathcal{K}\vartheta, \mathcal{K}v), \omega(\vartheta, v), \omega(\vartheta, \mathcal{K}\vartheta), \omega(v, \mathcal{K}v), \omega(\vartheta, \mathcal{K}v), \omega(\mathcal{K}\vartheta, v)) \leq 0 \quad (1)$$

for all $(\vartheta, v) \in \mathcal{R}$.

Now, we are equipped to state and prove our first main result as follows.

Theorem 4.2 *Let $(\mathcal{W}, d, \mathcal{R})$ be an RMS with w -distance ω and $\mathcal{K}: \mathcal{W} \rightarrow \mathcal{W}$. Suppose that the following conditions hold:*

- (A₁) $\mathfrak{X}(\mathcal{K}, \mathcal{R}) \neq \emptyset$;
- (A₂) $\mathcal{R}$ is $\mathcal{K}$ -closed and $\mathcal{K}$ -transitive;
- (A₃) $\mathcal{W}$ is $\mathcal{K}$ - $\mathcal{R}$ -complete;
- (A₄) $\mathcal{K}$ is a $\mathcal{R}_\omega$ -implicit contractive;
- (A₅) $\mathcal{K}$ is $\mathcal{R}$ -continuous.

Then there exists a point $\omega \in \text{Fix}(\mathcal{K})$. In addition, $\omega(\vartheta^, \vartheta^*) = 0$ provided $(\vartheta^*, \vartheta^*) \in \mathcal{R}$ holds.*

Proof Let $\theta_0 \in \mathfrak{X}(\mathcal{K}, \mathcal{R})$ be a point as given in (A₁). If $\mathcal{K}^n \theta_0 = \mathcal{K}^{n+1} \theta_0$ for some $n \in \mathbb{N} \cup \{0\}$, then there is nothing to prove. Construct a sequence $\{\theta_n\}$ of Picard iterates $\theta_n = \mathcal{K}^n(\theta_0)$ for all $n \in \mathbb{N}^*$.

Using (A₁)–(A₂), we have that $(\mathcal{K}\theta_0, \mathcal{K}^2\theta_0) \in \mathcal{R}$. Continuing this process inductively, we obtain

$$(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0) \in \mathcal{R} \quad (2)$$

for any $n \in \mathbb{N} \cup \{0\}$. Hence, $\{\theta_n\}$ is an $\mathcal{R}$ -preserving sequence.

Next, we show that

$$\lim_{n \rightarrow \infty} w(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0) = 0. \quad (3)$$

Using (1) for $\vartheta = \theta_{n-1}$, $\nu = \theta_n$,

$$\mathcal{H} \left(\begin{array}{l} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0), \omega(\mathcal{K}^{n-1} \theta_0, \mathcal{K}^n \theta_0), \omega(\mathcal{K}^{n-1} \theta_0, \mathcal{K}^n \theta_0), \\ \omega(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0), \omega(\mathcal{K}^{n-1} \theta_0, \mathcal{K}^{n+1} \theta_0), \omega(\mathcal{K}^n \theta_0, \mathcal{K}^n \theta_0) \end{array} \right) \leq 0.$$

Denoting $\varrho_n = \omega(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0)$ for all $n \in \mathbb{N}^*$ and applying $(\mathcal{H}_1)$ in the fifth variables, we have

$$\mathcal{H}(\varrho_n, \varrho_{n-1}, \varrho_{n-1}, \varrho_n, \varrho_{n-1} + \varrho_n, \omega(\mathcal{K}^n \theta_0, \mathcal{K}^n \theta_0)) \leq 0. \quad (4)$$

It follows from $(\mathcal{H}_{2a})$ that there is $\hbar \in [0, 1)$ such that

$$\varrho_n \leq \hbar \varrho_{n-1},$$

that is, the sequence $\{\varrho_n\}$ is a nonincreasing sequence of real numbers. Therefore there exists ζ such that

$$\lim_{n \rightarrow \infty} \varrho_n = \lim_{n \rightarrow \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0) = \zeta \quad \forall n \in \mathbb{N}.$$

Applying the limit in (4), by the continuity of $\mathcal{H}$, we get

$$\zeta \leq \lim_{n \rightarrow \infty} \varrho_n \leq \hbar \zeta,$$

a contradiction, and therefore $\zeta = 0$. Thus we conclude that $\lim_{n \rightarrow \infty} \varrho_n = \lim_{n \rightarrow \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0) = 0$. Similarly, from (i) $(\mathcal{K} \theta_0, \theta_0) \in \mathcal{R}$, using condition (ii), we get $(\theta_{n+1}, \theta_n) \in \mathcal{R}$ for all $n \in \mathbb{N}^*$. Using this conclusion and above arguments, it can be shown that

$$\lim_{n \rightarrow \infty} \omega(\mathcal{K}^{n+1} \theta_0, \mathcal{K}^n \theta_0) = 0. \quad (5)$$

Next, we show that $\{\mathcal{K}^n \theta_0\}$ is a Cauchy sequence in $\mathcal{W}$. For this we show that

$$\lim_{m, n \rightarrow \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^m \theta_0) = 0. \quad (6)$$

On the contrary, suppose that condition (6) does not hold. Then we can find $\delta > 0$ and increasing sequences $\{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty$ of positive integers with $m_k > n_k$ such that

$$\omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k} \theta_0) \geq \delta \quad \text{for all } k \in \{1, 2, 3, \dots\}. \quad (7)$$

By (3), there exists $k_0 \in \mathbb{N}$ such that $n_k > k_0$ implies that

$$\omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{n_k+1} \theta_0) < \delta.$$

In view of the two last inequalities, we observe that $m_k \neq n_{k+1}$. We may assume that m_k is the minimal index such that (7) holds, so that

$$\omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^r \theta_0) < \delta, \quad \text{for } r \in \{n_{k+1}, n_{k+2}, \dots, m_k - 1\}.$$

Now, making use of (7), we get

$$\begin{aligned} 0 < \delta &\leq \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k} \theta_0) \leq \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k-1} \theta_0) + \omega(\mathcal{K}^{m_k-1} \theta_0, \mathcal{K}^{m_k} \theta_0) \\ &< \delta + \omega(\mathcal{K}^{m_k-1} \theta_0, \mathcal{K}^{m_k} \theta_0). \end{aligned}$$

Thus,

$$\lim_{k \rightarrow \infty} \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k} \theta_0) = \delta. \quad (8)$$

Using the triangle inequality, we have

$$\begin{aligned} \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k} \theta_0) &\leq \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k+1} \theta_0) + \omega(\mathcal{K}^{m_k+1} \theta_0, \mathcal{K}^{m_k} \theta_0) \\ &\leq \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{n_k+1} \theta_0) + \omega(\mathcal{K}^{n_k+1} \theta_0, \mathcal{K}^{m_k+1} \theta_0) \\ &\quad + \omega(\mathcal{K}^{m_k+1} \theta_0, \mathcal{K}^{m_k} \theta_0). \end{aligned}$$

Taking the limit on both sides and making use of (3), (5) and (8), we obtain

$$\lim_{k \rightarrow \infty} \omega(\mathcal{K}^{n_k+1} \theta_0, \mathcal{K}^{m_k+1} \theta_0) \geq \delta. \quad (9)$$

Again, using the triangle inequality, we have

$$\begin{aligned}\omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0) &\leq \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k}\theta_0) + \omega(\mathcal{K}^{m_k}\theta_0, \mathcal{K}^{m_k+1}\theta_0) \\ &\leq \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{n_k}\theta_0) + \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{m_k}\theta_0) + \omega(\mathcal{K}^{m_k}\theta_0, \mathcal{K}^{m_k+1}\theta_0).\end{aligned}$$

Taking the limit on both sides and making use of (3), (5) and (8), we obtain

$$\lim_{k \rightarrow \infty} \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0) \leq \delta. \quad (10)$$

Combining (9) and (10), we have

$$\lim_{k \rightarrow \infty} \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0) = \delta. \quad (11)$$

Now, since $\mathcal{R}$ is transitive and since $\{\theta_n\}$ is an $\mathcal{R}$ -preserving sequence, we must have $(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{m_k}\theta_0) \in \mathcal{R}$ for all $r \in \mathbb{N}$. Therefore, on applying condition (1), we get

$$\mathcal{H} \left(\begin{array}{l} \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0), \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{m_k}\theta_0), \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{n_k+1}\theta_0), \\ \omega(\mathcal{K}^{m_k}\theta_0, \mathcal{K}^{m_k+1}\theta_0), \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{m_k+1}\theta_0), \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k}\theta_0) \end{array} \right) \leq 0.$$

Now applying $(\mathcal{H}_1)$ in the fifth and sixth variables, we have

$$\mathcal{H} \left(\begin{array}{l} \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0), \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{m_k}\theta_0), \\ \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{n_k+1}\theta_0), \omega(\mathcal{K}^{m_k}\theta_0, \mathcal{K}^{m_k+1}\theta_0), \\ \omega(\mathcal{K}^{n_k}\theta_0, \mathcal{K}^{n_k+1}\theta_0) + \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0), \\ \omega(\mathcal{K}^{n_k+1}\theta_0, \mathcal{K}^{m_k+1}\theta_0) + \omega(\mathcal{K}^{m_k+1}\theta_0, \mathcal{K}^{m_k}\theta_0) \end{array} \right) \leq 0.$$

Applying the limit and using the continuity of $\mathcal{H}$, we get

$$\mathcal{H}(\delta, \delta, 0, 0, \delta, \delta) \leq 0,$$

a contradiction to $(\mathcal{H}_3)$. Hence, $\{\mathcal{K}^n\theta_0\}$ must be a Cauchy sequence in $\mathcal{W}$.

From $\mathcal{R}$ -completeness of $\mathcal{W}$, there exists a point $\vartheta^* \in \mathcal{W}$ such that $\lim_{n \rightarrow \infty} \mathcal{K}^n\theta_0 = \vartheta^*$. We shall show that ϑ^* is a fixed point of $\mathcal{K}$.

Using the $\mathcal{R}$ -continuity of $\mathcal{K}$ (due to condition (A_5)), we have $\lim_{n \rightarrow \infty} \mathcal{K}\mathcal{K}^n\theta_0 = \mathcal{K}\vartheta^*$. Owing to the uniqueness of the limit, we obtain $\mathcal{K}\vartheta^* = \vartheta^*$.

Finally, assume that $\omega(\vartheta^*, \vartheta^*) > 0$. Then, by (1), we have

$$\mathcal{H} \left(\begin{array}{l} \omega(\mathcal{K}\vartheta^*, \mathcal{K}\vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \mathcal{K}\vartheta^*), \\ \omega(\mathcal{K}\vartheta^*, \vartheta^*), \omega(\mathcal{K}\vartheta^*, \vartheta^*), \omega(\vartheta^*, \mathcal{K}\vartheta^*) \end{array} \right) \leq 0$$

or

$$\mathcal{H} \left(\begin{array}{l} \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \\ \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*) \end{array} \right) \leq 0.$$

It follows from $(\mathcal{H}_{2a})$ and $(\mathcal{H}_1)$ (for $\zeta = \xi = \mu = \vartheta^*$) that there is $\hbar \in [0, 1)$ such that

$$\omega(\vartheta^*, \vartheta^*) \leq \hbar \omega(\vartheta^*, \vartheta^*) \quad (12)$$

a contradiction. Therefore, $\omega(\vartheta^*, \vartheta^*) = 0$. $\square$

Next, we have the following result.

Theorem 4.3 *The conclusion of Theorem 4.2 remains true if condition (A_5) is replaced with the following one:*

(A'_5) *For every $v \in \mathcal{W}$ with $v \neq \mathcal{K}v$, $\inf\{\omega(\vartheta, v) + \omega(\vartheta, \mathcal{K}\vartheta) \mid \vartheta \in \mathcal{W}\} > 0$.*

Proof Following the proof of Theorem 4.2, we observe that the sequence $\{\mathcal{K}^n \theta_0\}$ is a Cauchy sequence, and so there exists a point ϑ^* in $\mathcal{W}$ such that $\lim_{n \rightarrow \infty} \mathcal{K}^n \theta_0 = \vartheta^*$. Since $\lim_{m, n \rightarrow \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^m \theta_0) = 0$ for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon$ implies $\omega(\mathcal{K}^{N_\epsilon} \theta_0, \mathcal{K}^n \theta_0) < \epsilon$. Since $\lim_{n \rightarrow \infty} \mathcal{K}^n \theta_0 = \vartheta^*$ and $\omega(\vartheta, \cdot)$ is lower semi-continuous,

$$\omega(\mathcal{K}^{N_\epsilon} \theta_0, \vartheta^*) \leq \liminf_{n \rightarrow \infty} \omega(\mathcal{K}^{N_\epsilon} \theta_0, \mathcal{K}^n \theta_0) < \epsilon.$$

Therefore, $\omega(\mathcal{K}^{N_\epsilon} \theta_0, \vartheta^*) \leq \epsilon$. Set $\epsilon = 1/k$, $N_\epsilon = n_k$ so that

$$\lim_{k \rightarrow \infty} \omega(\mathcal{K}^{n_k} \theta_0, \vartheta^*) = 0.$$

Assume that $\mathcal{K}\vartheta^* \neq \vartheta^*$. Then, by hypothesis (v') , we have

$$\begin{aligned} 0 &< \inf\{\omega(\vartheta, \vartheta^*) + \omega(\vartheta, \mathcal{K}\vartheta) \mid \vartheta \in \mathcal{W}\} \\ &\leq \inf\{\omega(\mathcal{K}^{n_k} \theta_0, \vartheta^*) + \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{n_k+1} \theta_0) \mid n \in \mathbb{N}\} \rightarrow 0, \end{aligned}$$

which contradicts our assumption. Therefore, $\mathcal{K}\vartheta^* = \vartheta^*$.

The last conclusion is derived as in the proof of Theorem 4.2. $\square$

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 4.2 and 4.3.

To obtain uniqueness, we add the following additional conditions for the functions $\mathcal{H} \in \Omega$ and call them by Ω' : for all $\zeta, \xi > 0$,

$(\mathcal{H}_4)$ $\mathcal{H}(\zeta, \xi, 0, 0, \xi, \zeta) \leq 0$ implies that there exists $\bar{h} \in [0, 1)$ such that $\zeta \leq \varphi(\xi)$.

Theorem 4.4 *In addition to the hypotheses of Theorem 4.2 (or Theorem 4.3), if any of the following conditions is fulfilled:*

(I) *For all $\vartheta, v \in \mathcal{W}$, there exists $z \in \mathcal{W}$ such that*

$$\{(z, \mathcal{K}z), (z, \vartheta), (z, v)\} \subseteq \mathcal{R}; \quad (13)$$

(II) *The set $\mathcal{K}(\mathcal{W})$ is $\mathcal{R}$ -directed;*

(III) *$\mathcal{R}|_{\mathcal{K}(\mathcal{W})}$ is complete;*

(IV) *$\mathfrak{P}(\vartheta, v, \text{Fix}(\mathcal{K}), \mathcal{R}^s)$ is nonempty for each $\vartheta, v \in \text{Fix}(\mathcal{K})$,*

then $\mathcal{K}$ has a unique fixed point.

Proof In view of Theorem 4.2 (or Theorem 4.3), $\text{Fix}(\mathcal{T}) \neq \emptyset$ and $w(\vartheta, \vartheta) = 0$ for each $\vartheta \in \text{Fix}(\mathcal{T})$.

- *Assumption (I)*. Suppose that there exist distinct fixed points ϑ and ν of $\mathcal{K}$. We will consider the following two cases.

Case (A): ϑ and ν are $\mathcal{R}$ -comparable. Then $\mathcal{K}^n \vartheta = \vartheta$ and $\mathcal{K}^n \nu = \nu$ are comparable for $n = 0, 1, \dots$. Therefore, using condition (1),

$$\mathcal{H} \left(\begin{array}{c} \omega(\mathcal{K}^n \vartheta, \mathcal{K}^{n-1} \nu), \omega(\mathcal{K}^{n-1} \vartheta, \mathcal{K}^{n-1} \nu), \omega(\mathcal{K}^{n-1} \vartheta, \mathcal{K}^n \vartheta), \\ \omega(\mathcal{K}^{n-1} \nu, \mathcal{K}^n \nu), \omega(\mathcal{K}^{n-1} \vartheta, \mathcal{K}^n \nu), \omega(\mathcal{K}^n \vartheta, \mathcal{K}^{n-1} \nu) \end{array} \right) \leq 0$$

or

$$\mathcal{H} \left(\omega(\vartheta, \nu), \omega(\vartheta, \nu), 0, 0, \omega(\vartheta, \nu), \omega(\vartheta, \nu) \right) \leq 0$$

which contradict $(\mathcal{H}_3)$, then $w(\vartheta, \nu) = 0$. Since $w(\vartheta, \vartheta) = 0$, by using Lemma 2.2, we have $\vartheta = \nu$; i.e. the fixed point of $\mathcal{K}$ is unique.

Case (B): By assumption (I), there exists $z \in \mathcal{W}$, satisfying condition (13). Due to the $\mathcal{K}$ -closedness of $\mathcal{R}$, we get

$$(\mathcal{K}^{n-1} z, \vartheta) \in \mathcal{R}, \quad (\mathcal{K}^{n-1} z, \nu) \in \mathcal{R},$$

and, using (1), it follows that

$$\mathcal{H} \left(\begin{array}{c} \omega(\mathcal{K}^n z, \nu), \omega(\mathcal{K}^{n-1} z, \nu), \omega(\mathcal{K}^{n-1} z, \mathcal{K}^n z), \\ \omega(\nu, \mathcal{K} \nu), \omega(\mathcal{K}^{n-1} z, \mathcal{K} \nu), \omega(\mathcal{K}^n z, \nu) \end{array} \right) \leq 0. \quad (14)$$

Using $(z, \mathcal{K} z) \in \mathcal{R}$, similarly as in the proof of Theorem 4.2, it can be shown that $w(\mathcal{K}^{n-1} z, \mathcal{K}^n z) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, for n sufficiently large, from (14), we have

$$\mathcal{H} \left(\omega(\mathcal{K}^n z, \nu), \omega(\mathcal{K}^{n-1} z, \nu), 0, 0, \omega(\mathcal{K}^{n-1} z, \mathcal{K} \nu), \omega(\mathcal{K}^n z, \nu) \right) \leq 0. \quad (15)$$

It follows from $(\mathcal{H}_4)$ that there is $\hbar \in [0, 1)$ such that

$$\omega(\mathcal{K}^n z, \nu) \leq \hbar \omega(\mathcal{K}^{n-1} z, \nu).$$

It follows that the sequence $\{w(\mathcal{K}^n z, \nu)\}$ is nonincreasing. As earlier, we have

$$\lim_{n \rightarrow \infty} w(\mathcal{K}^n z, \nu) = 0.$$

Also, since $(z, \vartheta) \in \mathcal{R}$, proceeding as earlier, we can prove that

$$\lim_{n \rightarrow \infty} w(\mathcal{K}^n z, \vartheta) = 0,$$

and by using Lemma 2.2 we infer that $\nu = \vartheta$; i.e. the fixed point of $\mathcal{K}$ is unique.

- *Assumption (II)*. For any two fixed points ϑ, ν of $\mathcal{K}$, there must be an element $z \in \mathcal{K}(\mathcal{W})$ such that

$$(z, \vartheta) \in \mathcal{R} \quad \text{and} \quad (z, \nu) \in \mathcal{R}.$$

As $\mathcal{R}$ is $\mathcal{K}$ -closed, so for all $n \in \mathbb{N} \cup \{0\}$,

$$(\mathcal{K}^n z, \vartheta) \in \mathcal{R} \quad \text{and} \quad (\mathcal{K}^n z, v) \in \mathcal{R}.$$

In the line of proof of Case(B) (I), we obtain $\vartheta = v$, i.e. $\mathcal{K}$ has a unique fixed point.

- *Assumption (III)*. Suppose that ϑ^*, v^* are two fixed points of $\mathcal{K}$. Then we must have $(\vartheta^*, v^*) \in \mathcal{R}$ or $(v^*, \vartheta^*) \in \mathcal{R}$. For $(\vartheta^*, v^*) \in \mathcal{R}$, we obtain

$$\mathcal{H} \left(\begin{array}{l} \omega(\mathcal{K}\vartheta^*, \mathcal{K}v^*), \omega(\vartheta^*, v^*), \omega(\vartheta^*, \mathcal{K}\vartheta^*), \\ \omega(v^*, \mathcal{K}v^*), \omega(\vartheta^*, \mathcal{K}v^*), \omega(\mathcal{K}\vartheta^*, v^*) \end{array} \right) \leq 0,$$

that is,

$$\mathcal{H}(\omega(\vartheta^*, v^*), \omega(\vartheta^*, v^*), 0, 0, \omega(\vartheta^*, v^*), \omega(\vartheta^*, v^*)) \leq 0,$$

which contradicts $(\mathcal{H}_3)$, then $w(\vartheta^*, v^*) = 0$. Since $w(\vartheta^*, \vartheta^*) = 0$, by using Lemma 2.2, we have $\vartheta^* = v^*$; i.e. the fixed point of $\mathcal{K}$ is unique.

In a similar way, if $(v^*, \vartheta^*) \in \mathcal{R}$, we have $\vartheta^* = v^*$.

- *Assumption (IV)*. Suppose that ϑ^*, v^* are two fixed points of $\mathcal{K}$. Let $\{z_0, z_1, \dots, z_k\}$ be an $\mathcal{R}^s$ -path in $\text{Fix}(\mathcal{K})$ connecting ϑ^* and v^* . As in Case (I,A), it must be $z_{i-1} = z_i$ for $\square$ each $i = 1, 2, \dots, k$, and it follows that $\vartheta^* = v^*$.

By choosing $\mathcal{H} \in \Omega$ from Example 3.1–Example 3.2, we have the following consequences.

Corollary 4.5 *Let all the conditions of Theorem 4.2–Theorem 4.3 be satisfied, except that the assumption of orbitally ω -implicit contractive mapping for $\mathcal{H} \in \Omega$ is replaced with either of the forms*

(I)

$$[\omega(\mathcal{K}\vartheta, \mathcal{K}v)]^2 \leq a[\omega(\vartheta, v)]^2 - b \frac{[\omega(\vartheta, \mathcal{K}\vartheta)]^2 + [\omega(v, \mathcal{K}v)]^2}{1 + \omega(\vartheta, \mathcal{K}v) + \omega(\mathcal{K}\vartheta, v)}, \quad (16)$$

where $1/2 < a < 1$, $0 < b < 1/2$, or

(II)

$$[\omega(\mathcal{K}\vartheta, \mathcal{K}v)]^2 \leq a[\omega(\vartheta, v)]^2 - b \frac{[\omega(\vartheta, \mathcal{K}\vartheta)]^2 + [\omega(v, \mathcal{K}v)]^2}{1 + [\omega(\vartheta, \mathcal{K}v)]^2 + [\omega(\mathcal{K}\vartheta, v)]^2}, \quad (17)$$

where $1/2 < a < 1$, $0 < b < 1/2$.

Then $\text{Fix}(\mathcal{K})$ is a singleton.

5 Illustrations

In this part, we show how important our results are when compared to metric distances.

Example 5.1 (Inspired by [19]) Consider the set $\mathcal{W} = [0, \frac{3}{2}]$ with the usual metric d . Define a w -distance $\omega: \mathcal{W} \times \mathcal{W} \rightarrow [0, \infty)$ by $\omega(\vartheta, v) = v$ for all $\vartheta, v \in \mathcal{W}$ and the binary relation $\mathcal{R}$ by

$$(\vartheta, v) \in \mathcal{R} \quad \Leftrightarrow \quad v \leq 1 \quad \text{or} \quad \vartheta \leq 1.$$

Consider the self-mapping $\mathcal{K}$ on $\mathcal{W}$ given by

$$\mathcal{K}(\vartheta) = \begin{cases} \frac{\vartheta}{3}, & 0 \leq \vartheta \leq \frac{2}{3}; \\ 1 - \vartheta, & \frac{2}{3} < \vartheta < 1; \\ \frac{3}{4}, & \vartheta = 1; \\ \vartheta - \frac{1}{2}, & \vartheta > 1. \end{cases}$$

It is clear that $\mathcal{W}$ is $\mathcal{K}$ - $\mathcal{R}$ complete, $\mathfrak{X}(\mathcal{K}, \mathcal{R}) \neq \emptyset$ and $\mathcal{R}$ is $\mathcal{K}$ -closed.

Considering Example 3.2, we can define $\mathcal{H} \in \Omega$ as

$$\mathcal{H}(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - a q_2^2 - b \frac{q_3^2 + q_4^2}{1 + q_5^2 + q_6^2},$$

where $1/2 < a < 1$, $0 < b < 1/2$. Then (1) reduces to

$$[\omega(\mathcal{K}\vartheta, \mathcal{K}v)]^2 \leq a[\omega(\vartheta, v)]^2 - b \frac{[\omega(\vartheta, \mathcal{K}\vartheta)]^2 + [\omega(v, \mathcal{K}v)]^2}{1 + [\omega(\vartheta, \mathcal{K}v)]^2 + [\omega(\mathcal{K}\vartheta, v)]^2}, \quad (18)$$

where $1/2 < a < 1$, $0 < b < 1/2$.

We show that $\mathcal{K}$ satisfies (18). We divide the proof into three parts for $\vartheta \leq 1$. Similarly, we can show for $v \leq 1$.

Case 1: Let $\vartheta = 0$, then $v \in [0, \frac{3}{2}]$ such that $(\vartheta, v) \in \mathcal{R}$. Then

- Let $0 \leq v \leq \frac{2}{3}$. Then $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = v/3$, $\omega(\vartheta, v) = v$, $\omega(\vartheta, \mathcal{K}\vartheta) = 0$, $\omega(v, \mathcal{K}v) = v/3$, $\omega(\vartheta, \mathcal{K}v) = v/3$, $\omega(\mathcal{K}\vartheta, v) = v$. Therefore, condition (18) reduces to

$$\frac{v^2}{9} \leq a v^2 - b \frac{v^2}{2v^2 + 9}.$$

- Let $\frac{2}{3} < v < 1$, then $\mathcal{K}v \in (0, \frac{1}{3})$. Then $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = 1 - v$, $\omega(\vartheta, v) = v$, $\omega(\vartheta, \mathcal{K}\vartheta) = 0$, $\omega(v, \mathcal{K}v) = 1 - v$, $\omega(\vartheta, \mathcal{K}v) = 1 - v$, $\omega(\mathcal{K}\vartheta, v) = v$. Therefore, condition (18) reduces to

$$(1 - v)^2 \leq a v^2 - b \frac{(1 - v)^2}{2 + 2v^2 - 2v}.$$

- Let $v = 1$. Then $\omega(\mathcal{K}0, \mathcal{K}1) = \frac{3}{4}$, $\omega(\vartheta, v) = 1$, $\omega(\vartheta, \mathcal{K}\vartheta) = 0$, $\omega(v, \mathcal{K}v) = 3/4$, $\omega(\vartheta, \mathcal{K}v) = 3/4$, $\omega(\mathcal{K}\vartheta, v) = 1$. Therefore, condition (18) reduces to

$$\frac{9}{16} \leq a - b \frac{9}{41}.$$

- Let $v > 1$. Then we have $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = v - \frac{1}{2}$, $\omega(\vartheta, v) = v$, $\omega(\vartheta, \mathcal{K}\vartheta) = 0$, $\omega(v, \mathcal{K}v) = v - 1/2$, $\omega(\vartheta, \mathcal{K}v) = v - 1/2$, $\omega(\mathcal{K}\vartheta, v) = v$. Therefore, condition (18) reduces to

$$\left(v - \frac{1}{2}\right)^2 \leq a v^2 - b \frac{(v - 1/2)^2}{2v^2 - v + 5/4}.$$

Case 2: Let $\vartheta \in [0, 3/2]$ and $v = 0$. Then $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = 0$, and condition (18) is trivially true.

Case 3: Let $\vartheta \neq 0$. Then $v \leq 1$. Therefore, we have:

- Let $0 \leq v \leq \frac{2}{3}$, $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = v/3$, $\omega(\vartheta, v) = v$, $\omega(\vartheta, \mathcal{K}\vartheta) = \mathcal{K}\vartheta$, $\omega(v, \mathcal{K}v) = v/3$, $\omega(\vartheta, \mathcal{K}v) = v/3$, $\omega(\mathcal{K}\vartheta, v) = v$. Therefore, condition (18) reduces to

$$\frac{v^2}{9} \leq av^2 - b \frac{(\mathcal{K}\vartheta)^2 + v^2}{1 + v^2/9 + v^2}.$$

- Let $\frac{2}{3} < v < 1$, $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = 1 - v$, $\omega(\vartheta, v) = v$, $\omega(\vartheta, \mathcal{K}\vartheta) = \mathcal{K}\vartheta$, $\omega(v, \mathcal{K}v) = 1 - v$, $\omega(\vartheta, \mathcal{K}v) = 1 - v$, $\omega(\mathcal{K}\vartheta, v) = v$. Therefore, condition (18) reduces to

$$(1 - v)^2 \leq av^2 - b \frac{(\mathcal{K}\vartheta)^2 + (1 - v)^2}{2 + 2v^2 - 2v}.$$

- Let $v = 1$, $\omega(\mathcal{K}\vartheta, \mathcal{K}1) = \frac{3}{4}$, $\omega(\vartheta, v) = 1$, $\omega(\vartheta, \mathcal{K}\vartheta) = \mathcal{K}\vartheta$, $\omega(1, \mathcal{K}1) = 3/4$, $\omega(\vartheta, \mathcal{K}1) = 3/4$, $\omega(\mathcal{K}\vartheta, v) = 1$. Therefore, condition (18) reduces to

$$\frac{9}{16} \leq a - b \frac{(\mathcal{K}\vartheta)^2 + 9}{\frac{41}{16}}.$$

- Let $\vartheta \leq 1$ and $v > 1$. Then $\omega(\mathcal{K}\vartheta, \mathcal{K}v) = v - 1/2$, $\omega(\vartheta, v) = v$, $\omega(\vartheta, \mathcal{K}\vartheta) = \mathcal{K}\vartheta$, $\omega(v, \mathcal{K}v) = v - 1/2$, $\omega(\vartheta, \mathcal{K}v) = v - 1/2$, $\omega(\mathcal{K}\vartheta, v) = v$. Therefore, condition (18) reduces to

$$\left(v - \frac{1}{2}\right)^2 \leq av^2 - b \frac{(\mathcal{K}\vartheta)^2 + (v - 1/2)^2}{2v^2 - v + 5/4}.$$

It can be easily checked that the above cases hold true for $a = 9/10$, $b = 1/10$.

Thus $\mathcal{K}$ is an $\mathcal{R}_\omega$ -implicit contractive mapping.

Note that $\mathcal{K}$ is not $\mathcal{R}$ -continuous at $\vartheta = 1$ and $\vartheta = \frac{2}{3}$.

Finally, we will show that condition $(\mathcal{A}_5')$ of Theorem 4.3 holds true. If $v > 0$, we have $v \neq \mathcal{K}v$ so that

$$\inf\{\omega(\vartheta, v) + \omega(\vartheta, \mathcal{K}\vartheta) : \vartheta \in \mathcal{W}\} = \inf\{v + \mathcal{K}\vartheta : \vartheta \in \mathcal{W}\} > 0.$$

Thus, all the conditions of Theorem 4.3 are satisfied and $\text{Fix}(\mathcal{K}) = \{0\}$.

Remark 5.2 In the above example, it is noted that (18) is not true in a metric space, that is,

$$|\mathcal{K}\vartheta - \mathcal{K}v|^2 \leq a|\vartheta - v|^2 - b \frac{|\vartheta - \mathcal{K}\vartheta|^2 + |v - \mathcal{K}v|^2}{1 + |\vartheta - \mathcal{K}v|^2 + |\mathcal{K}\vartheta - v|^2}. \quad (19)$$

For instance, if $\vartheta = 1$, $v = 3/4$, (19) implies that

$$\frac{1}{4} \leq \frac{a}{16} - \frac{b}{5} = a \frac{1}{16} - b \frac{\frac{1}{16} + \frac{1}{4}}{1 + \frac{9}{16} + 0},$$

which is not true for any $1/2 < a < 1$, $0 < b < 1/2$.

Example 5.3 Consider the set $\mathcal{W} = \{0, 1, 2\}$ with the usual metric d , and order relation $\mathfrak{R}$ is defined as $(\vartheta, \nu) \in \mathcal{R} \Leftrightarrow \vartheta \leq \nu$. Define a w -distance $\omega: \mathcal{W} \times \mathcal{W} \rightarrow [0, \infty)$ by $\omega(\vartheta, \nu) = \nu$ for all $\vartheta, \nu \in \mathcal{W}$. Define a mapping $\mathcal{K}: \mathcal{W} \rightarrow \mathcal{W}$ by

$$\mathcal{K} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that $\mathcal{W}$ is $\mathcal{K} - \mathcal{R}$ is complete, $\mathfrak{X}(\mathcal{K}, \mathcal{R}) \neq \emptyset$ and $\mathcal{R}$ is $\mathcal{K}$ -closed.

Considering Example 3.1, we can define $\mathcal{H} \in \Omega$ as

$$\mathcal{H}(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 + b \frac{q_3^2 + q_4^2}{q_5 + q_6 + 1},$$

where $1/2 < a < 1$, $0 < b < 1/2$. Then (1) reduces to

$$[\omega(\mathcal{K}\vartheta, \mathcal{K}\nu)]^2 \leq a[\omega(\vartheta, \nu)]^2 - b \frac{[\omega(\vartheta, \mathcal{K}\vartheta)]^2 + [\omega(\nu, \mathcal{K}\nu)]^2}{1 + \omega(\vartheta, \mathcal{K}\nu) + \omega(\mathcal{K}\vartheta, \nu)}, \quad (20)$$

where $1/2 < a < 1$, $0 < b < 1/2$, $\vartheta \leq \nu$.

It is easy to check that $\mathcal{K}$ satisfies (20). Thus, by Theorem 4.2, $\text{Fix}(\mathcal{K}) = \{0\}$.

It is noted that (20) is not true in metric distance d . For instance, if $\vartheta = 1$, $\nu = 2$, (20) implies that

$$1 \leq a - \frac{b}{4} = a - b \frac{0 + 1}{1 + 0 + 2},$$

which is not true for any $1/2 < a < 1$, $0 < b < 1/2$.

Example 5.4 Consider the set $\mathcal{W} = [0, 1]$ with the usual metric d . Define a w -distance $\omega: \mathcal{W} \times \mathcal{W} \rightarrow [0, \infty)$ by $\omega(\vartheta, \nu) = \vartheta + \nu$ for all $\vartheta, \nu \in \mathcal{W}$ and the binary relation $\mathcal{R}$ by

$$\mathcal{R} = \left\{ (0, 0), (0, 1) \left(\frac{1}{4}, 1 \right), \left(\frac{1}{4}, 0 \right), \left(0, \frac{1}{4} \right), \left(\frac{1}{4}, \frac{1}{4} \right) \right\}.$$

Consider the self-mapping $\mathcal{K}$ on $\mathcal{W}$ given by

$$\mathcal{K}(\vartheta) = \begin{cases} 0, & 0 \leq \vartheta \leq \frac{1}{4} \\ \frac{1}{4}, & \frac{1}{4} < \vartheta \leq 1. \end{cases}$$

It is clear that $\mathcal{W}$ is $\mathcal{K} - \mathcal{R}$ is complete, $\mathfrak{X}(\mathcal{K}, \mathcal{R}) \neq \emptyset$ and $\mathcal{R}$ is $\mathcal{K}$ -closed.

Considering Example 3.2, we can define $\mathcal{H} \in \Omega$ as

$$\mathcal{H}(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 + b \frac{q_3^2 + q_4^2}{q_5^2 + q_6^2 + 1},$$

where $1/2 < a < 1$, $0 < b < 1/2$. Then (1) reduces to

$$[\omega(\mathcal{K}\vartheta, \mathcal{K}\nu)]^2 \leq a[\omega(\vartheta, \nu)]^2 - b \frac{[\omega(\vartheta, \mathcal{K}\vartheta)]^2 + [\omega(\nu, \mathcal{K}\nu)]^2}{1 + [\omega(\vartheta, \mathcal{K}\nu)]^2 + [\omega(\mathcal{K}\vartheta, \nu)]^2}, \quad (21)$$

where $1/2 < a < 1$, $0 < b < 1/2$.

We show that $\mathcal{K}$ satisfies (21).

Case 1: If $(\vartheta, \nu) \in \{(0, 0), (\frac{1}{4}, 0), (0, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4})\}$, then $\omega(\mathcal{K}\vartheta, \mathcal{K}\nu) = 0$, and condition (21) is trivially satisfied.

Case 2: If $(\vartheta, \nu) \in \{(0, 1), (\frac{1}{4}, 1)\}$.

- Let $(\vartheta, \nu) = (0, 1)$. Then $\omega(\mathcal{K}\vartheta, \mathcal{K}\nu) = 1/4$, $\omega(\vartheta, \nu) = 1$, $\omega(\vartheta, \mathcal{K}\vartheta) = 0$, $\omega(\nu, \mathcal{K}\nu) = 5/4$, $\omega(\vartheta, \mathcal{K}\nu) = 1/4$, $\omega(\mathcal{K}\vartheta, \nu) = 1$. Therefore, condition (21) reduces to

$$\frac{1}{16} \leq a - \frac{25c}{33}.$$

- Let $(\vartheta, \nu) = (\frac{1}{4}, 1)$. Then $\omega(\mathcal{K}\vartheta, \mathcal{K}\nu) = 1/4$, $\omega(\vartheta, \nu) = 5/4$, $\omega(\vartheta, \mathcal{K}\vartheta) = 1/4$, $\omega(\nu, \mathcal{K}\nu) = 5/4$, $\omega(\vartheta, \mathcal{K}\nu) = \frac{1}{2}$, $\omega(\mathcal{K}\vartheta, \nu) = 1$. Therefore, condition (21) reduces to

$$\frac{1}{16} \leq \frac{5a}{4} - \frac{13b}{18}.$$

It can be easily checked that the above cases hold true for $a = 3/4$, $b = 1/10$. Thus $\mathcal{K}$ is an $\mathcal{R}_\omega$ -implicit contractive mapping.

Finally, we will show that condition $(\mathcal{A}_5')$ of Theorem 4.3 holds true. If $\nu > 0$, we have $\nu \neq \mathcal{K}\nu$ so that

$$\inf\{\omega(\vartheta, \nu) + \omega(\vartheta, \mathcal{K}\vartheta) : \vartheta \in \mathcal{W}\} = \inf\{2\nu + \vartheta + \mathcal{K}\vartheta : \vartheta \in \mathcal{W}\} > 0.$$

Thus, all the conditions of Theorem 4.3 are satisfied and $\text{Fix}(\mathcal{K}) = \{0\}$.

Example 5.5 Consider the set $\mathcal{W} = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \dots\}$ with $d(\vartheta, \nu) = |\vartheta - \nu|$, where $(\vartheta, \nu) \in \mathcal{R} \Leftrightarrow \vartheta < \nu$. Define a w -distance $\omega : \mathcal{W} \times \mathcal{W} \rightarrow [0, +\infty)$ by $\omega(\vartheta, \nu) = \nu$ for all $\vartheta, \nu \in \mathcal{W}$.

Consider the self-mapping $\mathcal{K}$ on $\mathcal{W}$ given by

$$\mathcal{K}(\vartheta) = \begin{cases} \frac{1}{2}, & \text{if } \vartheta = 0 \\ \frac{1}{2^{n+1}}, & \text{if } \vartheta = \frac{1}{2^n}, n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Considering Example 3.1, (1) reduces to

$$[\omega(\mathcal{K}\vartheta, \mathcal{K}\nu)]^2 \leq a[\omega(\vartheta, \nu)]^2 - b \frac{[\omega(\vartheta, \mathcal{K}\vartheta)]^2 + [\omega(\nu, \mathcal{K}\nu)]^2}{1 + \omega(\vartheta, \mathcal{K}\nu) + \omega(\mathcal{K}\vartheta, \nu)}, \quad (22)$$

where $1/2 < a < 1$, $0 < b < 1/2$, $\vartheta < \nu$.

Consider two cases:

- If $\vartheta = 0$ and $\nu = 1/2^n$, $n \in \mathbb{N}$, then (22) reduces to

$$\frac{1}{2^{2n+2}} \leq a \frac{1}{2^{2n}} - b \frac{\frac{1}{4} + \frac{1}{2^{2n+2}}}{1 + \frac{1}{2^{n+1}} + \frac{1}{2^n}}$$

and is fulfilled for $1/2 < a < 1$, $0 < b < 1/2$.

- Let $\vartheta, \nu \in \{1/2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Consider $\vartheta = \frac{1}{2^m}$ and $\nu = \frac{1}{2^n}$ ($m > n$). Then inequality (22) has the form

$$\frac{1}{2^{2n+2}} \leq a \frac{1}{2^{2n}} - b \frac{\frac{1}{2^{2m+2}} + \frac{1}{2^{2n+2}}}{1 + \frac{1}{2^{n+1}} + \frac{1}{2^n}},$$

which is obviously true for $1/2 < a < 1$, $0 < b < 1/2$.

For instance, both of the above cases are true for $a = \frac{99}{100}$, $b = \frac{2}{100}$.

It can be noted that, for $0 \neq \mathcal{K}0$, we have

$$\inf \left\{ \omega \left(\frac{1}{2^n}, 0 \right) + \omega \left(\frac{1}{2^n}, \frac{1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} = \inf \left\{ \frac{1}{2^{n+1}} : n \in \mathbb{N} \right\} = 0.$$

Thus, all the conditions of Theorem 4.3 are satisfied except $(\mathcal{A}_5')$. Clearly, $\mathcal{K}$ has no fixed points in $\mathcal{W}$.

Remark 5.6 It is observed that for $0 \neq \mathcal{K}0$ in the metric space

$$\inf \left\{ d \left(\frac{1}{2^n}, 0 \right) + d \left(\frac{1}{2^n}, \frac{1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} = \inf \left\{ \frac{1}{2^n} + \frac{1}{2^{n+1}} : n \in \mathbb{N} \right\} = 0.$$

That is, condition $(\mathcal{A}_5')$ of Theorem 4.3 is also not satisfied in the metric space.

6 Application to nonlinear matrix equations

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n \times n$ Hermitian (resp. positive semi-definite, positive definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ be the set of all $n \times n$ matrices over $\mathbb{C}$. Denote by $s(\mathcal{U})$ any singular value of a matrix $\mathcal{U}$, and its trace norm by $s^+(\mathcal{U}) = \|\mathcal{U}\|$. For $\mathcal{C}, \mathcal{D} \in \mathcal{H}(n)$, $\mathcal{C} \geq \mathcal{D}$ (resp. $\mathcal{C} \succ \mathcal{D}$) will mean that the matrix $\mathcal{C} - \mathcal{D}$ is positive semi-definite (resp. positive definite).

Theorem 6.1 Consider the equation

$$\mathcal{U} = \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i, \quad (23)$$

where $\mathcal{Q} \in \mathcal{P}(n)$, $\mathcal{A}_i \in \mathcal{M}(n)$, $i = 1, \dots, k$, and the operator $\mathcal{G}: \mathcal{P}(n) \rightarrow \mathcal{P}(n)$ is continuous in the trace norm. Let, for some $M, N_1 \in \mathbb{R}$ and for any $\mathcal{U} \in \mathcal{P}(n)$ with $\|\mathcal{U}\| \leq M$, $s(\mathcal{G}(\mathcal{U})) \leq N_1$ hold for all singular values of $\mathcal{G}(\mathcal{U})$. Assume that:

- (I) $\|\mathcal{Q}\| \leq M - NN_1n$, where $\sum_{i=1}^k \|\mathcal{A}_i^*\| \|\mathcal{A}_i\| = N$;
- (II) For any $\mathcal{Q} \in \mathcal{P}(n)$ with $\|\mathcal{Q}\| \leq M$, $\sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{A}_i \geq \mathcal{O}$ holds;
- (III) There exist $1/2 < a < 1$, $0 < b < 1/2$ such that, for any $s(\mathcal{Q})$,

$$2NN_1 + 2s(\mathcal{Q}) \leq [\Upsilon(\mathcal{U}, \mathcal{V})]^{1/2} \quad (24)$$

holds for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}(n)$ with $\|\mathcal{U}\|, \|\mathcal{V}\| \leq M$, $\mathcal{U} \leq \mathcal{V}$ and

$\sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \neq \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i$, where

$$\begin{aligned} \Upsilon(\mathcal{U}, \mathcal{V}) &= a \left(|s^+(\mathcal{U})| + |s^+(\mathcal{V})| \right)^2 \\ &\quad - b \frac{(|s^+(\mathcal{U})| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)|)^2 + (|s^+(\mathcal{V})| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)|)^2}{1 + |s^+(\mathcal{U})| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)| + |s^+(\mathcal{V})|}. \end{aligned} \quad (25)$$

Then NME (23) has a unique solution $\widehat{\mathcal{U}} \in \mathcal{P}(n)$ with $\|\widehat{\mathcal{U}}\| \leq M$. Further, the solution can be obtained as the limit of the iterative sequence $\{\mathcal{U}_n\}$, where for $j \geq 0$,

$$\mathcal{U}_{j+1} = \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}_j) \mathcal{A}_i \quad (26)$$

and $\mathcal{U}_0$ is an arbitrary element of $\mathcal{P}(n)$ satisfying $\|\mathcal{U}_0\| \leq M$.

Proof Denote by $\Lambda := \{\mathcal{U} \in \mathcal{P}(n) : \|\mathcal{U}\| \leq M\}$ a closed subset of $\mathcal{P}(n)$. According to (II), any solution of (23) in Λ has to be positive definite. We have, for any $\mathcal{U} \in \Lambda$,

$$\begin{aligned} \left\| \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \right\| &\leq \|\mathcal{Q}\| + \left\| \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \right\| \\ &\leq \|\mathcal{Q}\| + \sum_{i=1}^k \|\mathcal{A}_i^*\| \|\mathcal{A}_i\| \|\mathcal{G}(\mathcal{U})\| = \|\mathcal{Q}\| + N \|\mathcal{G}(\mathcal{U})\|. \end{aligned} \quad (27)$$

Since all singular values of $\mathcal{U}$ satisfy $s(\mathcal{G}(\mathcal{U})) \leq N_1$, it follows that $\|\mathcal{G}(\mathcal{U})\| \leq N_1 n$. Thus, (27) implies

$$\left\| \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \right\| \leq \|\mathcal{Q}\| + NN_1 n \leq M - NN_1 n + NN_1 n = M.$$

Define now an operator $\mathcal{K} : \Lambda \rightarrow \Lambda$ by

$$\mathcal{K}(\mathcal{U}) = \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \quad \text{for all } \mathcal{U} \in \Lambda.$$

Also, define a binary relation

$$\mathcal{R} = \{(\mathcal{U}, \mathcal{V}) \in \Lambda \times \Lambda : \mathcal{U}\mathcal{V} \preceq \mathcal{U} \text{ or } \mathcal{U}\mathcal{V} \preceq \mathcal{V}\}.$$

It is clear that finding positive definite solution(s) of equation (23) is equivalent to finding fixed point(s) of $\mathcal{K}$. Notice that $\mathcal{K}$ is well defined, $\mathcal{R}$ -continuous and $\mathcal{R}$ is $\mathcal{K}$ -closed. Since

$$\sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{A}_i \succ 0$$

for some $\mathcal{Q} \in \Lambda$, we have $(\mathcal{Q}, \mathcal{K}(\mathcal{Q})) \in \mathcal{R}$ or $(\mathcal{K}(\mathcal{Q}), \mathcal{Q}) \in \mathcal{R}$, and hence $\Lambda(\mathcal{K}; \mathcal{R}) \neq \emptyset$.

Now, for $(\mathcal{U}, \mathcal{V}) \in \mathcal{R}$, we have

$$\begin{aligned} \|\mathcal{K}(\mathcal{U})\| + \|\mathcal{K}(\mathcal{V})\| &= \left\| \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \right\| + \left\| \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i \right\| \\ &\leq 2\|\mathcal{Q}\| + \left\| \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \right\| + \left\| \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i \right\| \end{aligned}$$

$$\begin{aligned}
&\leq 2\|\mathcal{Q}\| + \sum_{i=1}^k [\|\mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i\| + \|\mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i\|] \\
&\leq 2\|\mathcal{Q}\| + \sum_{i=1}^k \|\mathcal{A}_i^*\| \|\mathcal{A}_i\| [\|\mathcal{G}(\mathcal{U})\| + \|\mathcal{G}(\mathcal{V})\|] \\
&\leq 2\|\mathcal{Q}\| + N(\|\mathcal{G}(\mathcal{U})\| + \|\mathcal{G}(\mathcal{V})\|) \\
&\leq 2\|\mathcal{Q}\| + N(N_1 n + N_1 n) \\
&= 2\|\mathcal{Q}\| + 2NN_1 n.
\end{aligned}$$

Thus, for any $\mathcal{U}, \mathcal{V} \in \Lambda$ with $\mathcal{U} \leq \mathcal{V}$, we have

$$\|\mathcal{K}(\mathcal{U})\| + \|\mathcal{K}(\mathcal{V})\| \leq 2\|\mathcal{Q}\| + 2NN_1 n. \quad (28)$$

For some fixed $\mathcal{U}, \mathcal{V} \in \Lambda$ with $(\mathcal{U}, \mathcal{V}) \in \mathcal{R}$, from (24) and (25), we have

$$\begin{aligned}
&[2NN_1 + 2s(\mathcal{Q})]^2 \\
&\leq a(|s^+(\mathcal{U})| + |s^+(\mathcal{V})|)^2 \\
&\quad - b \frac{(|s^+(\mathcal{U})| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)|)^2 + (|s^+(\mathcal{V})| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)|)^2}{1 + |s^+(\mathcal{U})| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)| + |s^+(\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)| + |s^+(\mathcal{V})|} \\
&= a(\|\mathcal{U}\| + \|\mathcal{V}\|)^2 \\
&\quad - b \frac{(\|\mathcal{U}\| + \|\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i\|)^2 + (\|\mathcal{V}\| + \|\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i\|)^2}{1 + \|\mathcal{U}\| + \|\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i\| + \|\mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i\| + \|\mathcal{V}\|}.
\end{aligned}$$

The above relation holds for every singular value of $\mathcal{Q}$, so adding up, we obtain

$$\begin{aligned}
(2NN_1 n + 2\|\mathcal{Q}\|)^2 &\leq a(\|\mathcal{U}\| + \|\mathcal{V}\|)^2 \\
&\quad - b \frac{(\|\mathcal{U}\| + \|\mathcal{K}(\mathcal{U})\|)^2 + (\|\mathcal{V}\| + \|\mathcal{K}(\mathcal{V})\|)^2}{1 + (\|\mathcal{U}\| + \|\mathcal{K}(\mathcal{V})\|) + (\|\mathcal{K}(\mathcal{U})\| + \|\mathcal{V}\|)}.
\end{aligned}$$

Therefore, from (28) we get

$$\begin{aligned}
(\|\mathcal{K}(\mathcal{U})\| + \|\mathcal{K}(\mathcal{V})\|)^2 &\leq a(\|\mathcal{U}\| + \|\mathcal{V}\|)^2 \\
&\quad - b \frac{(\|\mathcal{U}\| + \|\mathcal{K}(\mathcal{U})\|)^2 + (\|\mathcal{V}\| + \|\mathcal{K}(\mathcal{V})\|)^2}{1 + (\|\mathcal{U}\| + \|\mathcal{K}(\mathcal{V})\|) + (\|\mathcal{K}(\mathcal{U})\| + \|\mathcal{V}\|)}. \quad (29)
\end{aligned}$$

Let $\omega: \Lambda \times \Lambda \rightarrow \mathbb{R}_+$ be defined by

$$\omega(\mathcal{U}, \mathcal{V}) = \|\mathcal{U}\| + \|\mathcal{V}\| \quad \text{for all } \mathcal{U}, \mathcal{V} \in \Lambda.$$

Then $(\Lambda, \|\cdot\|, \omega)$ is a complete relational metric space with the above ω -distance. It follows from (29) that

$$[\omega(\mathcal{K}(\mathcal{U}), \mathcal{K}(\mathcal{V}))]^2 \leq a[\omega(\mathcal{U}, \mathcal{V})]^2 - b \frac{[\omega(\mathcal{U}, \mathcal{K}(\mathcal{U}))]^2 + [\omega(\mathcal{V}, \mathcal{K}(\mathcal{V}))]^2}{1 + \omega(\mathcal{U}, \mathcal{K}(\mathcal{V})) + \omega(\mathcal{K}(\mathcal{U}), \mathcal{V})}. \quad (30)$$

Taking $\mathcal{H} \in \Omega$ as Example 3.1, (30) can be rewritten as

$$\mathcal{H} \begin{pmatrix} \omega(\mathcal{K}(\mathcal{U}), \mathcal{K}(\mathcal{V})), \omega(\mathcal{U}, \mathcal{V}), \omega(\mathcal{U}, \mathcal{K}(\mathcal{U})), \\ \omega(\mathcal{V}, \mathcal{K}(\mathcal{V})), \omega(\mathcal{U}, \mathcal{K}(\mathcal{V})), \omega(\mathcal{K}(\mathcal{U}), \mathcal{V}) \end{pmatrix} \leq 0.$$

Now, all the hypotheses of Theorem 4.2 are satisfied, and therefore there exists $\hat{\mathcal{X}} \in \mathcal{P}(n)$ such that $\mathcal{K}(\hat{\mathcal{X}}) = \hat{\mathcal{X}}$. Hence, matrix equation (23) has a solution in $\mathcal{P}(n)$. Furthermore, due to the existence of the least upper bound and the greatest lower bound for each pair $\mathcal{U}, \mathcal{V} \in \mathcal{K}(\mathcal{P}(n))$, we have $\mathcal{Y}(\mathcal{U}, \mathcal{V}; \mathcal{R}_{|\mathcal{K}(\mathcal{P}(n))}) \neq \emptyset$ for all $\mathcal{U}, \mathcal{V} \in \mathcal{K}(\mathcal{P}(n))$. Hence, using Theorem 4.4, $\mathcal{K}$ has a unique fixed point, and hence we conclude that matrix equation (23) has a unique solution in $\mathcal{P}(n)$. $\square$

7 Numerical analysis

Example 7.1 Consider NME (23) for $a = 0.75$, $b = 0.1$ with $\mathcal{G}(\mathcal{X}) = \mathcal{X}^{1/4}$, i.e.

$$\mathcal{X} = \mathcal{Q} + \mathcal{A}_1' * \mathcal{X}^{1/4} * \mathcal{A}_1 + \mathcal{A}_2' * \mathcal{X}^{1/4} * \mathcal{A}_2 + \mathcal{A}_3' * \mathcal{X}^{1/4} * \mathcal{A}_3, \quad (31)$$

where

$$\begin{aligned} \mathcal{A}_1 &= \begin{bmatrix} 0.0446 & 0.0600 & 0.1186 \\ 0.0483 & 0.0293 & 0.1212 \\ 0.1243 & 0.0490 & 0.0232 \end{bmatrix}, & \mathcal{A}_2 &= \begin{bmatrix} 0.0246 & 0.0319 & 0.0363 \\ 0.0517 & 0.0345 & 0.0405 \\ 0.0539 & 0.0526 & 0.0350 \end{bmatrix}, \\ \mathcal{A}_3 &= \begin{bmatrix} 0.1739 & 0.2720 & 0.0854 \\ 0.1455 & 0.0759 & 0.1644 \\ 0.3036 & 0.1138 & 0.1771 \end{bmatrix}, & \mathcal{Q} &= \begin{bmatrix} 9.2231 & 0.1203 & 0.1978 \\ 0.1203 & 9.1168 & 0.1645 \\ 0.1978 & 0.1645 & 9.2731 \end{bmatrix}. \end{aligned}$$

The conditions of Theorem 31 can be checked numerically by considering different particular values of matrices involved. For instance, it can be tested (and verified to be true) for

$$\mathcal{X} = \begin{bmatrix} 0.2230 & 0.1200 & 0.1978 \\ 0.1200 & 0.1079 & 0.1638 \\ 0.1978 & 0.1638 & 0.2730 \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} 9.0001 & 0.0003 & 0.0001 \\ 0.0003 & 9.0089 & 0.0007 \\ 0.0001 & 0.0007 & 9.0001 \end{bmatrix}.$$

To see the behaviour of convergence of the sequence $\{\mathcal{U}_n\}$ defined in (26), we take up three different initial values:

$$\mathcal{U}_0 = \begin{bmatrix} 0.0220 & 0.0048 & 0.0016 \\ 0.0048 & 0.0320 & 0.0040 \\ 0.0016 & 0.0040 & 0.0215 \end{bmatrix}, \quad \mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\|\mathcal{U}_0\| = 0.0755$, $\|\mathcal{V}_0\| = 1$, $\mathcal{W}_0 = \begin{bmatrix} 29.5737 & 0.9425 & 1.1894 \\ 0.9425 & 28.9075 & 0.8807 \\ 1.1894 & 0.8807 & 29.3474 \end{bmatrix}$ with $\|\mathcal{W}_0\| = 87.8286$.

After 10 successive iterations, the approximations of the unique PDS of NME (23) are the following:

$$\hat{\mathcal{U}} \approx \mathcal{U}_{10} = \begin{bmatrix} 9.5246 & 0.3142 & 0.3965 \\ 0.3142 & 9.3025 & 0.2936 \\ 0.3965 & 0.2936 & 9.4491 \end{bmatrix} \quad \text{with Error} = 7.0958 \times 10^{-16};$$

$$\hat{V} \approx V_{10} = \begin{bmatrix} 9.5246 & 0.3142 & 0.3965 \\ 0.3142 & 9.3025 & 0.2936 \\ 0.3965 & 0.2936 & 9.4491 \end{bmatrix} \quad \text{with Error} = 3.7139 \times 10^{-16};$$

$$\hat{W} \approx W_{10} = \begin{bmatrix} 9.5246 & 0.3142 & 0.3965 \\ 0.3142 & 9.3025 & 0.2936 \\ 0.3965 & 0.2936 & 9.4491 \end{bmatrix} \quad \text{with Error} = 2.2887 \times 10^{-16}.$$

It can be also verified that the elements of each sequence are order-preserving. The convergence behaviour (see Fig. 1) and the solution surface plot (see Fig. 2) are shown graphically.

Acknowledgements

This research is funded by the Foundation for Science and Technology Development of Ton Duc Thang University (FOSTECT), website: <http://fostect.tdtu.edu.vn>, under Grant FOSTECT.2019.14. The authors are grateful to the Basque Government for Grant IT1207-19. We are also indebted to the academic reviewers for their insightful comments, which assisted us in improving the text in some places.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details

¹Mathematics Division, SASL, VIT Bhopal University, Madhya Pradesh 466114, India. ²Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, 700000, Vietnam. ³Institute of Research and Development of Processes, University of the Basque Country, 48940 Leioa, Bizkaia, Spain.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 April 2021 Accepted: 3 October 2021 Published online: 18 October 2021

References

1. Alam, A., Imdad, M.: Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **17**(4), 693–702 (2015)
2. Aliouche, A., Djoudi, A.: Common fixed point theorems for mappings satisfying an implicit relation without decreasing assumption. *Hacet. J. Math. Stat.* **36**(1), 11–18 (2007)
3. Gopal, D., Agarwal, P., Kumam, P.: *Metric Structures and Fixed Point Theory*, pp. 67–102. CRC Press, Boca Raton (2021)
4. Kada, O., Suzuki, T., Takahashi, W.: Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Math. Jpn.* **44**, 381–391 (1996)
5. Kolman, B., Busby, R.C., Ross, S.: *Discrete Mathematical Structures*, 3rd edn. PHI Pvt. Ltd., New Delhi (2000)
6. Lakzian, H., Gopal, D., Sintunavarat, W.: New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations. *J. Fixed Point Theory Appl.* **18**(2), 251–266 (2016)
7. Lakzian, H., Rakočević, V., Aydi, H.: Extensions of Kannan contraction via w -distances. *Aequ. Math.* **93**(6), 1231–1244 (2019)
8. Lipschutz, S.: *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*. McGraw-Hill, New York (1964)
9. Maddux, R.D.: *Relation Algebras. Studies in Logic and Foundations of Mathematics*, vol. 150. Elsevier, Amsterdam (2006)
10. Matkowski, J.: Integrable solutions of functional equations. *Diss. Math.* **127**, 1–68 (1975)
11. Matkowski, J.: Fixed point theorems for mappings with a contractive iterate at a point. *Proc. Am. Math. Soc.* **62**, 344–348 (1977)
12. Mizoguchi, N., Takahashi, W.: Fixed point theorems for multivalued mappings on complete metric spaces. *J. Math. Anal. Appl.* **141**(1), 177–188 (1989)
13. Nieto, J.J., López, R.R.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **22**, 223–239 (2005)
14. Nieto, J.J., López, R.R.: Fixed point theorems in ordered abstract spaces. *Proc. Am. Math. Soc.* **135**, 2505–2517 (2007)
15. Popa, V., Mocanu, M.: Altering distance and common fixed points under implicit relations. *Hacet. J. Math. Stat.* **38**(3), 329–337 (2009)
16. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **132**, 1435–1443 (2004)
17. Rouzkard, F., Imdad, M., Gopal, D.: Some existence and uniqueness theorems on ordered metric spaces via generalized distances. *Fixed Point Theory Appl.* **2013**, 45 (2013)
18. Samet, B., Turinici, M.: Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications. *Commun. Math. Anal.* **13**, 82–97 (2012)
19. Senapati, T., Dey, L.K.: Relation-theoretic metrical fixed-point results via w -distance with applications. *J. Fixed Point Theory Appl.* **19**(4), 2945–2961 (2017)
20. Suzuki, T.: Several fixed point theorems in complete metric space. *Yokohama Math. J.* **44**, 61–72 (1997)
21. Turinici, M.: Abstract comparison principles and multivariable Gronwall-Bellman inequalities. *J. Math. Anal. Appl.* **117**, 100–127 (1986)
22. Turinici, M.: Fixed points for monotone iteratively local contractions. *Demonstr. Math.* **19**, 171–180 (1986)