Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The generalized projection methods in countably normed spaces

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Abstract

Let E be a Banach space with dual space E^*, and let K be a nonempty, closed, and convex subset of E. We generalize the concept of generalized projection operator "$\Pi_K : E \to K$" from uniformly convex uniformly smooth Banach spaces to uniformly convex uniformly smooth countably normed spaces and study its properties. We show the relation between J-orthogonality and generalized projection operator Π_K and give examples to clarify this relation. We introduce a comparison between the metric projection operator P_K and the generalized projection operator Π_K in uniformly convex uniformly smooth complete countably normed spaces, and we give an example explaining how to evaluate the metric projection P_K and the generalized projection Π_K in some cases of countably normed spaces, and this example illustrates that the generalized projection operator Π_K in general is a set-valued mapping. Also we generalize the generalized projection operator "$\pi_K : E^* \to K$" from reflexive Banach spaces to uniformly convex uniformly smooth countably normed spaces and study its properties in these spaces.

MSC: 46A04

Keywords: Countably normed space; Normalized duality mapping; J-orthogonality; Projection theorem in countably normed space; Metric projection operator; Generalized projection operator

1 Introduction

Let E be a Banach space with dual space E^*, and let K be a nonempty, closed, and convex subset of E. The metric projection operator $P_K : E \to K$ has been used in many topics of mathematics such as: fixed point theory, game theory, and variational inequalities. In 1996, Alber [1] introduced the generalized projection operators "$\Pi_K : E \to K$" and "$\pi_K : E^* \to K$" in uniformly convex and uniformly smooth Banach spaces, which are a natural extension of the classical metric projection operators of Hilbert spaces, and studied their properties in detail. Also, Alber [1] presented two of the most important applications of the generalized projection operators: solving variational inequalities by iterative projection methods and finding a common point of closed convex sets by iterative projection methods in Banach spaces. In 2005, Li [3] extended the generalized projection operator $\pi_K : E^* \to K$ from uniformly convex uniformly smooth Banach spaces to reflexive Banach spaces and studied the properties and applications of the generalized projection operator

π_K based on set-valued mappings. In this paper, we extend the concept of generalized projection operators "$\Pi_K : E \to K$" from uniformly convex uniformly smooth Banach spaces to uniformly convex uniformly smooth countably normed spaces and "$\pi_K : E^* \to K$" from reflexive Banach spaces to uniformly convex uniformly smooth countably normed spaces. Also, we show the relation between J-orthogonality and generalized projection operators and give examples to clarify these relations. We present a comparison between metric projection and generalized projection in uniformly convex uniformly smooth complete countably normed spaces.

2 Preliminaries

Definition 2.1 ([5, 6]) If E is a normed linear space, then:

(1) It is *uniformly convex* if for any $\varepsilon \in (0,2]$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

(2) It is *smooth* if $S(E) = \{x \in E : \|x\| = 1\}$ is the unit sphere of E and $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S(E)$.

(3) It is *uniformly smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S(E)$, where $S(E)$ is the unit sphere of E.

Definition 2.2 (The normalized duality mapping, [8, 11]) If E is a real Banach space with the norm $\| \ \|$, E^* is the dual space of E, and $\langle \cdot, \cdot \rangle$ is the duality pairing. Then *the normalized duality mapping* J from E to 2^{E^*} is defined by

$$Jx = \{ j \in E^* : \langle x, j \rangle = \|x\|^2 = \|j\|^2 \}.$$

The Hahn–Banach theorem guarantees that $Jx \neq \phi$ for every $x \in E$. If E is a smooth Banach space, then the normalized duality mapping is single-valued. We got the following example in [4] for the normalized duality mapping J in the uniformly convex and uniformly smooth Banach space ℓ^p with $p \in (1, \infty)$, we have $Jx = \|x\|_{\ell^p}^{2-p}\{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \dots\} \in \ell^q = \ell^{p*}$, where $x = \{x_1, x_2, \dots\} \in \ell^p$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.3 (Countably normed space, [4]) If E is a linear space equipped with a countable family of pairwise compatible norms, $\{\| \ \|_n, n \in \mathbb{N}\}$ is said to be *countably normed space*. We give an example for the countably normed space the space $\ell^{p+0} := \bigcap_n \ell^{p_n}$ for any choice of a monotonic decreasing sequence p_n converging to p for $1 < p < \infty$.

Remark 2.4 ([9]) For a countably normed space $(E, \{\| \ \|_n, n \in \mathbb{N}\})$, let the completion of E with respect to the norm $\| \ \|_n$ be E_n. We may assume that $\| \ \|_1 \leq \| \ \|_2 \leq \| \ \|_3 \leq \dots$ in any countably normed space, also we have $E \subset \cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_1$.

Proposition 2.5 ([4]) *Let $(E, \{\| \ \|_n, n \in \mathbb{N}\})$ be a countably normed space. Then E is complete if and only if $E = \bigcap_{n\in\mathbb{N}} E_n$. Each Banach space E_n has a dual, which is a Banach space denoted by E_n^* and the dual of the countably normed space E is given by $E^* = \bigcup_{n\in\mathbb{N}} E_n^*$, and we have the following inclusions*:

$$E_1^* \subset \cdots \subset E_n^* \subset E_{n+1}^* \subset \cdots \subset E^*.$$

Moreover, for $f \in E_n^*$, we have $\|f\|_n \geq \|f\|_{n+1}$ for all $n \in \mathbb{N}$.

In the following definition, we define uniformly convex uniformly smooth countably normed spaces "C.S.C.N.".

Definition 2.6 (C.S.C.N. space, [7]) A countably normed space $(E, \{\| \|_n, n \in \mathbb{N}\})$ is said to be *uniformly convex uniformly smooth* if $(E_n, \| \|_n)$ is uniformly convex uniformly smooth for all $n \in \mathbb{N}$.

Theorem 2.7 ([7]) *Let $(E, \{\| \|_n, n \in \mathbb{N}\})$ be a real uniformly convex complete countably normed space, and K be a nonempty convex proper subset of E such that K is closed in each E_n, then there exists a unique $\bar{x} \in K : \|x - \bar{x}\|_n = \inf_{y \in K} \|x - y\|_n$ for all $n \in \mathbb{N}$, and the metric projection $P_K : E \to K$ is defined by $P_K(x) = \bar{x}$.*

Definition 2.8 (Lyapunov functional, [2]) If E is a smooth Banach space and E^* is the dual space of E, then *Lyapunov functional* $\varphi : E \times E \to \mathbb{R}^+$ is defined by:

$$\varphi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in E$, where J is the normalized duality mapping from E to 2^{E^*}.

In the following the concept of the normalized duality mapping in smooth countably normed spaces "S.C.N." is introduced.

Definition 2.9 (The normalized duality mapping in S.C.N. spaces, [12]) Let $(E, \{\| \|_n, n \in \mathbb{N}\})$ be a smooth countably normed space such that E_n is the completion of E in $\| \|_n$ and $(E_n, \| \|_n)$ is a smooth Banach space for all $n \in \mathbb{N}$, so there exists a single-valued normalized duality mapping $J_n : E_n \to E_n^*$ with respect to $\| \|_n$ for all $n \in \mathbb{N}$. One understands that $\|J_n x\|_n$ is the E_n^*-norm and $\|x\|_n$ is the E_n-norm for all $n \in \mathbb{N}$.

The following multi-valued mapping is the *normalized duality mapping* of a smooth countably normed space as $J : E \to E^* = \bigcup_{n\in\mathbb{N}} E_n^*$ such that $J(x) = \{J_n x\}$, $\|J_n x\|_n = \|x\|_n$, $\langle x, J_n x\rangle = \|x\|_n^2$ $\forall n \in \mathbb{N}$.

Proposition 2.10 ([12]) *If $(E, \{\| \|_n, n \in \mathbb{N}\})$ is a real smooth uniformly convex complete countably normed space and K is a nonempty proper convex subset of E such that K is closed in each E_n, then $\bar{x} = P_K(x)$ is the metric projection of an arbitrary element $x \in E$ if and only if $\langle \bar{x} - y, J(x - \bar{x})\rangle \geq 0$, $\forall y \in K$, where J is the normalized duality mapping on E.*

Theorem 2.11 ([12]) *Let $(E, \{\| \|_n, n \in \mathbb{N}\})$ be a real smooth uniformly convex complete countably normed space and K be a nonempty proper convex subset of E such that K is closed in each E_n.*

Then $\bar{x} = P_K(x)$ is the metric projection of an arbitrary element $x \in E$ if and only if $\langle x - y, J_n(x - \bar{x})\rangle \geq \|x - \bar{x}\|_n^2$, $\forall y \in K$, $\forall n$.

Definition 2.12 (J-orthogonality in smooth countably normed spaces, [12]) Let $(E, \{\| \|_n, n \in \mathbb{N}\})$ be a smooth countably normed space, we say that an element $x \in E$ is *J-orthogonal* to an element $y \in E$ and write $x \perp^J y$ if $\langle y, J_n x\rangle = 0$, $\forall n$, i.e., $\langle y, Jx\rangle = 0$, where J is the normalized duality mapping of E.

Theorem 2.13 ([12]) *If* $(E, \{\|\ \|_n, n \in \mathbb{N}\})$ *is a real smooth uniformly convex complete countably normed space and* M *is a nonempty proper subspace of* E *such that* M *is closed in each* E_i*, then*

$$\forall x \in E \setminus M, \exists! \bar{x} \in M: \quad \|x - \bar{x}\|_i = \inf_{y \in M} \|x - y\|_i, \quad \forall i \quad \textit{if and only if} \quad x - \bar{x} \perp^J M.$$

3 Main results

In the following definition, we introduce the concept of the generalized projection operator "Π_K" in uniformly convex uniformly smooth countably normed spaces "C.S.C.N.".

Definition 3.1 (The generalized projection "Π_K" in C.S.C.N. spaces) Let $(E, \{\|\ \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space such that E_n is the completion of E in $\|\ \|_n$ and $(E_n, \|\ \|_n)$ is a uniformly convex uniformly smooth Banach space for all $n \in \mathbb{N}$, so there exists a single-valued injective normalized duality mapping $J_n : E_n \to E_n^*$ with respect to $\|\ \|_n$ for all $n \in \mathbb{N}$, and let K be a nonempty proper convex subset of E such that K is closed in each E_n for all n. Let $\phi_n(x, y)$ be Lyapunov functional with respect to $\|\ \|_n$, where $\phi_n : E_n \times E_n \to \mathbb{R}^+$ is defined as

$$\phi_n(x, y) = \|x\|_n^2 - 2\langle y, J_n x\rangle + \|y\|_n^2, \quad \forall n \in \mathbb{N},$$

so we have $\Pi^n{}_K : E_n \to K$ is defined as

$$\Pi^n{}_K(x) = x_{0n} \quad \Leftrightarrow \quad \phi_n(x, x_{0n}) = \inf_{y \in K} \phi_n(x, y).$$

We define the set-valued mapping $\Pi_K : E \to 2^K$ to be *the generalized projection operator*, where $\Pi_K(x) = \{\Pi^n{}_K(x)\} = \{x_{0n}\} \subseteq K$ such that

$$\phi_n(x, x_{0n}) = \inf_{y \in K} \phi_n(x, y), \quad \forall n.$$

Proposition 3.2 *Let* K *be a nonempty closed convex subset of a uniformly convex uniformly smooth countably normed space* E *and* $x \in E$*. Then*

$$\Pi_K(x) = \{x_{0i}\}$$

if and only if $\langle x_{0i} - y, J_i x - J_i x_{0i}\rangle \geq 0$, $\forall y \in K$, $\forall i$.

Proof "$\underline{\Rightarrow}$" Let $y \in K$ and let $\mu \in (0, 1)$, $\Pi_K(x) = \{x_{0i}\}$.
Then

$$\phi_i(x_{0i}, x) \leq \phi_i\big((1 - \mu)x_{0i} + \mu y, x\big) \quad \text{for all } i. \tag{$*$}$$

From ($*$) we have

$$\begin{aligned} 0 &\leq \big\|(1 - \mu)x_{0i} + \mu y\big\|_i^2 - 2\big\langle(1 - \mu)x_{0i} + \mu y, J_i x\big\rangle - \|x_{0i}\|_i^2 + 2\langle x_{0i}, J_i x\rangle \\ &= \big\|(1 - \mu)x_{0i} + \mu y\big\|_i^2 - \|x_{0i}\|_i^2 - 2\mu\langle y - x_{0i}, J_i x\rangle \end{aligned}$$

$$
\begin{aligned}
&\le 2\mu\bigl\langle y - x_{0i}, J_i((1-\mu)x_{0i} + \mu y\bigr\rangle - 2\mu\langle y - x_{0i}, J_i x\rangle \\
&= 2\mu\bigl\langle y - x_{0i}, J_i\bigl((1-\mu)x_{0i} + \mu y\bigr) - J_i x\bigr\rangle.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\mu\bigl\langle x_{0i} - y, J_i\bigl((1-\mu)x_{0i} + \mu y\bigr)\bigr\rangle \\
&\quad = \bigl\langle x_{0i} - \mu y - x_{0i} + \mu x_{0i}, J_i(1-\mu)x_{0i} + \mu y\bigr\rangle \\
&\quad = \bigl\langle x_{0i} - \bigl((1-\mu)x_{0i} + \mu y\bigr), J_i(1-\mu)x_{0i} + \mu y\bigr\rangle \\
&\quad \le \frac{1}{2}\bigl(\|x_{0i}\|_i^2 - \bigl\|(1-\mu)x_{0i} + \mu y\bigr\|_i^2\bigr).
\end{aligned}
$$

Taking the limit $\mu \to 0$, we get $\langle y - x_{0i}, J_i x_{0i} - J_i x\rangle \ge 0$.

Thus $\langle x_{0i} - y, J_i x - J_i x_{0i}\rangle \ge 0$ for all $y \in K$ and for all i.

"$\Leftarrow$" For any $y \in K$, we have

$$
\begin{aligned}
&\phi_i(y,x) - \phi_i(x_{0i},x) \\
&\quad = \|y\|_i^2 - 2\langle y, J_i x\rangle + \|x\|_i^2 - \|x_{0i}\|_i^2 + 2\langle x_{0i}, J_i x\rangle - \|x\|_i^2 \\
&\quad = \|y\|_i^2 - \|x_{0i}\|_i^2 - 2\langle y - x_{0i}, J_i x\rangle \\
&\quad \ge 2\langle y - x_{0i}, J_i x_{0i}\rangle - 2\langle y - x_{0i}, J_i x\rangle \\
&\quad = 2\langle y - x_{0i}, J_i x_{0i} - J_i x\rangle \ge 0, \quad \forall i.
\end{aligned}
$$

So $\Pi_K(x) = \{x_{0i}\}$. □

Proposition 3.3 *Let $(E, \{\|\ \|_i, i \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space and M be a nonempty proper subspace of E. Then $\Pi_M(x) = \{x_{0i}\}$ if and only if*

$$
\langle m, J_i x - J_i x_{0i}\rangle = 0 \quad \forall m \in M, \forall i.
$$

Proof "$\Rightarrow$" Suppose that $\Pi_M(x) = \{x_{0i}\}$. Since M is a subspace of E and using Proposition 3.2, we have

$$
\bigl\langle x_{0i} - (x_{0i} - m), J_i x - J_i x_{0i}\bigr\rangle = \langle m, J_i x - J_i x_{0i}\rangle \ge 0, \quad \forall m \in M, \forall i. \tag{1}
$$

Similarly,

$$
\bigl\langle x_{0i} - (x_{0i} + m), J_i x - J_i x_{0i}\bigr\rangle = \langle -m, J_i x - J_i x_{0i}\rangle \ge 0, \quad \forall m \in M, \forall i. \tag{2}
$$

From (1), (2) we get $\langle m, J_i x - J_i x_{0i}\rangle = 0$ $\forall m \in M$, $\forall i$.

"$\Leftarrow$" Suppose that $\langle m, J_i x - J_i x_{0i}\rangle = 0$ $\forall m \in M$, $\forall i$.

Using that M is a subspace of E, we have

$$
\langle x_{0i} - m, J_i x - J_i x_{0i}\rangle = 0, \quad \forall m \in M, \forall i.
$$

So,

$$\langle x_{0i} - m, J_i x - J_i x_{0i} \rangle \geq 0, \quad \forall m \in M, \forall i.$$

Thus $\Pi_M(x) = \{x_{0i}\}$. □

Example 3.4 For $\ell_{2+0} := \bigcap_{n\in\mathbb{N}} \ell_{2+\frac{1}{n}}$ is a uniformly convex uniformly smooth complete countably normed space with the norms

$$\| \ \|_3 \leq \| \ \|_{2.5} \leq \cdots \leq \| \ \|_{2+\frac{1}{n}} \leq \cdots,$$

for each $x = \{x_i\} \in \ell_{2+0}$,

$$J_n(x) = \|x\|^{-\frac{1}{n}}_{2+\frac{1}{n}} \left\{ x_i |x_i|^{\frac{1}{n}} \right\} \in \ell_{\frac{2n+1}{n+1}}, \quad \forall n.$$

Consider a closed subspace M of ℓ_{2+0} which is generated by $\{1, 0, 0, 0, \ldots\}$. Using Proposition 3.2 we get

$$\begin{aligned}
&\Pi_M(x) = \left\{ \{x_{0n}, 0, 0, \ldots\} \right\} \\
&\quad \Leftrightarrow \quad \left\langle \{t, 0, 0, \ldots\}, J_n x - J_n\{x_{0n}, 0, 0, \ldots\} \right\rangle = 0 \quad \forall t \in \mathbb{R}, \forall n \\
&\quad \Leftrightarrow \quad \left\langle \{t, 0, 0, \ldots\}, \|x\|^{-\frac{1}{n}}_{2+\frac{1}{n}} \left\{ x_1 |x_1|^{\frac{1}{n}}, x_2 |x_2|^{\frac{1}{n}}, \ldots \right\} \right\rangle \\
&\qquad\qquad = \left\langle \{t, 0, 0, \ldots\}, \|x_0\|^{-\frac{1}{n}}_{2+\frac{1}{n}} \left\{ x_{0n} |x_{0n}|^{\frac{1}{n}}, 0, 0, \ldots \right\} \right\rangle \\
&\quad \Leftrightarrow \quad \|x\|^{-\frac{1}{n}}_{2+\frac{1}{n}} x_1 |x_1|^{\frac{1}{n}} t = \|x_0\|^{-\frac{1}{n}}_{2+\frac{1}{n}} x_{0n} |x_{0n}|^{\frac{1}{n}} t = x_{0n} t \\
&\quad \Leftrightarrow \quad x_{0n} = \|x\|^{-\frac{1}{n}}_{2+\frac{1}{n}} x_1 |x_1|^{\frac{1}{n}}, \quad \forall n.
\end{aligned}$$

So $\Pi_M(x) = \{\{\|x\|^{-\frac{1}{n}}_{2+\frac{1}{n}} x_1 |x_1|^{\frac{1}{n}}, 0, 0, \ldots\}\}$, $\forall n$, hence we have a sequence of points.

Using Theorem 2.13 that is "$P_M(x) = \bar{x}$ if and only if $x - \bar{x} \perp^J M$", we get

$$\begin{aligned}
&P_M(x) = \bar{x} = \{\bar{x}_1, 0, 0, \ldots\} \\
&\quad \Leftrightarrow \quad \left\langle \{t, 0, 0, \ldots\}, J_n(x - \bar{x}) \right\rangle = \{0, 0, \ldots\}, \quad \forall t \in \mathbb{R}, \forall n \\
&\quad \Leftrightarrow \quad \left\langle \{t, 0, 0, \ldots\}, J_n\{x_1 - \bar{x}_1, x_2, x_3, \ldots, x_n, \ldots\} \right\rangle = \{0, 0, \ldots\} \\
&\quad \Leftrightarrow \quad \left\langle \{t, 0, 0, \ldots\}, \|x - \bar{x}\|^{-\frac{1}{n}}_{2+\frac{1}{n}} \left\{ |x_1 - \bar{x}_1|^{-\frac{1}{n}} (x_1 - \bar{x}_1), \ldots, x_i |x_i|^{\frac{1}{n}}, \ldots \right\} \right\rangle = \{0, 0, \ldots\} \\
&\quad \Leftrightarrow \quad \|x - \bar{x}\|^{-\frac{1}{n}}_{2+\frac{1}{n}} |x_1 - \bar{x}_1|^{-\frac{1}{n}} (x_1 - \bar{x}_1) t = 0, \quad \forall t \in \mathbb{R}, \forall n \\
&\quad \Leftrightarrow \quad \bar{x}_1 = x_1, \qquad P_M(x) = \bar{x} = \{x_1, 0, 0, \ldots\}.
\end{aligned}$$

So, for a metric projection we have only one point but for a generalized projection we have a sequence of points.

Remark 3.5 From Example 3.4 we observed that the metric projection and the generalized projection of a uniformly convex uniformly smooth complete countably normed space in general are distinct.

Remark 3.6 The space ℓ_{2+0} is a uniformly convex uniformly smooth complete countably normed space, so the metric projection is a single-valued mapping in it, see [10]. But the generalized projection in ℓ_{2+0} is still a set-valued mapping.

The following corollary gives a relation between the generalized projection and J-orthogonality in uniformly convex uniformly smooth countably normed spaces.

Corollary 3.7 *Let $(E, \{\| \ \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space and M be a nonempty proper subspace of E. Then $\Pi_M(x) = 0$ if and only if $x \perp^J M$.*

Proof By using Proposition 3.3, we get

$$\begin{aligned} \Pi_M(x) = 0 \quad &\Leftrightarrow \quad \langle m, J_i x\rangle = 0, \quad \forall m \in M, \forall i \\ &\Leftrightarrow \quad x \perp^J M. \end{aligned}$$

□

Example 3.8 For $\ell_{2+0} := \bigcap_{n\in\mathbb{N}} \ell_{2+\frac{1}{n}}$,

$$J_n(x) = \|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} \left\{x_i |x_i|^{\frac{1}{n}}\right\} \in \ell_{\frac{2n+1}{n+1}} \quad \forall n.$$

Consider a closed subspace M of ℓ_{2+0} which is generated by $\{1, 0, 0, 0, \ldots\}$.

$$\begin{aligned} &\Pi_M(x) = \{0, \ldots, 0, \ldots\} \\ &\quad \Leftrightarrow \quad x \perp^J M \\ &\quad \Leftrightarrow \quad \langle m, J_i x\rangle = 0 \quad \forall m \in M, \forall i \\ &\quad \Leftrightarrow \quad \left\langle \{m_1, 0, 0, \ldots\}, \|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} \left\{x_1 \|x_1\|^{\frac{1}{n}}, x_2 \|x_2\|^{\frac{1}{n}}, \ldots\right\}\right\rangle = \{0, 0, \ldots\} \\ &\quad \Leftrightarrow \quad x = \left[\{0, 1, 1, 1, \ldots\}\right]. \end{aligned}$$

Corollary 3.9 *Let $(E, \{\| \ \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space and M be a nonempty proper subspace of E. Then $\Pi_M(x)$ is homogeneous.*

Proof Let

$$\begin{aligned} \Pi_M(x) = \{x_{0i}\} \quad &\Leftrightarrow \quad \langle m, J_i x - J_i x_{0i}\rangle = 0 \quad \forall m \in M, \forall i \\ &\Leftrightarrow \quad \lambda\langle m, J_i x - J_i x_{0i}\rangle = 0 = \big\langle m, \lambda(J_i x - J_i x_{0i})\big\rangle, \quad \lambda \in \mathbb{R} \\ &\Leftrightarrow \quad \big\langle m, J_i(\lambda x) - J_i(\lambda x_{0i})\big\rangle = 0 \quad \forall m \in M, \forall i \\ &\Leftrightarrow \quad \Pi_M(\lambda x) = \lambda\{x_{0i}\} = \lambda \Pi_M(x). \end{aligned}$$

□

Proposition 3.10 *Let K be a nonempty closed convex subset of a uniformly convex uniformly smooth countably normed space E and $x \in E$. If $\Pi_K(x) = \{x_{0i}\}$, then*

$$\phi_i(y, x_{0i}) + \phi_i(x_{0i}, x) \le \phi_i(y, x), \quad \forall y \in K, \forall i.$$

Proof

$$\begin{aligned}
&\phi_i(y,x) - \phi_i(y, x_{0i}) + \phi_i(x_{0i}, x) \\
&\quad = -2\langle y, J_i x\rangle + 2\langle x_{0i}, J_i x\rangle + 2\big\langle y - x_{0i}, J_i(x_{0i})\big\rangle \\
&\quad = -2\big\langle y - x_{0i}, J_i(x)\big\rangle + 2\big\langle y - x_{0i}, J_i(x_{0i})\big\rangle \\
&\quad = 2\big\langle y - x_{0i}, J_i(x_{0i}) - J_i x\big\rangle \ge 0, \quad \forall y \in K, \forall i \text{ "using Proposition 3.2".}
\end{aligned}$$

□

Proposition 3.11 *Let $(E, \{\|\ \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space and M be a nonempty proper subspace of E, $x \in E$, $\Pi_M(x) = \{x_{0n}\}$. Then*

$$\phi_n(y, x_{0n}) + \phi_n(x_{0n}, x) = \phi_n(y, x), \quad \forall y \in M, \forall n.$$

Proof

$$\begin{aligned}
&\phi_n(y,x) - \phi_n(x_{0n}, x) - \phi_n(y, x_{0n}) \\
&\quad = -2\langle y, J_n x\rangle + 2\langle x_{0n}, J_n x\rangle + 2\langle y, J_n x_{0n}\rangle - 2\langle x_{0n}, x_{0n}\rangle \\
&\quad = 2\langle y - x_{0n}, J_n x_{0n} - J_n x\rangle = 0 \quad \forall y \in M, \forall n.
\end{aligned}$$

Thus $\phi_n(y, x_{0n}) + \phi_n(x_{0n}, x) = \phi_n(y, x)$ $\forall y \in M$, $\forall n$. □

Example 3.12 For $\ell_{2+0} := \bigcap_{n\in\mathbb{N}} \ell_{2+\frac{1}{n}}$,

$$J_n(x) = \|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} \big\{x_i |x_i|^{\frac{1}{n}}\big\} \in \ell_{\frac{2n+1}{n+1}} \quad \forall n.$$

Consider a closed subspace M of ℓ_{2+0} which is generated by $\{1, 0, 0, 0, \ldots\}$.

In Example 3.4, we got the generalized projection operator of $x \in \ell_{2+0}$ such that

$$\Pi_M(x) = \Big\{\big\{\|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} x_1 |x_1|^{\frac{1}{n}}, 0, 0, \ldots\big\}\Big\} \quad \forall n.$$

So, we get

$$\begin{aligned}
&\phi_n\Big(\{m, 0, \ldots\}, \big\{\|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} x_1 |x_1|^{\frac{1}{n}}, 0, 0, \ldots\big\}\Big) \\
&\qquad + \phi_n\Big(\big\{\|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} x_1 |x_1|^{\frac{1}{n}}, 0, 0, \ldots\big\}, \{x_1, x_2, \ldots\}\Big) \\
&\quad = m^2 - 2\big(\|x\|_{2+\frac{1}{n}}\big)^{-\frac{1}{n}} m x_1 \|x_1\|^{-\frac{1}{n}} + 2\big(\|x\|_{2+\frac{1}{n}}{}^{-\frac{1}{n}} x_1 \|x_1\|^{\frac{1}{n}}\big)^2 \\
&\qquad - 2\Big(\big(\|x\|_{2+\frac{1}{n}}\big)^{-\frac{1}{n}} x_1 \|x_1\|^{\frac{1}{n}}\Big)^2 + \big(\|x\|_{2+\frac{1}{n}}\big)^2 \\
&\quad = \phi_n\big(\{m, 0, 0, \ldots\}, \{x_1, x_2, \ldots\}\big) = \phi_n\big(\{m, 0, 0, \ldots\}, x\big) \quad \forall \{m, 0, 0, \ldots\} \in M, \forall n.
\end{aligned}$$

Remark 3.13 Proposition 2.10, Theorem 2.11, and Theorem 2.13 give relations between a metric projection operator and a normalized duality mapping in countably normed spaces. Proposition 3.2, Proposition 3.3, Corollary 3.7, and Proposition 3.10 give relations between the generalized projection operator and the normalized duality mapping in countably normed spaces, so we can get a useful comparison between the metric projection and the generalized projection in countably normed spaces.

In the following definition, we introduce the concept of generalized projection operator "π_K" in uniformly convex uniformly smooth countably normed spaces "C.S.C.N.".

Definition 3.14 (The generalized projection "π_K" in C.S.C.N. spaces) Let $(E, \{\| \ \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space such that E_n is the completion of E in $\| \ \|_n$ and $(E_n, \| \ \|_n)$ is a uniformly convex uniformly smooth Banach space for all $n \in \mathbb{N}$, and let K be a nonempty proper convex subset of E such that K is closed in each E_n for all n. Let $\varphi_n(f, y)$ be a Lyapunov functional with respect to $\| \ \|_n$, where $\varphi_n : E_n^* \times E_n \to \mathbb{R}^+$ is defined as

$$\varphi_n(f, y) = \|f\|_n^2 - 2\langle y, f\rangle + \|y\|_n^2, \quad \forall n \in \mathbb{N}.$$

Without being confused, one understands that $\|f\|_n$ is the E_n^*-norm and $\|y\|_n$ is the E_n-norm for all $n \in \mathbb{N}$. So we have

$$\pi^n{}_K : E_n^* \to K \quad \forall n \in \mathbb{N}$$

is defined as

$$\pi^n{}_K(f) = \bar{x}_n \quad \Leftrightarrow \quad \varphi_n(f, \bar{x}_n) = \inf_{y \in K} \varphi_n(f, y).$$

We define the set-valued mapping

$$\pi_K : E^* = \bigcup_{n \in \mathbb{N}} E_n^* \to 2^K$$

to be *the generalized projection operator of f*, where $f \in E_n^*$ for some n,

$$\pi_K(f) = \left\{\pi^n{}_K(f)\right\} = \{\bar{x}_n\} \subseteq K$$

such that $\varphi_n(f, \bar{x}_n) = \inf_{y \in K} \varphi_n(f, y)$.

Remark 3.15 Let $(E, \{\| \ \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space such that E_n is the completion of E in $\| \ \|_n$ and $(E_n, \| \ \|_n)$ is a uniformly convex uniformly smooth Banach space for all $n \in \mathbb{N}$, so there exists a single-valued injective normalized duality mapping $J_n : E_n \to E_n^*$ with respect to $\| \ \|_n$ and $J_n^* : E_n^* \to E_n^{**} = E_n$ since E_n is a reflexive Banach space for all $n \in \mathbb{N}$, and let K be a nonempty proper convex subset of E such that K is closed in each E_n for all n. Then $\Pi^n{}_K = \pi^n{}_K \circ J_n$ and $\pi^n{}_K = \Pi^n{}_K \circ J_n^*$, since $\Pi^n{}_K$ and $\pi^n{}_K$ are single-valued for all n. So $\Pi_K(x) = \{\Pi^n{}_K(x)\} = \{\pi^n{}_K \circ J_n(x)\}$ for all n and $\pi_K(f) = \{\pi^n{}_K(f)\} = \{\Pi^n{}_K \circ J_n^*(f)\}$, where $f \in E_n^*$ for some n.

Li studied and proved the properties of generalized projection operator π_K in reflexive Banach spaces, see [3], we extend and prove most of these properties of a generalized projection operator π_K in uniformly convex uniformly smooth countably normed spaces in the following theorem.

Theorem 3.16 *Let $(E, \{\| \|_n, n \in \mathbb{N}\})$ be a uniformly convex uniformly smooth countably normed space such that E_n is the completion of E in $\| \|_n$ and $(E_n, \| \|_n)$ is a uniformly convex uniformly smooth Banach space for all $n \in \mathbb{N}$, and let K be a nonempty proper convex subset of E such that K is closed in each E_n for all n, then the following properties hold*:

(1) *For any given $f \in E^*$, $\pi_K(f)$ is a convex subset of K.*

(2) *For any point $x \in K$ and any $J_i(x) \in J(x)$, where $J(x)$ is the normalized duality mapping of E, we have $x \in \pi_K(J_i(x))$, $\forall i$.*

(3) *π_K is monotone in E^* in some cases, that is, if $f_1, f_2 \in E^*$ where $f_1, f_2 \in E_i^*$ for some i, $x_1 \in \pi_K(f_1)$ and $x_2 \in \pi_K(f_2)$, we have*

$$\langle x_1 - x_2, f_1 - f_2 \rangle \geq 0.$$

(4) *For any given $f \in E^*$ such that $f \in E_i^*$ for some i, $x \in K$, if $J_i(x) \in J(x)$ such that*

$$\langle x - y, f - J_i(x) \rangle \geq 0 \quad \text{for all } y \in K, \text{then } x \in \pi_K(f).$$

(5) *If $x \in \pi_K(f)$ and $f \in E^*$ such that $f \in E_i^*$ for some i, $J_i(x) \in J(x)$, we have*

$$\langle x - y, f - J_i(x) \rangle \geq 0 \quad \text{for all } y \in K.$$

(6) *If $f_1, f_2 \in E^*$ such that $f_1, f_2 \in E_i^*$ for some i and $x \in (\pi_K(f_1) \cup \pi_K(f_2))$, we have $x \in \pi_K(\lambda f_1 + (1 - \lambda) f_2)$ for any $\lambda \in [0, 1]$, that is,*

$$\pi_K(f_1) \cup \pi_K(f_2) \subseteq \pi_K\bigl(\mathrm{co}(f_1, f_2)\bigr).$$

(7) *If $f \in E^*$ such that $f \in E_i^*$ for some i, $x \in \pi_K(f)$, the following inequality holds*:

$$\varphi_i(J_i x, y) \leq \varphi_i(f, y), \quad \forall y \in K.$$

Proof (1) Suppose $x_1, x_2 \in \pi_K(f)$, $f \in E_i^*$ for some i and $0 \leq \lambda \leq 1$, from the convexity property of the functional φ_i, we have

$$\begin{aligned} \varphi_i\bigl(f, \lambda x_1 + (1 - \lambda) x_2\bigr) &\leq \lambda \varphi_i(f, x_1) + (1 - \lambda) \varphi_i(f, x_2) \\ &= \lambda \inf_{y \in K} \varphi_i(f, y) + (1 - \lambda) \inf_{y \in K} \varphi_i(f, y) \\ &= \inf_{y \in K} \varphi_i(f, y). \end{aligned}$$

It implies $\lambda x_1 + (1 - \lambda) x_2 \in \pi_K(f)$. Hence $\pi_K(f)$ is a convex subset.

(2) $x \in \pi_K(J_i(x)) = \{{\pi^n}_K(J_i(x))\}$ for some n since, for $n = i$ "$J_i(x) \in E_i^*$", we have $\varphi_i(J_i x, x) = 0$, ${\pi^i}_K(J_i(x)) = x$, $\forall i$.

(3) If $f_1, f_2 \in E^*$ such that $f_1, f_2 \in E_i^*$ for some i, $x_1 \in \pi_K(f_1)$ and $x_2 \in \pi_K(f_2)$, we have

$$\varphi_i(f_1, x_1) \leq \varphi_i(f_1, y), \quad \forall y \in K,$$

and

$$\varphi_i(f_2, x_2) \leq \varphi_i(f_2, z), \quad \forall z \in K.$$

Let $y = x_2$ and $z = x_1$. Then

$$\varphi_i(f_1, x_1) \leq \varphi_i(f_1, x_2), \qquad \varphi_i(f_2, x_2) \leq \varphi_i(f_2, x_1),$$

and we have

$$\varphi_i(f_1, x_2) + \varphi_i(f_2, x_1) \geq \varphi_i(f_1, x_1) + \varphi_i(f_2, x_2).$$

From this we obtain the following relation:

$$\langle x_1, f_1\rangle + \langle x_2, f_2\rangle \geq \langle x_2, f_1\rangle + \langle x_1, f_2\rangle,$$

which is equivalent to $\langle x_1 - x_2, f_1 - f_2\rangle \geq 0$.

(4) For any given $f \in E^*$ such that $f \in E_i^*$ for some i, $x \in K$ if $J_i(x) \in J(x)$ such that

$$\bigl\langle x - y, f - J_i(x)\bigr\rangle \geq 0 \quad \text{for all } y \in K \text{ and for some } i.$$

Then we have

$$\begin{aligned}
\varphi_i(f, y) - \varphi_i(f, x) &= -2\langle y, f\rangle + \|y\|_i^2 + 2\langle x, f\rangle - \|x\|_i^2 \\
&= -2\langle y, f\rangle + 2\langle x, f\rangle - 2\langle x, J_i x\rangle + \|J_i x\|_i^2 + \|y\|_i^2 \\
&\geq -2\langle y, f\rangle + 2\langle x, f\rangle - 2\langle x, J_i x\rangle + 2\|J_i x\|_i \|y\|_i \\
&\geq -2\langle y, f\rangle + 2\langle x, f\rangle - 2\langle x, J_i x\rangle + 2\langle y, J_i x\rangle \\
&= 2\langle x - y, f - J_i x\rangle \geq 0, \quad \forall y \in K.
\end{aligned}$$

It implies $x \in \pi_K(f)$.

(5) If $x \in \pi_K(f)$ such that $f \in E_i^*$ for some i, $\lambda \in (0, 1]$, any $y \in K$, and using that K is convex, we get

$$\begin{aligned}
0 &\geq \varphi_i(f, x) - \varphi_i\bigl(f, \lambda y + (1 - \lambda)x\bigr) \\
&= 2\bigl\langle \lambda(y - x), f\bigr\rangle + \|x\|_i^2 - \bigl\|\lambda y + (1 - \lambda)x\bigr\|_i^2 \\
&= 2\bigl\langle \lambda(y - x), f - J_i\bigl(\lambda y + (1 - \lambda)x\bigr)\bigr\rangle + 2\bigl\langle \lambda(y - x), J_i\bigl(\lambda y + (1 - \lambda)x\bigr)\bigr\rangle \\
&\quad + \|x\|_i^2 - \bigl\|\lambda y + (1 - \lambda)x\bigr\|_i^2 \\
&= 2\bigl\langle \lambda(y - x), f - J_i\bigl(\lambda y + (1 - \lambda)x\bigr)\bigr\rangle + 2\bigl\langle \lambda y + (1 - \lambda)x\bigr\rangle - \bigl\langle x, J_i\bigl(\lambda y + (1 - \lambda)x\bigr)\bigr\rangle \\
&\quad + \|x\|_i^2 - \bigl\|\lambda y + (1 - \lambda)x\bigr\|_i^2
\end{aligned}$$

$$
\begin{aligned}
&= 2\langle \lambda(y-x), f - J_i(\lambda y + (1-\lambda)x)\rangle + 2\|\lambda y + (1-\lambda)x\|_i^2 \\
&\quad - 2\langle x, J_i(\lambda y + (1-\lambda)x)\rangle + \|x\|_i^2 - \|\lambda y + (1-\lambda)x\|_i^2 \\
&= 2\langle \lambda(y-x), f - J_i(\lambda y + (1-\lambda)x)\rangle + \|\lambda y + (1-\lambda)x\|_i^2 \\
&\quad - 2\langle x, J_i(\lambda y + (1-\lambda)x)\rangle + \|x\|_i^2 \\
&\geq 2\langle \lambda(y-x), f - J_i(\lambda y + (1-\lambda)x)\rangle.
\end{aligned}
$$

So, for some i, we have

$$\langle x-y, f - J_i(\lambda y + (1-\lambda)x)\rangle \geq 0, \quad \forall \lambda \in (0,1].$$

From the property

$$\|J_i(\lambda y + (1-\lambda)x)\|_i = \|\lambda y + (1-\lambda)x\|_i \leq \|x\|_i + \|y\|_i,$$

we get that the set $\{J_i(\lambda y + (1-\lambda)x) : \lambda \in (0,1]\}$ is bounded for any fixed $x, y \in K$, $\forall i$. Then there exists a subsequence $\{J_i(\lambda_n y + (1-\lambda_n)x)\}$ such that $\lambda_n \to 0$ and $J_i(\lambda_n y + (1-\lambda_n)x) \to \psi_i, \omega^*$-weakly with respect to $\|\ \|_i$, as $n \to \infty$ such that $\psi_i \in E^*$ $\forall i$. From the ω^*-convergence property, we have

$$
\begin{aligned}
\|\psi_i\|_i &\leq \lim_{n\to\infty} \inf \|J_i(\lambda_n y + (1-\lambda_n)x)\|_i \\
&= \lim_{n\to\infty} \inf \|\lambda_n y + (1-\lambda_n)x\|_i \leq \|x\|_i, \quad \forall i, \\
\langle z, \psi_i\rangle &= \lim_{n\to\infty} \langle z, J_i(\lambda_n y + (1-\lambda_n)x)\rangle, \quad \forall z \in K.
\end{aligned}
\tag{I}
$$

For $z = x$, we have

$$
\begin{aligned}
\langle x, \psi_i\rangle &= \lim_{n\to\infty} \langle x, J_i(\lambda_n y + (1-\lambda_n)x)\rangle \\
&= \lim_{n\to\infty} \langle \lambda_n(y-x), J_i(\lambda_n y + (1-\lambda_n)x)\rangle \\
&= \lim_{n\to\infty} \|\lambda_n y + (1-\lambda_n)x\|_i^2 = \|x\|_i^2.
\end{aligned}
\tag{II}
$$

Since $\|\psi_i\|_i \|x\|_i \geq \langle x, \psi_i\rangle = \|x\|_i^2$, it implies $\|\psi_i\|_i \geq \|x\|_i$. Here we may assume that $x \neq 0$. (It is easy to prove the second part if $x = 0$.) Combining with (I), (II), we get

$$\langle x, \psi_i\rangle = \|x\|_i^2 = \|\psi_i\|^2.$$

It yields that $\psi_i = J_i$. Applying the ω^*-convergence property again and using (I), we get that

$$\langle x-y, f - J_i(x)\rangle = \lim_{n\to\infty} \langle x-y, f - J_i(\lambda_n y + (1-\lambda_n)x)\rangle \geq 0, \quad \forall y \in K.$$

(6) If $f_1, f_2 \in E^*$ such that $f_1, f_2 \in E_i^*$ for some i and $x \in (\pi_K(f_1) \cup \pi_K(f_2))$, then by using property [6], we have $\langle x-y, f_1 - J_i(x)\rangle \geq 0$ and $\langle x-y, f_2 - J_i(x)\rangle \geq 0$, $\forall y \in K$. It implies

$$
\begin{aligned}
&\langle x-y, (\lambda f_1 + (1-\lambda)f_2) - J_i x\rangle \\
&\quad = \langle x-y, (\lambda f_1 + (1-\lambda)f_2) - (\lambda J_i(x) + (1-\lambda)J_i(x))\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\langle x-y, f_1 - J_i(x)\rangle + (1-\lambda)\langle x-y, f_2 - J_i(x)\rangle \\
&\geq 0, \quad \forall y \in K.
\end{aligned}
$$

Applying property [6] again, we obtain $x \in \pi_K(\lambda f_1 + (1-\lambda)f_2)$, that is,

$$\pi_K(f_1) \cup \pi_K(f_2) \subseteq \pi_K\bigl(\mathrm{co}(f_1, f_2)\bigr).$$

(7) If $f \in E^*$ such that $f \in E_i^*$ for some i. Let us rewrite property [6] in the form

$$\langle x-y, f\rangle \geq \bigl\langle x-y, J_i(x)\bigr\rangle, \quad \forall y \in K.$$

So we have

$$
\begin{aligned}
\langle y, f\rangle &\leq \bigl\langle y, J_i(x)\bigr\rangle + \langle x, f\rangle - \bigl\langle x, J_i(x)\bigr\rangle \\
&= \bigl\langle y, J_i(x)\bigr\rangle + \langle x, f\rangle - \|x\|_i^2.
\end{aligned}
$$

It is equivalent to the relation

$$
\begin{aligned}
&\|f\|_i^2 - 2\langle y, f\rangle + \|y\|_i^2 \\
&\quad \geq \|x\|_i^2 - 2\bigl\langle y, J_i(x)\bigr\rangle + \|y\|_i^2 + \|f\|_i^2 - 2\langle x, f\rangle + \|x\|_i^2.
\end{aligned}
$$

By observing the following equalities:

$$
\begin{aligned}
&\varphi_i\bigl(J_i(x), y\bigr) = \|x\|_i^2 - 2\bigl\langle y, J_i(x)\bigr\rangle + \|y\|_i^2, \\
&\varphi_i(f, y) = \|f\|_i^2 - 2\langle y, f\rangle + \|y\|_i^2, \\
&\varphi_i(f, x) = \|f\|_i^2 - 2\langle x, f\rangle + \|x\|_i^2,
\end{aligned}
$$

we get that

$$\varphi_i(J_i x, y) \leq \varphi_i(f, y) - \varphi_i(f, x), \quad \forall y \in K.$$

Consequently,

$$\varphi_i(J_i x, y) \leq \varphi_i(f, y), \quad \forall y \in K. \qquad \square$$

4 Conclusion

In this paper we extend the concept of the generalized projection operator "$\Pi_K : E \to K$" from uniformly convex uniformly smooth Banach spaces to uniformly convex uniformly smooth countably normed spaces and study its properties. We show the relation between J-orthogonality and generalized projection operator Π_K and give examples to clarify this relation. We introduce a comparison between the metric projection operator P_K and the generalized projection operator Π_K in uniformly convex uniformly smooth complete countably normed spaces, and we give an example explaining how to evaluate the metric projection P_K and the generalized projection Π_K in some cases of countably normed spaces, and this example illustrates that the generalized projection operator Π_K in general is a set-valued mapping. Also we generalize the generalized projection operator "$\pi_K : E^* \to K$" from reflexive Banach spaces to uniformly convex uniformly smooth countably normed spaces. We clarify that the properties of π_K in uniformly convex uniformly smooth countably normed spaces are closer to similarity with the properties of π_K in reflexive Banach spaces.

Acknowledgements
We are thankful to the reviewers for their careful reading and valuable comments, which considerably improved this paper.

Funding
The authors were supported financially while writing this article only from their own personal sources.

Abbreviations
S.C.N., smooth countably normed spaces; C.S.C.N., uniformly convex uniformly smooth countably normed spaces.

Availability of data and materials
Data sharing not applicable to this article.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 January 2021 Accepted: 23 September 2021 Published online: 11 October 2021

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