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On iterative methods for bilevel equilibrium problems

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Abstract

We use the notion of Halpern-type sequence recently introduced by the present authors to conclude two strong convergence theorems for solving the bilevel equilibrium problems proposed by Yuying et al. and some authors. Our result excludes some assumptions as were the cases in their results.

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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space. For a given bifunction $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, the *equilibrium problem* associated with g and a closed convex subset C of $\mathcal{H}$ is described as follows:

Find $x^* \in C$ such that

$$g(x^*, y) \geq 0 \quad \text{for all } y \in C.$$

In particular, the solution set of the problem is given by $\mathrm{EP}(g, C)$. This problem plays an important role in various branches in pure and applied sciences such as fixed point theory, optimization, and game theory. The formulation was studied by Blum and Ottelli [3] in 1994 and has been studied by many authors. In this paper, we consider the *bilevel equilibrium problem* associated with two bifunctions f and g and a closed convex subset C of $\mathcal{H}$:

Find $x^* \in \mathrm{EP}(g, C)$ such that

$$f(x^*, y) \geq 0 \quad \text{for all } y \in \mathrm{EP}(g, C).$$

So the bilevel equilibrium problem is the problem of finding $x^* \in \mathrm{EP}(f, \mathrm{EP}(g, C))$.

Yuying et al. [10] proposed two iterative methods to approximate a solution of the bilevel equilibrium problem. The purpose of this paper is to show that their two methods can be regarded as a particular case of the Halpern-type sequence introduced by Jaipranop and Saejung [6] with appropriate setting. Moreover, we obtain their results under weaker assumptions.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Denote by $\to$ and $\rightharpoonup$ the *strong convergence* and *weak convergence*, respectively, that is, $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if and only if $\lim_{n\to\infty}\|x_n - x\| = 0$ ($\lim_{n\to\infty}\langle x_n - x, y\rangle = 0$ for all $y \in \mathcal{H}$, resp.). For a given sequence $\{x_n\}$, let $\omega_w\{x_n\}$ denote the set of all weak cluster points of $\{x_n\}$, that is, $z \in \omega_w\{x_n\}$ if and only if $x_{n_k} \rightharpoonup z$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

For a closed convex subset C of $\mathcal{H}$, the *projection* $P_C : \mathcal{H} \to C$ is defined as follows:

$$P_C u := x^* \quad \Longleftrightarrow \quad x^* \in C \text{ satisfies } \|x^* - u\| = \min_{y\in C}\|u - y\|.$$

For a given $u \in \mathcal{H}$, it is not difficult to see that the point x^* above can be regarded as a solution of the equilibrium problem associated with the bifunction $g(x,y) := \langle u - x, x - y\rangle$ and a closed convex subset C. In fact, we have

$$P_C u := x^* \quad \Longleftrightarrow \quad x^* \in \mathrm{EP}(g, C) \quad \Longleftrightarrow \quad g(x^*, y) = \langle u - x^*, x^* - y\rangle \geq 0 \quad \text{for all } y \in C.$$

Definition 1 A function $G : \mathcal{H} \to \mathbb{R}$ is said to be

(1) *convex* if $G(\alpha x + (1-\alpha)y) \leq \alpha G(x) + (1-\alpha)G(y)$ for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$,

(2) *lower semicontinuous* if $G(y) \leq \liminf_{n\to\infty} G(y_n)$ whenever $\{y_n\}$ is a sequence in $\mathcal{H}$ such that $y_n \to y \in \mathcal{H}$,

(3) *weakly upper semicontinuous* if $\limsup_{n\to\infty} G(y_n) \leq G(y)$ whenever $\{y_n\}$ is a sequence in $\mathcal{H}$ such that $y_n \rightharpoonup y \in \mathcal{H}$,

(4) *subdifferentiable on* $\mathcal{H}$ if $\partial G(x) := \{w \in \mathcal{H} : \langle w, y - x\rangle \leq G(y) - G(x) \text{ for all } y \in \mathcal{H}\} \neq \varnothing$ for all $x \in \mathcal{H}$.

We recall the notion of a Halpern-type sequence introduced by the present authors [6].

Definition 2 Suppose that C and F are two nonempty closed convex subsets of $\mathcal{H}$ such that $F \subset C$. We say that a sequence $\{x_n\} \subset C$ is a *Halpern-type sequence with respect to* F if there exist a contraction $h : C \to C$, three sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ in C, and two sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$ such that the following conditions are satisfied:

(a) $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\beta_n < 1$;

(b) $x_{n+1} = \beta_n x_n + (1-\beta_n)w_n$ for all $n \geq 1$;

(c) $\max\{\|u_n - p\|, \|v_n - p\|\} \leq \|x_n - p\|$ for all $n \geq 1$ and $p \in F$;

(d) $\|w_n - p\| \leq \|y_n - p\|$ for all $n \geq 1$ and $p \in F$, where $y_n := \alpha_n h(u_n) + (1-\alpha_n)v_n$.

In this case, we also say that $\{x_n\}$ is a *Halpern sequence with respect to* F *associated with* $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, *and* h.

Remark 3 ([6]) Every Halpern-type sequence with respect to a nonempty closed convex set is bounded.

Theorem 4 ([6]) *Let C and F be two nonempty closed convex subsets of $\mathcal{H}$ such that $F \subset C$. Let $\{x_n\}$ be a Halpern-type sequence with respect to F associated with $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and h. Suppose, in addition, that $\lim_{n\to\infty} \alpha_n = 0$. Then there exists a unique element $x^* \in F$ such that $x^* = P_F h(x^*)$, and the following three statements are equivalent*:

(a) $x_n \to x^*$;

(b) $\omega_w\{v_n\} \subset F$;

(c) $\omega_w\{v_{n_k}\} \subset F$ *whenever* $\{x_{n_k}\}$ *is a subsequence of* $\{x_n\}$ *such that*

$$\lim_{k\to\infty} \left(\|v_{n_k} - p\| - \|x_{n_k} - p\| \right) = 0 \quad \textit{for some } p \in F.$$

The notion of a Halpern-type sequence was introduced in [6] to obtain a strong convergence theorem of an iterative sequence. This is different from the notion of a Fejér sequence (for more detail, we refer to [2, Chap. 5]), which is related to the weak convergence. In fact, there exists a Fejér sequence that is not strongly convergent.

3 Main results

Throughout this section, we assume that two bifunctions $f, g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfy the condition $f(x,x) = g(x,x) = 0$ for all $x \in \mathcal{H}$ and C is a closed convex subset of $\mathcal{H}$. To simplify the notation, we assume that

$$\Omega := \mathrm{EP}(g, C) \neq \varnothing \quad \text{and} \quad \Omega^* := \mathrm{EP}(f, \Omega).$$

3.1 Some preliminaries on equilibrium problems

We first prepare some tools in proving the main results in the next subsections. Let us recall the following conditions for a bifunction $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

(A1) G is *pseudomonotone on C with respect to* $\mathrm{EP}(G, C)$, that is, $G(y, x^*) \leq 0$ for all $(y, x^*) \in C \times \mathrm{EP}(G, C)$.

(A2) G is *β-strongly monotone on $\mathcal{H}$*, where $\beta > 0$, that is, $G(x,y) + G(y,x) \leq -\beta\|x - y\|^2$ for all $x, y \in \mathcal{H}$.

(A3) G is *Lipschitz-type continuous on $\mathcal{H}$ with constants $L_1, L_2 > 0$*, that is,

$$G(x,z) - G(x,y) - G(y,z) \leq L_1\|x - y\|^2 + L_2\|y - z\|^2 \quad \text{for all } x, y, z \in \mathcal{H}.$$

(A4) G is *jointly weakly continuous on $\mathcal{H} \times \mathcal{H}$*, that is, $G(x_n, y_n) \to G(x,y)$ whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in $\mathcal{H}$ such that $x_n \rightharpoonup x \in \mathcal{H}$ and $y_n \rightharpoonup y \in \mathcal{H}$.

(A5) For each $x \in \mathcal{H}$, $G(x, \cdot)$ is convex and lower semicontinuous on $\mathcal{H}$.

(A6) For each $y \in \mathcal{H}$, $G(\cdot, y)$ is weakly upper semicontinuous on $\mathcal{H}$.

Remark 5 ([2]) Let $G : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction, and let $x \in \mathcal{H}$.

(1) Suppose that $G(x, \cdot)$ is convex. Then $G(x, \cdot)$ is lower semicontinuous if and only if it is weakly lower semicontinuous.

(2) If $G(x, \cdot)$ is convex and lower semicontinuous on $\mathcal{H}$, then $G(x, \cdot)$ is subdifferentiable on $\mathcal{H}$, that is, the *subdifferential*
$\partial G(x, \cdot)(y) := \{z \in \mathcal{H} : G(x,y) + \langle z, w - y\rangle \leq G(x,w) \text{ for all } w \in \mathcal{H}\} \neq \varnothing$ for all $y \in \mathcal{H}$.
Moreover, $\partial G(x, \cdot)(y)$ is bounded for all $y \in \mathcal{H}$.

(3) $G(x, \cdot)$ satisfies (A5) $\iff$ $G(x, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$.

(4) If G satisfies (A4), then it satisfies (A6).

Lemma 6 ([7]) *Assume that* $g:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ *satisfies* (A1), (A5), *and either* (A3) *or* (A6). *Then* Ω *is closed and convex.*

Proof Set $\widehat{\Omega}:=\{\widehat{z}\in C: g(y,\widehat{z})\le 0 \text{ for all } y\in C\}$. It follows from (A5) and the closedness and convexity of C that $\widehat{\Omega}$ is closed and convex. It follows from (A1) that $\Omega\subset\widehat{\Omega}$. To complete the proof, we show that $\widehat{\Omega}\subset\Omega$. To see this, let $\widehat{z}\in\widehat{\Omega}$. Let $y\in C$ and $t\in(0,1)$. Since $(1-t)\widehat{z}+ty\in C$, we get $g((1-t)\widehat{z}+ty,\widehat{z})\le 0$.

Case 1: g satisfies (A3). Since g satisfies (A3) and (A5), we have

$$\begin{aligned}
0 &= g(\widehat{z},\widehat{z})\\
&\le g\bigl(\widehat{z},(1-t)\widehat{z}+ty\bigr)+g\bigl((1-t)\widehat{z}+ty,\widehat{z}\bigr)+(L_1+L_2)\bigl\|(1-t)\widehat{z}+ty-\widehat{z}\bigr\|^2\\
&\le tg(\widehat{z},y)+(L_1+L_2)t^2\|\widehat{z}-y\|^2.
\end{aligned}$$

Then $0\le g(\widehat{z},y)+(L_1+L_2)t\|\widehat{z}-y\|^2$. Letting $t\downarrow 0$ gives $0\le g(\widehat{z},y)$, that is, $\widehat{z}\in\Omega$. Hence $\widehat{\Omega}\subset\Omega$.

Case 2: g satisfies (A6). Since g satisfies (A5) and $g((1-t)\widehat{z}+ty,\widehat{z})\le 0$, we have

$$\begin{aligned}
0 &= g\bigl((1-t)\widehat{z}+ty,(1-t)\widehat{z}+ty\bigr)\\
&\le (1-t)g\bigl((1-t)\widehat{z}+ty,\widehat{z}\bigr)+tg\bigl((1-t)\widehat{z}+ty,y\bigr)\\
&\le tg\bigl((1-t)\widehat{z}+ty,y\bigr).
\end{aligned}$$

This implies that $0\le g((1-t)\widehat{z}+ty,y)$. Since g satisfies (A6), we have $0\le g(\widehat{z},y)$, that is, $\widehat{z}\in\Omega$. Hence $\widehat{\Omega}\subset\Omega$. ◻

Lemma 7 ([7]) *Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $x_1,x_2,y\in C$ and $\lambda>0$, and let $g:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be a bifunction such that $g(x,\cdot)$ is convex for all $x\in\mathcal{H}$. Then the following statements are equivalent.*

- $y=\operatorname{argmin}\{\lambda g(x_2,w)+\frac{1}{2}\|w-x_1\|^2 : w\in C\}$;
- $\langle x_1-y,w-y\rangle\le\lambda(g(x_2,w)-g(x_2,y))$ *for all* $w\in C$.

In particular, $x=\operatorname{argmin}\{\lambda g(x,w)+\frac{1}{2}\|w-x\|^2 : w\in C\}$ *if and only if* $g(x,w)\ge 0$ *for all* $w\in C$.

Recall that an element $x\in\mathcal{H}$ is a *fixed point* of a mapping $T:\mathcal{H}\to\mathcal{H}$ if $x=Tx$. We denote the set of all fixed points of T by $\operatorname{Fix}(T)$.

Lemma 8 *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $g:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be a bifunction satisfying* (A5). *For each $n\ge 1$, let $T_n:\mathcal{H}\to C$ be defined by*

$$T_n x := \operatorname{argmin}\left\{\lambda_n g(x,y)+\frac{1}{2}\|y-x\|^2 : y\in C\right\}$$

for $x\in\mathcal{H}$, where $0<\underline{\lambda}\le\lambda_n$. Then $\operatorname{Fix}(T_n)=\Omega$ for all $n\ge 1$. Moreover, suppose in addition that one of the following conditions holds:

(I) *g satisfies* (A4);

(II) *g satisfies* (A6), *and if $\{x'_n\},\{y'_n\}\subset\mathcal{H}$ are such that $x'_n-y'_n\to 0$, then $g(x'_n,y'_n)\to 0$.*

If $\{x_n\}\subset\mathcal{H}$ is a bounded sequence such that $x_n-T_nx_n\to 0$, then $\omega_w\{x_n\}\subset\Omega$.

Proof The first statement is trivial. To prove the "Moreover" part, suppose that $\{x_n\} \subset \mathcal{H}$ is a bounded sequence such that $x_n - T_n x_n \to 0$. Note that $\omega_w\{x_n\} = \omega_w\{T_n x_n\} \subset C$. We assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$ for some $x \in C$. By assumption we get $x_{n_k} - T_{n_k} x_{n_k} \to 0$ and $T_{n_k} x_{n_k} \rightharpoonup x$. From the definition of $T_{n_k} x_{n_k}$ with Lemma 7, we have

$$\langle x_{n_k} - T_{n_k} x_{n_k}, y - T_{n_k} x_{n_k} \rangle \le \lambda_{n_k} \bigl(g(x_{n_k}, y) - g(x_{n_k}, T_{n_k} x_{n_k}) \bigr) \quad \text{for all } y \in C.$$

Note that $x_{n_k} - T_{n_k} x_{n_k} \to 0$ and the bifunction g satisfies either (I) or (II). It follows from $0 < \underline{\lambda} \le \liminf_{k\to\infty} \lambda_{n_k}$ that $0 \le g(x, y)$ for all $y \in C$. Hence $\omega_w\{x_n\} \subset \Omega$. □

Finally, we prepare some tools for the bilevel equilibrium problems based on the diagonal subdifferential operators [5]. Suppose that $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bifunction such that $f(x, \cdot) : \mathcal{H} \to \mathbb{R}$ is convex and lower semicontinuous for each $x \in \mathcal{H}$. In particular, $\partial f(x, \cdot)(x) \neq \varnothing$ for all $x \in \mathcal{H}$. The *diagonal subdifferential operator* $S_f : \mathcal{H} \to 2^{\mathcal{H}}$ is the multivalued function defined by

$$S_f(x) := \partial f(x, \cdot)(x) \quad \text{for all } x \in \mathcal{H}.$$

We consider the following conditions.

(A7) S_f is *L-Lipschitz*, that is, $\|u - v\| \le L\|x - y\|$ for all $x, y \in \mathcal{H}$ and for all $(u, v) \in S_f(x) \times S_f(y)$.

(A8) The function $x \mapsto S_f(x)$ is bounded on each bounded subset of $\mathcal{H}$.

Remark 9 If S_f satisfies (A7), then S_f satisfies (A8), and S_f is a single-valued mapping. In this case the notation $S_f(x)$ is interpreted as an element rather than a singleton.

Lemma 10 *Assume that $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies* (A2) *with a constant β and* (A5). *Then S_f is strongly monotone with the constant β, that is,*

$$\langle u - v, x - y \rangle \ge \beta \|x - y\|^2$$

for all $x, y \in \mathcal{H}$ and for all $(u, v) \in S_f(x) \times S_f(y)$.

Proof Suppose that $x, y \in \mathcal{H}$ and $(u, v) \in S_f(x) \times S_f(y)$. It follows that

$$\langle u, y - x \rangle \le f(x, x) + \langle u, y - x \rangle \le f(x, y),$$
$$\langle v, x - y \rangle \le f(y, y) + \langle v, x - y \rangle \le f(y, x).$$

In particular, $\langle u - v, y - x \rangle \le -\beta \|x - y\|^2$, and this completes the proof. □

Lemma 11 *Assume that $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies* (A2), (A5), (A7), *and* $0 < \mu < 2\beta/L^2$. *Then $I - \mu S_f$ is a contraction, where I is the identity mapping.*

Proof From $0 < \mu < 2\beta/L^2$ we have $1 - \mu L^2(\frac{2\beta}{L^2} - \mu) \in (0, 1)$. Let $x, y \in \mathcal{H}$ and $u := S_f(x)$, $v := S_f(y)$. It follows from (A7) that

$$\bigl\| x - \mu u - (y - \mu v) \bigr\|^2 = \|x - y\|^2 - 2\mu \langle x - y, u - v \rangle + \mu^2 \|u - v\|^2$$

$$\le \|x-y\|^2 - 2\mu\beta\|x-y\|^2 + \mu^2L^2\|x-y\|^2$$
$$= \left(1-\mu L^2\left(\frac{2\beta}{L^2}-\mu\right)\right)\|x-y\|^2. \qquad \square$$

Lemma 12 *Assume that* $f : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ *satisfies* (A2), (A5), (A7) *and* $g : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ *satisfies* (A1), (A5), *and either* (A3) *or* (A6). *Let* $\Omega\neq\emptyset$ *and* $0<\mu<2\beta/L^2$. *Then* $\Omega^* = \{x^*\}$, *where* $x^* = P_\Omega(x^*-\mu S_f(x^*))$.

Proof From Lemma 11 we obtain that $P_\Omega\circ(I-\mu S_f) : \Omega\to\Omega$ is a contraction. By the Banach contraction principle and the completeness of Ω there exists $x^*\in\Omega$ such that $x^* = P_\Omega(x^*-\mu S_f(x^*))$. To show that $x^*\in\Omega^*$, let $y\in\Omega$. Note that $\langle x^*-\mu S_f(x^*)-x^*, x^*-y\rangle\geq 0$. This implies that $\langle S_f(x^*), y-x^*\rangle\geq 0$. It follows from the definition of S_f that $\langle S_f(x^*), y-x^*\rangle\leq f(x^*,y)$. In particular, $f(x^*,y)\geq 0$, and hence $x^*\in\Omega^*$. Suppose that there exists another element $x'\in\Omega^*$. It follows that $f(x^*,x')\geq 0$ and $f(x',x^*)\geq 0$. In particular, it follows from (A2) of f that

$$0\leq f\left(x^*,x'\right)+f\left(x',x^*\right)\leq -\beta\left\|x^*-x'\right\|^2.$$

Hence $x'=x^*$. This completes the proof. $\square$

3.2 On the algorithm of Halpern–Korpelevič type

In this subsection, we discuss the following assumption and algorithm.

Assumption 1 Let C be a nonempty closed convex subset of $\mathcal{H}$, let $g : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be a bifunction, and let $h : \mathcal{H}\to\mathcal{H}$ be a contraction. Assume that

(i) g satisfies (A1), (A3), (A5), and (A6);
(ii) A sequence $\{\alpha_n\}\subset(0,1)$ is such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^\infty\alpha_n=\infty$;
(iii) $0<\underline{\lambda}\leq\lambda_n\leq\overline{\lambda}<\min\{\frac{1}{2L_1},\frac{1}{2L_2}\}$;
(iv) A sequence $\{\beta_n\}\subset[0,1]$ is such that $\limsup_{n\to\infty}\beta_n<1$.

Algorithm 1 Let $\{x_n\}\subset\mathcal{H}$ be a sequence defined by

$$\begin{cases} x_1\in\mathcal{H} \text{ is arbitrarily chosen;} \\ y_n := \operatorname{argmin}\{\lambda_n g(x_n,y)+\frac{1}{2}\|y-x_n\|^2 : y\in C\}; \\ v_n := \operatorname{argmin}\{\lambda_n g(y_n,y)+\frac{1}{2}\|y-x_n\|^2 : y\in C\}; \\ x_{n+1} := \beta_n x_n+(1-\beta_n)(\alpha_n h(v_n)+(1-\alpha_n)v_n) \quad \text{for } n\geq 1. \end{cases}$$

Lemma 13 *Assume that* $g : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ *satisfies* (A1) *and* (A3) *with constants* L_1 *and* L_2 *and that* $g(x,\cdot) : \mathcal{H}\to\mathbb{R}$ *is convex for all* $x\in\mathcal{H}$. *Let* $x\in\mathcal{H}$ *and* $\lambda\in(0,\infty)$, *and let*

$$y := \operatorname{argmin}\left\{\lambda g(x,w)+\frac{1}{2}\|w-x\|^2 : w\in C\right\};$$
$$v := \operatorname{argmin}\left\{\lambda g(y,w)+\frac{1}{2}\|w-x\|^2 : w\in C\right\}.$$

If $p\in\Omega$, *then*

$$\|v-p\|^2\leq\|x-p\|^2-(1-2\lambda L_1)\|x-y\|^2-(1-2\lambda L_2)\|y-v\|^2.$$

Proof Let $p \in \Omega$. By the definitions of y and v it follows from Lemma 7 that

$$\|x-y\|^2 + \|y-v\|^2 - \|x-v\|^2 = 2\langle x-y, v-y\rangle \le 2\lambda\big(g(x,v) - g(x,y)\big);$$
$$\|x-v\|^2 + \|v-p\|^2 - \|x-p\|^2 = 2\langle x-v, p-v\rangle \le 2\lambda\big(g(y,p) - g(y,v)\big).$$

This implies that

$$\|v-p\|^2 \le \|x-p\|^2 - \|x-y\|^2 - \|y-v\|^2 + 2\lambda\big(g(x,v) - g(x,y) + g(y,p) - g(y,v)\big).$$

It follows from (A3) that there are $L_1, L_2 > 0$ such that

$$g(x,v) - g(x,y) - g(y,v) \le L_1\|x-y\|^2 + L_2\|y-v\|^2.$$

Note that $g(y,p) \le 0$. Hence

$$\|v-p\|^2 \le \|x-p\|^2 - (1-2\lambda L_1)\|x-y\|^2 - (1-2\lambda L_2)\|y-v\|^2.$$

This completes the proof. □

Lemma 14 *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction satisfying* (A1) *and* (A3) *with constants L_1 and L_2,* (A5), *and* (A6). *Let $\{x_n\}$ be a bounded sequence in $\mathcal{H}$, and let*

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\right\},$$
$$v_n := \operatorname{argmin}\left\{\lambda_n g(y_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\right\},$$

where $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ for all $n \ge 1$. If $\lim_{n\to\infty}(\|v_n - p\| - \|x_n - p\|) = 0$ for some $p \in \Omega$, then $\omega_w\{x_n\} \subset \Omega$.

Proof We assume that $\lim_{n\to\infty}(\|v_n - p\| - \|x_n - p\|) = 0$ for some $p \in \Omega$. Note that $\{x_n\}$ is bounded, and hence so is $\{v_n\}$. This implies that

$$\lim_{n\to\infty}\big(\|v_n - p\|^2 - \|x_n - p\|^2\big) = 0.$$

By Lemma 13 we have

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - (1-2\lambda_n L_1)\|x_n - y_n\|^2 - (1-2\lambda_n L_2)\|y_n - v_n\|^2.$$

Note that $\liminf_{n\to\infty}(1 - 2\lambda_n L_1) > 0$ and $\liminf_{n\to\infty}(1 - 2\lambda_n L_2) > 0$. It follows that $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|y_n - v_n\| = 0$. To show that $\omega_w\{x_n\} \subset \Omega$, let $x \in \omega_w\{x_n\}$. Then $x \in C$, and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. Let $y \in C$. It follows from Lemma 7 and the definitions of y_n and v_n that

$$\langle x_n - y_n, v_n - y_n\rangle \le \lambda_n\big(g(x_n, v_n) - g(x_n, y_n)\big),$$

$$\langle x_n - v_n, y - v_n\rangle \le \lambda_n\big(g(y_n, y) - g(y_n, v_n)\big).$$

In particular, $\liminf_{n\to\infty}(g(x_n, v_n) - g(x_n, y_n)) \ge 0$ and $\liminf_{n\to\infty}(g(y_n, y) - g(y_n, v_n)) \ge 0$. It follows from (A3) that

$$\limsup_{n\to\infty}\big(g(x_n, v_n) - g(x_n, y_n) - g(y_n, v_n)\big) \le L_1 \lim_{n\to\infty} \|x_n - y_n\|^2 + L_2 \lim_{n\to\infty} \|y_n - v_n\|^2 = 0.$$

This implies that $\liminf_{n\to\infty} g(y_n, y) \ge 0$. It follows from (A6) and $y_{n_k} \rightharpoonup x$ that $g(x, y) \ge 0$. This implies that $x \in \Omega$, and the proof is finished. □

We are ready to present the first main result of the paper.

Theorem 15 *Let $\{x_n\}$ be a sequence generated by Algorithm 1 satisfying Assumption 1. Then $\{x_n\}$ converges strongly to $x^* = P_\Omega h(x^*)$.*

Proof Let $p \in \Omega$. By Lemma 13 we have

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - (1 - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1 - 2\lambda_n L_2)\|y_n - v_n\|^2.$$

In particular, $\|v_n - p\| \le \|x_n - p\|$ for all $n \ge 1$, and hence $\{x_n\}$ is a Halpern sequence with respect to Ω associated with $\{\alpha_n\}$, $\{\beta_n\}$, $\{v_n\}$, $\{v_n\}$, $\{w_n\}$, and h, where $w_n := \alpha_n h(v_n) + (1 - \alpha_n)v_n$. Moreover, $\{x_n\}$ is bounded, and so is $\{v_n\}$ by Remark 3. Next, we prove that $\omega_w\{v_{n_k}\} \subset \Omega$ whenever $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty}(\|v_{n_k} - q\| - \|x_{n_k} - q\|) = 0$ for some $p \in \Omega$. To see this, let $\{x_{n_k}\}$ and $\{v_{n_k}\}$ be such subsequences. Note that $\lim_{k\to\infty} \|v_{n_k} - y_{n_k}\| = \lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0$. Hence $\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = 0$. By Lemma 14, $\omega_w\{v_{n_k}\} = \omega_w\{x_{n_k}\} \subset \Omega$, which leads to the conclusion that $x_n \to x^* = P_\Omega h(x^*)$ by Theorem 4. □

We now apply our theorem to improve Theorem 3.1 of Yuying et al. [10].

Theorem 16 *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $f, g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be two bifunctions. Assume that*

(i) *g satisfies* (A1), (A3), (A5), *and* (A6);
(ii) *f satisfies* (A2), (A5), *and* (A7);
(iii) $0 < \mu < 2\beta/L^2$;
(iv) *A sequence $\{\alpha_n\} \subset (0, 1)$ is such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*
(v) $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ *for all $n \ge 1$;*
(vi) $0 \le \beta_n \le 1 - \alpha_n$ *for all $n \ge 1$, and* $\limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\} \subset \mathcal{H}$ be a sequence defined by

$$\begin{cases} x_1 \in \mathcal{H} \text{ is arbitrarily chosen;} \\ y_n := \operatorname{argmin}\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\}; \\ v_n := \operatorname{argmin}\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\}; \\ u_n := S_f(v_n); \\ x_{n+1} := \beta_n x_n + (1 - \beta_n)v_n - \alpha_n \mu u_n \quad \text{for } n \ge 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - \mu S_f(x^*))$.

Proof Note that

$$\begin{aligned} x_{n+1} &:= \beta_n x_n + (1-\beta_n)v_n - \alpha_n \mu u_n \\ &= \beta_n x_n + \alpha_n(v_n - \mu u_n) + (1-\beta_n-\alpha_n)v_n \\ &= \beta_n x_n + (1-\beta_n)\left(\frac{\alpha_n}{1-\beta_n}(I-\mu S_f)v_n + \left(1-\frac{\alpha_n}{1-\beta_n}\right)v_n\right). \end{aligned}$$

By Lemma 11, $I-\mu S_f$ is a contraction. Note that $\frac{\alpha_n}{1-\beta_n} \in (0,1)$ for all $n \geq 1$, $\lim_{n\to\infty} \frac{\alpha_n}{1-\beta_n} = 0$, and $\sum_{n=1}^{\infty} \frac{\alpha_n}{1-\beta_n} = \infty$. From Theorem 15 we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - \mu S_f(x^*))$. □

Remark 17 Theorem 16 improves [10, Theorem 3.1] in the following ways.

(a) We exclude conditions (A6) and (A8) for the bifunction f and condition (A4) for g.

(b) The condition $\lim_{n\to\infty} \beta_n < 1$ is replaced by the *weaker* one $\limsup_{n\to\infty} \beta_n < 1$.

We now discuss two related results concerning the Halpern–Korpelevič algorithm. The first one is from [9, Theorem 3.2], which can be easily deduced from Theorem 15.

Corollary 18 ([9, Theorem 3.2 where $S := I$]) *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction satisfying* (A1), (A3), (A5), *and* (A6). *Assume that $F : \mathcal{H} \to \mathcal{H}$ is γ-Lipschitz continuous and β-strongly monotone. Assume that*

(i) *A sequence $\{\alpha_n\} \subset (0,1)$ is such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(ii) $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ *for all $n \geq 1$.*

Let $\{x_n\} \subset \mathcal{H}$ be the sequence defined by

$$\begin{cases} x_1 \in \mathcal{H} \text{ is arbitrarily chosen}; \\ y_n := \operatorname{argmin}\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\}; \\ v_n := \operatorname{argmin}\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\}; \\ x_{n+1} := v_n - \alpha_n F v_n \quad \text{for } n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - Fx^*)$.

Proof We can rewrite

$$x_{n+1} = \frac{\alpha_n}{\mu}(I - \mu F)v_n + \left(1 - \frac{\alpha_n}{\mu}\right)v_n.$$

Note that $I - \mu F$ is a contraction whenever $0 < \mu < 2\beta/\gamma^2$. From Theorem 15 with $\beta_n := 0$ and $h := I - \mu F$, we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - \mu F x^*) = P_\Omega(x^* - Fx^*)$. The latter equality holds because of the property of the projection P_Ω. □

The second result is from [1, Theorem 3.3], where $T := I$, $\beta_n := 0$ for $n \geq 1$, and $h(x) := x_1$ for $x \in \mathcal{H}$. To conclude the result, we need the assumption that the iterative sequence $\{x_n\}$ satisfies the condition $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Corollary 19 ([1, Theorem 3.3 with $T := I$]) *Let C be a nonempty closed convex subset of $\mathcal{H}$, let $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction satisfying* (A3) *and* (A5), *and suppose g is pseudomonotone on C, that is, $g(x,y) \geq 0 \Rightarrow g(y,x) \leq 0$ for all $x, y \in \mathcal{H}$. Assume that*

(i) *A sequence* $\{\alpha_n\} \subset (0,1)$ *is such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \lambda_n \leq \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ *and* $\lambda_n < \frac{1-\delta}{2L_1}$ *for all* $n \geq 1$, *where* $\delta \in (0,1)$.

Let $\{x_n\} \subset \mathcal{H}$ *be the sequence defined by*

$$\begin{cases} x_1 \in \mathcal{H} \text{ is arbitrarily chosen}; \\ y_n := \operatorname{argmin}\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\}; \\ v_n := \operatorname{argmin}\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\}; \\ x_{n+1} := \alpha_n x_1 + (1-\alpha_n) v_n \quad \text{for } n \geq 1. \end{cases}$$

If $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, *then* $\{x_n\}$ *converges strongly to* $x^* = P_\Omega x_1$.

As shown in [6], this result is not correct. Indeed, let $C = \mathcal{H} = \mathbb{R}$, and let $g(x,y) := \langle x, y-x\rangle$, $\alpha_n := 1/(n+1)$, $\lambda_n := 1/2^n$, and $x_1 := 1$. Then $\Omega = \{0\}$ and $x_n \to 1 \notin \Omega$.

3.3 On the extragradient-like algorithm with line search technique

In this subsection, we modify the algorithm to avoid the prior knowledge of the values L_1 and L_2 as was the case in the previous algorithm.

Assumption 2 Let C be a nonempty closed convex subset of $\mathcal{H}$, let $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction satisfying (A1), (A4), and (A5), and let $h : \mathcal{H} \to \mathcal{H}$ be a contraction. Assume that

(i) $\rho \in (0,2)$ and $\gamma \in (0,1)$;

(ii) A sequence $\{\alpha_n\} \subset (0,1)$ is such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ and $\{\xi_n\} \subset [\underline{\xi}, \overline{\xi}] \subset (0,2)$ for all $n \geq 1$.

For each $x \in C$ and $\lambda > 0$, let

$$y := \operatorname{argmin}\left\{\lambda g(x,w) + \frac{1}{2}\|w - x\|^2 : w \in C\right\}.$$

Suppose that $y \neq x$. It follows from [8] that there exists the smallest positive integer m such that

$$g(z,x) - g(z,y) \geq \frac{\rho}{2\lambda}\|x - y\|^2,$$

where $z := (1-\gamma^m)x + \gamma^m y$. Moreover, it was proved in [8] that $g(z,x) > 0$ and $0 \notin \partial g(z,\cdot)(x)$.

Algorithm 2 Let $\{x_n\} \subset C$ be a sequence defined by $x_1 \in C$ arbitrarily chosen, and let

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}.$$

If $y_n = x_n$, then $v_n := x_n$. If $y_n \neq x_n$, then (Armijo line search rule) find $m(n)$ as the smallest positive integer m satisfying

$$g\big((1-\gamma^m)x_n + \gamma^m y_n, x_n\big) - g\big((1-\gamma^m)x_n + \gamma^m y_n, y_n\big) \geq \frac{\rho}{2\lambda_n}\|x_n - y_n\|^2.$$

In particular, let $z_n := (1-\gamma^{m(n)})x_n + \gamma^{m(n)}y_n$. Choose $t_n \in \partial g(z_n,\cdot)(x_n)$ and $\sigma_n := g(z_n,x_n)/\|t_n\|^2$. Next,

$$\begin{aligned} v_n &:= P_C(x_n - \xi_n\sigma_n t_n);\\ x_{n+1} &:= P_C\bigl(\alpha_n h(v_n) + (1-\alpha_n)v_n\bigr) \quad \text{for } n \geq 1. \end{aligned}$$

Lemma 20 *Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $x \in C$, and let $\rho \in (0,2)$, $\gamma \in (0,1)$, and $\lambda \in (0,\infty)$. Assume that $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies* (A1), (A4), *and* (A5). *Let*

$$y := \operatorname{argmin}\left\{\lambda g(x,w) + \frac{1}{2}\|w-x\|^2 : w \in C\right\}.$$

Assume that $y \neq x$. Let m be the smallest positive integer such that

$$g\bigl((1-\gamma^m)x + \gamma^m y, x\bigr) - g\bigl((1-\gamma^m)x + \gamma^m y, y\bigr) \geq \frac{\rho}{2\lambda}\|x-y\|^2,$$

and let $z := (1-\gamma^m)x + \gamma^m y$. Let $t \in \partial g(z,\cdot)(x)$ and $\sigma := g(z,x)/\|t\|^2$. Then the following statements are true.

(i) *If $v := P_C(x - \xi\sigma t)$ where $\xi \in (0,2)$ and $p \in \Omega$, then*

$$\|v-p\|^2 \leq \|x-p\|^2 - \xi(2-\xi)\sigma^2\|t\|^2.$$

(ii) *If $v := P_{C\cap D}(x)$ where $D := \{w \in \mathcal{H} : \langle t, x-w\rangle \geq g(z,x)\}$ and $p \in \Omega$, then*

$$\|v-p\|^2 \leq \|x-p\|^2 - \sigma^2\|t\|^2.$$

Proof Let $p \in \Omega$. Since $g(z,x) > 0$ and $0 \notin \partial g(z,\cdot)(x)$, we have $\sigma > 0$. From (A1) and $z \in C$ we have $g(z,p) \leq 0$. Since $t \in \partial g(z,\cdot)(x)$, we get $\langle t, x-p\rangle \geq g(z,x) - g(z,p) \geq g(z,x)$.

(i) Since $\langle t, x-p\rangle \geq g(z,x) = \sigma\|t\|^2$ and $\sigma > 0$, we have

$$\begin{aligned} \|v-p\|^2 &\leq \|x-\xi\sigma t-p\|^2\\ &= \|x-p\|^2 - 2\xi\sigma\langle t, x-p\rangle + \xi^2\sigma^2\|t\|^2\\ &\leq \|x-p\|^2 - 2\xi\bigl(\sigma\|t\|\bigr)^2 + \xi^2\sigma^2\|t\|^2\\ &= \|x-p\|^2 - \xi(2-\xi)\sigma^2\|t\|^2. \end{aligned}$$

(ii) Since $C \cap D \subset D$, we have $P_{C\cap D}(x) = P_{C\cap D}(P_D x)$. Note that $P_D x = x - (g(z,x)/\|t\|^2)t = x - \sigma t$. It follows that $v := P_{C\cap D}(x) = P_{C\cap D}(P_D x) = P_{C\cap D}(x-\sigma t)$. Since $\langle t, x-p\rangle \geq g(z,x)$, we obtain $p \in D$. Hence $p \in C \cap D$. Following the proof of (i) with $\xi = 1$, we have

$$\|v-p\|^2 \leq \|x-\sigma t-p\|^2 \leq \|x-p\|^2 - \sigma^2\|t\|^2. \qquad \square$$

The following lemma is related to [8, Proposition 4.3].

Lemma 21 *Let $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfy* (A4) *and* (A5). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in $\mathcal{H}$, and let $t_n \in \partial g(z_n,\cdot)(x_n)$. Then $\{t_n\}$ is bounded.*

Proof Suppose that $\{t_n\}$ is not bounded. Passing to a suitable subsequence, we may assume that $\|t_{n_k}\| \to \infty$ and $x_{n_k} \rightharpoonup x$ and $z_{n_k} \rightharpoonup z$ for some $x, z \in \mathcal{H}$. Since $t_{n_k} \in \partial g(z_{n_k}, \cdot)(x_{n_k})$, we have $\langle t_{n_k}, w - x_{n_k}\rangle \le g(z_{n_k}, w) - g(z_{n_k}, x_{n_k})$ for all $w \in \mathcal{H}$. Let $y \in \mathcal{H}$. This implies that

$$g(z_{n_k}, x_{n_k}) - g(z_{n_k}, -y + x_{n_k}) \le \langle t_{n_k}, y\rangle \le g(z_{n_k}, y + x_{n_k}) - g(z_{n_k}, x_{n_k}).$$

It follows from (A4) that

$$\lim_{k\to\infty} \bigl(g(z_{n_k}, x_{n_k}) - g(z_{n_k}, -y + x_{n_k})\bigr) = g(z, x) - g(z, -y + x),$$
$$\lim_{k\to\infty} \bigl(g(z_{n_k}, y + x_{n_k}) - g(z_{n_k}, x_{n_k})\bigr) = g(z, y + x) - g(z, x).$$

In particular, there exists $M_y > 0$ such that $|\langle t_{n_k}, y\rangle| \le M_y$ for all $k \ge 1$. By the uniform boundedness principle the sequence $\{t_{n_k}\}$ is bounded, which is a contradiction. □

We can prove the following lemma as in the proof of [4, Lemma 4]. In fact, we can assume that $\{\lambda_n\} \subset (0, \overline{\lambda}] \subset (0, \infty)$ instead of $\lambda_n = 1$ for all $n \ge 1$.

Lemma 22 *Let $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfy* (A4) *and* (A5), *and let $\{\lambda_n\} \subset (0, \infty)$. Let $\{x_n\}$ be a sequence in C, and for each $n \ge 1$, let*

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}.$$

If $\{x_n\}$ and $\{\lambda_n\}$ are bounded, then so is $\{y_n\}$.

Proof Note that

$$\|x_n - y_n\|^2 = \langle x_n - y_n, x_n - y_n\rangle \le \lambda_n\bigl(g(x_n, x_n) - g(x_n, y_n)\bigr) = -\lambda_n g(x_n, y_n).$$

For $n \ge 1$, let $w_n \in S_g(x_n)$. Then

$$-g(x_n, y_n) \le \langle w_n, x_n - y_n\rangle \le \|w_n\|\|x_n - y_n\|.$$

In particular, $\|x_n - y_n\|^2 \le \lambda_n\|w_n\|\|x_n - y_n\|$, and hence $\|x_n - y_n\| \le \lambda_n\|w_n\|$. Since $\{x_n\}$ and $\{\lambda_n\}$ are bounded and $w_n \in S_g(x_n) := \partial g(x_n, \cdot)(x_n)$, it follows from Lemma 21 that $\{y_n\}$ is bounded. □

Lemma 23 *Let $\{x_n\}$ be a bounded sequence in C. Let $\rho \in (0, 2)$ and $\gamma \in (0, 1)$, and let $\{\lambda_n\} \subset (0, \overline{\lambda}] \subset (0, \infty)$ and $\{\xi_n\} \subset [\underline{\xi}, \overline{\xi}] \subset (0, 2)$. Assume that $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies* (A1), (A4), *and* (A5). *Define the sequence $\{y_n\}$ by*

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\right\}.$$

Assume that $y_n \ne x_n$ for all $n \ge 1$. Define the sequence $\{v_n\}$ as follows: find $m(n)$ as the smallest positive integer m satisfying

$$g\bigl((1 - \gamma^m)x_n + \gamma^m y_n, x_n\bigr) - g\bigl((1 - \gamma^m)x_n + \gamma^m y_n, y_n\bigr) \ge \frac{\rho}{2\lambda_n}\|x_n - y_n\|^2.$$

In particular, let $z_n := (1-\gamma^{m(n)})x_n + \gamma^{m(n)}y_n$. *Choose* $t_n \in \partial g(z_n,\cdot)(x_n)$ *and* $\sigma_n := g(z_n,x_n)/\|t_n\|^2$. *Next,*

$$v_n := P_C(x_n - \xi_n\sigma_n t_n).$$

If $\lim_{n\to\infty}(\|v_n-p\| - \|x_n-p\|) = 0$ *for some* $p\in\Omega$, *then*

(1) $\lim_{n\to\infty}\sigma_n\|t_n\| = 0$,

(2) $\lim_{n\to\infty}\|x_n-y_n\| = 0$.

Proof Assume that $\lim_{n\to\infty}(\|v_n-p\| - \|x_n-p\|) = 0$ for some $p\in\Omega$. By Lemma 20 we have

$$\|v_n-p\|^2 \le \|x_n-p\|^2 - \xi_n(2-\xi_n)\sigma_n^2\|t_n\|^2.$$

Since $\xi_n\in[\underline{\xi},\overline{\xi}]\subset(0,2)$ and $\{x_n\}$ is bounded, we have that $\{v_n\}$ is bounded, and hence

$$0 \le \xi_n(2-\xi_n)\sigma_n^2\|t_n\|^2 \le \|x_n-p\|^2 - \|v_n-p\|^2 \to 0.$$

This implies that $\lim_{n\to\infty}\sigma_n\|t_n\| = 0$. It follows from Lemma 22 that $\{y_n\}$ is bounded and so is $\{z_n\}$. So $\{t_n\}$ is bounded by Lemma 21. Since $g(z_n,\cdot)$ is convex, we have $0 = g(z_n,z_n) \le (1-\gamma^{m(n)})g(z_n,x_n) + \gamma^{m(n)}g(z_n,y_n)$, and hence $\gamma^{m(n)}(g(z_n,x_n) - g(z_n,y_n)) \le g(z_n,x_n)$. Thus

$$\frac{\gamma^{m(n)}\rho}{2\lambda_n}\|x_n-y_n\|^2 \le \gamma^{m(n)}\big(g(z_n,x_n) - g(z_n,y_n)\big) \le g(z_n,x_n) = \sigma_n\|t_n\|^2 \to 0.$$

Since $\lambda_n \le \overline{\lambda}$, we obtain

$$\lim_{n\to\infty}\gamma^{m(n)}\|x_n-y_n\|^2 = 0.$$

We will prove that $\lim_{n\to\infty}\|x_n-y_n\| = 0$ by contradiction. Suppose that there are $\varepsilon>0$ and a strictly increasing sequence $\{n_k\}$ such that $\|x_{n_k}-y_{n_k}\| \ge \epsilon$ for all $k\ge 1$.

Case 1: $\limsup_{k\to\infty}\gamma^{m(n_k)} > 0$. This implies that

$$\limsup_{k\to\infty}\gamma^{m(n_k)}\|x_{n_k}-y_{n_k}\|^2 > 0,$$

which is a contradiction.

Case 2: $\limsup_{k\to\infty}\gamma^{m(n_k)} = 0$. There is a further subsequence $\{n_{k_l}\}$ of $\{n_k\}$ such that $\lim_{l\to\infty}\gamma^{m(n_{k_l})} = 0$ and $x_{n_{k_l}} \rightharpoonup x$ and $y_{n_{k_l}} \rightharpoonup y$ for some $x,y\in\mathcal{H}$. We write $\overline{z}_n := (1-\gamma^{m(n)-1})x_n + \gamma^{m(n)-1}y_n$. In particular, we have

$$g(\overline{z}_{n_{k_l}},x_{n_{k_l}}) - g(\overline{z}_{n_{k_l}},y_{n_{k_l}}) < \frac{\rho}{2\lambda_{n_{k_l}}}\|x_{n_{k_l}}-y_{n_{k_l}}\|^2.$$

It follows from the definition of y_n and $x_n\in C$ that

$$\|x_n-y_n\|^2 \le -\lambda_n g(x_n,y_n).$$

This implies that

$$\left(1-\frac{\rho}{2}\right)\|x_{n_{k_l}}-y_{n_{k_l}}\|^2 < \lambda_{n_{k_l}}\left(g(\bar{z}_{n_{k_l}},y_{n_{k_l}})-g(\bar{z}_{n_{k_l}},x_{n_{k_l}})-g(x_{n_{k_l}},y_{n_{k_l}})\right).$$

Since $\lim_{l\to\infty}\gamma^{m(n_{k_l})}=0$ and $\gamma\in(0,1)$, it follows that $\lim_{l\to\infty}\gamma^{m(n_{k_l})-1}=0$. In particular, $\bar{z}_{n_{k_l}}\rightharpoonup x$. It follows from condition (A4) for g and $\limsup_{n\to\infty}\lambda_n\le\bar{\lambda}$ that

$$\lim_{l\to\infty}\|x_{n_{k_l}}-y_{n_{k_l}}\|=0,$$

which is a contradiction. □

As in the proof of Lemma 23 with $\xi_n := 1$ for all $n\ge 1$, we obtain an analog of the preceding lemma with $v_n := P_{C\cap D_n}(x_n)$.

Lemma 24 *Let $\{x_n\}$ be a bounded sequence in C. Let $\rho\in(0,2)$ and $\gamma\in(0,1)$, and let $\{\lambda_n\}\subset(0,\bar{\lambda}]\subset(0,\infty)$. Assume that $g:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ satisfies* (A1), (A4), *and* (A5). *Define the sequence $\{y_n\}$ by*

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n,w)+\frac{1}{2}\|w-x_n\|^2 : w\in C\right\}.$$

Assume that $y_n\ne x_n$ for all $n\ge 1$. Define the sequence $\{v_n\}$ as follows: find $m(n)$ as the smallest positive integer m satisfying

$$g\big((1-\gamma^m)x_n+\gamma^m y_n,x_n\big)-g\big((1-\gamma^m)x_n+\gamma^m y_n,y_n\big)\ge\frac{\rho}{2\lambda_n}\|x_n-y_n\|^2.$$

In particular, let $z_n := (1-\gamma^{m(n)})x_n+\gamma^{m(n)}y_n$. Choose $t_n\in\partial g(z_n,\cdot)(x_n)$ and $\sigma_n := g(z_n,x_n)/\|t_n\|^2$. Next,

$$v_n := P_{C\cap D_n}(x_n),$$

where $D_n := \{w\in\mathcal{H} : \langle t_n,x_n-w\rangle\ge g(z_n,x_n)\}$. If $\lim_{n\to\infty}(\|v_n-p\|-\|x_n-p\|)=0$ for some $p\in\Omega$, then $\lim_{n\to\infty}\sigma_n\|t_n\|=\lim_{n\to\infty}\|x_n-y_n\|=0$.

Theorem 25 *Let $\{x_n\}$ be a sequence generated by Algorithm 2 satisfying Assumption 2. Then $\{x_n\}$ converges strongly to $x^*=P_\Omega h(x^*)$.*

Proof Let $p\in\Omega$. We know that

$$\left\|P_C\big(\alpha_n h(v_n)+(1-\alpha_n)v_n\big)-p\right\|\le\left\|\big(\alpha_n h(v_n)+(1-\alpha_n)v_n\big)-p\right\|$$

for all $n\ge 1$. By Lemma 20 we have

$$\|v_n-p\|^2\le\|x_n-p\|^2-\xi_n(2-\xi_n)\big(\sigma_n\|t_n\|\big)^2,$$

where $\sigma_n := g(z_n,x_n)/\|t_n\|^2$ if $y_n\ne x_n$ and $\sigma_n=0$ otherwise. Since $\xi_n\in(0,2)$, we get $\|v_n-p\|\le\|x_n-p\|$ for all $n\ge 1$. Hence $\{x_n\}$ is a Halpern sequence with respect to Ω

associated with $\{\alpha_n\}$, $\{0\}$, $\{v_n\}$, $\{v_n\}$, $\{w_n\}$, and h, where $w_n := P_C(\alpha_n h(v_n) + (1-\alpha_n)v_n)$. Moreover, we have that $\{x_n\}$ is bounded, and so is $\{v_n\}$ by Remark 3.

Next, we prove that $\omega_w\{v_{n_k}\} \subset \Omega$ whenever $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty}(\|v_{n_k} - p\| - \|x_{n_k} - p\|) = 0$ for some $p \in \Omega$. To see this, let $\{x_{n_k}\}$ and $\{v_{n_k}\}$ be such subsequences. Without loss of generality, we assume that $y_{n_k} \neq x_{n_k}$ for all $k \geq 1$. Note that $\{x_{n_k}\}$ is bounded by Remark 3. From Lemma 23 we obtain $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$. By Lemma 8 we have

$$\omega_w\{x_{n_k}\} \subset \Omega.$$

Moreover, we have $\lim_{k\to\infty} \sigma_{n_k}\|t_{n_k}\| = 0$ by Lemma 23. Since $\{x_{n_k}\} \subset C$, we obtain $\|v_{n_k} - x_{n_k}\| = \|P_C(x_{n_k} - \xi_{n_k}\sigma_{n_k}t_{n_k}) - P_C x_{n_k}\| \leq \xi_{n_k}\sigma_{n_k}\|t_{n_k}\| \to 0$. Thus $\omega_w\{v_{n_k}\} = \omega_w\{x_{n_k}\} \subset \Omega$. Hence $x_n \to x^* = P_\Omega h(x^*)$ by Theorem 4. □

We now apply Theorem 25 to recover [10, Theorem 4.4].

Theorem 26 *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $f,g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be two bifunctions such that f satisfies* (A2), (A5), *and* (A7) *and g satisfies* (A1), (A4), *and* (A5). *Assume that*

(i) $0 < \mu < 2\beta/L^2$, $\rho \in (0,2)$, *and* $\gamma \in (0,1)$;
(ii) *A sequence* $\{\alpha_n\} \subset (0,1)$ *is such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $\lambda_n \in [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ *and* $\xi_n \in [\underline{\xi}, \overline{\xi}] \subset (0,2)$ *for all* $n \geq 1$.

Let $\{x_n\} \subset C$ be the sequence defined as follows: $x_1 \in C$ is arbitrarily chosen, and

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}.$$

If $y_n = x_n$, then $v_n := x_n$. If $y_n \neq x_n$, then (*Armijo line search rule*) *find $m(n)$ as the smallest positive integer m satisfying*

$$g\big((1-\gamma^m)x_n + \gamma^m y_n, x_n\big) - g\big((1-\gamma^m)x_n + \gamma^m y_n, y_n\big) \geq \frac{\rho}{2\lambda_n}\|x_n - y_n\|^2.$$

In particular, let $z_n := (1-\gamma^{m(n)})x_n + \gamma^{m(n)}y_n$. Choose $t_n \in \partial g(z_n,\cdot)(x_n)$ and $\sigma_n := g(z_n,x_n)/\|t_n\|^2$. Next,

$$\begin{aligned} v_n &:= P_C(x_n - \xi_n\sigma_n t_n);\\ u_n &:= S_f(v_n);\\ x_{n+1} &:= P_C(v_n - \alpha_n\mu u_n) \quad \text{for } n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - \mu S_f(x^*))$.

Proof We know that $I - \mu S_f$ is a contraction by Lemma 11. Note that

$$x_{n+1} = P_C\big(v_n - \alpha_n\mu S_f(v_n)\big) = P_C\big(\alpha_n(I - \mu S_f)v_n + (1-\alpha_n)v_n\big).$$

By Theorem 25 we have that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - \mu S_f(x^*))$. □

Remark 27 Theorem 26 improves [10, Theorem 4.4] in the following ways.

(a) We exclude conditions (A6) and (A8) for the bifunction f and condition (A6) for g as were the cases in [10, Theorem 4.4].

(b) We can replace the condition $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ by the weaker condition $\lim_{n\to\infty} \alpha_n = 0$. Moreover, the choice $\alpha_n := 1/\sqrt{n}$ is applicable in our result, but it is beyond the scope of [10, Theorem 4.4].

Next, we construct Algorithm 2a, which is the same as Algorithm 2, except that $v_n := P_C(x_n - \xi_n \sigma_n t_n)$ is replaced by $v_n := P_{C\cap D_n}(x_n)$ where $D_n := \{w \in \mathcal{H} : \langle t_n, x_n - w\rangle \geq g(z_n, x_n)\}$. We can conclude the same conclusion as follows.

Theorem 28 *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 2a *satisfying Assumption* 2. *Then* $\{x_n\}$ *converges strongly to* $x^* = P_\Omega h(x^*)$.

Proof First, since $t_n \in \partial g(z_n, \cdot)(x_n)$, we get $\langle t_n, x_n - z_n\rangle \geq g(z_n, x_n)$. Hence $z_n \in C \cap D_n$. It follows that

$$\begin{aligned}
\|v_n - x_n\| &= \left\|P_{C\cap D_n}(x_n) - x_n\right\| \\
&= \left\|P_{C\cap D_n}(x_n) - P_{C\cap D_n}(z_n) + z_n - x_n\right\| \\
&\leq \left\|P_{C\cap D_n}(x_n) - P_{C\cap D_n}(z_n)\right\| + \|z_n - x_n\| \\
&\leq 2\|x_n - z_n\| = 2\gamma^{m(n)}\|x_n - y_n\|.
\end{aligned}$$

We now follow the proof of Theorem 25 and prove that $\omega_w\{v_{n_k}\} \subset \Omega$ whenever $\{x_{n_k}\}$ is a subsequence if $\{x_n\}$ such that $\lim_{k\to\infty}(\|v_{n_k} - p\| - \|x_{n_k} - p\|) = 0$ for some $p \in \Omega$. To see this, let $\{x_{n_k}\}$ and $\{v_{n_k}\}$ be such subsequences. Without loss of generality, we assume that $y_{n_k} \neq x_{n_k}$ for all $k \geq 1$. Note that $\{x_{n_k}\}$ is bounded by Remark 3. From Lemma 24 we obtain $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$. By Lemma 8 we have

$$\omega_w\{x_{n_k}\} \subset \Omega.$$

Note that $\lim_{k\to\infty} \gamma^{m(n_k)}\|x_{n_k} - y_{n_k}\| = 0$. In particular, $\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = 0$, and hence $\omega_w\{v_{n_k}\} \subset \Omega$. This completes the proof. □

We now apply Theorem 28 to Theorem 4.4 of [9] where $S := I$.

Theorem 29 *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$, *and let* $g : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ *be a bifunction satisfying* (A1), (A4), *and* (A5). *Assume that* $F : \mathcal{H} \to \mathcal{H}$ *is* γ*-Lipschitz continuous and* β*-strongly monotone. Assume that*

(i) $\rho \in (0, 2)$ *and* $\gamma \in (0, 1)$;

(ii) *A sequence* $\{\alpha_n\} \subset (0, 1)$ *is such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, \infty)$ *for all* $n \geq 1$.

Let $\{x_n\} \subset C$ *be the sequence defined as follows*: $x_1 \in C$ *is arbitrarily chosen, and*

$$y_n := \operatorname{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}.$$

If $y_n = x_n$, then $v_n := x_n$. If $y_n \neq x_n$, then (Armijo line search rule) find $m(n)$ as the smallest positive integer m satisfying

$$g\big((1-\gamma^m)x_n + \gamma^m y_n, x_n\big) - g\big((1-\gamma^m)x_n + \gamma^m y_n, y_n\big) \geq \frac{\rho}{2\lambda_n}\|x_n - y_n\|^2.$$

In particular, let $z_n := (1-\gamma^{m(n)})x_n + \gamma^{m(n)}y_n$. Choose $t_n \in \partial g(z_n, \cdot)(x_n)$. Next,

$$\begin{aligned} &D_n := \big\{x \in \mathcal{H} : \langle t_n, x_n - x\rangle \geq g(z_n, x_n)\big\}; \\ &v_n := P_{C\cap D_n}(x_n); \\ &x_{n+1} := P_C(v_n - \alpha_n F v_n) \quad \text{for } n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - Fx^*)$.

Proof Note that $I - \mu F$ is a contraction whenever $0 < \mu < 2\beta/\gamma^2$. We can rewrite

$$x_{n+1} = P_C\left(\frac{\alpha_n}{\mu}(I-\mu F)v_n + \left(1 - \frac{\alpha_n}{\mu}\right)v_n\right).$$

From Theorem 28 with $h := I - \mu F$ we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x^* - \mu Fx^*) = P_\Omega(x^* - Fx^*)$. The latter equality holds because of the property of the projection P_Ω. □

Remark 30 Theorem 29 improves [9, Theorem 4.4] in the following ways.

- The condition $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$ is replaced by the weaker condition $\lim_{n\to\infty}\alpha_n = 0$.
- The condition $0 < \underline{\lambda} \leq \lambda_n \leq 1$ for all $n \geq 1$ is replaced by the weaker condition $\lambda_n \in [\underline{\lambda}, \overline{\lambda}] \subset (0,\infty)$ for all $n \geq 1$.

4 Conclusion

We apply the notion of a Halpern-type sequence introduced by the authors [6] for the problem of finding a solution of bilevel equilibrium problems. We can cover two recent results of Yuying et al. [10], where the first one uses the algorithm of Halpern–Korpelevič type, and the second one uses the extragradient-like algorithm with line search technique. The convergence results are established under weaker assumptions. The method used in this paper is simple and excludes some restrictions as were the cases in many results in the literature.

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