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Existence uniqueness of mild solutions for ψ-Caputo fractional stochastic evolution equations driven by fBm

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Abstract

In this paper, we investigate the existence uniqueness of mild solutions for a class of ψ-Caputo fractional stochastic evolution equations with varying-time delay driven by fBm, which seems to be the first theoretical result of the ψ-Caputo fractional stochastic evolution equations. Alternative conditions to guarantee the existence uniqueness of mild solutions are obtained using fractional calculus, stochastic analysis, fixed point technique, and noncompact measure method. Moreover, an example is presented to illustrate the effectiveness and feasibility of the obtained abstract results.

Keywords: ψ-Caputo fractional derivative; Stochastic evolution equations; Noncompact measure; Fractional Brownian motion

1 Introduction

Fractional differential equations grew to be a popular research topic for their wide applications in engineering, mathematics, physics, bio-engineering, and other applied sciences. Considerable work has been done in this area in recent years, both in theory and applications. Citing all papers and books of this field will be impossible. Therefore, here we only recommend the readers interested in this area refer to [1, 3, 8–12, 16, 21, 24, 27–29, 33–35] for more details on the theory and applications of fractional differential equations.

In particular, Kiryakova [18] proposed a theory of generalized fractional calculus (generalizations of fractional integrals and derivatives) and discussed its applications. We can find generalized fractional integrals and derivatives with specific functions $\psi(t)$ and with weights $w(t)$ in [18] and [25]. To be specific, it can be easily noticed that when $\psi(t) = t, w(t) = 1$, ψ-Riemann–Liouville fractional derivative coincides with the classical Riemann–Liouville fractional derivative, ψ-Caputo fractional derivative is actually the classical Caputo fractional derivative. When $\psi(t) = \ln t, w(t) = 1$, ψ-Riemann–Liouville fractional derivative coincides with the Hadamard fractional derivative. On the other hand, Almeida [2] studied some properties of ψ-Caputo fractional derivative by considering the Caputo fractional derivative of a function with respect to another function ψ. The advantage of this new definition of the fractional derivative is that a higher accuracy of the model could be achieved with the choice of a suitable function ψ. In addition, by applying Laplace transform and probability density functions, Suechoei and Sa Ngiamsunthorn [26] studied the local and global existence and uniqueness of mild solutions to the following fractional evolution equation of the form:

$$\begin{cases} {}^C_0D^{\psi}_{\alpha}x(t) = Ax(t) + f(t,x(t)), & t \in (0,b], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}^C_0D^{\psi}_{\alpha}$ considered in this work is in the sense of Caputo fractional derivative with respect to a function ψ which is more general than the classic Caputo fractional derivative. A is the infinitesimal generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space X, $f : [0,+\infty) \times X \to X$ is a given function satisfying some assumptions.

Stochastic differential equations have a wide variety of applications in many fields such as economics, finance, engineering, and social sciences, thus they are viewed as better tools for describing the real-life phenomena than ordinary differential equations since noise or stochastic perturbation is unavoidable in nature as well as in man-made systems, see [4, 6, 7, 22, 31, 32]. Recently, the stochastic differential equations driven by fBm have been investigated by many authors, see [5, 13, 15, 19, 20, 23, 30, 36] and the references therein.

However, it should be stressed that the existence uniqueness of mild solutions for ψ-Caputo fractional stochastic evolution equations is fairly scarce in contrast with the classical Caputo fractional stochastic evolution equations. This is greatly attributed to the relatively poor understanding of ψ-Caputo fractional derivative. In addition, many works focused on fractional stochastic evolution equations through various fixed point theorems when the corresponding semigroups are compact, which is convenient to obtain the corresponding compact resolvent operators. But for the case that the corresponding semigroups are noncompact, there are few results. Especially, there are no results considering the ψ-Caputo fractional stochastic differential equations driven by fBm. Therefore, inspired by the above discussions, the scope of this work is to study the existence uniqueness of mild solutions for the following ψ-Caputo fractional stochastic evolution equations with varying-time delay driven by fBm with the corresponding semigroup be compact or not:

$$\begin{cases} {}^C_0D^{\psi}_{\alpha}x(t) = Ax(t) + f(t,x(t-r(t))) + \sigma(t)\frac{dB^H_Q(t)}{dt}, & t \in J = [0,b], \\ x(t) = \phi(t), & t \in [-\tau,0], \end{cases} \tag{1.2}$$

where ${}^C_0D^{\psi}_{\alpha}$ is ψ-Caputo fractional derivative of order $\frac{1}{2} < \alpha \leq 1$; $x(\cdot)$ takes values in a separable Hilbert space X; $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ on a real separable Hilbert space X. Let Y be another separable Hilbert space. Let $\mathcal{L}(X,Y)$ denote the space of all bounded linear operators from X to Y. For convenience, we use the notation $\|\cdot\|$ to denote the norms in X, Y and $\mathcal{L}(X,Y)$ when no confusion possibly arises. $f : J \times X \to X$ is a function satisfying some specific assumptions given in (H_2). B^H is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$. The initial data $\phi \in C([-\tau,0],\mathcal{L}_2(\Omega,X))$, where $\mathcal{L}_2(\Omega,X)$ denote the collection of all strongly-measurable, square-integrable, X-valued random variables. Obviously, $\mathcal{L}_2(\Omega,X)$ is a Banach space equipped with the norm $\|x(\cdot)\|_{\mathcal{L}_2(\Omega,X)} = (E|x(\cdot)|)^{\frac{1}{2}}$. Let $\mathcal{C} := C([-r,b],\mathcal{L}_2(\Omega,X))$

be the Banach space of all continuous functions ξ from $[-\tau,b]$ into $\mathcal{L}_2(\Omega,X)$, equipped with the supremum norm $\|\xi\|_{\mathcal{C}}=\sup_{y\in[-\tau,b]}(E\|\xi(y)\|^2)^{\frac{1}{2}}$. In the sequel, $\mathcal{L}_2^0(\Omega,X)$ denotes the space of $\mathcal{F}_0$-measurable, X-valued, and square integrable stochastic process.

The paper is organized in the following way. In Sect. 2, we give some notations and useful concepts about fBm and fractional calculus. Section 3 aims to establish the existence of mild solutions for system (1.2) with Hurst parameter $H\in(1/2,1)$ by using the fixed point theorem and noncompact measure. In Sect. 4, we give an example to illustrate the application of the obtained abstract results. The conclusion is given in Sect. 5.

2 Preliminaries

In this section, we introduce some notations, definitions, preliminary facts for further convenience.

Before going further, we begin by recalling some basic facts about fBm and Wiener integral with respect to fBm.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space. Consider a time interval $[0,b]$ with arbitrary fixed horizon b, and let $\{B^H(t),t\in J\}$ be one-dimensional fBm with Hurst parameter $H\in(0,1)$. This means by definition that B^H is a continuous centered Gaussian process with covariance function

$$R_H(s,t)=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

In the rest of the paper, we always assume $\frac{1}{2}<H<1$. Consider the square integrable kernel given by

$$K_H(s,t)=c_H s^{\frac{1}{2}-H}\int_s^t (u-s)u^{H-\frac{1}{2}}\,du,$$

where $c_H=[\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}]^{\frac{1}{2}}$, $t>s$, $\beta(\cdot,\cdot)$ denotes the beta function. We take $K_H(s,t)=0$ for $t\leq s$, then it is easy to verify that

$$\frac{\partial K_H}{\partial t}(t,s)=c_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

We now consider an fBM $\{B^H(t),t\in[0,b]\}$. We denote by ξ the set of step functions on $[0,b]$. Let $\mathcal{H}$ be a Hilbert space defined as the closure of ξ with respect to the scalar product

$$\langle 1_{[0,t]},1_{[0,s]}\rangle_H=R_H(t,s).$$

The mapping $1_{[0,t]}\to\{B^H(t)\}$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\bar{\text{span}}^{L^2(\Omega)}\{B^H(t),t\in J\}$, and we will denote by $B^H(\varphi)$ the image of φ under this isometry.

Let us define the linear operator K_H^* from ξ to $L^2([0,b])$ by

$$\left(K_H^*\varphi\right)(s)=\int_s^t\varphi(t)\frac{\partial K_H}{\partial t}(t,s)\,dt.$$

Then K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,b])$.

Consider the process $\omega = \omega(t), t \in [0,b]$ defined by

$$\omega(t) = B^H\big(\big(K_H^*\big)^{-1}1_{[0,b])}\big).$$

Then ω is a Wiener process, and B^H has the integral representation

$$B^H(t) = \int_0^t K_H(t,s)\,d\omega(s).$$

Assume that there exists a complete orthogonal system $\{e_n\}_{n=1}^\infty$ in Y. Let $Q \in L(Y,Y)$ be an operator with finite trace $trQ = \sum_{n=1}^\infty \lambda_n < \infty(\lambda_n \geq 0)$ such that $Qe_n = \lambda_n e_n$. The infinite dimensional fBM on Y can be defined by using covariance operator Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\varphi e_n B_n^H(t),$$

where $B_n^H(t)$ are one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},P)$. Consider the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert–Schmidt operators $\varphi : Y \to X$. We recall that $\varphi \in L(Y,X)$ is called a Q-Hilbert–Schmidt operator if

$$\|\varphi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi e_n\|^2 < \infty, \tag{2.1}$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi,\psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^\infty \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Let $(\phi(s))_{s\in[0,b]}$ be a deterministic function with values in $\mathcal{L}_2^0(Y,X)$. The stochastic integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)\,dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\big(K_H^*(\phi e_n)\big)(s)\,dB_n(s).$$

Now we state some essential facts of fractional operators and Kuratowski's measure.

Definition 2.1 ([17] ψ-Riemann–Liouville fractional integral) Let $\alpha > 0$, f be an integrable function defined on $[a,b]$ and $\psi \in C^1([a,b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The ψ-Riemann–Liouville fractional integral operator of order α of a function f is defined by

$$\big({}_aI_\psi^\alpha f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} f(s)\psi'(s)\,ds. \tag{2.2}$$

It is obvious that when $\psi(t) = t$, (2.2) is the classical Riemann–Liouville fractional operator. When $\psi(t) = \ln t$, (2.2) is the Hadamard fractional operator.

Lemma 2.1 ([13]) *If $\varphi : [0,b] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^b \|\varphi(s)\|_{\mathcal{L}_2^0}^2 < \infty$, then the aforementioned sum in (2.1) is well defined as an X-valued random variable, and we have*

$$E\bigg\|\int_0^t \varphi(s)\,dB^H(s)\bigg\|^2 \leq c_0 H(2H-1)t^{2H-1} \int_0^t \big\|\varphi(s)\big\|_{\mathcal{L}_2^0}^2\,ds.$$

Definition 2.2 ([17] (ψ-Riemann–Liouville fractional derivative)) Let $n-1<\alpha<n, f$ be an integrable function defined on $[a,b]$ and $\psi\in C^1([a,b])$ be an increasing function with $\psi'(t)\neq 0$ for all $t\in[a,b]$. The ψ-Riemann–Liouville fractional derivative of order α of a function f is defined by

$$\left({}_aD^{\alpha}_{\psi}f\right)(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left({}_aI^{n-\alpha}_{\psi}f\right)(t)=\frac{(\frac{1}{\psi'(t)}\frac{d}{dt})^n}{\Gamma(n-\alpha)}\int_a^t\left(\psi(t)-\psi(s)\right)^{n-\alpha-1}f(s)\,ds,$$

where $n=\alpha+1$.

From the definition, when $\alpha=n\in\mathbb{N}$, we have $({}_aD^{\alpha}_{\psi}f)(t)=(\frac{1}{\psi'(t)}\frac{d}{dt})^nf(t)$.

Lemma 2.2 ([2])

(i) ${}_aI^{\alpha}_{\psi}(\psi(x)-\psi(a))^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$;

(ii) ${}_aD^{\alpha}_{\psi}(\psi(x)-\psi(a))^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$.

Definition 2.3 ((ψ-Caputo fractional derivative [17])) Let $n-1<\alpha<n, f\in C^n([a,b])$ and $\psi\in C^n([a,b])$ be an increasing function with $\psi'(t)\neq 0$ for all $t\in[a,b]$. The ψ-Caputo fractional derivative of order α of a function f is defined by

$$\left({}^C_aD^{\alpha}_{\psi}f\right)(t)=\left({}_aI^{n-\alpha}_{\psi}f^{[n]}\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\psi(t)-\psi(s)\right)^{n-\alpha-1}f^{[n]}(s)\psi'(s)\,ds,$$

where $n=[\alpha]+1$ and $f^{[n]}(t):=(\frac{1}{\psi'(t)}\frac{d}{dt})^nf(t)$ on $[a,b]$.

From the definition, it is clear that, when $\alpha=n\in\mathbb{N}$,

$${}^C_aD^{\alpha}_{\psi}f(t)=f^{[n]}(t).$$

Theorem 2.1 ([2]) *Let $f\in C^n([a,b])$ and $\alpha>0$. Then we have*

$${}_aI^{\alpha}_{\psi}\left({}^C_aD^{\alpha}_{\psi}f(t)\right)=f(t)-\sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{k!}\left(\psi(t)-\psi(a)\right)^k.$$

In particular, given $\alpha\in(0,1)$, we have ${}_aI^{\alpha}_{\psi}({}^C_aD^{\alpha}_{\psi}f(t))=f(t)-f(a)$.

Definition 2.4 ([32]) Let X be a Banach space and Ω_x be the bounded set of X. The Kuratowski measure of noncompactness is the map $\alpha:\Omega_x\to[0,\infty)$ defined by $\alpha(D)=\inf\{d>0:D\subseteq\bigcup_{i=1}^n D_i \text{ and } \operatorname{diam}(D_i)\leq d\}$, here $D\in\Omega_x$.

Lemma 2.3 ([33]) *The noncompact measure $\alpha(\cdot)$ satisfies*:

(i) *for all bounded subsets D_1, D_2 of X, $D_1\subseteq D_2$ implies $\alpha(D_1)\leq\alpha(D_2)$;*

(ii) *$\alpha\{\{x\}\cup D\}=\alpha(D)$ for every $x\in X$ and every nonempty subset $D\in X$;*

(iii) *$\alpha(D_1)=0$ if and only if D_1 is relatively compact in X;*

(iv) *$\alpha(D_1+D_2)\leq\alpha(D_1)+\alpha(D_2)$, where $D_1+D_2=\{x+y:x\in D_1,y\in D_2\}$;*

(v) *$\alpha(D_1\cup D_2)\leq\max\{\alpha(D_1),\alpha(D_2)\}$;*

(vi) *$\alpha(\lambda D)\leq|\lambda|\alpha(D)$ for any $\lambda\in\mathbb{R}$;*

(vii) *$\alpha(U+x)=\alpha(U)$ for any $x\in X$;*

(viii) *If the map $Q:D(Q)\subset H\to X$ is Lipschitz continuous with constant k, then $\alpha(Q(S))\leq k\alpha(S)$ for any bounded subset $S\subset D(Q)$, where X is a Banach space.*

For any $W \in C(I,X)$, we define $\int_0^t W(s)\,ds = \{\int_0^t u(s)\,ds : u \in W\}$ for $t \in J$, where $W(s) = \{u(s) \in X : u \in W\}$.

We use $\alpha(\cdot)$ and $\alpha_C(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded set of X and $C([-\tau,b],X)$, respectively. For any $D \subset C([-\tau,b],X)$ and $t \in [0,b]$, set $D(t) = \{u(t)|u \in D\}$, then $D(t) \subset X$. If $D \subset C([-\tau,b],X)$ is bounded, then $D(t)$ is bounded in X and $\alpha(D(t)) \leq \alpha_C(D)$.

Lemma 2.4 ([33]) *Let $D \subset C([-\tau,b],X)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $[-\tau,b]$, and $\alpha_C(D) = \max_{t\in[-\tau,b]} \alpha(D(t))$.*

Lemma 2.5 ([33]) *Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable set $D_0 \subset D$ such that $\alpha(D) \leq 2\alpha(D_0)$.*

Lemma 2.6 ([14]) *Let X be a Banach space. If $D = \{u_n\}_{n=1}^{\infty} \subset C([-\tau,b],X)$ is a countable set and there exists a function $m \in L^1([-\tau,b],\mathbb{R}^+)$ such that, for every $n \subset \mathbb{N}$,*

$$\left\|u_n(t)\right\| \leq m(t), \quad a.e.\ t \in [-\tau,b].$$

Then $\alpha(D(t))$ is Lebesgue integral on $[-\tau,b]$, and

$$\alpha\left(\left\{\int_0^b u_n(t)\,dt \middle| n \in \mathbb{N}\right\}\right) \leq 2\int_0^b \alpha\big(D(t)\big)\,dt.$$

Lemma 2.7 ((Sadovskii fixed point theorem [18])) *Let X be a Banach space. Assume that $D \subset X$ is a bounded closed and convex set on X and $Q : D \to D$ is a condensing operator. Then Q has at least one fixed point in D.*

For C_0 semigroup $\{S(t)\}_{t\geq 0}$, the following property will be used:

There is $M \geq 1$ such that

$$M := \sup_{t\in[0,+\infty)} S(t) < \infty. \tag{2.3}$$

Lemma 2.8 *The system*

$$\begin{cases} {}_0^C D_\alpha^\psi x(t) = Ax(t) + f(t, x(t - r(t))) + \sigma(t)\frac{dB_Q^H(t)}{dt}, & t \in J, \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{2.4}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) = \phi(0) &+ \frac{1}{\Gamma(\alpha)}\int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1}\big[Ax(s) + f\big(s, x\big(s - r(s)\big)\big)\big]\psi'(s)\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1}\psi'(s)\sigma(s)\,dB_Q^H(s), \quad t \in J. \end{aligned} \tag{2.5}$$

Proof We can readily obtain the result from Definition 2.3 and Theorem 2.1. Here we omit it. □

Lemma 2.9 *If*

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} \bigl[Ax(s) + f\bigl(s, x\bigl(s - r(s)\bigr)\bigr)\bigr] \psi'(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t \bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} \psi'(s)\sigma(s)\, dB_Q^H(s), \tag{2.6}$$

then we have

$$x(t) = S_\alpha^\psi(t,0)\phi(0) + \int_0^t \bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} T_\alpha^\psi(t,s) f\bigl(s, x\bigl(s - r(s)\bigr)\psi'(s)\, ds + \int_0^t \bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} T_\alpha^\psi(t,s)\sigma(s)\psi'(s)\, dB_Q^H(s), \tag{2.7}$$

where

$$S_\alpha^\psi(t,s)x = \int_0^\infty \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta\bigr) x\, d\theta,$$

$$T_\alpha^\psi(t,s)x = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta\bigr) x\, d\theta,$$

ϕ_α *is the probability density function defined on* $(0,\infty)$, *that is*, $\phi_\alpha(\theta) \geq 0, \theta \in (0,\infty)$ *and* $\int_0^\infty \phi_\alpha(\theta)\, d\theta = 1$.

Proof The proof is similar to the proof of Lemma 3.1 in [2], we can obtain the result by doing the necessary adjustments. We omit it here. □

Lemma 2.10 ([2]) *The operators* S_α^ψ *and* T_α^ψ *have the following properties*:

(i) *For any fixed* $t \geq s \geq 0, S_\alpha^\psi(t,s)$ *and* $T_\alpha^\psi(t,s)$ *are bounded linear operators with* $\|S_\alpha^\psi(t,s)(x)\| \leq M\|x\|$ *and* $\|T_\alpha^\psi(t,s)(x)\| \leq \frac{M}{\Gamma(\alpha)}\|x\|$ *for all* $x \in X$.

(ii) *The operators* S_α^ψ *and* T_α^ψ *are strongly continuous for all* $t \geq s \geq 0$, *that is, for every* $x \in X$ *and* $0 \leq s \leq t_1 < t_2 \leq b$, *we have*

$$\bigl\| S_\alpha^\psi(t_2,s)x - S_\alpha^\psi(t_1,s)x \bigr\| \to 0$$

and

$$\bigl\| T_\alpha^\psi(t_2,s)x - T_\alpha^\psi(t_1,s)x \bigr\| \to 0$$

as $t_1 - t_2 \to 0$.

3 Main results

In this section, we present and prove the existence of mild solutions for system (1.2). To develop our results, we first give the concept of mild solution for system (1.2).

Definition 3.1 An $\mathcal{F}_t$-adapted and measurable stochastic process $x \in \mathcal{L}_2^0(\Omega, X)$ is said to be a mild solution of system (1.2) if

(1) $x(t)$ is measurable, $\mathcal{F}_t$-adapted, and has càdlàg path on $0 \leq t \leq b$ almost everywhere;

(2) for $t \in [-\tau, 0], x(t) = \phi(t)$;

(3) for each $0 \leq t \leq b$, $x(t)$ satisfies the following integral equation:

$$x(t) = S_\alpha^\psi(t,0)\phi(0) + \int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t,s) f\big(s, x\big(s - r(s)\big)\psi'(s)\, ds + \int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t,s)\sigma(s)\psi'(s)\, dB_Q^H(s).$$

For further convenience, set $\frac{M^2}{\Gamma^2(\alpha)}\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1} = M_1$.

Before stating and proving the main results, we introduce the following hypotheses:

(H_0) Semigroup $S(t)$ is compact for each $t > 0$;

(H_1) Function $\psi(t) \in C^2(J, \mathbb{R})$ and $\psi''(t) > 0, \psi'(t) > 0$ for $\forall t \in J$;

(H_2) (2a) For each $x \in X$, the function $f(\cdot, x) : J \to X$ is strongly measurable with respect to t, and for each $t \in J$, the function $f(t, \cdot) : X \to X$ is continuous with respect to x;

(2b) There exist a continuous nondecreasing function $\mu : [0, \infty) \to (0, \infty)$ and constant L such that, for any $(t, x) \in J \times X$, we have

$$E\big\|f\big(t, x\big(t - r(t)\big)\big)\big\|^2 \leq L\big(1 + \mu\big(\|x\|_{\mathcal{C}}^2\big)\big), \qquad \liminf_{r\to\infty} \frac{\mu(r)}{r} = \Lambda < \infty;$$

(H_3) The function $\sigma : J \to \mathcal{L}_2^0(X, Y)$ satisfies

$$\sup_{t\in J}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2 < \infty, \quad \forall t \in J.$$

We define the operator $\Phi : \mathcal{C} \to \mathcal{C}$ as follows:

$$(\Phi x)(t) = S_\alpha^\psi(t,0)\phi(0) + \int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t,s) f\big(s, x\big(s - r(s)\big)\psi'(s)\, ds + \int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t,s)\sigma(s)\psi'(s)\, dB_Q^H(s).$$

According to assumptions (H_1) and (H_2) and Lemma 2.10, we can obtain

$$\begin{aligned}
&E\left\|\int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t,s) f\big(s, x\big(s - r(s)\big)\big)\psi'(s)\, ds\right\|^2 \\
&\quad\leq \frac{M^2}{\Gamma^2(\alpha)} E\left\|\int_0^t \big(\psi(t) - \psi(s)\big)^{\alpha-1} f\big(s, x\big(s - r(s)\big)\big)\psi'(s)\, ds\right\|^2 \\
&\quad\leq \frac{M^2}{\Gamma^2(\alpha)} \int_0^t E\big\|\big(\psi(t) - \psi(s)\big)^{\alpha-1} f\big(s, x\big(s - r(s)\big)\big)\psi'(s)\big\|^2\, ds \\
&\quad\leq \frac{M^2}{\Gamma^2(\alpha)} \int_0^t \big(\psi(t) - \psi(s)\big)^{2\alpha-2} E\big\|f\big(s, x\big(s - r(s)\big)\big)\big\|^2 \big(\psi'(s)\big)^2\, ds \\
&\quad\leq \frac{M^2}{\Gamma^2(\alpha)} \psi'(b) \frac{(\psi(t) - \psi(s))^{2\alpha-1}}{2\alpha - 1} L(1 + \mu\big(\|x\|_{\mathcal{C}}^2\big) \\
&\quad\leq \frac{M^2}{\Gamma^2(\alpha)} \psi'(b) \frac{(\psi(b) - \psi(0))^{2\alpha-1}}{2\alpha - 1} L(1 + \mu\big(\|x\|_{\mathcal{C}}^2\big).
\end{aligned}$$

Now, from assumptions (H_1) and (H_3), we can get

$$\begin{aligned}
&E\left\| \int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1} T_\alpha^\psi(t,s)\sigma(s)\psi'(s)\,dB_Q^H(s)\right\|^2 \\
&\quad\leq \frac{M^2}{\Gamma^2(\alpha)} E\left\| \int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1}\sigma(s)\psi'(s)\,dB_Q^H(s)\right\|^2 \\
&\quad\leq c_0 H(2H-1)t^{2H-1}\frac{M^2}{\Gamma^2(\alpha)}\int_0^t E\bigl\|\bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1}\sigma(s)\psi'(s)\bigr\|^2\,ds \\
&\quad\leq c_0 H(2H-1)t^{2H-1}\frac{M^2}{\Gamma^2(\alpha)}\int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{2\alpha-2}E\bigl\|\sigma(s)\bigr\|^2\bigl(\psi'(s)\bigr)^2\,ds \\
&\quad\leq c_0 H(2H-1)b^{2H-1}\frac{M^2}{\Gamma^2(\alpha)}\psi'(b)\frac{(\psi(t)-\psi(s))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\bigl\|\sigma(s)\bigr\|^2_{\mathcal{L}_2^0} \\
&\quad\leq c_0 H(2H-1)b^{2H-1}\frac{M^2}{\Gamma^2(\alpha)}\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\bigl\|\sigma(s)\bigr\|^2_{\mathcal{L}_2^0}.
\end{aligned}$$

In the following, we give the first existence result for system (1.2) with the corresponding semigroup be compact.

Theorem 3.1 *Suppose that hypotheses* (H_0)*–*(H_3) *hold, then system* (1.2) *has at least one mild solution defined on* J *provided that* $3M_1L\Lambda < 1$.

Proof Denote $B_q = \{x\in\mathcal{C}, \|x\|_{\mathcal{C}}^2 \leq q\}$, obviously, B_q is a bounded, closed, convex set in $\mathcal{C}$. We divide the proof into three steps.

Step 1. We shall show that there exists a constant $r = r(a)$ such that $\Phi(B_r)\subset B_r$.

In fact, if it is not true, then for each positive constant r there exists some $\hat{x}\in B_r$ such that $\Phi(\hat{x})\notin B_r$, i.e.,

$$\begin{aligned}
r &< E\bigl\|\Psi(\hat{x})\bigr\|^2 \\
&\leq 3E\bigl\|S_\alpha^\psi(t,0)\phi(0)\bigr\|^2 \\
&\quad + 3E\left\|\int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1}T_\alpha^\psi(t,s)\psi'(s)f\bigl(s,\hat{x}\bigl(s-r(s)\bigr)\bigr)\,ds\right\|^2 \\
&\quad + 3E\left\|\int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1}T_\alpha^\psi(t,s)\psi'(s)\sigma(s)\,dB_Q^H(s)\right\|^2 \\
&\leq 3M^2\phi^2(0) + 3\frac{M^2}{\Gamma^2(\alpha)}\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}L(1+\mu\bigl(\|\hat{x}\|_{\mathcal{C}}^2\bigr) \\
&\quad + 3c_0H(2H-1)b^{2H-1}\frac{M^2}{\Gamma^2(\alpha)}\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\bigl\|\sigma(s)\bigr\|^2_{\mathcal{L}_2^0}.
\end{aligned}$$

Dividing both sides by r and taking $r\to\infty$, we get

$$3\frac{M^2}{\Gamma^2(\alpha)}\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}L\Lambda = 3M_1L\Lambda > 1,$$

which is a contradiction to the hypotheses of Theorem 3.1. Thus, there exists r such that Φ maps B_r into itself.

Step 2. Φ is continuous on B_r.

For any $x_n, x \in B_r, n = 1, 2, \ldots$, with $\lim_{n\to\infty} \|x_n - x\|_{\mathcal{C}}^2 = 0$, we get $\lim_{n\to\infty} x_n(t) = x(t)$ for $t \in J$. Thus, by assumption (H_2), we can easily get Φ is continuous on B_r.

Step 3. Φ is a completely continuous operator.

We subdivide Step 3 into three claims.

Claim 1. Φ maps bounded sets into uniformly bounded sets in $\mathcal{C}$.

Actually, we only need to show that there exists a positive constant Δ such that, for each $x \in B_r$, one has $\|\Phi x\|_{\mathcal{C}} \le \Delta$. As a matter of fact, for each $t \in J$, Step 1 enables us to obtain this assertion.

Claim 2. $\Phi(B_r)$ is equicontinuous on B_r.

For $\forall x \in B_r$, let $0 = t_1 < t_2 \le b$. Taking (H_2), (H_3) and the strong continuity of $\{S_\alpha^\psi(t)\}_{t\ge 0}$ into account, we get as $t_2 \to 0$,

$$
\begin{aligned}
&E\big\|(\Phi x)(t_2) - (\Phi x)(0)\big\|^2 \\
&\quad\le 3E\big\|S_\psi^\alpha(t_2, 0) - S_\psi^\alpha(0, 0)\big\|^2 \\
&\qquad + 3E\bigg\|\int_0^{t_2} \big(\psi(t_2) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t_2 - s) f\big(s, x\big(s - r(s)\big)\big)\psi'(s)\,ds\bigg\|^2 \\
&\qquad + 3E\bigg\|\int_0^{t_2} \big(\psi(t_2) - \psi(s)\big)^{\alpha-1} T_\alpha^\psi(t_2 - s)\sigma(s)\psi'(s)\,dB_Q^H(s)\bigg\|^2 \\
&\quad\le 3E\big\|S_\psi^\alpha(t_2, 0) - S_\psi^\alpha(0, 0)\big\|^2 \\
&\qquad + 3\frac{M^2}{\Gamma^2(\alpha)}\frac{(\psi(t_2) - \psi(0))^{2\alpha-1}}{2\alpha - 1}\psi'(b)L(1 + \mu\big(\|x\|_{\mathcal{C}}^2\big) \\
&\qquad + 3\frac{M^2}{\Gamma^2(\alpha)}\frac{(\psi(t_2) - \psi(0))^{2\alpha-1}}{2\alpha - 1}\psi'(b)\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2 \to 0.
\end{aligned}
$$

For $0 < t_1 < t_2 \le b$, from the strong continuity of $\{T_\alpha^\psi(t)\}_{t\ge 0}$, there exist arbitrarily small constants $\delta, \tau > 0$ such that as long as $|t_2 - t_1| < \delta$, we have $\|T_\alpha^\psi(t_1) - T_\alpha^\psi(t_2)\| < \tau$. Then, for $\forall x \in B_r$, we can obtain

$$
\begin{aligned}
&E\big\|(\Phi x)(t_2) - (\Phi x)(t_1)\big\|^2 \\
&\quad\le 7E\big\|S_\psi^\alpha(t_2, 0) - S_\psi^\alpha(t_1, 0)\big\|^2 \\
&\qquad + 7E\bigg\|\int_0^{t_1} \big[\big(\psi(t_2) - \psi(s)\big)^{\alpha-1} - \big(\psi(t_1) - \psi(s)\big)^{\alpha-1}\big] T_\alpha^\psi(t_2 - s) \\
&\qquad \times f\big(s, x\big(s - r(s)\big)\big)\psi'(s)\,ds\bigg\|^2 \\
&\qquad + 7E\bigg\|\int_0^{t_1} \big[\big(\psi(t_2) - \psi(s)\big)^{\alpha-1} - \big(\psi(t_1) - \psi(s)\big)^{\alpha-1}\big] T_\alpha^\psi(t_2 - s) \\
&\qquad \times \sigma(s)\psi'(s)\,dB_Q^H(s)\bigg\|^2 \\
&\qquad + 7E\bigg\|\int_0^{t_1} \big(\psi(t_1) - \psi(s)\big)^{\alpha-1}\big[T_\alpha^\psi(t_2 - s) - T_\alpha^\psi(t_1 - s)\big] \\
&\qquad \times f\big(s, x\big(s - r(s)\big)\big)\psi'(s)\,ds\bigg\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 7E\left\| \int_0^{t_1} \big(\psi(t_1)-\psi(s)\big)^{\alpha-1}\big[T_\alpha^\psi(t_2-s)-T_\alpha^\psi(t_1-s)\big]\sigma(s)\psi'(s)\,dB_Q^H(s)\right\|^2 \\
&+ 7E\left\| \int_{t_1}^{t_2} \big(\psi(t_2)-\psi(s)\big)^{\alpha-1} T_\alpha^\psi(t_2-s) f\big(s,x\big(s-r(s)\big)\big)\psi'(s)\,ds\right\|^2 \\
&+ 7E\left\| \int_{t_1}^{t_2} \big(\psi(t_2)-\psi(s)\big)^{\alpha-1} T_\alpha^\psi(t_2-s)\sigma(s)\psi'(s)\,dB_Q^H(s)\right\|^2 \\
\le{}& 7E\big\|S_\psi^\alpha(t_2,0)-S_\psi^\alpha(t_1,0)\big\|^2 \\
&+ 7\int_0^{t_1} \big\|\big(\psi(t_2)-\psi(s)\big)^{\alpha-1}-\big(\psi(t_1)-\psi(s)\big)^{\alpha-1}\big\|^2 \big\|T_\alpha^\psi(t_2-s)\big\|^2 \\
&\times E\big\|f\big(s,x\big(s-r(s)\big)\big)\big\|^2\big(\psi'(s)\big)^2\,ds \\
&+ 7\int_0^{t_1} \big\|\big(\psi(t_2)-\psi(s)\big)^{\alpha-1}-\big(\psi(t_1)-\psi(s)\big)^{\alpha-1}\big\|^2 \big\|T_\alpha^\psi(t_2-s)\big\|^2 \\
&\times \big\|\sigma(s)\big\|^2\big(\psi'(s)\big)^2\,dB_Q^H(s) \\
&+ 7\big\|T_\alpha^\psi(t_2-s)-T_\alpha^\psi(t_1-s)\big\|^2\psi'(b)\frac{(\psi(b))^{2\alpha-1}}{2\alpha-1}L\big(1+\mu\big(\|x\|_{\mathcal{C}}^2\big)\big) \\
&+ 7\big\|T_\alpha^\psi(t_2-s)-T_\alpha^\psi(t_1-s)\big\|^2\psi'(b)\frac{(\psi(b))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2 \\
&+ 7\frac{M^2}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{2(\alpha-1)}E\big\|f\big(s,x\big(s-r(s)\big)\big)\big\|^2\big(\psi'(s)\big)^2\,ds \\
&+ 7\frac{M^2}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{2(\alpha-1)}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2\big(\psi'(s)\big)^2\,dB_Q^H(s) := 7\sum_{i=1}^7 I_i,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= E\big\|S_\psi^\alpha(t_2,0)-S_\psi^\alpha(t_1,0)\big\|^2, \\
I_2 &= \int_0^{t_1} \big\|\big(\psi(t_2)-\psi(s)\big)^{\alpha-1}-\big(\psi(t_1)-\psi(s)\big)^{\alpha-1}\big\|^2 \big\|T_\alpha^\psi(t_2-s)\big\|^2 \\
&\quad \times E\big\|f\big(s,x\big(s-r(s)\big)\big)\big\|^2\big(\psi'(s)\big)^2\,ds, \\
I_3 &= \int_0^{t_1} \big\|\big(\psi(t_2)-\psi(s)\big)^{\alpha-1}-\big(\psi(t_1)-\psi(s)\big)^{\alpha-1}\big\|^2 \big\|T_\alpha^\psi(t_2-s)\big\|^2 \\
&\quad \times \big\|\sigma(s)\big\|^2\big(\psi'(s)\big)^2\,dB_Q^H(s), \\
I_4 &= \big\|T_\alpha^\psi(t_2-s)-T_\alpha^\psi(t_1-s)\big\|^2\psi'(b)\frac{(\psi(b))^{2\alpha-1}}{2\alpha-1}L\big(1+\mu\big(\|x\|_{\mathcal{C}}^2\big)\big), \\
I_5 &= \big\|T_\alpha^\psi(t_2-s)-T_\alpha^\psi(t_1-s)\big\|^2\psi'(b)\frac{(\psi(b))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2, \\
I_6 &= \frac{M^2}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}\big(\psi(t_2)-\psi(s)\big)^{2(\alpha-1)}E\big\|f\big(s,x\big(s-r(s)\big)\big)\big\|^2\big(\psi'(s)\big)^2\,ds, \\
I_7 &= \frac{M^2}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}\big(\psi(t_2)-\psi(s)\big)^{2(\alpha-1)}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2\big(\psi'(s)\big)^2\,dB_Q^H(s).
\end{aligned}
$$

We next verify if each term tends to 0 as $t_2-t_1\to 0$.

For I_1, from the strong continuity of $\{S_\alpha^\psi(t)\}_{t\geq 0}$, we can draw the conclusion.

For I_2, letting $t_2 - t_1 \to 0$ leads to

$$
\begin{aligned}
I_2 \leq{} & \int_0^{t_1} \bigl[\bigl(\psi(t_2) - \psi(s)\bigr)^{\alpha-1} - \bigl(\psi(t_1) - \psi(s)\bigr)^{\alpha-1}\bigr]\psi'(s)\,ds \\
& \times \int_0^{t_1} \bigl[\bigl(\psi(t_2) - \psi(s)\bigr)^{\alpha-1} - \bigl(\psi(t_1) - \psi(s)\bigr)^{\alpha-1}\bigr]\bigl\| T_\alpha^\psi(t_2 - s)\bigr\|^2 \\
& \times E\bigl\| f\bigl(s, x\bigl(s - r(s)\bigr)\bigr)\bigr\|^2 \psi'(s)\,ds \\
\leq{} & \frac{M^2}{\Gamma^2(\alpha)} \biggl(\frac{(\psi(t_2) - \psi(t_1))^\alpha - (\psi(t_2))^\alpha + (\psi(t_1))^\alpha}{\alpha}\biggr)^2 L(1 + \mu\bigl(\|x\|_C^2\bigr) \to 0.
\end{aligned}
$$

Similarly, we can get I_3 tends to zero as $t_2 - t_1 \to 0$.

The strong continuity of $\{T_\alpha^\psi(t)\}_{t\geq 0}$ leads to $\| T_\alpha^\psi(t_2 - s) - T_\alpha^\psi(t_1 - s)\|^2 \to 0$ as $t_2 - t_1 \to 0$, thus I_4, I_5 tend to 0 as $t_2 - t_1 \to 0$.

In addition, we can derive that, as $t_2 - t_1 \to 0$,

$$
\begin{aligned}
I_6 &\leq \frac{M^2}{\Gamma^2(\alpha)} \int_{t_1}^{t_2} \bigl(\psi(t_2) - \psi(s)\bigr)^{2(\alpha-1)} E\bigl\| f\bigl(s, x\bigl(s - \nu(s)\bigr)\bigr)\bigr\|^2 \bigl(\psi'(s)\bigr)^2\,ds \\
&\leq \frac{M^2}{\Gamma^2(\alpha)} \psi'(b) \frac{(\psi(t_2) - \psi(t_1))^{2\alpha-1}}{2\alpha - 1} L(1 + \mu\bigl(\|x\|_C^2\bigr) \to 0.
\end{aligned}
$$

Similarly, we can obtain $I_7 \to 0$ as $t_2 - t_1 \to 0$.

Therefore, we derive that Φ is equicontinuous on B_r.

Claim 3. $V(t) = \{(\Phi x)(t), x \in B_r\}$ is relatively compact in X.

Let $0 < t \leq b$ be fixed, for $\forall \lambda \in (0, t)$ and $\forall \delta > 0, x \in B_r$, define an operator

$$
\begin{aligned}
& \bigl(\Phi^{\lambda,\delta} x\bigr)(t) \\
&\quad = \int_0^\infty \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta\bigr)\phi(0)\,d\theta \\
&\qquad + \alpha \int_0^{t-\lambda} \int_\delta^\infty \theta\bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta\bigr) \\
&\qquad \times f\bigl(s, x\bigl(s - r(s)\bigr)\bigr)\psi'(s)\,d\theta\,ds \\
&\qquad + \alpha \int_0^{t-\lambda} \int_\delta^\infty \theta\bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta\bigr) \\
&\qquad \times \sigma(s)\psi'(s)\,d\theta\,dB_Q^H(s) \\
&\quad = \int_0^\infty \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta\bigr)\phi(0)\,d\theta \\
&\qquad + \alpha S\bigl(\lambda^\alpha \theta\bigr) \int_0^{t-\lambda} \int_\delta^\infty \theta\bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta - \lambda^\alpha \theta\bigr) \\
&\qquad \times f\bigl(s, x\bigl(s - r(s)\bigr)\bigr)\psi'(s)\,d\theta\,ds \\
&\qquad + \alpha S\bigl(\lambda^\alpha \theta\bigr) \int_0^{t-\lambda} \int_\delta^\infty \theta\bigl(\psi(t) - \psi(s)\bigr)^{\alpha-1} \phi_\alpha(\theta) S\bigl(\bigl(\psi(t) - \psi(s)\bigr)^\alpha \theta - \lambda^\alpha \theta\bigr) \\
&\qquad \times \sigma(s)\psi'(s)\,d\theta\,dB_Q^H(s).
\end{aligned}
$$

From the compactness of $S(\lambda^\alpha\delta), \lambda^\alpha\delta > 0$, we obtain that, for $\forall\lambda \in (0,t)$ and $\forall\delta > 0$, the set $V^{\varepsilon,\delta}(t) = \{(\Phi^{\lambda,\delta}x)(t), x \in B_r\}$ is relatively compact in X.

Moreover, for each $x \in B_r$, from (H_1)–(H_3), we have

$$\begin{aligned}
&E\big\|(\Phi x)(t) - \big(\Phi^{\lambda,\delta}x\big)(t)\big\|^2 \\
&\quad= E\bigg\|\int_0^t\int_0^\delta \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big) \\
&\qquad\times f\big(s,x\big(s-r(s)\big)\big)\psi'(s)\,ds \\
&\qquad+\int_0^t\int_\delta^\infty \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)f\big(s,x\big(s-r(s)\big)\big)\psi'(s)\,ds \\
&\qquad+\int_0^t\int_0^\delta \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)\sigma(s)\psi'(s)\,dB_Q^H(s) \\
&\qquad+\int_0^t\int_\delta^\infty \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)\sigma(s)\psi'(s)\,dB_Q^H(s) \\
&\qquad-\int_0^{t-\lambda}\int_\delta^\infty \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)f\big(s,x\big(s-r(s)\big)\big)\psi'(s)\,ds \\
&\qquad-\int_0^{t-\lambda}\int_\delta^\infty \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)\sigma(s)\psi'(s)\,dB_Q^H(s)\bigg\|^2 \\
&\quad\le 4\alpha^2E\bigg\|\int_0^t\int_0^\delta \theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big) \\
&\qquad\times f\big(s,x\big(s-r(s)\big)\big)\psi'(s)\,ds\bigg\|^2 \\
&\qquad+4\alpha^2E\bigg\|\int_0^t\int_0^\delta \theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)\sigma(s)\psi'(s)\,dB_Q^H(s)\bigg\|^2 \\
&\qquad+4\alpha^2E\bigg\|\int_{t-\lambda}^t\int_\delta^\infty \theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big) \\
&\qquad\times f\big(s,x\big(s-r(s)\big)\big)\psi'(s)\,ds\bigg\|^2 \\
&\qquad+4\alpha^2E\bigg\|\int_{t-\lambda}^t\int_\delta^\infty \theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\phi_\alpha(\theta)S\big(\big(\psi(t)-\psi(s)\big)^\alpha\theta\big)\sigma(s)\psi'(s)\,dB_Q^H(s)\bigg\|^2 \\
&\quad\le 4\alpha^2M^2\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}L\big(1+\mu\big(\|x\|_C^2\big)\big)\bigg(\int_0^\delta\theta\phi_\alpha(\theta)\,d\theta\bigg)^2 \\
&\qquad+4\alpha^2M^2\psi'(b)\frac{(\psi(b)-\psi(0))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2\bigg(\int_0^\delta\theta\phi_\alpha(\theta)\,d\theta\bigg)^2 \\
&\qquad+4\alpha^2M^2\frac{1}{\Gamma^2(\alpha+1)}\psi'(b)\frac{(\psi(t)-\psi(t-\lambda))^{2\alpha-1}}{2\alpha-1}L\big(1+\mu\big(\|x\|_C^2\big)\big) \\
&\qquad+4\alpha^2M^2\frac{1}{\Gamma^2(\alpha+1)}\psi'(b)\frac{(\psi(t)-\psi(t-\lambda))^{2\alpha-1}}{2\alpha-1}\sup_{t\in J}\big\|\sigma(s)\big\|_{\mathcal{L}_2^0}^2,
\end{aligned}$$

where we have used the equality

$$\int_0^\infty \theta^\xi\phi_\alpha(\theta)\,d\theta = \int_0^\infty \frac{1}{\theta^{\alpha\xi}}\psi_\alpha(\theta)\,d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\alpha\xi)}, \quad \xi\in[0,1].$$

The right-hand side of the above inequality tends to 0 as $\lambda, \delta \to 0$. So we can obtain $E\|(\Phi x)(t) - (\Phi^{\lambda,\delta} x)(t)\|^2 \to 0$ as $\lambda, \delta \to 0^+$. Since there are relatively compact sets arbitrarily close to the set $V(t) = \{(\Phi x)(t), x \in B_r\}$, we consequently derive that $V(t) = \{(\Phi x)(t), x \in B_r\}$ is also a relatively compact set in X.

From Claims 1–3 and the Arzola–Ascoli theorem, we deduce that Φ is a completely continuous map, then Schauder's fixed point theorem enables us to claim that the operator equation $\Phi x = x$ has at least one fixed point on B_r which is just a mild solution for system (1.2). The proof is complete. □

Compared with Theorem 4.1 in [26], the condition imposed on f is easier to be satisfied in this theorem.

To establish the existence results when the associated C_0-semigroup is not necessary compact, we first require the following assumptions where B_r is still defined as in Theorem 3.1.

(H_0') $S(t)$ is continuous in the uniform operator topology for $t \geq 0$, and $\{S(t)\}_{t\geq 0}$ is uniformly bounded, i.e., there exists $M > 1$ such that $\sup_{t\in[0,+\infty)} |S(t)| < M$;

(H_4) There exists a positive function $L_f \in L_1(J, \mathbb{R}^+)$ such that, for $\forall x, y \in \mathcal{C}$,

$$E\big\|f\big(t, x\big(t - r(t)\big)\big) - f\big(t, y\big(t - r(t)\big)\big)\big\|^2 \leq L_f(t) E\big\|x\big(t - r(t)\big) - y\big(t - r(t)\big)\big\|^2, \quad \forall t \in J.$$

Next, we present our second existence uniqueness result for system (1.2) based on the Banach contraction principle in the case that semigroup $\{S(t)\}_{t>0}$ is not necessary compact.

Theorem 3.2 *Suppose that hypotheses* (H_0'), (H_1)–(H_4) *hold, then system* (1.2) *has a unique mild solution on* B_r *provided that* $M_1\|L_f(t)\|_{L^1(J,\mathbb{R}^+)} < 1$.

Proof We omit the proof here since it can be easily verified. □

Remark 3.1 The function σ is independent of $x(t), t \in (-\tau, b]$. From the functional point of view, we have $\alpha(\int_0^T (\psi(t) - \psi(s))^{\alpha-1} T_\alpha^\psi (t - s)\sigma(s)\psi'(s)\, dB_Q^H(s)) = 0$.

To give our last existence results, we require the following assumptions where Φ, B_r are still defined as in Theorem 3.1.

(H_5) There exists a positive function $m_f \in L_1(J, \mathbb{R}^+)$ such that, for any bounded closed subset $D \in B_r$, such that $\alpha(f(t, D(t)) \leq m_f(t)\alpha(D(t))$;

(H_6)

$$\frac{2M}{\Gamma(\alpha + 1)} \big(\psi(b) - \psi(0)\big)^\alpha \big\|m_f(t)\big\|_{L^1(J,\mathbb{R}^+)} < 1.$$

To end this section, we shall present our last existence uniqueness theorem for system (1.2).

Theorem 3.3 *Suppose that hypotheses* (H_0'), (H_1)–(H_3), (H_5), (H_6) *hold, then system* (1.2) *has at least one mild solution on* B_r.

Proof Set

$$(\Phi x)(t) := \sum_{i=1}^{3}(\Phi_i x_n)(t),$$

where

$$\begin{aligned}
(\Phi_1 x_n)(t) &= S_\alpha^\psi(t,0)\phi_n(0),\\
(\Phi_2 x_n)(t) &= \int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1} T_\alpha^\psi(t,s) f\bigl(s,x_n\bigl(s-r(s)\bigr)\bigr)\psi'(s)\,ds,\\
(\Phi_3 x_n)(t) &= \int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1} T_\alpha^\psi(t,s)\sigma(s)\psi'(s)\,dB_Q^H(s).
\end{aligned}$$

In what follows, we will prove that $\Phi : B_r \to B_r$ is a condensing operator.

For any $D \subset B_r$, by Lemma 2.7, there exists a countable set $D_0 = \{u_n\} \subset D$ such that

$$\alpha_C\bigl(\Phi(D)\bigr) \le 2\alpha_C\bigl(\Phi(D_0)\bigr). \tag{3.1}$$

Since $\Phi(D_0) \subset \Phi(B_r)$ is equicontinuous, we get from Lemma 2.4 that $\alpha_C(\Phi(D_0)) = \max_{t\in J}\alpha(\Phi(D_0)(t))$.

Obviously, we can derive

$$0 \le \alpha\bigl(\Phi_1(D_0)(t)\bigr) = \alpha\bigl(S_\alpha^\psi(t,0)\phi(0)\bigr) = 0.$$

Thus $\alpha(\Phi_1 x_n)(t) = 0$.

By Lemma 2.6 and (H_5), we have

$$\begin{aligned}
\alpha\bigl(\Phi_2(D_0)(t)\bigr) &= \alpha\biggl(\int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1} T_\alpha^\psi(t,s) f\bigl(s,x_n\bigl(s-r(s)\bigr)\bigr)\psi'(s)\,ds\biggr)\\
&\le \frac{2M}{\Gamma(\alpha)}\int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1}\psi'(s)\alpha\bigl(f\bigl(s,x_n\bigl(s-r(s)\bigr)\bigr)\,ds\\
&\le \frac{2M}{\Gamma(\alpha)}\int_0^t \bigl(\psi(t)-\psi(s)\bigr)^{\alpha-1}\psi'(s)m_f(t)\alpha\bigl(D(t)\bigr)\,ds\\
&\le \frac{2M}{\Gamma(\alpha+1)}\bigl(\psi(t)-\psi(s)\bigr)^{\alpha}\|m_f(t)\|_{L^1(J,\mathbb{R}^+)}\alpha\bigl(D(t)\bigr)\\
&\le \frac{2M}{\Gamma(\alpha+1)}\bigl(\psi(b)-\psi(0)\bigr)^{\alpha}\|m_f(t)\|_{L^1(J,\mathbb{R}^+)}\alpha\bigl(D(t)\bigr).
\end{aligned}$$

By Remark 3.1, we have $\alpha(\Phi_3(D_0)(t)) = 0$. Thus, by Lemma 2.3 and the above inequalities, we have

$$\begin{aligned}
&\alpha\bigl(\Phi(D_0)(t)\bigr)\\
&\quad\le \alpha\bigl(\Phi_1(D_0)(t)\bigr) + \alpha\bigl(\Phi_2(D_0)(t)\bigr) + \alpha\bigl(\Phi_3(D_0)(t)\bigr)\\
&\quad\le \frac{2M}{\Gamma(\alpha+1)}\bigl(\psi(b)-\psi(0)\bigr)^{\alpha}\|m_f(t)\|_{L^1(J,\mathbb{R}^+)}\alpha\bigl(D(t)\bigr).
\end{aligned} \tag{3.2}$$

Hence, from (3.1), (3.2), and assumption (H_6), we deduce that

$$\begin{aligned}\alpha\big(\Phi(D)(t)\big) &< \frac{2M}{\Gamma(\alpha+1)}\big(\psi(b)-\psi(0)\big)^{\alpha}\big\|m_f(t)\big\|_{L^1(J,\mathbb{R}^+)}\alpha\big(D(t)\big)\\ &< \alpha\big(D(t)\big).\end{aligned}$$

Thus, $\Phi : B_r \to B_r$ is a condensing operator. It follows from Lemma 2.7 that Φ has at least one fixed point in B_r which is actually a mild solution of system (1.2). This completes the proof of Theorem 3.3. □

Remark 3.2 Having compared Theorem 3.1 and Theorem 3.2 with Theorem 3.3, we know that one can replace the strong restriction condition H_0 with H_0' on the semigroup $\{S(t)\}_{t\geq 0}$ by applying the noncompact measure method. This seems a totally new result in contrast with earlier works on fractional stochastic evolution equations. Furthermore, the obtained results can been applied to fractional stochastic partial differential equations of parabolic type.

Remark 3.3 The above theorems provide existence results of system (1.2) in the case $\psi(t) \in C^2(J,\mathbb{R})$ and $\psi''(t) > 0, \psi'(t) > 0, \forall t \in J$, which is relatively restrictive. In fact, when $\psi(t) \in C^2(J,\mathbb{R})$ and $\psi''(t) < 0, \psi'(t) > 0, \forall t \in J$, we can also establish the corresponding existence results of system (1.2) with appropriate modifications of the hypotheses.

4 An example

As an application of our obtained results, we consider the following ψ-Caputo fractional stochastic evolution equations driven by fBm:

$$\begin{cases} {}^{C}_{0}D^{\alpha}_{\psi}x(t,z) = x_{zz}(t,z) + c_1 x(\frac{t}{2},z) + e^{-t}\frac{dB^H_Q(t)}{dt}, & t \in J = [0,1], z \in [0,\pi],\\ x(t,0) = x(t,\pi) = 0, & t \in J = [0,1],\\ x(t,z) = \phi(t,z), & t \in [-\tau,0], z \in [0,\pi], \end{cases} \tag{4.1}$$

where ${}^{C}_{0}D^{\alpha}_{\psi}$ is the ψ-Caputo fractional derivative of order $\frac{3}{4}$, $\psi(t) = e^t, f(t,x) = c_1 x(\frac{t}{2},z)$, $\sigma(t) = e^{-t}$, $r(t) = \frac{t}{2}$.

We choose the space $X = Y = L^2[0,\pi]$. Define an operator A by $Av = v''$ with the domain $D(A) = \{v \in X : v, v'$ absolutely continuous, $v'' \in X, v(0) = v(\pi) = 0\}$. Then A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ which is compact, analytic, and self-adjoint. With the above choices of A, f, σ, system (4.1) can be rewritten into the abstract form of system (1.2).

Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2, n \in N$, and the corresponding orthogonal eigenvectors are given by $e_n(z) = \sqrt{\frac{2}{\pi}}\sin(nz)$. Then $Az = \sum_{n=1}^{\infty} n^2\langle z, e_n\rangle e_n$. In addition, we know that for each $v \in X, S(t)v = \sum_{n=1}^{\infty} e^{-n^2 t}\langle v, e_n\rangle e_n$, in particular, $S(\cdot)$ is a uniformly stable semigroup and $\|S(t)\| \leq e^{-t} \leq 1 := M$.

Assume $B_r = \{x | x \in X, E\|x\|^2 \leq r\}$, then for $\forall t \in [0,1], x \in B_r$, we have

$$E\big\|f(t,x)\big\|^2 \leq c_1^2 E\bigg\|x\bigg(\frac{t}{2},z\bigg)\bigg\|^2 \leq 1 + c_1^2 E\bigg\|x\bigg(\frac{t}{2},z\bigg)\bigg\|^2 \leq 1 + c_1^2 r := L\big(1+\mu(r)\big);$$

$$\lim_{r\to\infty}\inf\frac{\mu(r)}{r}=\Lambda:=c_1^2,\qquad \sup_{t\in J}\left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0}=\sup_{t\in J}e^{-t}\le 1;$$
$$M_1=\frac{M^2}{\Gamma^2(\alpha)}\frac{(\psi(1)-\psi(0))^{2\alpha-1}}{2\alpha-1}\psi'(1)=\frac{2}{\Gamma^2(\frac{3}{4})}(e-1))^{\frac{1}{2}}e.$$

Therefore, (H_0)–(H_3) are satisfied with $M=1, L=1, \mu(r)=c_1^2r, \Lambda=c_1^2, M_1=\frac{2}{\Gamma^2(\frac{3}{4})}(e-1))^{\frac{1}{2}}e$. On the other hand, we can choose arbitrary constant c_1 such that $c_1<(\frac{1}{\frac{6}{\Gamma^2(\frac{3}{4})}(e-1))^{\frac{1}{2}}e})^{\frac{1}{2}}$, then we have $3M_1L\Lambda<1$, this implies that all assumptions of Theorem 3.1 are satisfied. Hence, from Theorem 3.1, we can claim that system (4.1) admits at least one mild solution on $[0,1]$.

5 Conclusion

The aim of this manuscript was to achieve sufficient conditions to ensure the existence and uniqueness of mild solutions for a class of ψ-Caputo fractional stochastic evolution equations with varying-time delay driven by fBm using the fixed point technique, noncompact measure method, and stochastic analysis when the associated C_0-semigroup is compact or not. The obtained results generalized the classical Caputo fractional derivative case. Also, we provided an example to illustrate our results. In addition, one interesting question is to study simultaneous finite dimensional exact and approximate controllability (finite-approximate controllability) of ψ-Caputo fractional stochastic differential inclusions driven by fBm or other stochastic noise which will be treated in the future.

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