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On discrete weighted Hardy type inequalities and properties of weighted discrete Muckenhoupt classes

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Abstract

In this paper, first we prove some new refinements of discrete weighted inequalities with negative powers on finite intervals. Next, by employing these inequalities, we prove that the self-improving property (*backward propagation property*) of the weighted discrete Muckenhoupt classes holds. The main results give exact values of the limit exponents as well as the new constants of the new classes. As an application, we establish the self-improving property (*forward propagation property*) of the discrete Gehring class.

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1 Introduction

The study of regularity and boundedness of the discrete operators on ℓ^p and higher summability of sequences was considered in the literature; see for example [2, 19–21, 28, 29] and the references cited therein. We also refer the reader to the papers [5–8, 28, 29, 33–35] for more results in the discrete spaces. In this paper, we confine ourselves to proving the self-improving property of the weighted discrete Muckenhoupt classes and then derive the self-improving property of the discrete Gehring classes. For the structure and relations between classical Muckenhoupt and Gehring classes (in integral forms) and their applications, we refer the reader to [1, 3, 9–11, 13–18, 22–26], [27, 30], and [36]. Throughout the paper, we assume that $1 < p < \infty$ and $\mathbb{I} \subseteq \mathbb{Z}_+$ is a fixed interval and the cardinality of a set $J \subset \mathbb{I}$ will be denoted by $|J|$. A discrete weight v defined on $\mathbb{Z}_+ = \{1, 2, \dots\}$ is a sequence $v = \{v(n)\}_{n=1}^\infty$ of nonnegative real numbers. A sequence of positive real numbers λ will be considered the weight in the space $\ell_\lambda^p(\mathbb{Z}_+)$ and the norm will be considered as

$$\|v\|_{\ell_\lambda^p(\mathbb{Z}_+)} := \left(\sum_{n=1}^{\infty} |v(n)|^p \lambda(n) \right)^{1/p} < \infty.$$

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A discrete nonnegative weight ν belongs to the discrete Muckenhoupt class $\mathcal{A}^p(A)$ on the interval $\mathbb{I} \subseteq \mathbb{Z}_+$ for $p > 1$ and $A > 1$ if the inequality

$$\left(\frac{1}{|J|} \sum_{k \in J} \nu(k) \right) \left(\frac{1}{|J|} \sum_{k \in J} \nu^{\frac{-1}{p-1}}(k) \right)^{p-1} \leq A \quad (1.1)$$

holds for every subinterval $J \subset \mathbb{I}$. For a given exponent $p > 1$, we define the $\mathcal{A}^p$ -norm of the discrete weight ν by the following quantity:

$$[\nu]_{\mathcal{A}^p} := \sup_{J \subset \mathbb{I}} \left(\frac{1}{|J|} \sum_{k \in J} \nu(k) \right) \left(\frac{1}{|J|} \sum_{k \in J} \nu^{\frac{-1}{p-1}}(k) \right)^{p-1}, \quad (1.2)$$

where the supremum is taken over all intervals $J \subset \mathbb{I}$. By fixing a constant $A > 1$, the pair of real numbers (p, A) defines a discrete Muckenhoupt class $\mathcal{A}^p(A)$ as follows:

$$\nu \in \mathcal{A}^p(A) \iff [\nu]_{\mathcal{A}^p} \leq A,$$

and we will refer to A as the $\mathcal{A}^p$ -constant of the class. Note that from Hölder's inequality $[\nu]_{\mathcal{A}^p} \geq 1$ holds for all $1 < p < \infty$ and the following inclusion is true:

$$\text{if } 1 < p \leq q < \infty, \text{ then } \mathcal{A}^p \subset \mathcal{A}^q \text{ and } [\nu]_{\mathcal{A}^q} \leq [\nu]_{\mathcal{A}^p}.$$

In [32], the authors proved that $\nu \in \mathcal{A}^p$, $1 < p < \infty$ if and only if $\nu^{-\frac{1}{p-1}} \in \mathcal{A}^{p'}$, where p' is the conjugate of p , and if $\nu \in \mathcal{A}^p$ ($1 < p < \infty$), then $\nu \in \mathcal{A}^q$ for every $q > p$ and that $\nu^\alpha \in \mathcal{A}^p$ for any $0 < \alpha < 1$. Böttcher and Seybold [4] proved that if $\nu \in \mathcal{A}^p$ then there is $\varepsilon = \varepsilon_{p,\nu}$ such that $\nu^r \in \mathcal{A}^p$ for all $r \in (1 - \varepsilon, 1 + \varepsilon)$. Böttcher and Seybold also proved the *self-improving* property for the discrete class $\mathcal{A}^p$ of weights which states that: if $\nu \in \mathcal{A}^p(\mathcal{C})$, then there exist constants $\delta > 0$ and $\mathcal{C}_1 > 0$ such that $\nu \in \mathcal{A}^{p-\delta}(\mathcal{C}_1)$, or equivalently

$$\mathcal{A}^p(\mathcal{C}) \subset \mathcal{A}^{p-\delta}(\mathcal{C}_1) \quad (1.3)$$

holds, without obtaining the exact values of δ and $\mathcal{C}_1$.

For a given exponent $q > 1$ and a constant $\mathcal{K} > 1$, a discrete nonnegative weight ν belongs to the discrete Gehring class $\mathcal{G}^q(\mathcal{K})$ (or satisfies a reverse Hölder inequality) on the interval $\mathbb{I} \subset \mathbb{Z}_+$ if for every subinterval $J \subseteq \mathbb{I}$ we have

$$\left(\frac{1}{|J|} \sum_{k \in J} \nu^q(k) \right)^{\frac{1}{q}} \leq \mathcal{K} \frac{1}{|J|} \sum_{k \in J} \nu(k).$$

For a given exponent $q > 1$, we define the $\mathcal{G}^q$ -norm of ν as

$$[\nu]_{\mathcal{G}^q} := \sup_{J \subset \mathbb{I}} \left[\left(\frac{1}{|J|} \sum_{k \in J} \nu(k) \right)^{-1} \left(\frac{1}{|J|} \sum_{k \in J} \nu^q(k) \right)^{\frac{1}{q}} \right]^{\frac{q}{q-1}}, \quad (1.4)$$

where the supremum is taken over all intervals $J \subset \mathbb{I}$ and represents the best constant for which the $\mathcal{G}^q$ -condition holds true independently on the interval $J \subseteq \mathbb{I}$. We say that ν is a

discrete Gehring class $\mathcal{G}^q$ -weight if its $\mathcal{G}^q$ -norm is finite, i.e.,

$$\nu \in \mathcal{G}^q \iff [\nu]_{\mathcal{G}^q} < \infty. \quad (1.5)$$

Note that by Hölder's inequality, $[\nu]_{\mathcal{G}^q} \geq 1$ holds for all $1 < q < \infty$ and the following inclusion is true:

$$\text{if } 1 < p \leq q < \infty, \text{ then } \mathcal{G}^q \subset \mathcal{G}^p \text{ and } 1 \leq [\nu]_{\mathcal{G}^p} \leq [\nu]_{\mathcal{G}^q}. \quad (1.6)$$

In [4], Böttcher and Seybold proved that if $\nu \in \mathcal{A}^p(\mathcal{C})$, then there exist constants $\delta > 0$ and $\mathcal{K}_1 < \infty$ depending only on p and ν such that

$$\frac{1}{|J|} \sum_{k \in J} \nu^{p(1+\varepsilon)}(k) \leq \mathcal{C}_1 \left(\frac{1}{|J|} \sum_{k \in J} \nu^p(k) \right)^{1+\varepsilon} \quad (1.7)$$

for all $\varepsilon \in [0, \delta]$ and all J of the form $|J| = 2^r$ with $r \in \mathbb{N}$ (the set of natural numbers). Note that inequality (1.7) is the reverse of the Hölder inequality

$$\left(\frac{1}{|J|} \sum_{k \in J} \nu^p(k) \right)^{1+\varepsilon} \leq \frac{1}{|J|} \sum_{k \in J} \nu^{p(1+\varepsilon)}(k),$$

so the result of Böttcher and Seybold is: if $\nu \in \mathcal{A}^p(\mathcal{C})$, then $\nu \in \mathcal{G}^q(\mathcal{C}_1)$, and then

$$\mathcal{A}^p(\mathcal{C}) \subset \mathcal{G}^q(\mathcal{C}_1). \quad (1.8)$$

This shows that every Muckenhoupt weight belongs to some Gehring class (*a transition property*). A discrete nonnegative weight ν belongs to the discrete Muckenhoupt class $\mathcal{A}^1(A)$ on a fixed interval $\mathbb{I} \subset \mathbb{Z}_+$ for $p > 1$ and $A > 1$ if the inequality

$$\frac{1}{|J|} \sum_{k \in J} \nu(k) \leq A \nu(k) \quad \text{for all } k \in J \quad (1.9)$$

holds for every subinterval $J \subset \mathbb{I}$ and $|J|$ is the cardinality of the set J . In [29], the authors proved that if ν is a nonnegative, nonincreasing sequence satisfying (1.9) for $A > 1$, then the inequality

$$\frac{1}{|J|} \sum_{k \in J} \nu^p(k) \leq A_1 \left(\frac{1}{|J|} \sum_{k \in J} \nu(k) \right)^p \quad \text{for } J \subset \mathbb{I} \quad (1.10)$$

holds for $p \in [1, A/(A-1))$, where $A_1 = \eta^p A/(A-p(A-1))$. From (1.10), we see that if $\nu \in \mathcal{A}^1(A)$, then $\nu \in \mathcal{G}^p(A_1)$, and so

$$\mathcal{A}^1(A) \subset \mathcal{G}^p(A_1) \quad \text{for } p < A/(A-1). \quad (1.11)$$

That is, every Muckenhoupt $\mathcal{A}^1$ -weight belongs to some Gehring class (*a transition property*). Later, in [31], the authors improved these results by giving sharp explicit values of the exponents p and A_1 .

Now, we give the definition of the weighted Muckenhoupt class $\mathcal{A}_\lambda^p(\mathcal{C})$, with a weight λ , that will be considered in this paper. A nonnegative discrete weight ν defined on a fixed interval $\mathbb{I}$ is called an $\mathcal{A}_\lambda^p(\mathcal{C})$ -Muckenhoupt weight for $p > 1$ if there exists a constant $C < \infty$ such that

$$\left(\frac{1}{\Lambda(J)} \sum_{k \in J} \lambda(k) \nu(k) \right) \left(\frac{1}{\Lambda(J)} \sum_{k \in J} \lambda(k) \nu^{-\frac{1}{p-1}}(k) \right)^{p-1} \leq C \quad (1.12)$$

for every subinterval $J \subset \mathbb{I}$, where $\Lambda(J) = \sum_{k \in J} \lambda(k)$. The smallest constant C satisfying (1.12) is called the $\mathcal{A}_\lambda^p$ -norm of the weight ν and is denoted by $[\nu]_{\mathcal{A}_\lambda^p}$. For a given fixed constant $C > 1$ if the weight ν belongs to $\mathcal{A}_\lambda^p(\mathcal{C})$, then $[\nu]_{\mathcal{A}_\lambda^p} \leq C$. Note that from Hölder's inequality $[\nu]_{\mathcal{A}_\lambda^p} \geq 1$ for all $1 < p < \infty$ and the following inclusion is true:

$$\text{if } 1 < p \leq q < \infty, \text{ then } \mathcal{A}_\lambda^p \subset \mathcal{A}_\lambda^q \quad \text{and} \quad [\nu]_{\mathcal{A}_\lambda^q} \leq [\nu]_{\mathcal{A}_\lambda^p}.$$

Our aim in this paper is to first prove some new refinements of discrete weighted inequalities with negative powers on finite intervals, then we use these inequalities to prove that the self-improving property (*backward propagation property*) of the weighted discrete Muckenhoupt classes holds. In particular, we prove that if $\nu \in \mathcal{A}_\lambda^p(\mathcal{C})$, then there exist constants $\delta, C_1 > 0$ such that $\nu \in \mathcal{A}_\lambda^{p-\delta}(\mathcal{C}_1)$. As an application, we deduce the self-improving property of the discrete Gehring class, i.e., we prove that if $\nu \in \mathcal{G}^p(\mathcal{K})$, then there exist constants $\epsilon, K_1 > 0$ such that $\nu \in \mathcal{G}^{p+\epsilon}(\mathcal{K}_1)$. The technique used in this paper allows us to get exact values of the exponents (δ and ϵ) and also exact values of the constants C_1 and K_1 . These values correspond to the sharp values obtained by Nikolidakis (see [26] and [25]) in the continuous case.

The paper is organized as follows: In Sect. 2, we prove some fundamental inequalities of Hardy's type with negative powers that will be needed in the proof of the main results. In Sect. 3, we prove the main results and show how to obtain the self-improving property of the discrete Gehring class by using the discrete weighted Muckenhoupt class.

2 Main inequalities

In this section, we prove some fundamental inequalities that shall be used in the proof of the main results. Throughout the paper, we assume that the weights in study are nonnegative sequences defined and use the conventions $0 \cdot \infty = 0$ and $0/0 = 0$ and $\sum_{k=m}^b \gamma(k) = 0$, whenever $m > b$. We fix an interval $J \subset \mathbb{Z}_+$ and consider subintervals J of the form $\{1, 2, \dots, N\}$ for $1 < N \leq \infty$ and assume that λ is a positive sequence defined on J . The classical Hölder inequality asserts that

$$\sum_{n=1}^N |u(n) \nu(n)| \leq \left[\sum_{n=1}^N |u(n)|^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^N |\nu(n)|^q \right]^{\frac{1}{q}}, \quad (2.1)$$

where u and ν are real sequences defined on J , $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The summation by parts formula reads

$$\sum_{k=1}^n \Delta u(k) \nu(k+1) = u(k) \nu(k) \Big|_{k=1}^{n+1} - \sum_{k=1}^n u(k) \Delta \nu(k), \quad (2.2)$$

where Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$. For any sequence $v: J \rightarrow \mathbb{R}^+$ (a nonnegative sequence), we define the operator (weighted Hardy operator) $\mathcal{H}_\lambda v: J \rightarrow \mathbb{R}^+$ by

$$\mathcal{H}_\lambda v(n) = \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k)v(k) \quad \text{for all } n \in J, \quad (2.3)$$

where $\Lambda(n) = \sum_{k=1}^n \lambda(k)$. From the definition of $\mathcal{H}_\lambda v$, we see that if v is nondecreasing, then

$$\mathcal{H}_\lambda v(n) = \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k)v(k) \leq \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k)v(n) = v(n).$$

Also, we have by using the above inequality that

$$\Delta(\mathcal{H}_\lambda v(n)) = \frac{\lambda(n)[\Lambda(n)v(n) - \sum_{k=1}^n \lambda(k)v(k)]}{\Lambda(n)\Lambda(n+1)} \geq 0.$$

From these two facts, we have the following properties of $\mathcal{H}_\lambda v$.

Lemma 2.1

- (i) If v is nondecreasing, then $\mathcal{H}_\lambda v(n) \leq v(n)$.
- (ii) If v is nondecreasing, then so is $\mathcal{H}_\lambda v$.

Remark 2.1 As a consequence of Lemma 2.1, we notice that if v is nonnegative and nondecreasing, then $\mathcal{H}_\lambda v^q \leq v^q$. We also notice from Lemma 2.1 that if v is nonnegative and nondecreasing, then $\mathcal{H}_\lambda v^q$ is also nonnegative and nondecreasing for $q > 1$.

Also, from the definition of $\mathcal{H}_\lambda$, we see that if v is nonincreasing, then

$$\mathcal{H}_\lambda v(n) = \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k)v(k) \geq \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k)v(n) = v(n).$$

Also, we have by using the above inequality that

$$\Delta \mathcal{H}_\lambda v(n) = \frac{\lambda(n)[\Lambda(n)v(n) - \sum_{k=1}^n \lambda(k)v(k)]}{\Lambda(n)\Lambda(n+1)} \leq 0.$$

From these two facts, we have the following properties of $\mathcal{H}_\lambda v$.

Lemma 2.2

- (i) If v is nonincreasing, then $\mathcal{H}_\lambda v(n) \geq v(n)$.
- (ii) If v is nonincreasing, then so is $\mathcal{H}_\lambda v$.

Remark 2.2 As a consequence of Lemma 2.2, we notice that if v is nonnegative and nonincreasing, then $\mathcal{H}_\lambda v^q \geq v^q$. We also notice from Lemma 2.2 that if v is nonnegative and nonincreasing, then $\mathcal{H}_\lambda v^q$ is also nonnegative and nonincreasing for $q > 1$.

Theorem 2.1 Assume that v is a nondecreasing weight. If $p \geq q > 0$, then for any $n \in J$, we have that

$$\sum_{k=1}^n \lambda(k) v^{q/p}(k) [\mathcal{H}_\lambda v(k)]^{-p-\frac{q}{p}} \leq \left(\frac{p+1}{p}\right)^{q/p} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p}. \quad (2.4)$$

Proof First we consider the case when $p = q$ and prove that the inequality

$$\sum_{k=1}^n \lambda(k) v(k) [\mathcal{H}_\lambda v(k)]^{-p-1} \leq \left(\frac{p+1}{p}\right) \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p}$$

holds for $p > 0$. For brevity, we write $\Phi(k) = \mathcal{H}_\lambda v(k)$. Now, by employing summation by parts with $\Delta u(k) = \lambda(k)$, $v(k+1) = (\Phi(k))^{-p}$, we have

$$u(k) = \Lambda(k-1), \quad \Delta v(k) = \Delta(\Phi(k-1))^{-p},$$

and then we obtain

$$\begin{aligned} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} &= \Lambda(k-1) \Phi^{-p}(k-1) \Big|_1^{n+1} - \sum_{k=1}^n \Lambda(k-1) \Delta \Phi^{-p}(k-1) \\ &= \Lambda(n) (\Phi(n))^{-p} - \sum_{k=1}^n \Lambda(k-1) \Delta \Phi^{-p}(k-1). \end{aligned} \quad (2.5)$$

By applying the inequality

$$\gamma z^{\gamma-1}(y-z) \leq y^\gamma - z^\gamma \leq \gamma y^{\gamma-1}(y-z), \quad \text{for } y \geq z > 0, \gamma \geq 1 \text{ or } \gamma < 0, \quad (2.6)$$

with $\gamma = -p < 0$, $y = \Phi(k)$ and $z = \Phi(k-1)$, and taking into account that the nondecreasing property of v implies that $\Phi(k) = \mathcal{H}_\lambda v(k)$ is also nondecreasing and then $y \geq z$, then we have

$$\Delta \Phi^{-p}(k-1) \leq -p(\Phi(k))^{-p-1} \Delta \Phi(k-1), \quad (2.7)$$

where

$$\begin{aligned} \Delta \Phi(k-1) &= \Delta \left(\frac{A(k-1)}{\Lambda(k-1)} \right) = \frac{A(k)}{\Lambda(k)} - \frac{A(k-1)}{\Lambda(k-1)} \\ &= \frac{\Lambda(k) \lambda(k) v(k) - \lambda(k) A(k)}{\Lambda(k-1) \Lambda(k)} \\ &= \frac{\lambda(k) v(k)}{\Lambda(k-1)} - \frac{\lambda(k) A(k)}{\Lambda(k-1) \Lambda(k)} \\ &= \frac{\lambda(k) v(k)}{\Lambda(k-1)} - \frac{\lambda(k) \Phi(k)}{\Lambda(k-1)}. \end{aligned} \quad (2.8)$$

By substituting (2.7) and (2.8) into (2.5), we obtain

$$\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \geq \Lambda(n) (\Phi(n))^{-p} + p \sum_{k=1}^n (\Phi(k))^{-p-1} [\lambda(k) v(k) - \lambda(k) \Phi(k)], \quad (2.9)$$

from which we have

$$\begin{aligned}\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} &\geq p \sum_{k=1}^n (\Phi(k))^{-p-1} [\lambda(k)v(k) - \lambda(k)\Phi(k)] \\ &= p \sum_{k=1}^n \lambda(k)v(k) (\Phi(k))^{-p-1} - p \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p}.\end{aligned}$$

This implies that

$$\sum_{k=1}^n v(k) \lambda(k) (\Phi(k))^{-p-1} \leq \left(\frac{p+1}{p} \right) \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p}, \quad (2.10)$$

which proves inequality (2.4) when $p = q$. Now, consider the case when $p \neq q$ and fix $q \in (0, p)$. Then, by applying Hölder's inequality with exponents $p/q > 1$ and $p/(p-q)$ and a weight λ and then by using (2.10), we get that

$$\begin{aligned}\sum_{k=1}^n \lambda(k) v^{q/p}(k) (\Phi(k))^{-p-\frac{q}{p}} &= \sum_{k=1}^n \lambda(k) [v^{q/p}(k) (\Phi(k))^{-q-\frac{q}{p}}] (\Phi(k))^{-p+q} \\ &\leq \left[\sum_{k=1}^n \lambda(k) v(k) (\Phi(k))^{-p-1} \right]^{q/p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{1-\frac{q}{p}} \\ &\leq \left(\frac{p+1}{p} \right)^{q/p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{q/p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{1-\frac{q}{p}} \\ &= \left(\frac{p+1}{p} \right)^{q/p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p}.\end{aligned}$$

Using the substitution $\Phi(k) = \mathcal{H}_\lambda v(k)$, we obtain the desired inequality (2.4). The proof is complete. $\square$

Theorem 2.2 Assume that v is a nondecreasing weight. If $p \geq q > 0$, then

$$\sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p} \leq \left(\frac{p+1}{p} \right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) [\mathcal{H}_\lambda v(k)]^{-p+q}. \quad (2.11)$$

Proof The elementary inequality (see Elliott [12]) states that

$$py^{p+1} - (p+1)y^p \geq -1 \quad (2.12)$$

for every $y \geq 0$ and $p > 0$, then we deduce that

$$py - (p+1) \geq -y^{-p},$$

from which we obtain

$$y^{-p} + py \geq p + 1 \quad \text{for every } y \geq 0 \text{ and } p > 0.$$

By employing the last inequality with $y = y_1/y_2$, where $y_1, y_2 > 0$, we get that

$$y_1^{-p} + py_1y_2^{-p-1} \geq (p+1)y_2^{-p}. \quad (2.13)$$

Let

$$y_1 = \left(\frac{p}{p+1}\right)^{1+q/p} v^{q/p}(k) (\mathcal{H}_\lambda v(k))^{1-q/p}, \quad y_2 = \left(\frac{p}{p+1}\right) (\mathcal{H}_\lambda v(k)),$$

then

$$\begin{aligned} y_1^{-p} &:= \left(\frac{p}{p+1}\right)^{-p-q} v^{-q}(k) (\mathcal{H}_\lambda v(k))^{-p+q}, \\ y_2^{-p} &:= \left(\frac{p}{p+1}\right)^{-p} (\mathcal{H}_\lambda v(k))^{-p}, \\ y_1 y_2^{-p-1} &:= \left(\frac{p}{p+1}\right)^{-p+q/p} v^{q/p}(k) (\mathcal{H}_\lambda v(k))^{-p-q/p}. \end{aligned}$$

By using these values in (2.13), we get that

$$\begin{aligned} (p+1) \left(\frac{p}{p+1}\right)^{-p} (\mathcal{H}_\lambda v(k))^{-p} &\leq \left(\frac{p}{p+1}\right)^{-p-q} v^{-q}(k) (\mathcal{H}_\lambda v(k))^{-p+q} \\ &\quad + p \left(\frac{p}{p+1}\right)^{-p+q/p} v^{q/p}(k) (\mathcal{H}_\lambda v(k))^{-p-q/p}. \end{aligned}$$

Thus, we have

$$\begin{aligned} (p+1) (\mathcal{H}_\lambda v(k))^{-p} &\leq \left(\frac{p+1}{p}\right)^q v^{-q}(k) (\mathcal{H}_\lambda v(k))^{-p+q} + p \left(\frac{p}{p+1}\right)^{q/p} v^{q/p}(k) (\mathcal{H}_\lambda v(k))^{-p-q/p}. \end{aligned}$$

By multiplying both sides by $\lambda(k)$ and summing from 1 to n , we have

$$\begin{aligned} (p+1) \sum_{k=1}^n \lambda(k) (\mathcal{H}_\lambda v(k))^{-p} &\leq \left(\frac{p+1}{p}\right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) (\mathcal{H}_\lambda v(k))^{-p+q} \\ &\quad + p \left(\frac{p+1}{p}\right)^{-q/p} \sum_{k=1}^n \lambda(k) v^{q/p}(k) (\mathcal{H}_\lambda v(k))^{-p-q/p}. \end{aligned} \quad (2.14)$$

By applying Theorem 2.1 on the last term and simplifying, (2.14) becomes

$$\sum_{k=1}^n \lambda(k) (\mathcal{H}_\lambda v(k))^{-p} \leq \left(\frac{p+1}{p}\right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) (\mathcal{H}_\lambda v(k))^{-p+q},$$

which is the desired inequality (2.11). The proof is complete. $\square$

As a special case, when $p = q = r$, $r > 0$, we obtain the following result.

Theorem 2.3 *Assume that v is a nondecreasing weight. If $r > 0$, we get that*

$$\sum_{k=1}^n \lambda(k) (\mathcal{H}_\lambda v(k))^{-r} \leq \left(\frac{r+1}{r} \right)^r \sum_{k=1}^n \lambda(k) v^{-r}(k). \quad (2.15)$$

Theorem 2.4 *Assume that v is a nondecreasing weight. If $p \geq q > 0$, then*

$$\begin{aligned} \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) v^{q/p}(k) [\mathcal{H}_\lambda v(k)]^{-p-\frac{q}{p}} &\leq \left(\frac{p+1}{p} \right)^{q/p} \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p} \\ &\quad - \frac{q}{p^2} \left(\frac{p+1}{p} \right)^{q/p-1} [\mathcal{H}_\lambda v(n)]^{-p}. \end{aligned} \quad (2.16)$$

Proof We proceed as in the proof of Theorem 2.1 to obtain (2.9), which writes

$$\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \geq \Lambda(n) (\Phi(n))^{-p} + p \sum_{k=1}^n \lambda(k) v(k) (\Phi(k))^{-p-1} - p \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p}.$$

By simplifying, we conclude that

$$\begin{aligned} 0 &< \sum_{k=1}^n \lambda(k) v(k) (\Phi(k))^{-p-1} \\ &\leq \frac{p+1}{p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} - \frac{1}{p} \Lambda(n) (\Phi(n))^{-p}. \end{aligned} \quad (2.17)$$

Fix $q \in (0, p)$, by applying Hölder's inequality with exponents $p/q > 1$ and $p/(p-q)$ and a weight λ , and then by using inequality (2.17), we obtain

$$\begin{aligned} &\sum_{k=1}^n \lambda(k) v^{q/p}(k) (\Phi(k))^{-p-(q/p)} \\ &= \sum_{k=1}^n \lambda(k) [v^{q/p}(k) (\Phi(k))^{-q-(q/p)}] (\Phi(k))^{-p+q} \\ &\leq \left[\sum_{k=1}^n \lambda(k) v(k) (\Phi(k))^{-p-1} \right]^{q/p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{1-(q/p)} \\ &\leq \left[\frac{p+1}{p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} - \frac{1}{p} \Lambda(n) (\Phi(n))^{-p} \right]^{q/p} \\ &\quad \times \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{1-(q/p)}. \end{aligned} \quad (2.18)$$

Now, in order to complete the proof, we use the inequality

$$(u+v)^\gamma \leq u^\gamma + \gamma u^{\gamma-1} v, \quad \text{where } 0 < \gamma < 1, \quad (2.19)$$

which is a variant of the well-known Bernoulli inequality. This inequality is valid for all $u \geq 0$ and $u + v \geq 0$ or $u > 0$ and $u + v > 0$, and equality holds if and only if $v = 1$. Now, by employing (2.19) with $\gamma = q/p < 1$,

$$u := \frac{p+1}{p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \quad \text{and} \quad v := -\frac{1}{p} \Lambda(n) (\Phi(n))^{-p},$$

and noting that (see (2.17))

$$\frac{p+1}{p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} - \frac{1}{p} \Lambda(n) (\Phi(n))^{-p} > 0,$$

we get

$$\begin{aligned} & \left[\frac{p+1}{p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} - \frac{1}{p} \Lambda(n) (\Phi(n))^{-p} \right]^{q/p} \\ & \leq \left(\frac{p+1}{p} \right)^{q/p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{q/p} \\ & \quad - \frac{q}{p} \left(\frac{p+1}{p} \right)^{q/p-1} \times \frac{1}{p} \Lambda(n) (\Phi(n))^{-p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{q/p-1} \\ & = \left(\frac{p+1}{p} \right)^{q/p} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{q/p} \\ & \quad - \frac{q}{p^2} \left(\frac{p+1}{p} \right)^{q/p-1} \left[\sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} \right]^{q/p-1} \Lambda(n) (\Phi(n))^{-p}. \end{aligned}$$

By substituting the last inequality into (2.18), we obtain

$$\begin{aligned} & \sum_{k=1}^n \lambda(k) v^{q/p}(k) (\Phi(k))^{-p-q/p} \\ & \leq \left(\frac{p+1}{p} \right)^{q/p} \sum_{k=1}^n \lambda(k) (\Phi(k))^{-p} - \frac{q}{p^2} \left(\frac{p+1}{p} \right)^{(q/p)-1} \Lambda(n) (\Phi(n))^{-p}, \end{aligned}$$

from which we obtain the desired inequality (2.16). The proof is complete. $\square$

Theorem 2.5 Assume that v is a nondecreasing weight. If $p \geq q > 0$, then

$$\begin{aligned} & \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p} \\ & \leq \left(\frac{p+1}{p} \right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) [\mathcal{H}_\lambda v(k)]^{-p+q} - \frac{q}{p+1} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{-p}. \end{aligned} \quad (2.20)$$

Proof We proceed as in the proof of Theorem 2.2 to obtain

$$\begin{aligned} & \left(\frac{p+1}{p}\right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) [\mathcal{H}_\lambda v(k)]^{-p+q} \\ & \quad + p \left(\frac{p}{p+1}\right)^{q/p} \sum_{k=1}^n \lambda(k) v^{q/p}(k) [\mathcal{H}_\lambda v(k)]^{-p-q/p} \\ & \geq (p+1) \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p}. \end{aligned} \quad (2.21)$$

By applying Theorem 2.4, we get

$$\begin{aligned} & \sum_{k=1}^n \lambda(k) v^{q/p}(k) [\mathcal{H}_\lambda v(k)]^{-p-q/p} \\ & \leq \left(\frac{p+1}{p}\right)^{q/p} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p} - \frac{q}{p^2} \left(\frac{p+1}{p}\right)^{q/p-1} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{-p}, \end{aligned}$$

and then (2.21) becomes

$$\begin{aligned} & \left(\frac{p+1}{p}\right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) [\mathcal{H}_\lambda v(k)]^{-p+q} + p \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p} \\ & \quad - \frac{q}{p} \left(\frac{p}{p+1}\right) \Lambda(n) [\mathcal{H}_\lambda v(n)]^{-p} \\ & \geq (p+1) \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p}. \end{aligned}$$

By combining similar terms, we obtain

$$\begin{aligned} & \left(\frac{p+1}{p}\right)^q \sum_{k=1}^n \lambda(k) v^{-q}(k) [\mathcal{H}_\lambda v(k)]^{-p+q} - \frac{q}{p+1} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{-p} \\ & \geq \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-p}, \end{aligned} \quad (2.22)$$

which is the desired inequality (2.20). The proof is complete. $\square$

The following lemma plays a crucial rule in establishing our main results.

Lemma 2.3 *Let $p > 1$, v be nonincreasing weight on $\mathbb{I}$, and let $\mathcal{H}_\lambda v$ be defined by (2.3). If $\gamma > 1$, then*

$$\frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) \left[v(k) (\mathcal{H}_\lambda v(k))^{\gamma-1} - \frac{\gamma-1}{\gamma} (\mathcal{H}_\lambda v(k))^\gamma \right] \leq \frac{1}{\gamma} (\mathcal{H}_\lambda v(n))^\gamma \quad (2.23)$$

holds for all $n \in \mathbb{I}$.

Proof Let $k \in \mathbb{I}$. Since $\lambda(k)v(k) = \Delta[\Lambda(k-1)\mathcal{H}_\lambda v(k-1)]$ and $\lambda(k) = \Delta\Lambda(k-1)$, then we have

$$\begin{aligned} & \lambda(k)v(k)(\mathcal{H}_\lambda v(k))^{\gamma-1} - \frac{\gamma-1}{\gamma}\lambda(k)(\mathcal{H}_\lambda v(k))^\gamma \\ &= \Delta[\Lambda(k-1)\mathcal{H}_\lambda v(k-1)](\mathcal{H}_\lambda v(k))^{\gamma-1} - \frac{\gamma-1}{\gamma}(\mathcal{H}_\lambda v(k))^\gamma \Delta\Lambda(k-1), \end{aligned}$$

and by using the product rule

$$\Delta u(k)v(k) = u(k)\Delta v(k) + v(k+1)\Delta u(k), \quad (2.24)$$

once with $u(k) = \Lambda(k-1)\mathcal{H}_\lambda v(k-1)$ and $v(k) = (\mathcal{H}_\lambda v(k-1))^{\gamma-1}$ and once with $u(k) = \Lambda(k-1)$ and $v(k) = (\mathcal{H}_\lambda v(k-1))^\gamma$, we have that

$$\begin{aligned} & \lambda(k)v(k)(\mathcal{H}_\lambda v(k))^{\gamma-1} - \frac{\gamma-1}{\gamma}\lambda(k)(\mathcal{H}_\lambda v(k))^\gamma \\ &= \Delta[\Lambda(k-1)(\mathcal{H}_\lambda v(k-1))^\gamma] - \Lambda(k-1)\mathcal{H}_\lambda v(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^{\gamma-1} \\ &\quad - \frac{\gamma-1}{\gamma}\Delta[\Lambda(k-1)(\mathcal{H}_\lambda v(k-1))^\gamma] + \frac{\gamma-1}{\gamma}\Lambda(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^\gamma \\ &= \frac{1}{\gamma}\Delta[\Lambda(k-1)(\mathcal{H}_\lambda v(k-1))^\gamma] - \Lambda(k-1)\mathcal{H}_\lambda v(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^{\gamma-1} \\ &\quad + \frac{\gamma-1}{\gamma}\Lambda(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^\gamma. \end{aligned} \quad (2.25)$$

Since v is nonincreasing, then so is $\mathcal{H}_\lambda v$ (or equivalently, $\Delta\mathcal{H}_\lambda v < 0$) and again by applying the product rule (2.24) with $u(k) = \mathcal{H}_\lambda v(k-1)$ and $v(k) = (\mathcal{H}_\lambda v(k-1))^{\gamma-1}$, we have

$$\begin{aligned} & -\Lambda(k-1)\mathcal{H}_\lambda v(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^{\gamma-1} + \frac{\gamma-1}{\gamma}\Lambda(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^\gamma \\ &\leq -\Lambda(k-1)\mathcal{H}_\lambda v(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^{\gamma-1} + \Lambda(k-1)\Delta(\mathcal{H}_\lambda v(k-1))^\gamma \\ &= \Lambda(k-1)(\mathcal{H}_\lambda v(k))^{\gamma-1}\Delta\mathcal{H}_\lambda v(k-1) < 0. \end{aligned} \quad (2.26)$$

By using (2.25) and (2.26), we have

$$\lambda(k)v(k)(\mathcal{H}_\lambda v(k))^{\gamma-1} - \frac{\gamma-1}{\gamma}\lambda(k)(\mathcal{H}_\lambda v(k))^\gamma \leq \frac{1}{\gamma}\Delta[\Lambda(k-1)(\mathcal{H}_\lambda v(k-1))^\gamma],$$

and by summing from $k = 1$ to n and multiplying by $\Lambda(k)$, we obtain

$$\begin{aligned} & \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) \left[v(k)(\mathcal{H}_\lambda v(k))^{\gamma-1} - \frac{\gamma-1}{\gamma}(\mathcal{H}_\lambda v(k))^\gamma \right] \\ &\leq \frac{1}{\gamma} \frac{1}{\Lambda(n)} \sum_{k=1}^n \Delta[\Lambda(k-1)(\mathcal{H}_\lambda v(k-1))^\gamma] \\ &= \frac{1}{\gamma} \frac{1}{\Lambda(n)} \Lambda(n)(\mathcal{H}_\lambda v(n))^\gamma = \frac{1}{\gamma}(\mathcal{H}_\lambda v(n))^\gamma. \end{aligned}$$

This is the required result. This completes the proof. $\square$

If we assume that v is nondecreasing and replace v with $v^{\frac{-1}{p-1}}$ in Lemma 2.3, we obtain the following result which will be needed in proving the main results in the next section.

Lemma 2.4 *Assume that v is a nondecreasing weight and $\mathcal{H}_\lambda v$ is defined as in (2.3) for sequence $\lambda(n)$ of positive numbers. If $1 < \gamma < \infty$, then for all $n \in J$ we have*

$$\begin{aligned} & \frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) \left[(v(k))^{\frac{-1}{p-1}} (\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k))^{\gamma-1} - \frac{\gamma-1}{\gamma} (\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k))^\gamma \right] \\ & \leq \frac{1}{\gamma} [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(n)]^\gamma. \end{aligned} \quad (2.27)$$

3 Main results

Now, we will state and prove the main results which deal with the self-improving property of the weighted Muckenhoupt class $\mathcal{A}_\lambda^p$, with exact values of the constant and the limit of exponents.

Theorem 3.1 *Assume that $p > 1$ and v is a nondecreasing weight and λ is a positive weight on the interval J . If $v \in \mathcal{A}_\lambda^p(C)$, then $v \in \mathcal{A}_\lambda^q(C_1)$ for any $q \in (q_0, p]$, where q_0 is the unique root of the equation*

$$\frac{p-q_0}{p-1} (Cq_0)^{\frac{1}{p-1}} = 1. \quad (3.1)$$

Furthermore, the constant C_1 is given by

$$C_1 := \left(\frac{q-1}{p-1} \frac{C^{\frac{1}{p-1}}}{\Gamma^{p,q}(C)} \right)^{q-1}, \quad (3.2)$$

where $\Gamma^{p,q}(C) := 1 - (Cq)^{\frac{1}{p-1}} (\frac{p-q}{p-1}) > 0$.

Proof By applying Lemma 2.4 with $\gamma = (p-1)/(q-1) > 1$ for $p > q > 1$, we obtain

$$\begin{aligned} & \frac{p-1}{q-1} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k)]^{\frac{p-q}{q-1}} - \frac{p-q}{q-1} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k)]^{\frac{p-1}{q-1}} \\ & \leq \Lambda(n) [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(n)]^{\frac{p-1}{q-1}}. \end{aligned}$$

Since $v \in \mathcal{A}_\lambda^p(C)$, we see that

$$\mathcal{H}_\lambda v(n) [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(n)]^{p-1} \leq C \quad \text{for } C > 1. \quad (3.3)$$

Moreover, from (3.3), we obtain

$$\begin{aligned} & \sum_{k=1}^n \lambda(k) \left[\frac{p-1}{p-q} (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k)]^{\frac{p-q}{q-1}} - [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k)]^{\frac{p-1}{q-1}} \right] \\ & \leq \frac{q-1}{p-q} C^{\frac{1}{q-1}} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}}. \end{aligned} \quad (3.4)$$

Consider the function

$$\phi(\beta) = \frac{p-1}{p-q} \alpha \beta^{\frac{p-q}{q-1}} - \beta^{\frac{p-1}{q-1}} \quad \text{for } \beta > \alpha,$$

then

$$\phi'(\beta) = \frac{p-1}{q-1} \beta^{\frac{1}{q-1}(p-2q+1)} \alpha - \frac{p-1}{q-1} \beta^{\frac{p-q}{q-1}} \leq \frac{p-1}{q-1} \beta^{\frac{1}{q-1}(p-q)} - \frac{p-1}{q-1} \beta^{\frac{p-q}{q-1}} = 0.$$

That is, ϕ is decreasing for all $\beta > \alpha$. Now, choose

$$\alpha = (v(k))^{\frac{-1}{p-1}}, \quad \beta = \mathcal{H}_\lambda v^{\frac{-1}{p-1}}, \quad \text{and} \quad \zeta = C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}}.$$

Since v is nondecreasing, we see that $v^{\frac{-1}{p-1}}$ is nonincreasing, and then from Lemma 2.2 we have that $\mathcal{H}_\lambda v^{\frac{-1}{p-1}} \geq v^{\frac{-1}{p-1}}$, which implies that $\alpha \leq \beta$. Also, by using (3.3), we see that $\beta \leq \zeta$. Thus, we have $\alpha \leq \beta \leq \zeta$, and then

$$\Phi(\alpha) \geq \Phi(\beta) \geq \Phi(\zeta) \quad \text{for } \alpha \leq \beta \leq \zeta.$$

This implies that

$$\Phi(\mathcal{H}_\lambda v^{\frac{-1}{p-1}}) \geq \Phi(C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v]^{\frac{-1}{q-1}}),$$

that is,

$$\begin{aligned} & \sum_{k=1}^n \lambda(k) \left[\frac{p-1}{p-q} (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k)]^{\frac{p-q}{q-1}} - [\mathcal{H}_\lambda v^{\frac{-1}{p-1}}(k)]^{\frac{p-1}{q-1}} \right] \\ & \geq \sum_{k=1}^n \lambda(k) \left[\frac{p-1}{p-q} (v(k))^{\frac{-1}{p-1}} [C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v]^{\frac{-1}{q-1}}]^{\frac{p-q}{q-1}} - [C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v]^{\frac{-1}{q-1}}]^{\frac{p-1}{q-1}} \right] \\ & = \sum_{k=1}^n \lambda(k) \left[\frac{p-1}{p-q} (v(k))^{\frac{-1}{p-1}} [C^{-1} \mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} - C^{\frac{1}{q-1}} [\mathcal{H}_\lambda v(k)]^{\frac{-1}{q-1}} \right]. \end{aligned}$$

From the last inequality and (3.4), we get that

$$\begin{aligned} & \frac{p-1}{p-q} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [C^{-1} \mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} \\ & \leq \frac{q-1}{p-q} C^{\frac{1}{q-1}} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}} + C^{\frac{1}{q-1}} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{\frac{-1}{q-1}}. \end{aligned}$$

This leads to

$$\begin{aligned} & \frac{p-1}{p-q} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} \\ & \leq \frac{q-1}{p-q} C^{\frac{1}{q-1}} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}} + C^{\frac{1}{p-1}} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{\frac{-1}{q-1}}. \end{aligned} \quad (3.5)$$

By replacing p and q with $1/(q-1) > 0$ and $1/(p-1) > 0$, respectively, in inequality (2.11), we obtain

$$\sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-\frac{1}{q-1}} \leq q^{\frac{1}{p-1}} \sum_{k=1}^n \lambda(k) v^{\frac{-1}{p-1}}(k) [\mathcal{H}_\lambda v(k)]^{-\frac{1}{q-1} + \frac{1}{p-1}}. \quad (3.6)$$

By combining (3.5) and (3.6) and simplifying, we have

$$\begin{aligned} & \frac{p-1}{p-q} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} - (Cq)^{\frac{1}{p-1}} \sum_{k=1}^n \lambda(k) v^{\frac{-1}{p-1}}(k) [\mathcal{H}_\lambda v(k)]^{-\frac{1}{q-1} + \frac{1}{p-1}} \\ & \leq \frac{q-1}{p-q} C^{\frac{1}{p-1}} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}}, \end{aligned}$$

from which we obtain

$$\begin{aligned} & \left[1 - \frac{p-q}{p-1} (Cq)^{\frac{1}{p-1}} \right] \left(\frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) v^{\frac{-1}{p-1}}(k) [\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} \right) \\ & \leq \frac{q-1}{p-1} C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}}. \end{aligned} \quad (3.7)$$

Let $\psi(x) = 1 - \phi(p, x)$, where

$$\phi(x, p) = \left(\frac{p-x}{p-1} \right) (Cx)^{\frac{1}{p-1}}.$$

Clearly, $\psi(p) = 1 > 0$ and since $-\phi(p, x)$ is a strictly increasing function for positive values of x , then the same holds for $\psi(x)$ which will vanish for a certain value $q_0 < p$ given by the unique positive solution of the equation $\phi(p, x) = 1$. Then $\psi(q_0) = 1$, and

$$\psi(x) > 0 \quad \Leftrightarrow \quad \phi(p, x) < 1.$$

Thus, we have $\psi(q) > 0$ for $q \in (q_0, p]$, where q_0 is the unique root of the equation

$$\frac{p-q_0}{p-1} (Cq_0)^{\frac{1}{p-1}} = 1.$$

Since v is nondecreasing, then we get (from Lemma 2.1) that

$$\mathcal{H}_\lambda v(k) \leq v(k), \quad (3.8)$$

and since $q-1 < p-1$, then (3.8) gives

$$[\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} \geq (v(k))^{\frac{1}{p-1} - \frac{1}{q-1}}.$$

By using this in (3.7), we have

$$\begin{aligned} & \left[1 - \frac{p-q}{p-1} (Cq)^{\frac{1}{p-1}} \right] \left(\frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{q-1}} \right) \\ & \leq \frac{q-1}{p-1} C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}}, \end{aligned} \quad (3.9)$$

which implies that

$$\mathcal{H}_\lambda v(n) \left(\mathcal{H}_\lambda v^{-\frac{1}{q-1}}(k) \right)^{q-1} \leq C_1.$$

That is $v \in \mathcal{A}_\lambda^q(C_1)$. The proof is complete. $\square$

Now, we will refine the above result by improving the constant that appears in (3.2).

Theorem 3.2 Assume that $p > 1$ and v is nondecreasing on the interval J . If $v \in \mathcal{A}_\lambda^p(C)$, then $v \in \mathcal{A}_\lambda^q(C_2)$ for any $q \in (q_0, p]$, where q_0 is the unique root of equation (3.1). Furthermore the constant C_2 is given by

$$C_2 := \left(\frac{p}{q} \left(\frac{q-1}{p-1} \right)^2 \frac{C^{\frac{1}{p-1}}}{\Gamma^{p,q}(C)} \right)^{q-1},$$

where $\Gamma^{p,q}(C) := 1 - (Cq)^{\frac{1}{p-1}} \left(\frac{p-q}{p-1} \right) > 0$.

Proof We proceed as in Theorem 3.1, and by replacing p and q with $1/(q-1)$ and $1/(p-1)$ in (2.22), we obtain

$$\begin{aligned} \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{-\frac{1}{q-1}} &\leq q^{\frac{1}{p-1}} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v(k)]^{-\frac{1}{q-1} + \frac{1}{p-1}} \\ &\quad - \frac{q-1}{q(p-1)} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{-\frac{1}{q-1}}. \end{aligned} \quad (3.10)$$

From (3.5), we have

$$\begin{aligned} \frac{p-1}{p-q} C^{-\frac{1}{p-1}} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} &- \frac{q-1}{p-q} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}} \\ &\leq \sum_{k=1}^n \lambda(k) [\mathcal{H}_\lambda v(k)]^{\frac{-1}{q-1}}. \end{aligned} \quad (3.11)$$

Now, by combining (3.10) and (3.11), multiplying by $C^{\frac{1}{p-1}}$, and rearranging the terms, we have

$$\begin{aligned} \frac{p-1}{p-q} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} \\ - (Cq)^{\frac{1}{p-1}} \sum_{k=1}^n \lambda(k) v^{\frac{-1}{p-1}}(k) [\mathcal{H}_\lambda v(k)]^{-\frac{1}{q-1} + \frac{1}{p-1}} \\ \leq \frac{q-1}{p-q} C^{\frac{1}{p-1}} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}} - C^{\frac{1}{p-1}} \frac{q-1}{q(p-1)} \Lambda(n) [\mathcal{H}_\lambda v(n)]^{-\frac{1}{q-1}}, \end{aligned}$$

from which we obtain

$$\left[1 - \frac{p-q}{p-1} (Cq)^{\frac{1}{p-1}} \right] \left(\frac{1}{\Lambda(n)} \sum_{k=1}^n \lambda(k) (v(k))^{\frac{-1}{p-1}} [\mathcal{H}_\lambda v(k)]^{\frac{1}{p-1} - \frac{1}{q-1}} \right)$$

$$\begin{aligned}
&\leq \left[\frac{q-1}{p-1} - \frac{(p-q)(q-1)}{q(p-1)^2} \right] C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}} \\
&= \frac{p}{q} \left(\frac{q-1}{p-1} \right)^2 C^{\frac{1}{p-1}} [\mathcal{H}_\lambda v(n)]^{\frac{-1}{q-1}}.
\end{aligned} \tag{3.12}$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence is omitted. The proof is complete. $\square$

Now, we show that the self-improving property of the weighted Muckenhoupt class $\mathcal{A}_\lambda^p(\mathcal{C})$ can be applied to prove the self-improving property of the Gehring class with a sharp value of the limit of exponents.

Lemma 3.1 *Let us fix $q > 1$, $\mathcal{K} > 1$ and assume that v is nonincreasing on the interval J . If $v \in \mathcal{G}^q(\mathcal{K})$, then $v \in \mathcal{G}^s(\mathcal{K}_1)$ for some known constant $\mathcal{K}_1$ and any $s \in [q, q^*)$, where the limit of the exponent $q^* = q^*(q, \mathcal{K})$ is given by the unique positive solution of the equation*

$$\mathcal{K}^q \left(\frac{x-q}{x} \right) \left(\frac{x}{x-1} \right)^q = 1. \tag{3.13}$$

This lemma says that if a weight satisfies a reverse Hölder inequality (belongs to a Gehring class $\mathcal{G}^q(\mathcal{K})$ for some exponent p , then it satisfies a reverse Hölder inequality for a slightly larger exponent q . This lemma was proved in [29]. An alternative way involves exploiting the correspondence between a weighted Muckenhoupt class and a reverse Hölder class. This in fact provides a simple proof of the self-improving property of $\mathcal{G}^q(\mathcal{K})$ as follows: Assume that $v \in \mathcal{G}^q(\mathcal{K})$, i.e., the condition

$$\left(\frac{1}{|J|} \sum_{k \in J} v^q(k) \right)^{1/q} \leq \mathcal{K} \left(\frac{1}{|J|} \sum_{k \in J} v(k) \right) \quad \text{for all } J \subset \mathbb{I} \tag{3.14}$$

holds. This condition can be rewritten in the form

$$\left(\frac{1}{|J|} \sum_{k \in J} v^{q-\frac{1}{p-1}}(k) \left(\frac{1}{v(k)} \right)^{\frac{-1}{p-1}} \right)^{p-1} \leq \mathcal{K}^{q(p-1)} \left(\frac{1}{|J|} \sum_{k \in J} v(k) \right)^{q(p-1)}. \tag{3.15}$$

By taking $p = q/(q-1)$, we have from (3.15) that

$$\begin{aligned}
&\left(\frac{1}{\Lambda(J)} \sum_{k \in J} v(k) \left(\frac{1}{v(k)} \right) dx \right) \left(\frac{1}{\Lambda(J)} \sum_{k \in J} v(k) \left(\frac{1}{v(k)} \right)^{\frac{-1}{p-1}} \right)^{p-1} \\
&\leq \mathcal{K}^{q(p-1)} \frac{|J|^p}{\Lambda^p(J)} \left(\frac{\Lambda(J)}{|J|} \right)^p = \mathcal{K}^p \quad \text{for all } J \subset \mathbb{I},
\end{aligned} \tag{3.16}$$

which is a weighted $\mathcal{A}_v^p(\mathcal{K}^p)$ condition for v^{-1} with respect to the weight v and $q = p/(p-1)$. This shows that if $v \in \mathcal{G}^q(\mathcal{K})$ then $v^{-1} \in \mathcal{A}_v^p(\mathcal{C})$ (which is nondecreasing) with $\mathcal{C} = \mathcal{K}^p$ where $q = p/(p-1)$. From Theorem 3.1 the inequality

$$\left(\frac{1}{\Lambda(J)} \sum_{k \in J} v(x) \left(\frac{1}{v(x)} \right) \right) \left(\frac{1}{\Lambda(J)} \sum_{k \in J} v(k) \left(\frac{1}{v(k)} \right)^{\frac{-1}{q-1}} \right)^{q-1} \leq \mathcal{C}_1$$

holds for any $q \in (q_0, p]$, where q_0 is the unique root of the equation

$$\frac{p-x}{p-1} \mathcal{K}_{\frac{p}{p-1}} x^{\frac{1}{p-1}} = 1.$$

By using the transformation $x \rightarrow x/(x-1)$, we then get that

$$\left(\frac{(p-1)x-p}{(p-1)(x-1)} \right) \mathcal{K}_{\frac{p}{p-1}} \left(\frac{x}{x-1} \right)^{\frac{1}{p-1}} = 1,$$

i.e.,

$$\left(\frac{(p-1)x-p}{(p-1)x} \right) \mathcal{K}_{\frac{p}{p-1}} \left(\frac{x}{x-1} \right)^{\frac{p}{p-1}} = 1.$$

Using the value of $q = p/(p-1)$ and $p = q/(q-1)$, we have that

$$\mathcal{K}^q \left(\frac{x-q}{x} \right) \left(\frac{x}{x-1} \right)^q = 1,$$

which is (3.13).

Conclusion 1 In this paper, we were able to prove some new refinements of discrete weighted inequalities with negative powers on finite intervals, by employing which we were able to prove that the self-improving property of the weighted discrete Muckenhoupt classes holds. The main results gave exact values of the limit exponents as well as new constants of the new classes. These values correspond to the sharp values obtained by Nikolidakis (see [26] and [25]) in the continuous case. As an application, we established the self-improving property (forward propagation property) of the discrete Gehring class.

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Authors' contributions

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