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Fixed point theorems in modular *G*-metric spaces

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Abstract
We prove the existence and uniqueness of fixed points of some generalized contractible operators defined on modular *G*-metric spaces and also prove the modular *G*-continuity of such operators. Furthermore, we prove that some generalized weakly compatible contractive operators in modular *G*-metric spaces have a unique fixed point. Our results extend, generalize, complement and include several known results as special cases.

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1 Introduction

Banach in 1922 [16] proved his famous contraction mapping principle in a complete metric space. Since then this result of Banach has been characterized in different spaces such as those in quasi-metric, multiplicative metric space, b-metric, D-metric space, D^*-metric spaces, G-metrics, F-metric spaces, modular and modular G-metric spaces. Some other results in b-metric space and control metric space can be found in [33, 64] and the references therein and [6, 56].

In 1966, Gahler in [24], introduced 2-metric spaces, and Dhage in [23] extended the work in [24] in which D-metric spaces were introduced. These authors claimed that their results generalized the concept of metric spaces. The nonnegative real function D is called a D-metric on X. The set X together with such a generalized metric D is called a generalized metric space, or D-metric space, and denoted by (X, D) and some of their topological properties have been studied by Dhage.

The concept of G-metric space was introduced by Mustafa and Sims in [45] and they pointed out that the fundamental topology properties of D-metric spaces introduced by Dhage were incorrect. To ameliorate the drawback about D-metric spaces, Mustafa and Sims in [46] introduced a generalization of metric spaces, which they called G-metric spaces and proved some fixed point theorems. Samet *et al.* [62] and Jleli-Samet [29] noticed that some fixed point theorems in the context of a G-metric space in the literature can be deduced directly from some existing results in the setting of a quasi-metric and metric space. In fact, if the contraction condition of the fixed point theorem on a G-metric space can be reduced to two variables instead of three variables, then one can construct an equivalent fixed point theorems in the setup of a usual metric space. Precisely, in [29, 62] the authors noticed that $2d(x,y) = G(x,y,y)$ forms a quasi-metric. Hence, if one can transform the contraction condition of existence results in a G-metric space in such terms, $G(x,y,y)$, then the related fixed point results become the known fixed point results in the context of a quasi-metric space. Sequel to these arguments about the concept of generalized metric space, Karapinar and Agarwal [30] noticed that the techniques used in [29, 62] are valid if the contraction condition in the statement of the theorem(s) can be reduced into two variables. Furthermore, Karapinar and Agarwal [30] proved some fixed point theorems in the context of a G-metric space for which the techniques used in [29, 62] are inapplicable.

The reduction method introduced by Samet *et al.* [62] and Jleli and Samet [29] reported that most of fixed point results in the context of G-metric space, defined by Mustafa and Sims, can be derived from the usual fixed point theorems on the usual metric space. This enabled [12] *et al.* to prove some fixed point theorems in the framework of G-metric space that contradicted the ideas of Samet *et al.* [62] and Jleli and Samet [29] showing that not all fixed point results can be obtained from the existence results in the context of associated metric space.

Shatanawi *et al.* [63] utilized the concept of Ω-distance in the sense of Saadati to introduce new fixed and common fixed point results for mappings of cyclic form, through the concept of G-metric space in sense of Mustafa and Sims [46]. Shatanawi *et al.* [63] pointed out that the method of Jleli and Samet cannot be applied to their results. Some authors have obtained some fixed point theorems in the structure of G-metric spaces and established that they cannot be obtained from the existing results in the context of allied metric spaces and do not meet the remarks of Samet *et al.* [62] and Jleli and Samet [29]. Other interesting results about fixed point theorems in G-metric spaces can be found in [1–3, 7, 9, 10, 25, 32, 39–41, 43, 44, 47, 48, 65] while [38] investigated common fixed points of weakly compatible mappings in G-metric spaces.

Mustafa and Jaradat in their recent paper [42] produced an example to show that D^*-metric need not be G-metric as well as the G-metric need not be D^*-metric. With all these results, in this paper, we will give some results in the setting of modular G-metric which cannot be reduced to its allied modular metric spaces.

Modular function space and its theory were introduced by Nakano in [51] in line with the theory of order spaces. Musielak and Orlicz redefined and generalized the concept in [36, 57] and proved some fixed point theorems. The notion of a modular metric on an arbitrary set and the corresponding modular space, which is more general than a metric space, were introduced and studied in a landmark paper by Chistyakov in [20]. Also fixed point theorems in modular metric spaces and their applications have been dealt with in [21] and [22]. Many authors have introduced and generalized, studied fixed point theorems and their applications in modular metric spaces such as those in [4, 5, 11, 13, 17, 18, 20, 21, 27, 28, 34, 35, 49, 54, 61] in modular metric spaces; see [13, 18, 21, 52, 55] and the references therein and some other interesting results of fixed point theorems in metric spaces appearing in [7, 8, 16, 17, 26, 31, 34, 37, 50, 58, 59, 61] and [60].

Since then, many authors such as [4, 5, 11, 13, 18, 19, 27, 28, 35, 49] have studied and improved some results of fixed point theory in the setting of modular metric spaces, while

[52, 55] studied iterative approximation of fixed point of multi-valued ρ-quasi-contractive and multi-valued ρ-quasi-nonexpansive mappings in modular function spaces, respectively.

Following the notion of modular metric space in [5], Definition 1.1 in [14] can be interpreted in terms of modular metric spaces as follows. Let X be a nonempty set, and let $\omega^G : (0,\infty) \times X \times X \times X \to [0,\infty]$ be a function defined by $\omega^G_\lambda(x,y,z) = \omega_\lambda(x,y) + \omega_\lambda(y,z) + \omega_\lambda(x,z)$ for all $x,y,z \in X$ and $\lambda > 0$. Then (X,ω^G) is a modular G-metric space. In this case, for all $\lambda \in (0,\infty)$, $\omega^G_\lambda(x,y,z)$ can be interpreted as the perimeter of the triangle of vertices x, y and z. Now axiom (1) means that with one point we cannot have a positive perimeter for all $\lambda \in (0,\infty)$, and axiom (2) is equivalent to the fact that the distance between two different points can never be zero for all $\lambda \in (0,\infty)$. Meanwhile, as the perimeter of a triangle for all $\lambda \in (0,\infty)$ cannot depend on the order in which we consider its vertices, we have axiom (4) and axiom (5) is an extension of the triangle inequality using a fourth vertex for all $\lambda \in (0,\infty)$. By axiom (3), we see that the length of an edge of a triangle is less than or equal to its semi-perimeter, i.e. $\omega_\lambda(x,y) \leq \frac{\omega_\lambda(x,y)+\omega_\lambda(y,z)+\omega_\lambda(x,z)}{2}$ for all $x,y,z \in X$ and $\lambda > 0$, which is the famous Hero formula.

In 2013, the concept of modular G-metric spaces were introduced by Azadifar *et al.* in [14]. They obtained some fixed point theorems of contractive mappings defined on modular G-metric spaces. The existence of fixed point of contractive mapping defined on modular G-metric spaces was proved, where the completeness is replaced with weaker conditions.

Azadifar *et al.* [15], used the theory they developed in [14] to prove the existence and uniqueness of common fixed point of a pair of weakly compatible mappings satisfying the Φ-map in modular G-metric spaces.

Very recently, Okeke and Francis [53] proved the existence and uniqueness of a fixed point of mappings satisfying Geraghty-type contractions in the setting of preordered modular G-metric spaces. The results were applied in solving nonlinear Volterra–Fredholm-type integral equations.

In the present paper, we introduce the concept of Δ_3-type condition in modular G-metric spaces. We will prove the existence and uniqueness of fixed point of some generalized contractible operators defined on modular G-metric spaces satisfying a Δ_3-type condition and also prove the modular G-continuity of such operators in modular G-metric spaces. Our results extend, generalize, complement and include several known results as special cases, including the results of [14] and [15].

2 Preliminaries

Here we shall define the modular G-metric space following Azadifar *et al.* [14].

Definition 2.1 ([14]) Let X be a nonempty set, and let $\omega^G : (0,\infty) \times X \times X \times X \to [0,\infty]$ be a function satisfying;

(1) $\omega^G_\lambda(x,y,z) = 0$ for all $x,y,z \in X$ and $\lambda > 0$ if $x = y = z$,

(2) $\omega^G_\lambda(x,x,y) > 0$ for all $x,y \in X$ and $\lambda > 0$ with $x \neq y$,

(3) $\omega^G_\lambda(x,x,y) \leq \omega^G_\lambda(x,y,z)$ for all $x,y,z \in X$ and $\lambda > 0$ with $z \neq y$,

(4) $\omega^G_\lambda(x,y,z) = \omega^G_\lambda(x,z,y) = \omega^G_\lambda(y,z,x) = \cdots$ for all $\lambda > 0$ (symmetry in all three variables),

(5) $\omega^G_{\lambda+\mu}(x,y,z) \leq \omega^G_\lambda(x,a,a) + \omega^G_\mu(a,y,z)$, for all $x,y,z,a \in X$ and $\lambda, \nu > 0$,

then the function $\omega_\lambda^G(\cdot,\cdot)$ is called a modular G-metric on X and the pair (X,ω^G) is called a modular G-metric space.

Definition 2.2 Let X be a nonempty set, and let $\omega^G : (0,\infty)\times X\times X\times X \to [0,\infty]$ is said to be convex modular (G-metric) on a set X if it satisfies the conditions (1) and (4) of the Definition 2.1 as well as the axiom $\omega^G_{\lambda+\mu}(x,y,z) \leq \frac{\lambda}{\lambda+\mu}\omega^G_\lambda(x,a,a) + \frac{\mu}{\lambda+\mu}\omega^G_\mu(a,y,z)$, for all $\lambda > 0$, $\mu > 0$ and $x,y,z,a \in X_{\omega^G}$.

Definition 2.3 Let X be a nonempty set, and let $\omega^G_\lambda : (0,\infty)\times X\times X\times X \to [0,\infty]$ be a function, then it is non-symmetric if ω^G_λ is not symmetric, for all $\lambda > 0$.

Remark 2.1
(a) If $x = a$, then condition (5) of Definition 2.1 above becomes
$\omega^G_{\lambda+\mu}(a,y,z) \leq \omega^G_\mu(a,y,z)$.
(b) Condition (5) of Definition 2.1 is called the rectangle inequality.
In this paper, we will take X_{ω^G} to be a modular G-metric space.

Definition 2.4 ([14]) Let (X,ω^G) be a modular G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_{ω^G} is modular G-convergent to x, if it converges to x in the topology $\tau(\omega^G_\lambda)$.

A function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called modular G-continuous if $\omega^G_\lambda(x_n,x,x) \to 0$ then $\omega^G_\lambda(Tx_n,Tx,Tx) \to 0$, for all $\lambda > 0$.

Remark 2.2 We see that a function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called modular G-continuous if $\omega^G_\lambda(x,x_n,x_n) \to 0$ then $\omega^G_\lambda(Tx,Tx_n,Tx_n) \to 0$, for all $\lambda > 0$.

Definition 2.5 ([14]) Let (X,ω^G) be a modular G-metric space, then the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ is said to be modular G-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega^G_\lambda(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \geq n_\epsilon$ and $\lambda > 0$.

A modular G-metric space X_{ω^G} is said to be modular G-complete if every modular G-Cauchy sequence in X_{ω^G} is modular G-convergent in X_{ω^G}.

Definition 2.6 ([15]) Let g and h be single-valued self-mappings on a set X. If $w = gx = hx$ for some $x \in X$, then x is called a coincidence point of g and h, and w is called a point of coincidence of g and h.

Definition 2.7 ([15]) A pair of maps g and h is called weakly compatible pair if they commute at coincidence point.

Definition 2.8 ([31]) Let T, S be two self-mappings on a nonempty set X. Then:
(i) $x \in X$ is called a fixed point of T if $Tx = x$.
(ii) $x \in X$ is called a coincidence point of T and S if $Tx = Sx$.
(iii) $x \in X$ is called a common fixed point of T and S if $Tx = Sx = x$.
(iv) $x \in X$ is called a commuting point of T, L if $TLx = LTx$.

Following [4], we give the definition of being modular G-bounded as follows.

Definition 2.9 We say that $M \subseteq X_{\omega^G}$ is G-bounded provided that $\delta_{\omega^G}(M) = \sup\{\omega^G_\lambda(a,b,c); a,b,c \in M < \infty\}$.

Proposition 2.1 ([15]) *Let g and h be weakly compatible self-mappings on a set X. If g and h have a unique point of coincidence $w = gx = hx$, then w is the unique common fixed point of g and h.*

Remark 2.3 Observe that, if $\lim_{n\to\infty} \omega_\alpha^G(x_n, x_n, x) = 0$, for some $\alpha > 0$, then $\lim_{n\to\infty} \omega_\alpha^G(x_n, x_n, x) = 0$ may not necessarily hold for all $\alpha > 0$.

Definition 2.10 Let (X, ω^G) be a modular G-metric space, we say that ω satisfy the Δ_3-condition if $\lim_{n\to\infty} \omega_\alpha^G(x_n, x_n, x) = 0$, for some $\alpha > 0$ implies that $\lim_{n\to\infty} \omega_\alpha^G(x_n, x_n, x) = 0$ for all $\alpha > 0$, $x_n \subseteq X_{\omega^G}$.

We next introduce the Δ_3-type condition which will play a crucial role in the proofs of our results in this paper.

Definition 2.11 Let (X, ω^G) be a modular G-metric space, for all $\lambda > 0$, we say that ω satisfies a Δ_3-type condition if for $\alpha > 0$, there exists $C_\alpha > 0$ such that $\omega_{\frac{\lambda}{\alpha}}^G(x, y, z) \le C_\alpha \omega_\lambda^G(x, y, z)$, for all $x, y, z \in X$ and for any $\lambda > 0$ and x is distinct from y, z.

Remark 2.4 If $\alpha = 2$, then $\omega_{\frac{\lambda}{2}}^G(x, y, z) \le C_2 \omega_\lambda^G(x, y, z)$, for some $C_2 > 0$.

Proposition 2.2 ([14]) *Let (X, ω^G) be a modular G-metric space, for any $x, y, z, a \in X$, it follows that*:

(1) *If $\omega_\lambda^G(x, y, z) = 0$ for all $\lambda > 0$, then $x = y = z$.*
(2) $\omega_\lambda^G(x, y, z) \le \omega_{\frac{\lambda}{2}}^G(x, x, y) + \omega_{\frac{\lambda}{2}}^G(x, x, z)$ *for all $\lambda > 0$.*
(3) $\omega_\lambda^G(x, y, y) \le 2\omega_{\frac{\lambda}{2}}^G(x, x, y)$ *for all $\lambda > 0$.*
(4) $\omega_\lambda^G(x, y, z) \le \omega_{\frac{\lambda}{2}}^G(x, a, z) + \omega_{\frac{\lambda}{2}}^G(a, y, z)$ *for all $\lambda > 0$.*
(5) $\omega_\lambda^G(x, y, z) \le \frac{2}{3}(\omega_{\frac{\lambda}{2}}^G(x, y, a) + \omega_{\frac{\lambda}{2}}^G(x, a, z) + \omega_{\frac{\lambda}{2}}^G(a, y, z))$ *for all $\lambda > 0$.*
(6) $\omega_\lambda^G(x, y, z) \le \omega_{\frac{\lambda}{2}}^G(x, a, a) + \omega_{\frac{\lambda}{4}}^G(y, a, a) + \omega_{\frac{\lambda}{4}}^G(z, a, a)$ *for all $\lambda > 0$.*

Proposition 2.3 ([14]) *Let (X, ω^G) be a modular G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then the following are equivalent*:

(1) *$\{x_n\}_{n\in\mathbb{N}}$ is ω^G-convergent to x,*
(2) *$\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$, i.e. $\{x_n\}_{n\in\mathbb{N}}$ converges to x relative to the modular metric ω_λ,*
(3) *$\omega_\lambda^G(x_n, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$,*
(4) *$\omega_\lambda^G(x_n, x, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$,*
(5) *$\omega_\lambda^G(x_m, x_n, x) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$.*

3 Main results

We state our main results as follows.

Theorem 3.1 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists $\lambda > 0$ and $\rho \in (0, 1)$ such that*

$$\omega_\lambda^G(Tx, Ty, Tz) \le \rho K_\lambda(x, y, z) \quad \textit{for all } x, y, z \in X_{\omega^G}, \lambda > 0, \tag{3.1}$$

where

$$K_\lambda(x,y,z) = \max\left\{\begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,z,Ty), \omega_\lambda^G(Tx,y,z), \omega_\lambda^G(x,Tx,y), \omega_\lambda^G(y,z,Ty), \\ \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G(x,Ty,Ty), \dfrac{\omega_\lambda^G(y,z,Ty)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)}, \\ \omega_\lambda^G(x,Tx,z), \omega_\lambda^G(y,Ty,Ty), \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)}, \omega_\lambda^G(y,Tx,Ty), \\ \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,Tx,z)}, \omega_\lambda^G\bigl(y,T^2x,Ty\bigr), \\ \omega_\lambda^G(y,Tz,Tz), \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)}, \omega_\lambda^G(z,Tx,Tx), \\ \omega_\lambda^G(z,Tz,Tz), \omega_\lambda^G(z,Tx,Ty), \omega_\lambda^G\bigl(z,T^2x,Tz\bigr), \omega_\lambda^G\bigl(Tx,T^2x,Ty\bigr), \\ \omega_\lambda^G\bigl(Tx,T^2x,Tz\bigr), \dfrac{\omega_\lambda^G(x,z,Ty)[1+\omega_\lambda^G(x,z,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)}, \\ \dfrac{\omega_\lambda^G(T^2x,Ty,Tz)[1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(Tx,y,Tz)]}{1+\omega_\lambda^G(Tx,y,Tz)+\omega_\lambda^G(T^2x,Ty,Tz)} \end{array}\right\}.$$

Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u).

Proof If $x=y=z\in X_{\omega^G}$, then, for $\lambda>0$, $\omega_\lambda^G(Tx,Tx,Tx)\leq \rho K_\lambda(x,x,x)$. But $K_\lambda(x,x,x)\geq 0$ implies that $0\leq \rho K_\lambda(x,x,x)$, so that $K_\lambda(x,x,x)\geq 0$ and $\rho\neq 0$. Suppose that $K_\lambda(x,x,x)=0$, then there is nothing to prove because, $Tx=x$. In fact, $\omega_\lambda^G(x,x,x)=0$. This is condition (1) of Definition 2.1, for $\lambda>0$, so that

$$K_\lambda(x,x,x) = \max\left\{\begin{array}{l} \omega_\lambda^G(x,x,x), \omega_\lambda^G(x,x,Tx), \omega_\lambda^G(Tx,x,x), \omega_\lambda^G(x,Tx,x), \omega_\lambda^G(x,x,Tx), \\ \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G(x,Tx,Tx), \dfrac{\omega_\lambda^G(x,x,Tx)[1+\omega_\lambda^G(x,Tx,x)]}{1+\omega_\lambda^G(x,x,x)}, \\ \omega_\lambda^G(x,Tx,x), \omega_\lambda^G(x,Tx,Tx), \dfrac{\omega_\lambda^G(x,Tx,x)[1+\omega_\lambda^G(x,Tx,x)]}{1+\omega_\lambda^G(x,x,x)}, \\ \omega_\lambda^G(x,Tx,Tx), \dfrac{\omega_\lambda^G(x,Tx,x)[1+\omega_\lambda^G(x,x,x)]}{1+\omega_\lambda^G(x,x,x)+\omega_\lambda^G(x,Tx,x)}, \omega_\lambda^G\bigl(x,T^2x,Tx\bigr), \\ \omega_\lambda^G(x,Tx,Tx), \dfrac{\omega_\lambda^G(x,Tx,x)[1+\omega_\lambda^G(x,x,Tx)]}{1+\omega_\lambda^G(Tx,x,x)+\omega_\lambda^G(x,x,Tx)}, \omega_\lambda^G(x,Tx,Tx), \\ \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G\bigl(x,T^2x,Tx\bigr), \\ \omega_\lambda^G\bigl(Tx,T^2x,Tx\bigr), \omega_\lambda^G\bigl(Tx,T^2x,Tx\bigr), \dfrac{\omega_\lambda^G(x,x,Tx)[1+\omega_\lambda^G(x,x,Tx)]}{1+\omega_\lambda^G(Tx,x,x)+\omega_\lambda^G(x,x,Tx)}, \\ \dfrac{\omega_\lambda^G(T^2x,Tx,Tx)[1+\omega_\lambda^G(Tx,x,x)+\omega_\lambda^G(Tx,x,Tx)]}{1+\omega_\lambda^G(Tx,x,Tx)+\omega_\lambda^G(T^2x,Tx,Tx)} \end{array}\right\}.$$

Now, we assume that $K_\lambda(x,x,x) \geq 0$ and $x \neq y = z$. Let $x_0 \in X_{\omega^G}$ be arbitrary. We generate the sequence of iteration of T based on $x_0 \in X_{\omega^G}$ as follows:

$$\begin{aligned} Tx_0 &= x_1 \\ Tx_1 &= x_2 \\ &\vdots \\ Tx_n &= x_{n+1} \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_0 is a fixed point of T. Now for all $n \in \mathbb{N}$, $x_{n+1} \neq x_n$ and $\lambda > 0$, take $x = x_n$ and $y = x_{n+1} = z$, then we have $\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) = \omega_\lambda^G(Tx, Ty, Tz) = \omega_\lambda^G(Tx_n, Tx_{n+1}, Tx_{n+1})$, so that inequality (3.1) becomes

$$\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \rho K_\lambda(x_n, x_{n+1}, x_{n+1}) \quad \forall x_n, x_{n+1} \in X_{\omega^G}, \lambda > 0, \tag{3.3}$$

where

$$\begin{aligned} &K_\lambda(x_n, x_{n+1}, x_{n+1}) \\ &= \max \left\{ \begin{aligned} &\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, Tx_{n+1}), \omega_\lambda^G(Tx_n, x_{n+1}, x_{n+1}), \\ &\omega_\lambda^G(x_n, Tx_n, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, Tx_{n+1}), \omega_\lambda^G(x_n, Tx_n, Tx_n), \\ &\omega_\lambda^G(x_n, Tx_{n+1}, Tx_{n+1}), \frac{\omega_\lambda^G(x_{n+1}, x_{n+1}, Tx_{n+1})[1 + \omega_\lambda^G(x_n, Tx_n, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})}, \\ &\omega_\lambda^G(x_n, Tx_n, x_{n+1}), \omega_\lambda^G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ &\frac{\omega_\lambda^G(x_n, Tx_{n+1}, x_{n+1})[1 + \omega_\lambda^G(x_n, Tx_n, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})}, \\ &\omega_\lambda^G(x_{n+1}, Tx_n, Tx_{n+1}), \frac{\omega_\lambda^G(x_n, Tx_{n+1}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+1}, Tx_n, x_{n+1})}, \\ &\omega_\lambda^G\bigl(x_{n+1}, T^2x_n, Tx_{n+1}\bigr), \omega_\lambda^G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ &\frac{\omega_\lambda^G(x_n, Tx_{n+1}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, Tx_{n+1})]}{1 + \omega_\lambda^G(Tx_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_n, x_{n+1}, Tx_{n+1})}, \\ &\omega_\lambda^G(x_{n+1}, Tx_n, Tx_n), \omega_\lambda^G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ &\omega_\lambda^G(x_{n+1}, Tx_n, Tx_{n+1}), \omega_\lambda^G\bigl(x_{n+1}, T^2x_n, Tx_{n+1}\bigr), \omega_\lambda^G\bigl(Tx_n, T^2x_n, Tx_{n+1}\bigr), \\ &\omega_\lambda^G\bigl(Tx_n, T^2x_n, Tx_{n+1}\bigr), \frac{\omega_\lambda^G(x_n, x_{n+1}, Tx_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, Tx_{n+1})]}{1 + \omega_\lambda^G(Tx_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_n, x_{n+1}, Tx_{n+1})}, \\ &\frac{\omega_\lambda^G(T^2x_n, Tx_{n+1}, Tx_{n+1})[1 + \omega_\lambda^G(Tx_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(Tx_n, x_{n+1}, Tx_{n+1})]}{1 + \omega_\lambda^G(Tx_n, x_{n+1}, Tx_{n+1}) + \omega_\lambda^G(T^2x_n, Tx_{n+1}, Tx_{n+1})} \end{aligned} \right\} \end{aligned}$$

$$= \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+2}), \dfrac{\omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})}, \\ \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \dfrac{\omega_\lambda^G(x_n, x_{n+2}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})}, \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \\ \dfrac{\omega_\lambda^G(x_n, x_{n+2}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1})}, \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \dfrac{\omega_\lambda^G(x_n, x_{n+2}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})}, \\ \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \dfrac{\omega_\lambda^G(x_n, x_{n+1}, x_{n+2})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})}, \\ \dfrac{\omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2})[1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}) + \omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2})} \end{array} \right\}.$$

On simplifying, we get

$$\begin{aligned} & K_\lambda(x_n, x_{n+1}, x_{n+1}) \\ &= \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+2}), \omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2}) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \end{array} \right\}. \end{aligned}$$

We will examine this in five different cases as follows.

Case 1. If $K_\lambda(x_n, x_{n+1}, x_{n+1}) = \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})$, then

$$\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \le \rho\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \tag{3.4}$$

for $\rho \in (0, 1)$.

Case 2. If $K_\lambda(x_n,x_{n+1},x_{n+1}) = \omega_\lambda^G(x_n,x_{n+1},x_{n+2})$, then using conditions (1), (4) of Definition 2.1, conditions (3), (6) of Proposition 2.2 and Definition 2.11 twice, we get

$$\begin{aligned}
\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) &\leq \rho\omega_\lambda^G(x_n,x_{n+1},x_{n+2}) \\
&\leq \rho\big[\omega_{\frac{\lambda}{2}}^G(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_{n+1},x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_{n+1},x_{n+1},x_{n+2})\big] \\
&= \rho\big[\omega_{\frac{\lambda}{2}}^G(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_{n+1},x_{n+1},x_{n+2})\big] \\
&\leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + C_4\omega_\lambda^G(x_{n+1},x_{n+1},x_{n+2})\big) \\
&\leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 2C_4\omega_{\frac{\lambda}{2}}^G(x_{n+1},x_{n+2},x_{n+2})\big) \\
&\leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 2C_2C_4\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2})\big),
\end{aligned}$$

which implies that

$$\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 2C_2C_4\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2})\big),$$

hence, we have

$$\begin{aligned}
&\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \leq \frac{\rho C_2}{1-2\rho C_2C_4}\omega_\lambda^G(x_n,x_{n+1},x_{n+1}), \\
&\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \leq k_j\omega_\lambda^G(x_n,x_{n+1},x_{n+1}),
\end{aligned} \tag{3.5}$$

where $k_j = \frac{\rho C_2}{1-j\rho C_2C_4} < 1$, so that $C_2C_4 \in (0,\frac{1}{j\rho})$, where $j = 2$.

Case 3. If $K_\lambda(x_n,x_{n+1},x_{n+1}) = \omega_\lambda^G(x_n,x_{n+2},x_{n+2})$, then using conditions (3), (6) of Proposition 2.2 and Definition 2.11, we have

$$\begin{aligned}
\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) &\leq \rho\omega_\lambda^G(x_n,x_{n+2},x_{n+2}) \\
&\leq \rho\big[\omega_{\frac{\lambda}{2}}^G(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_{n+2},x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_{n+2},x_{n+1},x_{n+1})\big] \\
&= \rho\big(\omega_{\frac{\lambda}{2}}^G(x_n,x_{n+1},x_{n+1}) + 2\omega_{\frac{\lambda}{4}}^G(x_{n+1},x_{n+1},x_{n+2})\big) \\
&\leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 2C_4\omega_\lambda^G(x_{n+1},x_{n+1},x_{n+2})\big) \\
&\leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 4C_4\omega_{\frac{\lambda}{2}}^G(x_{n+1},x_{n+2},x_{n+2})\big) \\
&\leq \rho\big(C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 4C_2C_4\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2})\big),
\end{aligned}$$

which implies that

$$\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \leq \frac{\rho C_2}{1-4\rho C_2C_4}\omega_\lambda^G(x_n,x_{n+1},x_{n+1}),$$

hence, we have

$$\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \leq k_j\omega_\lambda^G(x_n,x_{n+1},x_{n+1}), \tag{3.6}$$

where $k_j = \frac{\rho C_2}{1-j\rho C_2C_4} < 1$, so that $C_2C_4 \in (0,\frac{1}{j\rho})$, where $j = 4$.

Case 4. If $K_\lambda(x_n,x_{n+1},x_{n+1}) = \omega_\lambda^G(x_{n+1},x_{n+1},x_{n+2})$, then using condition (6) of Proposition 2.2 and Definition 2.11, inequality (3.3) becomes

$$\begin{aligned}
&\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \\
&\quad\le \rho\bigl(\omega_{\frac{\lambda}{2}}^G(x_n,x_n,x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_n,x_n,x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_n,x_n,x_{n+2})\bigr) \\
&\quad\le \rho\bigl(C_2\omega_\lambda^G(x_{n+1},x_n,x_n) + C_4\omega_\lambda^G(x_{n+1},x_n,x_n) + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad= \rho\bigl((C_4+C_2)\omega_\lambda^G(x_{n+1},x_n,x_n) + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad\le \rho\bigl((C_4+C_2)\bigl[\omega_{\frac{\lambda}{2}}^G(x_{n+1},x_{n+2},x_{n+2}) + \omega_{\frac{\lambda}{4}}^G(x_n,x_{n+2},x_{n+2}) \\
&\qquad + \omega_{\frac{\lambda}{4}}^G(x_n,x_{n+2},x_{n+2})\bigr] + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad\le \rho\bigl((C_2+C_4)\bigl(C_2\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + 2C_4\omega_\lambda^G(x_n,x_{n+2},x_{n+2})\bigr) \\
&\qquad + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad\le \rho\bigl((C_2+C_4)\bigl(C_2\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + 2C_4\bigl(\omega_{\frac{\lambda}{2}}^G(x_n,x_n,x_{n+2}) \\
&\qquad + \omega_{\frac{\lambda}{2}}^G(x_n,x_n,x_{n+2})\bigr)\bigr) + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad= \rho\bigl((C_2+C_4)\bigl(C_2\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + 2C_4\bigl(2\omega_{\frac{\lambda}{2}}^G(x_n,x_n,x_{n+2})\bigr)\bigr) \\
&\qquad + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad\le \rho\bigl((C_2+C_4)\bigl(C_2\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + 2C_4\bigl(2C_2\omega_\lambda^G(x_n,x_n,x_{n+2})\bigr)\bigr) \\
&\qquad + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad= \rho\bigl((C_2+C_4)\bigl(C_2\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + 4C_4C_2\omega_\lambda^G(x_n,x_n,x_{n+2})\bigr) \\
&\qquad + C_4\omega_\lambda^G(x_{n+2},x_n,x_n)\bigr) \\
&\quad= \rho\bigl(C_2(C_2+C_4)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + C_4\bigl(1+4C_2(C_2+C_4)\bigr)\omega_\lambda^G(x_n,x_n,x_{n+2})\bigr) \\
&\quad\le \rho\bigl(C_2(C_2+C_4)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + C_4\bigl(1+4C_2(C_2+C_4)\bigr) \\
&\qquad \times \bigl[\omega_{\frac{\lambda}{2}}^G(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{4}}^G(x_{n+1},x_{n+1},x_{n+2})\bigr]\bigr) \\
&\quad\le \rho\bigl(C_2(C_2+C_4)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + C_4\bigl(1+4C_2(C_2+C_4)\bigr) \\
&\qquad \times \bigl[C_2\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + C_4\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + C_4\omega_\lambda^G(x_{n+1},x_{n+1},x_{n+2})\bigr]\bigr) \\
&\quad= \rho\bigl(C_2(C_2+C_4)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + C_4\bigl(1+4C_2(C_2+C_4)\bigr) \\
&\qquad \times \bigl[(C_2+C_4)\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + C_4\omega_\lambda^G(x_{n+1},x_{n+1},x_{n+2})\bigr]\bigr) \\
&\quad\le \rho\bigl(C_2(C_2+C_4)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + C_4\bigl(1+4C_2(C_2+C_4)\bigr) \\
&\qquad \times \bigl[(C_2+C_4)\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 2C_4\omega_{\frac{\lambda}{2}}^G(x_{n+1},x_{n+2},x_{n+2})\bigr]\bigr) \\
&\quad\le \rho\bigl(C_2(C_2+C_4)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) + C_4\bigl(1+4C_2(C_2+C_4)\bigr) \\
&\qquad \times \bigl[(C_2+C_4)\omega_\lambda^G(x_n,x_{n+1},x_{n+1}) + 2C_4C_2\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2})\bigr]\bigr) \\
&\quad= \rho\bigl(C_2(C_2+C_4) + 2C_4^2C_2\bigl(1+4C_2(C_2+C_4)\bigr)\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \\
&\qquad + C_4(C_2+C_4)\bigl(1+4C_2(C_2+C_4)\bigr)\omega_\lambda^G(x_n,x_{n+1},x_{n+1})\bigr).
\end{aligned} \tag{3.7}$$

Therefore, it follows from inequality (3.7) that

$$\omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2}) \leq k_3 \omega_{\lambda}^{G}(x_n,x_{n+1},x_{n+1}), \tag{3.8}$$

where $k_3 := \frac{\rho C_4(C_2+C_4)(1+4C_2(C_2+C_4))}{1-j\rho(C_2(C_2+C_4)+2C_4^2C_2(1+4C_2(C_2+C_4)))} < 1$ and $(C_2(C_2+C_4)+2C_4^2C_2(1+4C_2(C_2+C_4))) \in (0,\frac{1}{j\rho})$ for $j=1$.

Case 5. If $K_{\lambda}(x_n,x_{n+1},x_{n+1}) = \omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2})$, then inequality (3.3) becomes

$$\omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2}) \leq \rho \omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2}), \tag{3.9}$$

and $\rho \in (0,1)$.

Now, we take $h = \max\{\rho,k_j,k_3\}$, for $j=1,2,4,\ldots$, so that

$$\begin{aligned}\omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2}) &\leq h\omega_{\lambda}^{G}(x_n,x_{n+1},x_{n+1})\\ &\leq h^2\omega_{\lambda}^{G}(x_{n-1},x_n,x_n)\\ &\leq h^3\omega_{\lambda}^{G}(x_{n-2},x_{n-1},x_{n-1})\\ &\ \ \vdots\\ &\leq h^n\omega_{\lambda}^{G}(x_0,x_1,x_1).\end{aligned} \tag{3.10}$$

But $\sum_{n\in\mathbb{N}} h^n < +\infty$. Now $\sum_{n\in\mathbb{N}} \omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2}) \leq \omega_{\lambda}^{G}(x_0,x_1,x_1)\sum_{n\in\mathbb{N}} h^n < +\infty$ for all $\lambda > 0$. Suppose that $m,n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Observe that, for any arbitrary ϵ, using the rectangle inequality repeatedly and condition (2) of Proposition 2.2, we have

$$\begin{aligned}\omega_{\lambda}^{G}(x_n,x_m,x_m) &\leq \omega_{\frac{\lambda}{m-n}}^{G}(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{m-n}}^{G}(x_{n+1},x_{n+2},x_{n+2}) + \omega_{\frac{\lambda}{m-n}}^{G}(x_{n+2},x_{n+3},x_{n+3})\\ &\quad + \omega_{\frac{\lambda}{m-n}}^{G}(x_{n+3},x_{n+4},x_{n+4}) + \cdots + \omega_{\frac{\lambda}{m-n}}^{G}(x_{m-1},x_m,x_m)\\ &\leq \omega_{\frac{\lambda}{m}}^{G}(x_n,x_{n+1},x_{n+1}) + \omega_{\frac{\lambda}{m}}^{G}(x_{n+1},x_{n+2},x_{n+2}) + \omega_{\frac{\lambda}{m}}^{G}(x_{n+2},x_{n+3},x_{n+3})\\ &\quad + \omega_{\frac{\lambda}{m}}^{G}(x_{n+3},x_{n+4},x_{n+4}) + \cdots + \omega_{\frac{\lambda}{m}}^{G}(x_{m-1},x_m,x_m)\\ &\leq \sum_{n=N}\omega_{\lambda}^{G}(x_{n+1},x_{n+2},x_{n+2})\\ &< \epsilon,\end{aligned} \tag{3.11}$$

for all $m > n \geq N$ for some $N \in \mathbb{N}$. As ϵ is arbitrary, we have

$$\omega_{\lambda}^{G}(x_n,x_m,x_m) = 0 \quad \text{as } n,m \to \infty \quad \text{or} \quad \lim_{n,m\to\infty}\omega_{\lambda}^{G}(x_n,x_m,x_m) = 0. \tag{3.12}$$

For $n,m,k \in \mathbb{N}$, condition (2) of Proposition 2.2 implies that

$$\omega_{\lambda}^{G}(x_n,x_m,x_k) \leq \omega_{\frac{\lambda}{2}}^{G}(x_n,x_m,x_m) + \omega_{\frac{\lambda}{2}}^{G}(x_k,x_m,x_m), \tag{3.13}$$

so that on taking the limit of both sides of inequality (3.13) as $n, m, l \to \infty$ and by applying Definition 2.11 and Eq. (3.12), we get

$$\begin{aligned}\lim_{n,m,k\to\infty} \omega^G_\lambda(x_n, x_m, x_k) &\le \lim_{n,m\to\infty} \omega^G_{\frac{\lambda}{2}}(x_n, x_m, x_m) + \lim_{k,m\to\infty} \omega^G_{\frac{\lambda}{2}}(x_k, x_m, x_m)\\ &\le C_2 \lim_{n,m\to\infty} \omega^G_\lambda(x_n, x_m, x_m) + C_2 \lim_{k,m\to\infty} \omega^G_\lambda(x_k, x_m, x_m)\\ &= C_2\Big(\lim_{n,m\to\infty} \omega^G_\lambda(x_n, x_m, x_m) + \lim_{k,m\to\infty} \omega^G_\lambda(x_k, x_m, x_m)\Big); \end{aligned} \tag{3.14}$$

thus we have

$$\lim_{n,m,k\to\infty} \omega^G_\lambda(x_n, x_m, x_k) = 0 \quad \forall\lambda > 0. \tag{3.15}$$

Equation (3.15) confirmed that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence. The completeness of (X, ω^G) implies that, for any $\lambda > 0$, $\lim_{n,m\to\infty} \omega^G_\lambda(x_n, x_m, u) = 0$, i.e. for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega^G_\lambda(x_n, x_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \ge n_0$, which implies that $\lim_{n\to\infty} x_n \to u$. Suppose, if possible, that $Tu \ne u$, i.e. $\omega^G_\lambda(u, Tu, Tu) > 0$, then from inequality (3.1), with $x = x_n$, $y = u = z$, we have

$$\omega^G_\lambda(x_{n+1}, Tu, Tu) = \omega^G_\lambda(Tx_n, Tu, Tu) \le \rho K_\lambda(x_n, u, u) \quad \text{for all } x_n, u, \in X_{\omega^G}, \lambda > 0, \tag{3.16}$$

so that

$$\begin{aligned} &K_\lambda(x_n, u, u)\\ &= \max\left\{\begin{array}{l} \omega^G_\lambda(x_n, u, u), \omega^G_\lambda(x_n, u, Tu), \omega^G_\lambda(Tx_n, u, u), \omega^G_\lambda(x_n, Tx_n, u),\\ \omega^G_\lambda(u, u, Tu), \omega^G_\lambda(x_n, Tx_n, Tx_n), \omega^G_\lambda(x_n, Tu, Tu),\\ \dfrac{\omega^G_\lambda(u, u, Tu)[1 + \omega^G_\lambda(x_n, Tx_n, u)]}{1 + \omega^G_\lambda(x_n, u, u)}, \omega^G_\lambda(x_n, Tx_n, u), \omega^G_\lambda(u, Tu, Tu),\\ \dfrac{\omega^G_\lambda(x_n, Tu, u)[1 + \omega^G_\lambda(x_n, Tx_n, u)]}{1 + \omega^G_\lambda(x_n, u, u)}, \omega^G_\lambda(u, Tx_n, Tu),\\ \dfrac{\omega^G_\lambda(x_n, Tu, u)[1 + \omega^G_\lambda(x_n, u, u)]}{1 + \omega^G_\lambda(x_n, u, u) + \omega^G_\lambda(u, Tx_n, u)}, \omega^G_\lambda\big(u, T^2x_n, Tu\big),\\ \omega^G_\lambda(u, Tu, Tu), \dfrac{\omega^G_\lambda(x_n, Tu, u)[1 + \omega^G_\lambda(x_n, u, Tu)]}{1 + \omega^G_\lambda(Tx_n, u, u) + \omega^G_\lambda(x_n, u, Tu)}, \omega^G_\lambda(u, Tx_n, Tx_n),\\ \omega^G_\lambda(u, Tu, Tu), \omega^G_\lambda(u, Tx_n, Tu), \omega^G_\lambda\big(u, T^2x_n, Tu\big), \omega^G_\lambda\big(Tx_n, T^2x_n, Tu\big),\\ \omega^G_\lambda\big(Tx_n, T^2x_n, Tu\big), \dfrac{\omega^G_\lambda(x_n, u, Tu)[1 + \omega^G_\lambda(x_n, u, Tu)]}{1 + \omega^G_\lambda(Tx_n, u, u) + \omega^G_\lambda(x_n, u, Tu)},\\ \dfrac{\omega^G_\lambda(T^2x_n, Tu, Tu)[1 + \omega^G_\lambda(Tx_n, u, u) + \omega^G_\lambda(Tx_n, u, Tu)]}{1 + \omega^G_\lambda(Tx_n, u, Tu) + \omega^G_\lambda(T^2x_n, Tu, Tu)} \end{array}\right\}. \end{aligned}$$

Thus

$$
\begin{aligned}
&K_\lambda(x_n,u,u)\\
&= \max\left\{\begin{array}{l}
\omega_\lambda^G(x_n,u,u), \omega_\lambda^G(x_n,u,Tu), \omega_\lambda^G(x_{n+1},u,u), \omega_\lambda^G(x_n,x_{n+1},u),\\
\omega_\lambda^G(u,u,Tu), \omega_\lambda^G(x_n,x_{n+1},x_{n+1}), \omega_\lambda^G(x_n,Tu,Tu),\\
\dfrac{\omega_\lambda^G(u,u,Tu)[1+\omega_\lambda^G(x_n,x_{n+1},u)]}{1+\omega_\lambda^G(x_n,u,u)}, \omega_\lambda^G(x_n,x_{n+1},u), \omega_\lambda^G(u,Tu,Tu),\\
\dfrac{\omega_\lambda^G(x_n,Tu,u)[1+\omega_\lambda^G(x_n,x_{n+1},u)]}{1+\omega_\lambda^G(x_n,u,u)}, \omega_\lambda^G(u,x_{n+1},Tu),\\
\dfrac{\omega_\lambda^G(x_n,Tu,u)[1+\omega_\lambda^G(x_n,u,u)]}{1+\omega_\lambda^G(x_n,u,u)+\omega_\lambda^G(u,x_{n+1},u)}, \omega_\lambda^G(u,x_{n+2},Tu), \omega_\lambda^G(u,Tu,Tu),\\
\dfrac{\omega_\lambda^G(x_n,Tu,u)[1+\omega_\lambda^G(x_n,u,Tu)]}{1+\omega_\lambda^G(x_{n+1},u,u)+\omega_\lambda^G(x_n,u,Tu)}, \omega_\lambda^G(u,x_{n+1},x_{n+1}),\\
\omega_\lambda^G(u,Tu,Tu), \omega_\lambda^G(u,x_{n+1},Tu), \omega_\lambda^G(u,x_{n+2},Tu), \omega_\lambda^G(x_{n+1},x_{n+2},Tu),\\
\omega_\lambda^G(x_{n+1},x_{n+2},Tu), \dfrac{\omega_\lambda^G(x_n,u,Tu)[1+\omega_\lambda^G(x_n,u,Tu)]}{1+\omega_\lambda^G(x_{n+1},u,u)+\omega_\lambda^G(x_n,u,Tu)},\\
\dfrac{\omega_\lambda^G(x_{n+2},Tu,Tu)[1+\omega_\lambda^G(x_{n+1},u,u)+\omega_\lambda^G(x_{n+1},u,Tu)]}{1+\omega_\lambda^G(x_{n+1},u,Tu)+\omega_\lambda^G(x_{n+2},Tu,Tu)}
\end{array}\right\}.
\end{aligned}
$$

Since $\lim_{n\to\infty} x_n \to u \in X_\omega$,

$$
\begin{aligned}
&K_\lambda(u,u,u)\\
&= \max\left\{\begin{array}{l}
\omega_\lambda^G(u,u,u), \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,u,u),\\
\omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,Tu,Tu), \dfrac{\omega_\lambda^G(u,u,Tu)[1+\omega_\lambda^G(u,u,u)]}{1+\omega_\lambda^G(u,u,u)},\\
\omega_\lambda^G(u,u,u), \omega_\lambda^G(u,Tu,Tu), \dfrac{\omega_\lambda^G(u,Tu,u)[1+\omega_\lambda^G(u,u,u)]}{1+\omega_\lambda^G(u,u,u)},\\
\omega_\lambda^G(u,u,Tu), \dfrac{\omega_\lambda^G(u,Tu,u)[1+\omega_\lambda^G(u,u,u)]}{1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,u)}, \omega_\lambda^G(u,u,Tu),\\
\omega_\lambda^G(u,Tu,Tu), \dfrac{\omega_\lambda^G(u,Tu,u)[1+\omega_\lambda^G(u,u,Tu)]}{1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,Tu)}, \omega_\lambda^G(u,u,u),\\
\omega_\lambda^G(u,Tu,Tu), \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,Tu),\\
\dfrac{\omega_\lambda^G(u,u,Tu)[1+\omega_\lambda^G(u,u,Tu)]}{1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,Tu)},\\
\dfrac{\omega_\lambda^G(u,Tu,Tu)[1+\omega_\lambda^G(u,u,u)+\omega_\lambda^G(u,u,Tu)]}{1+\omega_\lambda^G(u,u,Tu)+\omega_\lambda^G(u,Tu,Tu)}
\end{array}\right\},
\end{aligned}
$$

which gives

$$K_\lambda(u,u,u) = \max\left\{\begin{array}{l} \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,u,Tu), \\ \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,u,u), \\ \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,u), \\ \omega_\lambda^G(u,Tu,Tu), \omega_\lambda^G(u,u,Tu), \\ \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,Tu,Tu), \\ \omega_\lambda^G(u,Tu,u), \omega_\lambda^G(u,u,Tu), \\ \omega_\lambda^G(u,Tu,u), \omega_\lambda^G(u,u,Tu), \\ \omega_\lambda^G(u,Tu,Tu), \omega_\lambda^G(u,Tu,u), \\ \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,Tu,Tu), \\ \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,Tu), \\ \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,u,Tu), \\ \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,Tu,Tu) \end{array}\right\},$$

for which by condition (4) of Definition 2.1, we have

$$K_\lambda(u,u,u) = \max\bigl\{\omega_\lambda^G(u,u,u), \omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,Tu,Tu)\bigr\} \tag{3.17}$$

or

$$K_\lambda(u,u,u) = \max\bigl\{\omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,Tu,Tu)\bigr\}. \tag{3.18}$$

The modular G-metric space is G-continuous in each variable for $\lambda > 0$ and all the elements of $K_\lambda(x_n,u,u)$ converge to $\omega_\lambda^G(u,u,u)$, $\omega_\lambda^G(u,u,Tu)$ and $\omega_\lambda^G(u,Tu,Tu)$ for $\lambda > 0$. By condition (4) of Proposition 2.2 and Definition 2.11, we have $\omega_\lambda^G(u,u,Tu) \leq 2\omega_{\frac{\lambda}{2}}^G(u,Tu,Tu) \leq 2C_2\omega_\lambda^G(u,Tu,Tu)$ by Definition 2.11, inequality (3.16) gives

$$\begin{aligned} \omega_\lambda^G(u,Tu,Tu) &\leq \rho\max\bigl\{\omega_\lambda^G(u,u,Tu), \omega_\lambda^G(u,Tu,Tu)\bigr\} \\ &\leq 2\rho\omega_{\frac{\lambda}{2}}^G(u,Tu,Tu) \\ &\leq 2C_2\rho\omega_\lambda^G(u,Tu,Tu), \quad C_2 \in \left(0, \frac{1}{2\rho}\right), \end{aligned} \tag{3.19}$$

which implies that $\omega_\lambda^G(u,Tu,Tu) \leq 0$ for all $\lambda > 0$, a contradiction. Hence, $Tu = u$.

We now show that T has a unique fixed point. Suppose that there exists $v \in X_{\omega^G}$ such that $Tv = v$ is another fixed point of T in X_{ω^G}, so that $u \neq v$; that is $\omega_\lambda^G(u,v,v) > 0$. Indeed suppose, if possible, otherwise, that, for all $\lambda > 0$, from inequality (3.1), we have

$$\omega_\lambda^G(u,v,v) = \omega_\lambda^G(Tu,Tv,Tv) \leq \rho K_\lambda(u,v,v) \quad \text{for all } u,v \in X_{\omega^G}, \lambda > 0, \tag{3.20}$$

where

$$
\begin{aligned}
&K_\lambda(u,v,v)\\
&= \max\left\{
\begin{aligned}
&\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,v,Tv),\omega_\lambda^G(Tu,v,v),\omega_\lambda^G(u,Tu,v),\omega_\lambda^G(v,v,Tv),\\
&\omega_\lambda^G(u,Tu,Tu),\omega_\lambda^G(u,Tv,Tv),\frac{\omega_\lambda^G(v,v,Tv)[1+\omega_\lambda^G(u,Tu,v)]}{1+\omega_\lambda^G(u,v,v)},\\
&\omega_\lambda^G(u,Tu,v),\omega_\lambda^G(v,Tv,Tv),\frac{\omega_\lambda^G(u,Tv,v)[1+\omega_\lambda^G(u,Tu,v)]}{1+\omega_\lambda^G(u,v,v)},\\
&\omega_\lambda^G(v,Tu,Tv),\frac{\omega_\lambda^G(u,Tv,v)[1+\omega_\lambda^G(u,v,v)]}{1+\omega_\lambda^G(u,v,v)+\omega_\lambda^G(v,Tu,v)},\omega_\lambda^G\big(v,T^2u,Tv\big),\\
&\omega_\lambda^G(v,Tv,Tv),\frac{\omega_\lambda^G(u,Tv,v)[1+\omega_\lambda^G(u,v,Tv)]}{1+\omega_\lambda^G(Tu,v,v)+\omega_\lambda^G(u,v,Tv)},\omega_\lambda^G(v,Tu,Tu),\\
&\omega_\lambda^G(v,Tv,Tv),\omega_\lambda^G(v,Tu,Tv),\omega_\lambda^G\big(v,T^2u,Tv\big),\omega_\lambda^G\big(Tu,T^2u,Tv\big),\\
&\omega_\lambda^G\big(Tu,T^2u,Tv\big),\frac{\omega_\lambda^G(u,v,Tv)[1+\omega_\lambda^G(u,v,Tv)]}{1+\omega_\lambda^G(Tu,v,v)+\omega_\lambda^G(u,v,Tv)},\\
&\frac{\omega_\lambda^G(T^2u,Tv,Tv)[1+\omega_\lambda^G(Tu,v,v)+\omega_\lambda^G(Tu,v,Tv)]}{1+\omega_\lambda^G(Tu,v,Tv)+\omega_\lambda^G(T^2u,Tv,Tv)}
\end{aligned}
\right\}\\
&= \max\left\{
\begin{aligned}
&\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,u,v),\omega_\lambda^G(v,v,v),\omega_\lambda^G(u,u,u),\\
&\omega_\lambda^G(u,v,v),\frac{\omega_\lambda^G(v,v,v)[1+\omega_\lambda^G(u,u,v)]}{1+\omega_\lambda^G(u,v,v)},\omega_\lambda^G(u,u,v),\omega_\lambda^G(v,v,v),\\
&\frac{\omega_\lambda^G(u,v,v)[1+\omega_\lambda^G(u,u,v)]}{1+\omega_\lambda^G(u,v,v)},\omega_\lambda^G(v,u,v),\frac{\omega_\lambda^G(u,v,v)[1+\omega_\lambda^G(u,v,v)]}{1+\omega_\lambda^G(u,v,v)+\omega_\lambda^G(v,u,v)},\\
&\omega_\lambda^G(v,u,v),\omega_\lambda^G(v,v,v),\frac{\omega_\lambda^G(u,v,v)[1+\omega_\lambda^G(u,v,v)]}{1+\omega_\lambda^G(u,v,v)+\omega_\lambda^G(u,v,v)},\omega_\lambda^G(v,u,u),\\
&\omega_\lambda^G(v,v,v),\omega_\lambda^G(v,u,v),\omega_\lambda^G(v,u,v),\omega_\lambda^G(u,u,v),\omega_\lambda^G(u,u,v),\\
&\frac{\omega_\lambda^G(u,v,v)[1+\omega_\lambda^G(u,v,v)]}{1+\omega_\lambda^G(u,v,v)+\omega_\lambda^G(u,v,v)},\frac{\omega_\lambda^G(u,v,v)[1+\omega_\lambda^G(u,v,v)+\omega_\lambda^G(u,v,v)]}{1+\omega_\lambda^G(u,v,v)+\omega_\lambda^G(u,v,v)}
\end{aligned}
\right\}.
\end{aligned}
$$

Using condition (4) of Definition 2.1, we clearly see that

$$
\begin{aligned}
K_\lambda(u,v,v) &= \max\left\{
\begin{aligned}
&\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,u,v),\omega_\lambda^G(v,v,v),\\
&\omega_\lambda^G(u,u,u),\omega_\lambda^G(u,v,v),\omega_\lambda^G(v,v,v),\omega_\lambda^G(u,u,v),\omega_\lambda^G(v,v,v),\\
&\omega_\lambda^G(u,v,v),\omega_\lambda^G(v,u,v),\omega_\lambda^G(u,v,v),\omega_\lambda^G(v,u,v),\omega_\lambda^G(v,v,v),\\
&\omega_\lambda^G(u,v,v),\omega_\lambda^G(v,u,u),\omega_\lambda^G(v,v,v),\omega_\lambda^G(v,u,v),\omega_\lambda^G(v,u,v),\\
&\omega_\lambda^G(u,u,v),\omega_\lambda^G(u,u,v),\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,v,v)
\end{aligned}
\right\}\\
&= \max\big\{\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,u,v)\big\}.
\end{aligned}
$$

Thus from inequality (3.20)

$$
\omega_\lambda^G(u,v,v)\le\rho\max\big\{\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,u,v)\big\}. \tag{3.21}
$$

Case 1. If $\max\{\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,u,v)\}$ is $\omega_\lambda^G(u,v,v)$, then inequality (3.21) becomes

$$\omega_\lambda^G(u,v,v)\le\rho\omega_\lambda^G(u,v,v), \tag{3.22}$$

which implies that $\omega_\lambda^G(u,v,v)\le 0$, $\rho\in(0,1)$.

Case 2. If $\max\{\omega_\lambda^G(u,v,v),\omega_\lambda^G(u,u,v)\}$ is $\omega_\lambda^G(u,u,v)$, then using condition (3) of Proposition 2.2 and Definition 2.11, inequality (3.21) becomes

$$\begin{aligned}\omega_\lambda^G(u,v,v)&\le\rho\omega_\lambda^G(u,u,v)\\&\le 2\rho\omega_{\frac{\lambda}{2}}^G(u,v,v)\\&\le 2C_2\rho\omega_\lambda^G(u,v,v),\end{aligned}\tag{3.23}$$

which implies that $\omega_\lambda^G(u,v,v)\le 0$, $C_2\in(0,\frac{1}{2\rho})$. Case 1 and Case 2 contradict our initial claim that $\omega_\lambda^G(u,v,v)>0$ for all $\lambda>0$ hence T has a unique fixed point, i.e. $u=v$.

To see that T is modular G-continuous at u, let $\{x_n\}_{n\in\mathbb{N}}\subseteq X_{\omega^G}$ be a sequence such that $x_n\to u$, then, by taking $x=u$, $y=x_n=z$, inequality (3.1) becomes

$$\omega_\lambda^G(Tu,Tx_n,Tx_n)\le\rho K_\lambda(u,x_n,x_n)\quad\Longrightarrow\quad\omega_\lambda^G(u,Tx_n,Tx_n)\le\rho K_\lambda(u,x_n,x_n), \tag{3.24}$$

where

$$\begin{aligned}&K_\lambda(u,x_n,x_n)\\&=\max\left\{\begin{aligned}&\omega_\lambda^G(u,x_n,x_n),\omega_\lambda^G(u,x_n,Tx_n),\omega_\lambda^G(Tu,x_n,x_n),\omega_\lambda^G(u,Tu,x_n),\omega_\lambda^G(x_n,x_n,Tx_n),\\&\omega_\lambda^G(u,Tu,Tu),\omega_\lambda^G(u,Tx_n,Tx_n),\frac{\omega_\lambda^G(x_n,x_n,Tx_n)[1+\omega_\lambda^G(u,Tu,x_n)]}{1+\omega_\lambda^G(u,x_n,x_n)},\\&\omega_\lambda^G(u,Tu,x_n),\omega_\lambda^G(x_n,Tx_n,Tx_n),\frac{\omega_\lambda^G(u,Tx_n,x_n)[1+\omega_\lambda^G(u,Tu,x_n)]}{1+\omega_\lambda^G(u,x_n,x_n)},\\&\omega_\lambda^G(x_n,Tu,Tx_n),\frac{\omega_\lambda^G(u,Tx_n,x_n)[1+\omega_\lambda^G(u,x_n,x_n)]}{1+\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(x_n,Tu,x_n)},\\&\omega_\lambda^G\big(x_n,T^2u,Tx_n\big),\omega_\lambda^G(x_n,Tx_n,Tx_n),\frac{\omega_\lambda^G(u,Tx_n,x_n)[1+\omega_\lambda^G(u,x_n,Tx_n)]}{1+\omega_\lambda^G(Tu,x_n,x_n)+\omega_\lambda^G(u,x_n,Tx_n)},\\&\omega_\lambda^G(x_n,Tu,Tu),\omega_\lambda^G(x_n,Tx_n,Tx_n),\omega_\lambda^G(x_n,Tu,Tx_n),\omega_\lambda^G\big(x_n,T^2u,Tx_n\big),\\&\omega_\lambda^G\big(Tu,T^2u,Tx_n\big),\omega_\lambda^G\big(Tu,T^2u,Tx_n\big),\frac{\omega_\lambda^G(u,x_n,Tx_n)[1+\omega_\lambda^G(u,x_n,Tx_n)]}{1+\omega_\lambda^G(Tu,x_n,x_n)+\omega_\lambda^G(u,x_n,Tx_n)},\\&\frac{\omega_\lambda^G(T^2u,Tx_n,Tx_n)[1+\omega_\lambda^G(Tu,x_n,x_n)+\omega_\lambda^G(Tu,x_n,Tx_n)]}{1+\omega_\lambda^G(Tu,x_n,Tx_n)+\omega_\lambda^G(T^2u,Tx_n,Tx_n)}\end{aligned}\right\}\end{aligned}$$

$$= \max \left\{ \begin{array}{l} \omega_\lambda^G(u,x_n,x_n), \omega_\lambda^G(u,x_n,Tx_n), \omega_\lambda^G(u,x_n,x_n), \omega_\lambda^G(u,u,x_n), \omega_\lambda^G(x_n,x_n,Tx_n), \\ \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,Tx_n,Tx_n), \dfrac{\omega_\lambda^G(x_n,x_n,Tx_n)[1+\omega_\lambda^G(u,u,x_n)]}{1+\omega_\lambda^G(u,x_n,x_n)}, \\ \omega_\lambda^G(u,u,x_n), \omega_\lambda^G(x_n,Tx_n,Tx_n), \dfrac{\omega_\lambda^G(u,Tx_n,x_n)[1+\omega_\lambda^G(u,u,x_n)]}{1+\omega_\lambda^G(u,x_n,x_n)}, \\ \omega_\lambda^G(x_n,u,Tx_n), \dfrac{\omega_\lambda^G(u,Tx_n,x_n)[1+\omega_\lambda^G(u,x_n,x_n)]}{1+\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(x_n,u,x_n)}, \\ \omega_\lambda^G(x_n,u,Tx_n), \omega_\lambda^G(x_n,Tx_n,Tx_n), \dfrac{\omega_\lambda^G(u,Tx_n,x_n)[1+\omega_\lambda^G(u,x_n,Tx_n)]}{1+\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(u,x_n,Tx_n)}, \\ \omega_\lambda^G(x_n,u,u), \omega_\lambda^G(x_n,Tx_n,Tx_n), \omega_\lambda^G(x_n,u,Tx_n), \omega_\lambda^G(x_n,u,Tx_n), \\ \omega_\lambda^G(u,u,Tx_n), \omega_\lambda^G(u,u,Tx_n), \dfrac{\omega_\lambda^G(u,x_n,Tx_n)[1+\omega_\lambda^G(u,x_n,Tx_n)]}{1+\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(u,x_n,Tx_n)}, \\ \dfrac{\omega_\lambda^G(u,Tx_n,Tx_n)[1+\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(u,x_n,Tx_n)]}{1+\omega_\lambda^G(u,x_n,Tx_n)+\omega_\lambda^G(u,Tx_n,Tx_n)} \end{array} \right\}.$$

Since $Tu = u \implies T^2u = T(Tu) = Tu = u$ and $\frac{\omega_\lambda^G(u,u,x_n)}{\omega_\lambda^G(u,x_n,x_n)} \leq 1$, we have

$$K_\lambda(u,x_n,x_n) = \max \left\{ \begin{array}{l} \omega_\lambda^G(u,x_n,x_n), \omega_\lambda^G(u,x_n,Tx_n), \omega_\lambda^G(u,x_n,x_n), \\ \omega_\lambda^G(u,u,x_n), \omega_\lambda^G(x_n,x_n,Tx_n), \omega_\lambda^G(u,u,u), \\ \omega_\lambda^G(u,Tx_n,Tx_n), \omega_\lambda^G(x_n,x_n,Tx_n), \omega_\lambda^G(u,u,x_n), \\ \omega_\lambda^G(x_n,Tx_n,Tx_n), \omega_\lambda^G(u,Tx_n,x_n), \omega_\lambda^G(x_n,u,Tx_n), \\ \omega_\lambda^G(u,Tx_n,x_n), \omega_\lambda^G(x_n,u,Tx_n), \omega_\lambda^G(x_n,Tx_n,Tx_n), \\ \omega_\lambda^G(u,Tx_n,x_n), \omega_\lambda^G(x_n,u,u), \omega_\lambda^G(x_n,Tx_n,Tx_n), \\ \omega_\lambda^G(x_n,u,Tx_n), \omega_\lambda^G(x_n,u,Tx_n), \omega_\lambda^G(u,u,Tx_n), \\ \omega_\lambda^G(u,u,Tx_n), \omega_\lambda^G(u,x_n,Tx_n), \omega_\lambda^G(u,Tx_n,Tx_n) \end{array} \right\}.$$

So,

$$K_\lambda(u,x_n,x_n) = \max \left\{ \begin{array}{l} \omega_\lambda^G(u,x_n,x_n), \omega_\lambda^G(u,x_n,Tx_n), \\ \omega_\lambda^G(u,u,x_n), \omega_\lambda^G(x_n,x_n,Tx_n), \\ \omega_\lambda^G(u,u,u), \omega_\lambda^G(u,Tx_n,Tx_n), \\ \omega_\lambda^G(x_n,Tx_n,Tx_n), \omega_\lambda^G(u,u,Tx_n) \end{array} \right\},$$

$$K_\lambda(u,x_n,x_n) = \max \left\{ \begin{array}{l} \omega_\lambda^G(u,x_n,x_n), \omega_\lambda^G(u,x_n,Tx_n), \omega_\lambda^G(u,u,x_n), \omega_\lambda^G(x_n,x_n,Tx_n), \\ \omega_\lambda^G(u,Tx_n,Tx_n), \omega_\lambda^G(x_n,Tx_n,Tx_n), \omega_\lambda^G(u,u,Tx_n) \end{array} \right\}.$$

Now we consider the following cases.

Case 1. If $K_\lambda(u,x_n,x_n)=\omega^G_\lambda(u,x_n,x_n)$, then inequality (3.24) becomes

$$\omega^G_\lambda(u,Tx_n,Tx_n)\le\rho\omega^G_\lambda(u,x_n,x_n), \tag{3.25}$$

for $\rho\in(0,1)$.

Case 2. If $K_\lambda(u,x_n,x_n)=\omega^G_\lambda(u,x_n,Tx_n)$, using conditions (1), (5) of Definition 2.1, conditions (6), (3) of Proposition 2.2 and Definition 2.11, then inequality (3.24) becomes

$$\begin{aligned}
\omega^G_\lambda(u,Tx_n,Tx_n)&\le\rho\omega^G_\lambda(u,x_n,Tx_n)\\
&\le\rho\bigl\{\omega^G_{\frac{\lambda}{2}}(u,x_n,x_n)+\omega^G_{\frac{\lambda}{4}}(x_n,x_n,x_n)+\omega^G_{\frac{\lambda}{4}}(x_n,x_n,Tx_n)\bigr\}\\
&=\rho\bigl\{\omega^G_{\frac{\lambda}{2}}(u,x_n,x_n)+\omega^G_{\frac{\lambda}{4}}(x_n,x_n,Tx_n)\bigr\}\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+C_4\omega^G_\lambda(x_n,x_n,Tx_n)\bigr\}\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+2C_4\omega^G_{\frac{\lambda}{2}}(x_n,Tx_n,Tx_n)\bigr\}\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+2C_2C_4\omega^G_\lambda(x_n,Tx_n,Tx_n)\bigr\}\\
&\quad\text{but } \omega^G_\lambda(x_n,Tx_n,Tx_n)\le\omega^G_{\frac{\lambda}{2}}(x_n,u,u)+\omega^G_{\frac{\lambda}{2}}(u,Tx_n,Tx_n)\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+2C_2C_4\bigl(\omega^G_{\frac{\lambda}{2}}(x_n,u,u)+\omega^G_{\frac{\lambda}{2}}(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+2C_2C_4\bigl(C_2\omega^G_\lambda(x_n,u,u)+C_2\omega^G_\lambda(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+2C_2^2C_4\bigl(2\omega^G_{\frac{\lambda}{2}}(u,x_n,x_n)+\omega^G_\lambda(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\le\rho\bigl\{C_2\omega^G_\lambda(u,x_n,x_n)+2C_2^2C_4\bigl(2C_2\omega^G_\lambda(u,x_n,x_n)+\omega^G_\lambda(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&=\rho\bigl(C_2\omega^G_\lambda(u,x_n,x_n)+4C_2^3C_4\omega^G_\lambda(u,x_n,x_n)+2C_2^2C_4\omega^G_\lambda(u,Tx_n,Tx_n)\bigr)\\
&=\rho\bigl(C_2\bigl(1+4C_2^2C_4\bigr)\omega^G_\lambda(u,x_n,x_n)+2C_2^2C_4\omega^G_\lambda(u,Tx_n,Tx_n)\bigr),
\end{aligned}$$

so that

$$\omega^G_\lambda(u,Tx_n,Tx_n)\le\frac{\rho C_2(1+4C_2^2C_4)}{1-2\rho C_2^2C_4}\omega^G_\lambda(u,x_n,x_n), \tag{3.26}$$

therefore, we have

$$\omega^G_\lambda(u,Tx_n,Tx_n)\le\theta_j\omega^G_\lambda(u,x_n,x_n), \tag{3.27}$$

where $\theta_j=\frac{\rho C_2(1+4C_2^2C_4)}{1-j\rho C_2^2C_4}<1$ and $C_2^2C_4\in(0,\frac{1}{j\rho})$, where $j=2$.

Case 3. If $K_\lambda(u,x_n,x_n)=\omega^G_\lambda(u,u,x_n)$, by condition (3) of Proposition 2.2 and Definition 2.11, then inequality (3.24) becomes

$$\begin{aligned}
\omega^G_\lambda(u,Tx_n,Tx_n)&\le\rho\omega^G_\lambda(u,u,x_n)\\
&\le 2\rho\omega^G_{\frac{\lambda}{2}}(u,x_n,x_n)
\end{aligned}$$

$$\begin{aligned}&\leq 2\rho C_2\omega_\lambda^G(u,x_n,x_n)\\&\leq \delta_j\omega_\lambda^G(u,x_n,x_n),\end{aligned}\tag{3.28}$$

where $\delta_j = j\rho C_2 < 1$ and $C_2 \in (0, \frac{1}{j\rho})$ for $j = 2$.

Case 4. If $K_\lambda(u,x_n,x_n) = \omega_\lambda^G(x_n,x_n,Tx_n)$, using condition (3) of Proposition 2.2, condition (5) of Definition 2.1 and Definition 2.11, then inequality (3.24) becomes

$$\begin{aligned}\omega_\lambda^G(u,Tx_n,Tx_n) &\leq \rho\omega_\lambda^G(x_n,x_n,Tx_n)\\&\leq 2\rho\omega_{\frac{\lambda}{2}}^G(x_n,Tx_n,Tx_n)\\&\leq 2C_2\rho\omega_\lambda^G(x_n,Tx_n,Tx_n)\\&\leq 2C_2\rho\big(\omega_{\frac{\lambda}{2}}^G(x_n,u,u)+\omega_{\frac{\lambda}{2}}^G(u,Tx_n,Tx_n)\big)\\&\leq 2C_2^2\rho\big(\omega_\lambda^G(x_n,u,u)+\omega_\lambda^G(u,Tx_n,Tx_n)\big)\\&\leq 2C_2^2\rho\big(2\omega_{\frac{\lambda}{2}}^G(u,x_n,x_n)+\omega_\lambda^G(u,Tx_n,Tx_n)\big)\\&\leq 2C_2^2\rho\big(2C_2\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(u,Tx_n,Tx_n)\big),\end{aligned}$$

so that

$$\omega_\lambda^G(u,Tx_n,Tx_n)\leq \beta_j\omega_\lambda^G(u,x_n,x_n),\tag{3.29}$$

where $\beta_j = \frac{4\rho C_2^3}{1-j\rho C_2^2} < 1$ and $C_2^2 \in (0, \frac{1}{j\rho})$ for $j = 2$.

Case 5. If $K_\lambda(u,x_n,x_n) = \omega_\lambda^G(u,Tx_n,Tx_n)$, using condition (6) of Proposition 2.2 and Case 4, then inequality (3.24) becomes

$$\begin{aligned}\omega_\lambda^G(u,Tx_n,Tx_n) &\leq \rho\omega_\lambda^G(u,Tx_n,Tx_n)\\&\leq \rho\big(\omega_{\frac{\lambda}{2}}^G(u,x_n,x_n)+\omega_{\frac{\lambda}{4}}^G(x_n,x_n,Tx_n)+\omega_{\frac{\lambda}{4}}^G(x_n,x_n,Tx_n)\big)\\&= \rho\big(\omega_{\frac{\lambda}{2}}^G(u,x_n,x_n)+2\omega_{\frac{\lambda}{4}}^G(x_n,x_n,Tx_n)\big)\\&\leq \rho\big(C_2\omega_\lambda^G(u,x_n,x_n)+2C_4\omega_\lambda^G(x_n,x_n,Tx_n)\big)\\&\leq \rho\big(C_2\omega_\lambda^G(u,x_n,x_n)+6C_4C_2^3\omega_\lambda^G(u,x_n,x_n)\\&\quad+4C_4C_2^2\omega_\lambda^G(x_n,Tx_n,Tx_n)\big).\end{aligned}\tag{3.30}$$

Therefore, from inequality (3.30), we get

$$\omega_\lambda^G(x_n,Tx_n,Tx_n)\leq \frac{\rho C_2(1+6C_4C_2^2)}{1-4\rho C_4C_2^2}\omega_\lambda^G(u,x_n,x_n),\tag{3.31}$$

so that $C_2^2C_4 \in (0, \frac{1}{j\rho})$, where $j = 4$.

Case 6. If $K_\lambda(u,x_n,x_n) = \omega_\lambda^G(x_n,Tx_n,Tx_n)$, using property (5) of Definition 2.1, condition (3) of Proposition 2.2 and Definition 2.11, then inequality (3.24) becomes

$$\begin{aligned}\omega_\lambda^G(u,Tx_n,Tx_n) &\leq \rho\omega_\lambda^G(x_n,Tx_n,Tx_n)\\&\leq \rho\big(\omega_{\frac{\lambda}{2}}^G(x_n,u,u)+\omega_{\frac{\lambda}{2}}^G(u,Tx_n,Tx_n)\big)\end{aligned}$$

$$\begin{aligned}
&\leq \rho C_2\bigl(\omega_\lambda^G(x_n,u,u)+\omega_\lambda^G(u,Tx_n,Tx_n)\bigr)\\
&\leq \rho C_2\bigl(2\omega_{\frac{\lambda}{2}}^G(u,x_n,x_n)+\omega_\lambda^G(u,Tx_n,Tx_n)\bigr)\\
&\leq \rho C_2\bigl(2C_2\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(u,Tx_n,Tx_n)\bigr),
\end{aligned}$$

so that

$$\omega_\lambda^G(u,Tx_n,Tx_n)\leq \frac{2\rho C_2^2}{1-\rho C_2}\omega_\lambda^G(u,x_n,x_n),\quad \text{for some } C_2>0, \tag{3.32}$$

so that $C_2\in(0,\frac{1}{j\rho})$, where $j=1$.

Case 7. If $K_\lambda(u,x_n,x_n)=\omega_\lambda^G(u,u,Tx_n)$, using conditions (6), (3) of Proposition 2.2 and Definition 2.11, condition (5) of Definition 2.1, then inequality (3.24) becomes

$$\begin{aligned}
&\omega_\lambda^G(u,Tx_n,Tx_n)\\
&\quad\leq \rho\omega_\lambda^G(u,u,Tx_n)\\
&\quad\leq \rho\bigl\{\omega_{\frac{\lambda}{2}}^G(u,x_n,x_n)+\omega_{\frac{\lambda}{4}}^G(u,x_n,x_n)+\omega_{\frac{\lambda}{4}}^G(x_n,x_n,Tx_n)\bigr\}\\
&\quad\leq \rho\bigl\{C_2\omega_\lambda^G(u,x_n,x_n)+C_4\omega_\lambda^G(u,x_n,x_n)+C_4\omega_\lambda^G(x_n,x_n,Tx_n)\bigr\}\\
&\quad= \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+C_4\omega_\lambda^G(x_n,x_n,Tx_n)\bigr\}\\
&\quad\leq \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+2C_4\omega_{\frac{\lambda}{2}}^G(x_n,Tx_n,Tx_n)\bigr\}\\
&\quad\leq \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+2C_2C_4\omega_\lambda^G(x_n,Tx_n,Tx_n)\bigr\}\\
&\quad\leq \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+2C_2C_4\bigl(\omega_{\frac{\lambda}{2}}^G(x_n,u,u)+\omega_{\frac{\lambda}{2}}^G(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\quad\leq \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+2C_2^2C_4\bigl(\omega_\lambda^G(x_n,u,u)+\omega_\lambda^G(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\quad\leq \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+2C_2^2C_4\bigl(2\omega_{\frac{\lambda}{2}}^G(u,x_n,x_n)+\omega_\lambda^G(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\quad\leq \rho\bigl\{(C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+2C_2^2C_4\bigl(2C_2\omega_\lambda^G(u,x_n,x_n)+\omega_\lambda^G(u,Tx_n,Tx_n)\bigr)\bigr\}\\
&\quad= \rho\bigl((C_2+C_4)\omega_\lambda^G(u,x_n,x_n)+4C_2^3C_4\omega_\lambda^G(u,x_n,x_n)+2C_2^2C_4\omega_\lambda^G(u,Tx_n,Tx_n)\bigr)\\
&\quad= \rho\bigl((C_2+C_4+4C_2^3C_4)\omega_\lambda^G(u,x_n,x_n)+2C_2^2C_4\omega_\lambda^G(u,Tx_n,Tx_n)\bigr),
\end{aligned} \tag{3.33}$$

so that inequality (3.33) becomes

$$\omega_\lambda^G(u,Tx_n,Tx_n)\leq \eta_j\omega_\lambda^G(u,x_n,x_n), \tag{3.34}$$

where $\eta_j=\frac{\rho(C_2+C_4(1+4C_2^3))}{1-2\rho C_2^2C_4}<1$, and $C_2^2C_4\in(0,\frac{1}{j\rho})$, where $j=2$. Since $x_n\to u$, $n\to\infty$, in all the cases, we have $Tx_n=u=Tu$ as $n\to\infty$, showing that T is modular G-continuous at the fixed point u. □

Remark 3.1 If the statement of Theorem 3.1 holds without non-symmetric condition and the maximum in inequality (3.1) is $\omega_\lambda^G(x,y,z)$, then inequality (3.1) becomes

$$\omega_\lambda^G(Tx,Ty,Tz)\leq \rho\omega_\lambda^G(x,y,z),\quad \forall x,y,z\in X_{\omega^G},\rho\in(0,1),\lambda>0. \tag{3.35}$$

This generalized Theorem 3.2 in [35].

Remark 3.2 If the statement of Theorem 3.1 hold without Δ_3-type conditions and the maximum in inequality (3.1) becomes

$$\omega_\lambda^G(Tx,Ty,Tz) \le \rho K_\lambda(x,y,z) \quad \text{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.36}$$

where

$$K_\lambda(x,y,z) = \max\left\{\begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,Tx,y), \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G(x,Ty,Ty), \omega_\lambda^G(x,Tx,z), \\ \omega_\lambda^G(y,Ty,Ty), \omega_\lambda^G(y,Tx,Ty), \omega_\lambda^G(y,T^2x,Ty), \omega_\lambda^G(y,Tz,Tz), \\ \omega_\lambda^G(z,Tx,Tx), \omega_\lambda^G(z,Tz,Tz), \omega_\lambda^G(z,Tx,Ty), \omega_\lambda^G(z,T^2x,Tz), \\ \omega_\lambda^G(Tx,T^2x,Ty), \omega_\lambda^G(Tx,T^2x,Tz) \end{array}\right\}.$$

Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u).This is Theorem 3.1 in [30], Theorem 2.1 in [12] in the setting of modular G-metric space.

Remark 3.3 Remark 3.2 can be extended a little as follows: If the statement of Theorem 3.1 holds without Δ_3-type conditions and the maximum in inequality (3.1) becomes

$$\omega_\lambda^G(Tx,Ty,Tz) \le \rho K_\lambda(x,y,z) \quad \text{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.37}$$

where

$$K_\lambda(x,y,z) = \max\left\{\begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,z,Ty), \omega_\lambda^G(Tx,y,z), \\ \omega_\lambda^G(x,Tx,y), \omega_\lambda^G(y,z,Ty), \omega_\lambda^G(x,Tx,Tx), \\ \omega_\lambda^G(x,Ty,Ty), \omega_\lambda^G(x,Tx,z), \omega_\lambda^G(y,Ty,Ty), \\ \omega_\lambda^G(y,Tx,Ty), \omega_\lambda^G(y,T^2x,Ty), \omega_\lambda^G(y,Tz,Tz), \\ \omega_\lambda^G(z,Tx,Tx), \omega_\lambda^G(z,Tz,Tz), \omega_\lambda^G(z,Tx,Ty), \\ \omega_\lambda^G(z,T^2x,Tz), \omega_\lambda^G(Tx,T^2x,Ty), \omega_\lambda^G(Tx,T^2x,Tz) \end{array}\right\},$$

then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u). This is an extension of Theorem 3.1 in [30] in the setting of modular G-metric space.

We give the following examples to support Theorem 3.1.

Example 3.1 Suppose that $X = [0,1] \cup \{\infty\} \subseteq \bar{\mathbb{R}}$ with the modular G-metric for $\lambda > 0$ defined by $\omega_\lambda^G(x,y,z) = \frac{G(x,y,z)}{\lambda}$ so that $G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$ for all $x,y,z \in X_{\omega^G}$. Define a self map $T : X_{\omega^G} \to X_{\omega^G}$ by $Tx = \frac{x^2}{2}$, $\forall x \in X_{\omega^G}$, then:

(i) (X,ω^G) is a G-complete modular G-metric space. It suffices to show that any arbitrary sequence we pick in modular G-metric space, i.e. $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ will be a modular G-Cauchy sequence; that is if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega^G_\lambda(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \geq n_\epsilon$ and $\lambda > 0$. Recall that a modular G-metric space X_{ω^G} is said to be G-complete if every G-Cauchy sequence in X_{ω^G} is G-convergent in X_{ω^G}. Indeed, let (X,ω^G) be a modular G-metric space and let $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ be a sequence that converges to $u \in X_{\omega^G}$. Let $\epsilon > 0$ be arbitrary. By definition of the modular G-metric and for all $\lambda > 0$, there exists $n_\epsilon \in \mathbb{N}$, $C_2 > 0$ such that $\omega^G_\lambda(x_n,x_m,u) < \frac{\epsilon}{3C_2}$ for all $n,m \geq n_\epsilon$ and$\omega^G_\lambda(x_n,x_n,u) < \frac{\epsilon}{3C_2^2}$ for all $n,m \geq n_\epsilon$ for all $\lambda > 0$. Using conditions (4), (5) of Definition 2.1 and condition (3) of Proposition 2.2 and Definition 2.11 or in particular the Remark 2.4, we have, for all $n,m,l > n_\epsilon$,

$$\begin{aligned}
\omega^G_\lambda(x_n,x_m,x_l) &\leq \omega^G_{\frac{\lambda}{2}}(x_n,u,u) + \omega^G_{\frac{\lambda}{2}}(u,x_m,x_l) \\
&\leq C_2\omega^G_\lambda(x_n,u,u) + C_2\omega^G_\lambda(u,x_m,x_l) \\
&= C_2\big[\omega^G_\lambda(x_n,u,u) + \omega^G_\lambda(u,x_m,x_l)\big] \\
&\leq C_2\big[2\omega^G_{\frac{\lambda}{2}}(x_n,x_n,u) + \omega^G_\lambda(u,x_m,x_l)\big] \\
&\leq C_2\big[2C_2\omega^G_\lambda(x_n,x_n,u) + \omega^G_\lambda(u,x_m,x_l)\big] \\
&= 2C_2^2\omega^G_\lambda(x_n,x_n,u) + C_2\omega^G_\lambda(u,x_m,x_l) \\
&< 2C_2^2\frac{\epsilon}{3C_2^2} + C_2\frac{\epsilon}{3C_2} \\
&= \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{3.38}$$

Thus $\omega^G_\lambda(x_n,x_m,x_l) < \epsilon$ for all $n,m,l > n_\epsilon$ and $\lambda > 0$. Therefore, $\{x_n\}_{n\geq 1}$ is a modular G-Cauchy sequence. Let $\{x_n\}_{n\geq 1}$ be a modular G-Cauchy sequence. Since every modular G-Cauchy sequence is modular G-bounded, $\{x_n\}_{n\geq 1}$ is modular G-bounded. Since every modular G-bounded sequence has a limit point, $\{x_n\}_{n\geq 1}$ has a limit point u. Furthermore, we show that $\{x_n\}_{n\geq 1}$ converges to u. Let $\epsilon > 0$ be given. Since $\{x_n\}_{n\geq 1}$ is a modular G-Cauchy sequence, there exists a positive integer, m, and $C_2 > 0$ such that $\omega^G_\lambda(x_n,x_m,x_m) < \frac{\epsilon}{3C_2}$ $\forall n \geq m$ and for all $\lambda > 0$. Since u is a limit point of $\{x_n\}_{n\geq 1}$, every neighborhood of u contains infinitely many times of $\{x_n\}_{n\geq 1}$ which implies that $x_n \in (u - \frac{\epsilon}{3C_2}, u + \frac{\epsilon}{3C_2})$ for infinitely many values of n. In particular, we can find a positive integer $k > m$, $\lambda > 0$ and $C_4 > 0$ such that $\omega^G_\lambda(x_k,x_k,u) < \frac{\epsilon}{3C_4}$ $\forall k \geq m$ i.e. $x_k \in (u - \frac{\epsilon}{3C_4}, u + \frac{\epsilon}{3C_4})$. Also, since $k > m$, we have $\omega^G_\lambda(x_k,x_m,x_m) < \frac{\epsilon}{3C_4}$. Now by condition (6) of Proposition 2.2 and Definition 2.11 we have

$$\begin{aligned}
\omega^G_\lambda(x_n,x_n,u) &\leq \omega^G_{\frac{\lambda}{2}}(x_n,x_m,x_m) + \omega^G_{\frac{\lambda}{4}}(x_m,x_m,x_k) + \omega^G_{\frac{\lambda}{2}}(x_k,x_k,u) \\
&\leq C_2\omega^G_\lambda(x_n,x_m,x_m) + C_4\omega^G_\lambda(x_m,x_m,x_k) + C_4\omega^G_\lambda(x_k,x_k,u) \\
&< C_2\frac{\epsilon}{3C_2} + C_4\frac{\epsilon}{3C_4} + C_4\frac{\epsilon}{3C_4} \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{3.39}$$

Thus $\omega_\lambda^G(x_n, x_n, u) < \epsilon$ for all $n \geq m$ and $\lambda > 0$, showing that $\{x_n\}_{n\geq 1}$ converges to $u \in X_{\omega^G}$. Hence (X, ω^G) is a modular G-complete modular G-metric space.

(ii) T satisfies inequality (3.1) of Theorem 3.1. Indeed, take $|x| \leq \frac{1}{2}$, $|y| \leq \frac{1}{2}$ and $|z| \leq \frac{1}{2}$. By Definition of the modular G-metric, we get

$$\begin{aligned}
\omega_\lambda^G(Tx, Ty, Tz) &= \frac{1}{\lambda}\max\bigl\{|Tx - Ty|, |Ty - Tz|, |Tz - Tx|\bigr\} \quad \forall x, y, z \in X_{\omega^G} \\
&= \max\left\{\left|\frac{x^2}{2} - \frac{y^2}{2}\right|, \left|\frac{y^2}{2} - \frac{z^2}{2}\right|, \left|\frac{x^2}{2} - \frac{z^2}{2}\right|\right\} \\
&= \frac{1}{2}\max\bigl\{\left|x^2 - y^2\right|, \left|y^2 - z^2\right|, \left|x^2 - z^2\right|\bigr\} \\
&= \frac{1}{2}\max\bigl\{|x + y||x - y|, |y + z||y - z|, |x + z||x - z|\bigr\} \\
&\leq \frac{1}{2}\max\bigl\{\bigl(|x| + |y|\bigr)|x - y|, \bigl(|y| + |z|\bigr)|y - z|, \bigl(|x| + |z|\bigr)|x - z|\bigr\} \\
&\leq \frac{1}{2}\max\bigl\{|x - y|, |y - z|, |x - z|\bigr\} \\
&= \frac{1}{2}\omega_\lambda^G(x, y, z) \\
&\leq \frac{1}{2}K_\lambda(x, y, z),
\end{aligned} \tag{3.40}$$

where

$$K_\lambda(x, y, z) = \max\left\{\begin{array}{l}
\omega_\lambda^G(x, y, z), \omega_\lambda^G(x, z, Ty), \omega_\lambda^G(Tx, y, z), \omega_\lambda^G(x, Tx, y), \omega_\lambda^G(y, z, Ty), \\
\omega_\lambda^G(x, Tx, Tx), \omega_\lambda^G(x, Ty, Ty), \dfrac{\omega_\lambda^G(y, z, Ty)[1 + \omega_\lambda^G(x, Tx, y)]}{1 + \omega_\lambda^G(x, y, z)}, \\
\omega_\lambda^G(x, Tx, z), \omega_\lambda^G(y, Ty, Ty), \dfrac{\omega_\lambda^G(x, Ty, z)[1 + \omega_\lambda^G(x, Tx, y)]}{1 + \omega_\lambda^G(x, y, z)}, \\
\omega_\lambda^G(y, Tx, Ty), \dfrac{\omega_\lambda^G(x, Ty, z)[1 + \omega_\lambda^G(x, y, z)]}{1 + \omega_\lambda^G(x, y, z) + \omega_\lambda^G(y, Tx, z)}, \omega_\lambda^G\bigl(y, T^2x, Ty\bigr), \\
\omega_\lambda^G(y, Tz, Tz), \dfrac{\omega_\lambda^G(x, Ty, z)[1 + \omega_\lambda^G(x, y, Ty)]}{1 + \omega_\lambda^G(Tx, y, z) + \omega_\lambda^G(x, z, Ty)}, \omega_\lambda^G(z, Tx, Tx), \\
\omega_\lambda^G(z, Tz, Tz), \omega_\lambda^G(z, Tx, Ty), \omega_\lambda^G\bigl(z, T^2x, Tz\bigr), \omega_\lambda^G\bigl(Tx, T^2x, Ty\bigr), \\
\omega_\lambda^G\bigl(Tx, T^2x, Tz\bigr), \dfrac{\omega_\lambda^G(x, z, Ty)[1 + \omega_\lambda^G(x, z, Ty)]}{1 + \omega_\lambda^G(Tx, y, z) + \omega_\lambda^G(x, z, Ty)}, \\
\dfrac{\omega_\lambda^G(T^2x, Ty, Tz)[1 + \omega_\lambda^G(Tx, y, z) + \omega_\lambda^G(Tx, y, Tz)]}{1 + \omega_\lambda^G(Tx, y, Tz) + \omega_\lambda^G(T^2x, Ty, Tz)}
\end{array}\right\}.$$

(iii) T has a unique fixed point at $x = 0$ and 2. To see this, we know that $Tx = x$ is the fixed point of T. Therefore, we have

$$Tx = x \quad \Longrightarrow \quad Tx - x = 0$$

$$\begin{aligned} &\Longrightarrow \quad \frac{x^2}{2} - x = 0 \\ &\Longrightarrow \quad x^2 - 2x = 0 \\ &\Longrightarrow \quad x(x-2) = 0 \\ &\Longrightarrow \quad x = 0 \quad \text{or} \quad x = 2. \end{aligned} \tag{3.41}$$

To check, we get for $x = 0$, $T(0) = \frac{0^2}{2} = 0$ and for $x = 2$, $T(2) = \frac{2^2}{2} = 2$.

(iv) T is modular G-continuous at the fixed point $u \in X_{\omega^G}$. Indeed, to avoid duplication of proof, if we take a sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{x_n\}_{n\geq 1}$ converges to $u = \frac{x^2}{2}$, then it follows from inequalities (3.24) to (3.34) that

$$\omega_\lambda^G(u, Tx_n, Tx_n) \longrightarrow 0 \quad \forall \lambda > 0, \text{as } n \to \infty. \tag{3.42}$$

This shows that $Tx_n = u = Tu$ as $n \to \infty$.

Example 3.2 Let $X = \mathbb{R} \cup \{\infty\} \subseteq \bar{\mathbb{R}}$. Define $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by $G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\}$ for all distinct $x,y,z \in \mathbb{R}$ and 0 for $x = y = z$. For any $\lambda > 0$, let $\omega_\lambda^G(x,y,z) = \frac{G(x,y,z)}{\lambda}$ for all $x,y,z \in \mathbb{R}$. Define a map $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \frac{\sin^2(x)}{6}$ $\forall x \in \mathbb{R}$.

Indeed, the real line endowed with the above modular G-metric is a complete modular G-metric space and it follows directly from (i) of Example 3.1 above. T has a trivial fixed point at $x = 0$ and $x = \arcsin(6)$, i.e. we know that $Tw = w$ is the fixed point of T and take $w = \sin(x)$. Therefore, we have

$$\begin{aligned} Tw = w \quad &\Longrightarrow \quad Tw - w = 0 \\ &\Longrightarrow \quad \frac{w^2}{6} - w = 0 \\ &\Longrightarrow \quad w^2 - 6w = 0 \\ &\Longrightarrow \quad w(w-6) = 0 \\ &\Longrightarrow \quad x = 0 \quad \text{or} \quad x = \arcsin(6), \end{aligned} \tag{3.43}$$

and it is G-continuous at $u \in X_{\omega^G}$ say. To see this, if we take a sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{x_n\}_{n\geq 1}$ converges to u, then it follows from inequalities (3.24) to (3.34) that

$$\omega_\lambda^G(u, Tx_n, Tx_n) \longrightarrow 0 \quad \forall \lambda > 0, \text{as } n \to \infty. \tag{3.44}$$

This shows that $Tx_n = u = Tu$ as $n \to \infty$.

Lastly, for all $x,y,z \in \mathbb{R}$ and taking $|\sin(z)| < |z|$,

$$\begin{aligned} G(Tx, Ty, Tz) &= \max\bigl\{|Tx - Ty|, |Ty - Tz|, |Tx - Tz|\bigr\} \\ &= \frac{1}{6}\max\bigl\{\bigl|\sin^2(x) - \sin^2(y)\bigr|, \bigl|\sin^2(y) - \sin^2(z)\bigr|, \bigl|\sin^2(x) - \sin^2(z)\bigr|\bigr\} \\ &= \frac{1}{6}\max\left\{\begin{array}{l} \bigl|\sin(x+y)\sin(x-y)\bigr|, \bigl|\sin(y+z)\sin(y-z)\bigr|, \\ \bigl|\sin(x+z)\sin(x-z)\bigr| \end{array}\right\} \\ &\leq \frac{1}{6}\max\bigl\{|x-y|, |y-z|, |x-z|\bigr\}, \end{aligned}$$

$$\omega_\lambda^G(Tx, Ty, Tz) \le \frac{1}{6} K_\lambda(x, y, z),$$

where

$$K_\lambda(x,y,z) = \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,z,Ty), \omega_\lambda^G(Tx,y,z), \omega_\lambda^G(x,Tx,y), \omega_\lambda^G(y,z,Ty), \\ \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G(x,Ty,Ty), \dfrac{\omega_\lambda^G(y,z,Ty)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)}, \\ \omega_\lambda^G(x,Tx,z), \omega_\lambda^G(y,Ty,Ty), \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)}, \\ \omega_\lambda^G(y,Tx,Ty), \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,Tx,z)}, \omega_\lambda^G(y,T^2x,Ty), \\ \omega_\lambda^G(y,Tz,Tz), \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)}, \omega_\lambda^G(z,Tx,Tx), \\ \omega_\lambda^G(z,Tz,Tz), \omega_\lambda^G(z,Tx,Ty), \omega_\lambda^G(z,T^2x,Tz), \\ \omega_\lambda^G(Tx,T^2x,Ty), \omega_\lambda^G(Tx,T^2x,Tz), \dfrac{\omega_\lambda^G(x,z,Ty)[1+\omega_\lambda^G(x,z,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)}, \\ \dfrac{\omega_\lambda^G(T^2x,Ty,Tz)[1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(Tx,y,Tz)]}{1+\omega_\lambda^G(Tx,y,Tz)+\omega_\lambda^G(T^2x,Ty,Tz)} \end{array} \right\}.$$

Therefore, all the conditions of Theorem 3.1 are satisfied.

Remark 3.4 With some ω^G elements in Theorem 3.1, for $\lambda > 0$, we have

$$\omega_\lambda^G(Tx,Ty,Tz) \le k \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,Tx,Tx), \omega_\lambda^G(y,Ty,Ty), \omega_\lambda^G(z,Tz,Tz), \\ \omega_\lambda^G(x,Ty,Ty), \omega_\lambda^G(z,Ty,Ty), G(z,Tx,Tx) \end{array} \right\}, \tag{3.45}$$

where $k \in (0,1)$. Then T has a unique fixed point say u and is modular G-continuous at u. This is Theorem 2.1 in [47] in the setting of modular G-metric space.

Its variant form in modular G-metric space is of the right form, for any $\lambda > 0$, $m = 1, 2, \dots, r$, then we have

$$\omega_\lambda^G(T^m x, T^m y, T^m z) \le k \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,T^m x,T^m x), \omega_\lambda^G(y,T^m y,T^m y), \\ \omega_\lambda^G(z,T^m z,T^m z), \omega_\lambda^G(x,T^m y,T^m y), \\ \omega_\lambda^G(z,T^m y,T^m y), G(z,T^m x,T^m x) \end{array} \right\}, \tag{3.46}$$

where $k \in (0,1)$. Then T has a unique fixed point (say u) and is modular G-continuous at u. This is Corollary 2.3 in [47] in the setting of modular G-metric space.

Remark 3.5 If the statement of Theorem 3.1 holds and

$$k\max\left\{\begin{array}{l}\omega_\lambda^G(x,y,z),\omega_\lambda^G(x,z,Ty),\omega_\lambda^G(Tx,y,z),\omega_\lambda^G(x,Tx,y),\omega_\lambda^G(y,z,Ty),\omega_\lambda^G(x,Tx,Tx),\\ \omega_\lambda^G(x,Ty,Ty),\dfrac{\omega_\lambda^G(y,z,Ty)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)},\omega_\lambda^G(x,Tx,z),\omega_\lambda^G(y,Ty,Ty),\\ \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)},\omega_\lambda^G(y,Tx,Ty),\dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,Tx,z)},\\ \omega_\lambda^G(y,T^2x,Ty),\omega_\lambda^G(y,Tz,Tz),\dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)},\omega_\lambda^G(z,Tx,Tx),\\ \omega_\lambda^G(z,Tz,Tz),\omega_\lambda^G(z,Tx,Ty),\omega_\lambda^G(z,T^2x,Tz),\\ \omega_\lambda^G(Tx,T^2x,Ty),\omega_\lambda^G(Tx,T^2x,Tz),\dfrac{\omega_\lambda^G(x,z,Ty)[1+\omega_\lambda^G(x,z,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)},\\ \dfrac{\omega_\lambda^G(T^2x,Ty,Tz)[1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(Tx,y,Tz)]}{1+\omega_\lambda^G(Tx,y,Tz)+\omega_\lambda^G(T^2x,Ty,Tz)}\end{array}\right\}$$

$$= k\left\{\omega_\lambda^G(x,Tx,Tx)+\omega_\lambda^G(y,Ty,Ty)+\omega_\lambda^G(z,Tz,Tz)\right\} \tag{3.47}$$

for all $x,y,z\in M\subseteq X_{\omega^G}$. This is Theorem 3.6 of [14].

Corollary 3.2 *Let (X,ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4\in(0,\frac{1}{4\rho})$ and let $T:X_{\omega^G}\to X_{\omega^G}$ be a mapping. Suppose that there exists $\lambda>0$, and $k\in(0,1)$ such that*

$$\begin{aligned}&\omega_\lambda^G(Tx,Ty,Tz)\\ &\le k\max\left\{\begin{array}{l}\omega_\lambda^G(x,y,z),\omega_\lambda^G(x,z,Ty),\omega_\lambda^G(Tx,y,z),\omega_\lambda^G(x,Tx,y),\omega_\lambda^G(y,z,Ty),\\ \omega_\lambda^G(x,Tx,Tx),\omega_\lambda^G(x,Ty,Ty),\dfrac{\omega_\lambda^G(y,z,Ty)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)},\\ \omega_\lambda^G(x,Tx,z),\omega_\lambda^G(y,Ty,Ty),\dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)},\\ \omega_\lambda^G(y,Tx,Ty),\dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,Tx,z)},\omega_\lambda^G(y,T^2x,Ty),\\ \omega_\lambda^G(y,Tz,Tz),\dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)},\omega_\lambda^G(z,Tx,Tx),\\ \omega_\lambda^G(z,Tz,Tz),\omega_\lambda^G(z,Tx,Ty),\omega_\lambda^G(z,T^2x,Tz),\\ \omega_\lambda^G(Tx,T^2x,Ty),\omega_\lambda^G(Tx,T^2x,Tz),\dfrac{\omega_\lambda^G(x,z,Ty)[1+\omega_\lambda^G(x,z,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)},\\ \dfrac{\omega_\lambda^G(T^2x,Ty,Tz)[1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(Tx,y,Tz)]}{1+\omega_\lambda^G(Tx,y,Tz)+\omega_\lambda^G(T^2x,Ty,Tz)}\end{array}\right\}\\ &= k\left\{\begin{array}{l}\omega_\lambda^G(x,y,z)+\omega_\lambda^G(x,Tx,Tx)+\omega_\lambda^G(y,Ty,Ty)+\omega_\lambda^G(z,Tz,Tz)\\ \quad+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(y,T^2x,Ty)+\omega_\lambda^G(y,Tz,Tz)+\omega_\lambda^G(z,Tx,Tx)\\ \quad+\omega_\lambda^G(z,T^2x,Tz)+\omega_\lambda^G(Tx,T^2x,Ty)+\omega_\lambda^G(Tx,T^2x,Tz)\end{array}\right\}\end{aligned} \tag{3.48}$$

for all $x, y, z \in M \subseteq X_{\omega^G}$, $\lambda > 0$. Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u).

Proposition 3.3 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists $\lambda > 0$, and $\rho \in (0,1)$ such that*

$$\begin{aligned} &\omega_\lambda^G(Tx, Ty, Tz) \\ &\quad \leq \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(Tx, y, z), \omega_\lambda^G(z, Tx, Tx), \\ \dfrac{\omega_\lambda^G(T^2x, Ty, Tz)[1 + \omega_\lambda^G(Tx, y, z) + \omega_\lambda^G(Tx, y, Tz)]}{1 + \omega_\lambda^G(Tx, y, Tz) + \omega_\lambda^G(T^2x, Ty, Tz)} \end{array} \right\} \end{aligned} \tag{3.49}$$

for all $x, y, z \in M \subseteq X_{\omega^G}$, $\lambda > 0$. Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u).

Proof Let $x_0 \in X_{\omega^G}$ be arbitrary. We generate the sequence of iteration of T based on $x_0 \in X_{\omega^G}$ as follows:

$$\begin{aligned} &Tx_0 = x_1 \\ &Tx_1 = x_2 \\ &\quad\vdots \\ &Tx_n = x_{n+1} \end{aligned} \tag{3.50}$$

for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_0 is a fixed point of T. Now for all $n \in \mathbb{N}$, $x_{n+1} \neq x_n$, so that $\omega_\lambda^G(n_n, x_{n+1}, x_{n+1}) > 0$ and for $\lambda > 0$, take $x = x_n$ and $y = x_{n+1} = z$, then we have $\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) = \omega_\lambda^G(Tx, Ty, Tz) = \omega_\lambda^G(Tx_n, Tx_{n+1}, Tx_{n+1})$ so that $\forall x_n, x_{n+1} \in X_{\omega^G}$, $\lambda > 0$, inequality (3.49) becomes

$$\begin{aligned} &\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad \leq \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(Tx_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, Tx_n, Tx_n), \\ \dfrac{\omega_\lambda^G(T^2x_n, Tx_{n+1}, Tx_{n+1})[1 + \omega_\lambda^G(Tx_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(Tx_n, x_{n+1}, Tx_{n+1})]}{1 + \omega_\lambda^G(Tx_n, x_{n+1}, Tx_{n+1}) + \omega_\lambda^G(T^2x_n, Tx_{n+1}, Tx_{n+1})} \end{array} \right\} \\ &\quad = \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \\ \dfrac{\omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2})[1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}) + \omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2})} \end{array} \right\}. \end{aligned}$$

Therefore, we have $\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \leq 0$. For $m > n \in \mathbb{N}$, we get

$$\lim_{n,m \to \infty} \omega_\lambda^G(x_n, x_m, x_m) = 0. \tag{3.51}$$

For $n, m, k \in \mathbb{N}$, condition (2) of Proposition 2.2 implies that

$$\omega_\lambda^G(x_n, x_m, x_k) \leq \omega_{\frac{\lambda}{2}}^G(x_n, x_m, x_m) + \omega_{\frac{\lambda}{2}}^G(x_k, x_m, x_m), \tag{3.52}$$

so that on taking the limit of both sides of inequality (3.52) as $n, m, l \to \infty$ and by applying Definition 2.11 and Eq. (3.51), we get

$$\begin{aligned}\lim_{n,m,k\to\infty} \omega_\lambda^G(x_n, x_m, x_k) &\leq \lim_{n,m\to\infty} \omega_{\frac{\lambda}{2}}^G(x_n, x_m, x_m) + \lim_{k,m\to\infty} \omega_{\frac{\lambda}{2}}^G(x_k, x_m, x_m) \\ &\leq C_2 \lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, x_m) + C_2 \lim_{k,m\to\infty} \omega_\lambda^G(x_k, x_m, x_m) \\ &= C_2\Big(\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, x_m) + \lim_{k,m\to\infty} \omega_\lambda^G(x_k, x_m, x_m)\Big), \end{aligned} \tag{3.53}$$

thus we have

$$\lim_{n,m,k\to\infty} \omega_\lambda^G(x_n, x_m, x_k) = 0. \tag{3.54}$$

Equation (3.54) shows that $\{x_n\}_{n\in\mathbb{N}}$ is a modular G-Cauchy sequence. By Theorem 3.1, T has a unique fixed point in X_{ω^G} and also is modular G-continuous at the fixed point u. □

Proposition 3.4 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists $\lambda > 0$, and $\rho \in (0,1)$ such that*

$$\begin{aligned}&\omega_\lambda^G(Tx, Ty, Tz) \\ &\quad\leq \rho \max\left\{\begin{array}{c}\omega_\lambda^G(x,y,z), \omega_\lambda^G(Tx,y,z), \omega_\lambda^G(z, Tx, Tx), \\ \dfrac{\omega_\lambda^G(T^2x, Ty, Tz)[1 + \omega_\lambda^G(Tx,y,z) + \omega_\lambda^G(Tx,y,Tz)]}{1 + \omega_\lambda^G(Tx,y,Tz) + \omega_\lambda^G(T^2x,Ty,Tz)}\end{array}\right\}. \end{aligned} \tag{3.55}$$

Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u).

Proof From Proposition 3.3, $\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \rho\omega_\lambda^G(x_n, x_{n+1}, x_{n+1})$, so that

$$\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \rho\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \tag{3.56}$$

therefore,

$$\begin{aligned}\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \rho\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}) \\ &\leq \rho^2\omega_\lambda^G(x_{n-1}, x_n, x_n) \\ &\leq \rho^3\omega_\lambda^G(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\ \ \vdots \\ &\leq \rho^n\omega_\lambda^G(x_0, x_1, x_1). \end{aligned} \tag{3.57}$$

But $\sum_{n\in\mathbb{N}} \rho^n < +\infty$. Now $\sum_{n\in\mathbb{N}} \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \omega_\lambda^G(x_0, x_1, x_1) \sum_{n\in\mathbb{N}} \rho^n < +\infty$ for all $\lambda > 0$. Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Observe that, for any arbitrary ϵ, using the rectangle inequality repeatedly and condition (2) of Proposition 2.2, we have

$$\begin{aligned}\omega_\lambda^G(x_n,x_m,x_m) &\le \omega^G_{\frac{\lambda}{m-n}}(x_n,x_{n+1},x_{n+1}) + \omega^G_{\frac{\lambda}{m-n}}(x_{n+1},x_{n+2},x_{n+2}) + \omega^G_{\frac{\lambda}{m-n}}(x_{n+2},x_{n+3},x_{n+3}) \\ &\quad + \omega^G_{\frac{\lambda}{m-n}}(x_{n+3},x_{n+4},x_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{m-n}}(x_{m-1},x_m,x_m) \\ &\le \omega^G_{\frac{\lambda}{m}}(x_n,x_{n+1},x_{n+1}) + \omega^G_{\frac{\lambda}{m}}(x_{n+1},x_{n+2},x_{n+2}) + \omega^G_{\frac{\lambda}{m}}(x_{n+2},x_{n+3},x_{n+3}) \\ &\quad + \omega^G_{\frac{\lambda}{m}}(x_{n+3},x_{n+4},x_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{m}}(x_{m-1},x_m,x_m) \\ &\le \sum_{n=N} \omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \\ &< \epsilon, \end{aligned} \tag{3.58}$$

for all $m > n \ge N$ for some $N \in \mathbb{N}$. As ϵ is arbitrary, we have

$$\omega_\lambda^G(x_n,x_m,x_m) = 0 \quad \text{as } n,m \to \infty \quad \text{or} \quad \lim_{n,m\to\infty} \omega_\lambda^G(x_n,x_m,x_m) = 0. \tag{3.59}$$

For $n,m,k \in \mathbb{N}$, condition (2) of Proposition 2.2 implies that

$$\omega_\lambda^G(x_n,x_m,x_k) \le \omega^G_{\frac{\lambda}{2}}(x_n,x_m,x_m) + \omega^G_{\frac{\lambda}{2}}(x_k,x_m,x_m), \tag{3.60}$$

so that on taking the limit of both sides of inequality (3.60) as $n,m,l \to \infty$ and by applying Definition 2.11 and Eq. (3.59), we get

$$\begin{aligned}\lim_{n,m,k\to\infty} \omega_\lambda^G(x_n,x_m,x_k) &\le \lim_{n,m\to\infty} \omega^G_{\frac{\lambda}{2}}(x_n,x_m,x_m) + \lim_{k,m\to\infty} \omega^G_{\frac{\lambda}{2}}(x_k,x_m,x_m) \\ &\le C_2 \lim_{n,m\to\infty} \omega_\lambda^G(x_n,x_m,x_m) + C_2 \lim_{k,m\to\infty} \omega_\lambda^G(x_k,x_m,x_m) \\ &= C_2\Bigl(\lim_{n,m\to\infty} \omega_\lambda^G(x_n,x_m,x_m) + \lim_{k,m\to\infty} \omega_\lambda^G(x_k,x_m,x_m)\Bigr); \end{aligned} \tag{3.61}$$

thus we have

$$\lim_{n,m,k\to\infty} \omega_\lambda^G(x_n,x_m,x_k) = 0. \tag{3.62}$$

Equation (3.62) shows that $\{x_n\}_{n\in\mathbb{N}}$ is a modular G-Cauchy sequence. By Theorem 3.1, T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point u. □

Proposition 3.5 *Let (X,ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0,\frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some positive integer $m \ge 1$, $\lambda > 0$, and $\rho \in (0,1)$ such that*

$$\begin{aligned}&\omega_\lambda^G\bigl(T^m x, T^m y, T^m z\bigr) \\ &\quad \le \rho \max \left\{ \begin{array}{l} \omega_\lambda^G\bigl(T^m x, y, z\bigr), \omega_\lambda^G\bigl(z, T^m x, T^m x\bigr), \\ \dfrac{\omega_\lambda^G(T^m x, T^m y, T^m z)[1 + \omega_\lambda^G(T^m x, y, z) + \omega_\lambda^G(T^m x, y, T^m z)]}{1 + \omega_\lambda^G(T^m x, y, T^m z) + \omega_\lambda^G(T^m x, T^m y, T^m z)} \end{array} \right\}. \end{aligned} \tag{3.63}$$

Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at the fixed point (say u) for some positive integer $m \geq 1$.

Proof For $m = 2$, we have

$$\omega_\lambda^G\big(T^2x, T^2y, T^2z\big) \leq \rho \max \left\{ \begin{array}{c} \omega_\lambda^G\big(T^2x, y, z\big), \omega_\lambda^G\big(z, T^2x, T^2x\big), \\ \dfrac{\omega_\lambda^G(T^2x, T^2y, T^2z)[1 + \omega_\lambda^G(T^2x, y, z) + \omega_\lambda^G(T^2x, y, T^2z)]}{1 + \omega_\lambda^G(T^2x, y, T^2z) + \omega_\lambda^G(T^2x, T^2y, T^2z)} \end{array} \right\}. \tag{3.64}$$

Let $x_0 \in X_{\omega^G}$ be arbitrary. We generate the sequence of iteration of T based on $x_0 \in X_{\omega^G}$ as follows:

$$\begin{aligned} Tx_0 &= x_1 \\ Tx_1 &= x_2 \\ &\vdots \\ Tx_n &= x_{n+1} \end{aligned} \tag{3.65}$$

for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_0 is a fixed point of T. Now for all $n \in \mathbb{N}$, $x_{n+1} \neq x_n$ and $\lambda > 0$, take $x = x_n$ and $y = x_{n+1} = z$, then we have $\omega_\lambda^G(x_{n+2}, x_{n+3}, x_{n+3}) = \omega_\lambda^G(T^2x, T^2y, T^2z) = \omega_\lambda^G(T^2x_n, T^2x_{n+1}, T^2x_{n+1})$ so that $\forall x_n, x_{n+1} \in X_\omega$, $\lambda > 0$, inequality (3.64) becomes

$$\omega_\lambda^G(x_{n+2}, x_{n+3}, x_{n+3}) \leq \rho \max \left\{ \begin{array}{c} \omega_\lambda^G\big(T^2x_n, x_{n+1}, x_{n+1}\big), \omega_\lambda^G\big(x_{n+1}, T^2x_n, T^2x_n\big), \\ \dfrac{\omega_\lambda^G(T^2x_n, T^2x_{n+1}, T^2x_{n+1})[1 + \omega_\lambda^G(T^2x_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(T^2x_n, x_{n+1}, T^2x_{n+1})]}{1 + \omega_\lambda^G(T^2x_n, x_{n+1}, T^2x_{n+1}) + \omega_\lambda^G(T^2x_n, T^2x_{n+1}, T^2x_{n+1})} \end{array} \right\}.$$

So,

$$= \rho \max \left\{ \begin{array}{c} \omega_\lambda^G(x_{n+2}, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \dfrac{\omega_\lambda^G(x_{n+2}, x_{n+3}, x_{n+3})[1 + \omega_\lambda^G(x_{n+2}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+2}, x_{n+1}, x_{n+3})]}{1 + \omega_\lambda^G(x_{n+2}, x_{n+1}, x_{n+3}) + \omega_\lambda^G(x_{n+2}, x_{n+3}, x_{n+3})} \end{array} \right\}.$$

Proposition 3.4 shows that T has a unique fixed point in X_{ω^G}. For $m = 3, 4, \dots, k$, $T^m u$ has a unique fixed point u say, that is $T^m u = u$. Now we can see that $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$. Suppose that Tu is another fixed point for $T^m u$ and by uniqueness of the limit, $Tu = u$. By Theorem 3.1, $T^m u$ is modular G-continuous at say u, i.e. $T^m x_n = u = T^m u$ for some positive integer $m \geq 1$. □

Corollary 3.6 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some positive integer $m \geq 1$, $\lambda > 0$, and $\rho \in (0, 1)$ such that*

$$\omega_\lambda^G\big(T^m x, T^m y, T^m z\big) \leq \rho K_\lambda(x, y, z) \quad \textit{for all } x, y, z \in X_{\omega^G}, \lambda > 0, \tag{3.66}$$

where

$$K_\lambda(x,y,z) = \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,z,T^m y), \omega_\lambda^G(T^m x,y,z), \omega_\lambda^G(x,T^m x,y), \\ \omega_\lambda^G(y,z,T^m y), \omega_\lambda^G(x,T^m x,T^m x), \omega_\lambda^G(x,T^m y,T^m y), \\ \frac{\omega_\lambda^G(y,z,T^m y)[1+\omega_\lambda^G(x,T^m x,y)]}{1+\omega_\lambda^G(x,y,z)}, \omega_\lambda^G(x,T^m x,z), \omega_\lambda^G(y,T^m y,T^m y), \\ \frac{\omega_\lambda^G(x,T^m y,z)[1+\omega_\lambda^G(x,T^m x,y)]}{1+\omega_\lambda^G(x,y,z)}, \omega_\lambda^G(y,T^m x,T^m y), \\ \frac{\omega_\lambda^G(x,T^m y,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,T^m x,z)}, \omega_\lambda^G(y,T^m x,T^m y), \\ \omega_\lambda^G(y,T^m z,T^m z), \frac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,T^m y)]}{1+\omega_\lambda^G(T^m x,y,z)+\omega_\lambda^G(x,z,T^m y)}, \\ \omega_\lambda^G(z,T^m x,T^m x), \omega_\lambda^G(z,T^m z,T^m z), \omega_\lambda^G(z,T^m x,T^m y), \\ \omega_\lambda^G(z,T^m x,T^m z), \omega_\lambda^G(T^m x,T^m x,T^m y), \omega_\lambda^G(T^m x,T^m x,T^m z), \\ \frac{\omega_\lambda^G(x,z,T^m y)[1+\omega_\lambda^G(x,z,T^m y)]}{1+\omega_\lambda^G(T^m x,y,z)+\omega_\lambda^G(x,z,T^m y)}, \\ \frac{\omega_\lambda^G(T^m x,T^m y,T^m z)[1+\omega_\lambda^G(T^m x,y,z)+\omega_\lambda^G(T^m x,y,T^m z)]}{1+\omega_\lambda^G(T^m x,y,T^m z)+\omega_\lambda^G(T^m x,T^m y,T^m z)} \end{array} \right\}.$$

Then T has a unique fixed point in X_{ω^G} and T^m is modular G-continuous at the fixed point (say u) for some positive integer $m \geq 1$.

Proof $T^m u$ has a unique fixed point u say, that is $T^m u = u$. Now we see that $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$. But Tu is another fixed point for $T^m u$ and by uniqueness of the limit, $Tu = u$. By Theorem 3.1, $T^m u$ is modular G-continuous at say u, i.e. $T^m x_n = u = T^m u$. □

Corollary 3.7 *Let (X,ω^G) be a G-complete modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0,\frac{1}{4\rho})$ and let $T: X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some positive integer $m \geq 1$, $\lambda > 0$, and $\rho \in (0,1)$ such that*

$$\omega_\lambda^G(T^m x, T^m y, T^m z) \leq \rho\omega_\lambda^G(x,y,z) \quad \text{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.67}$$

then T has a unique fixed point in X_{ω^G} and T^m is modular G-continuous at the fixed point (say u) for some positive integer $m \geq 1$.

Proof Following Corollary 3.6, $T^m u$ has a unique fixed point u say, that is $T^m u = u$. Now we see that $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$. But Tu is another fixed point for $T^m u$ and by uniqueness of the limit, $Tu = u$. By Theorem 3.1, $T^m u$ is modular G-continuous at say u, i.e. $T^m x_n = u = T^m u$. □

Remark 3.6 The variant of Remark 3.5 above reads: let (X,ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in$

$(0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some positive integer $m \geq 1$, $\lambda > 0$, and $k \in (0,1)$ such that

$$\omega_\lambda^G\big(T^m x, T^m y, T^m z\big) \leq k Q_\lambda(x,y,z) \quad \text{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.68}$$

where

$$\begin{aligned} &Q_\lambda(x,y,z) \\ &= k \max \left\{ \begin{aligned} &\omega_\lambda^G(x,y,z), \omega_\lambda^G\big(x,z,T^m y\big), \omega_\lambda^G\big(T^m x,y,z\big), \omega_\lambda^G\big(x,T^m x,y\big), \omega_\lambda^G\big(y,z,T^m y\big), \\ &\omega_\lambda^G\big(x,T^m x,T^m x\big), \omega_\lambda^G\big(x,T^m y,T^m y\big), \frac{\omega_\lambda^G(y,z,T^m y)[1+\omega_\lambda^G(x,T^m x,y)]}{1+\omega_\lambda^G(x,y,z)}, \\ &\omega_\lambda^G\big(x,T^m x,z\big), \omega_\lambda^G\big(y,T^m y,T^m y\big), \frac{\omega_\lambda^G(x,T^m y,z)[1+\omega_\lambda^G(x,T^m x,y)]}{1+\omega_\lambda^G(x,y,z)}, \\ &\omega_\lambda^G\big(y,T^m x,T^m y\big), \omega_\lambda^G\big(y,T^m x,T^m y\big), \frac{\omega_\lambda^G(x,T^m y,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,T^m x,z)}, \\ &\omega_\lambda^G\big(y,T^m z,T^m z\big), \frac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,T^m y)]}{1+\omega_\lambda^G(T^m x,y,z)+\omega_\lambda^G(x,z,T^m y)}, \\ &\omega_\lambda^G\big(z,T^m x,T^m x\big), \omega_\lambda^G\big(z,T^m z,T^m z\big), \omega_\lambda^G\big(z,T^m x,T^m y\big), \\ &\omega_\lambda^G\big(z,T^m x,T^m z\big), \omega_\lambda^G\big(T^m x,T^m x,T^m y\big), \\ &\omega_\lambda^G\big(T^m x,T^m x,T^m z\big), \frac{\omega_\lambda^G(x,z,T^m y)[1+\omega_\lambda^G(x,z,T^m y)]}{1+\omega_\lambda^G(T^m x,y,z)+\omega_\lambda^G(x,z,T^m y)}, \\ &\frac{\omega_\lambda^G(T^m x,T^m y,T^m z)[1+\omega_\lambda^G(T^m x,y,z)+\omega_\lambda^G(T^m x,y,T^m z)]}{1+\omega_\lambda^G(T^m x,y,T^m z)+\omega_\lambda^G(T^m x,T^m y,T^m z)} \end{aligned} \right\} \\ &= k\big\{\omega_\lambda^G\big(x,T^m x,T^m x\big)+\omega_\lambda^G\big(y,T^m y,T^m y\big)+\omega_\lambda^G\big(z,T^m z,T^m z\big)\big\}, \end{aligned} \tag{3.69}$$

for all $x,y,z \in A \subseteq X_\omega$, $m = 1,2,\dots,k$. Then T has a unique fixed point for some positive integer $m \geq 1$. This is a variant form of Theorem 3.6 in [14].

Proposition 3.8 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : \mathfrak{A} \subseteq X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some integer $m \geq 1$, $\lambda > 0$, and $k \in (0,1)$ such that*

$$\omega_\lambda^G\big(T^m x, T^m y, T^m z\big) \leq k B_\lambda(x,y,z) \quad \textit{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.70}$$

where

$$\begin{aligned} &B_\lambda(x,y,z) \\ &= k \left\{ \begin{aligned} &\omega_\lambda^G(x,y,z)+\omega_\lambda^G\big(x,T^m x,T^m x\big)+\omega_\lambda^G\big(y,T^m y,T^m y\big)+\omega_\lambda^G\big(z,T^m z,T^m z\big) \\ &\quad+\omega_\lambda^G\big(T^m x,y,z\big)+\omega_\lambda^G\big(y,T^m x,T^m y\big)+\omega_\lambda^G\big(y,T^m z,T^m z\big)+\omega_\lambda^G\big(z,T^m x,T^m x\big) \\ &\quad+\omega_\lambda^G\big(z,T^m x,T^m z\big)+\omega_\lambda^G\big(T^m x,T^m x,T^m y\big)+\omega_\lambda^G\big(T^m x,T^m x,T^m z\big) \end{aligned} \right\} \end{aligned}$$

for all $x, y, z \in \mathfrak{A} \subseteq X_{\omega^G}$, Then T has a unique fixed point for some positive integer $m \geq 1$. This is a variant form of Theorem 3.6 *in* [14].

Theorem 3.9 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists $\lambda > 0$ and $\rho \in (0,1)$ such that*

$$\omega_\lambda^G(Tx, Ty, Tz) \leq \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,z,Ty), \omega_\lambda^G(Tx,y,z), \\ \omega_\lambda^G(x,Tx,y), \omega_\lambda^G(y,z,Ty), \omega_\lambda^G(x,Tx,Tx), \\ \omega_\lambda^G(x,Ty,Ty), \omega_\lambda^G(x,Tx,z), \omega_\lambda^G(y,Ty,Ty), \\ \omega_\lambda^G(y,Tx,Ty), \omega_\lambda^G(y,T^2x,Ty), \omega_\lambda^G(y,Tz,Tz), \\ \omega_\lambda^G(z,Tx,Tx), \omega_\lambda^G(z,Tz,Tz), \omega_\lambda^G(z,Tx,Ty), \\ \omega_\lambda^G(z,T^2x,Tz), \omega_\lambda^G(Tx,T^2x,Ty), \omega_\lambda^G(Tx,T^2x,Tz) \end{array} \right\}. \tag{3.71}$$

Then T has a unique fixed point in X_{ω^G} and is modular G-continuous at its fixed point (say u).

Proof Following Theorem 3.1, $Tu = u$ and hence T has a unique fixed point in X_{ω^G}. Also $Tx_n = u = Tu$ as $n \to \infty$. □

Corollary 3.10 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some positive integer, $m \geq 1$, $\lambda > 0$, and $\rho \in (0,1)$ such that*

$$\begin{aligned} &\omega_\lambda^G(T^m x, T^m y, T^m z) \\ &\quad \leq \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \omega_\lambda^G(x,z,T^m y), \omega_\lambda^G(T^m x,y,z), \\ \omega_\lambda^G(x,T^m x,y), \omega_\lambda^G(y,z,T^m y), \omega_\lambda^G(x,T^m x,T^m x), \\ \omega_\lambda^G(x,T^m y,T^m y), \omega_\lambda^G(x,T^m x,z), \omega_\lambda^G(y,T^m y,T^m y), \\ \omega_\lambda^G(y,T^m x,T^m y), \omega_\lambda^G(y,T^m x,T^m y), \omega_\lambda^G(y,T^m z,T^m z), \\ \omega_\lambda^G(z,T^m x,T^m x), \omega_\lambda^G(z,T^m z,T^m z), \omega_\lambda^G(z,T^m x,T^m y), \\ \omega_\lambda^G(z,T^m x,T^m z), \omega_\lambda^G(T^m x,T^m x,T^m y), \omega_\lambda^G(T^m x,T^m x,T^m z) \end{array} \right\}. \end{aligned} \tag{3.72}$$

Then T has a unique fixed point in X_{ω^G} and T^m is modular G-continuous at its fixed point (say u) for some positive integer $m \geq 1$.

Proof $T^m u$ has a unique fixed point u say, that is $T^m u = u$. Now we see that $Tu = T(T^m u) = T^{m+1} u = T^m(Tu)$. But Tu is another fixed point for $T^m u$ and by uniqueness of the limit, $Tu = u$. By Corollary 3.6, $T^m u$ is modular G-continuous at u, i.e. $T^m x_n = u = T^m u$ for some positive integer $m \geq 1$. □

Corollary 3.11 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping.*

Suppose that there exists $\lambda > 0$ *and* $\rho \in (0,1)$ *such that*

$$\omega_\lambda^G(Tx, Ty, Tz) \leq \rho K_\lambda(x,y,z) \quad \textit{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.73}$$

where

$$K_\lambda(x,y,z) = \max\left\{\begin{array}{l} \omega_\lambda^G(x,y,z), \dfrac{\omega_\lambda^G(y,z,Ty)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)}, \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,Tx,y)]}{1+\omega_\lambda^G(x,y,z)}, \\ \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,z)]}{1+\omega_\lambda^G(x,y,z)+\omega_\lambda^G(y,Tx,z)}, \dfrac{\omega_\lambda^G(x,Ty,z)[1+\omega_\lambda^G(x,y,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)}, \\ \dfrac{\omega_\lambda^G(x,z,Ty)[1+\omega_\lambda^G(x,z,Ty)]}{1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(x,z,Ty)}, \\ \dfrac{\omega_\lambda^G(T^2x,Ty,Tz)[1+\omega_\lambda^G(Tx,y,z)+\omega_\lambda^G(Tx,y,Tz)]}{1+\omega_\lambda^G(Tx,y,Tz)+\omega_\lambda^G(T^2x,Ty,Tz)} \end{array}\right\}.$$

Then T *has a unique fixed point in* X_{ω^G} *and is modular G-continuous at its fixed point (say* u*).*

Proof Following Theorem 3.1, we have $\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) = \omega_\lambda^G(Tx,Ty,Tz) = \omega_\lambda^G(Tx_n, Tx_{n+1}, Tx_{n+1})$, so that inequality (3.73) becomes

$$\omega_\lambda^G(x_{n+1},x_{n+2},x_{n+2}) \leq \rho K_\lambda(x_n,x_{n+1},x_{n+1}) \quad \forall x_n, x_{n+1} \in X_{\omega^G}, \lambda > 0, \tag{3.74}$$

where

$$K_\lambda(x_n,x_{n+1},x_{n+1}) = \max\left\{\begin{array}{l} \omega_\lambda^G(x_n,x_{n+1},x_{n+1}), \dfrac{\omega_\lambda^G(x_{n+1},x_{n+1},Tx_{n+1})[1+\omega_\lambda^G(x_n,Tx_n,x_{n+1})]}{1+\omega_\lambda^G(x_n,x_{n+1},x_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n,Tx_{n+1},x_{n+1})[1+\omega_\lambda^G(x_n,Tx_n,x_{n+1})]}{1+\omega_\lambda^G(x_n,x_{n+1},x_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n,Tx_{n+1},x_{n+1})[1+\omega_\lambda^G(x_n,x_{n+1},x_{n+1})]}{1+\omega_\lambda^G(x_n,x_{n+1},x_{n+1})+\omega_\lambda^G(x_{n+1},Tx_n,x_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n,Tx_{n+1},x_{n+1})[1+\omega_\lambda^G(x_n,x_{n+1},Tx_{n+1})]}{1+\omega_\lambda^G(Tx_n,x_{n+1},x_{n+1})+\omega_\lambda^G(x_n,x_{n+1},Tx_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n,x_{n+1},Tx_{n+1})[1+\omega_\lambda^G(x_n,x_{n+1},Tx_{n+1})]}{1+\omega_\lambda^G(Tx_n,x_{n+1},x_{n+1})+\omega_\lambda^G(x_n,x_{n+1},Tx_{n+1})}, \\ \dfrac{\omega_\lambda^G(T^2x_n,Tx_{n+1},Tx_{n+1})[1+\omega_\lambda^G(Tx_n,x_{n+1},x_{n+1})+\omega_\lambda^G(Tx_n,x_{n+1},Tx_{n+1})]}{1+\omega_\lambda^G(Tx_n,x_{n+1},Tx_{n+1})+\omega_\lambda^G(T^2x_n,Tx_{n+1},Tx_{n+1})} \end{array}\right\}$$

$$= \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \dfrac{\omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n, x_{n+2}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n, x_{n+2}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1})]}{1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1})}, \\ \dfrac{\omega_\lambda^G(x_n, x_{n+2}, x_{n+1})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})}, \\ \dfrac{\omega_\lambda^G(x_n, x_{n+1}, x_{n+2})[1 + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_n, x_{n+1}, x_{n+2})}, \\ \dfrac{\omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2})[1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}) + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2})]}{1 + \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}) + \omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2})} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_n, x_{n+2}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+2}), \\ \omega_\lambda^G(x_{n+2}, x_{n+2}, x_{n+2}) \end{array} \right\}.$$

By Theorem 3.1, T has a unique fixed point in X_{ω^G} and is modular G-continuous at its fixed point u. □

Corollary 3.12 *Let (X, ω^G) be a G-complete non-symmetric modular G-metric space satisfying a Δ_3-type condition, such that $C_2C_4 \in (0, \frac{1}{4\rho})$ and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping. Suppose that there exists some positive integer, $m \geq 1$, $\lambda > 0$, and $\rho \in (0,1)$ such that*

$$\omega_\lambda^G\big(T^m x, T^m y, T^m z\big) \leq \rho W_\lambda(x,y,z) \quad \text{for all } x,y,z \in X_{\omega^G}, \lambda > 0, \tag{3.75}$$

where

$$W_\lambda(x,y,z)$$

$$= \max \left\{ \begin{array}{l} \omega_\lambda^G(x,y,z), \dfrac{\omega_\lambda^G(y,z,T^m y)[1 + \omega_\lambda^G(x, T^m x, y)]}{1 + \omega_\lambda^G(x,y,z)}, \\ \dfrac{\omega_\lambda^G(x, T^m y, z)[1 + \omega_\lambda^G(x, T^m x, y)]}{1 + \omega_\lambda^G(x,y,z)}, \\ \dfrac{\omega_\lambda^G(x, T^m y, z)[1 + \omega_\lambda^G(x,y,z)]}{1 + \omega_\lambda^G(x,y,z) + \omega_\lambda^G(y, T^m x, z)}, \dfrac{\omega_\lambda^G(x, T^m y, z)[1 + \omega_\lambda^G(x,y,T^m y)]}{1 + \omega_\lambda^G(T^m x, y, z) + \omega_\lambda^G(x, z, T^m y)}, \\ \dfrac{\omega_\lambda^G(x, z, T^m y)[1 + \omega_\lambda^G(x, z, T^m y)]}{1 + \omega_\lambda^G(T^m x, y, z) + \omega_\lambda^G(x, z, T^m y)}, \\ \dfrac{\omega_\lambda^G(T^m x, T^m y, T^m z)[1 + \omega_\lambda^G(T^m x, y, z) + \omega_\lambda^G(T^m x, y, T^m z)]}{1 + \omega_\lambda^G(T^m x, y, T^m z) + \omega_\lambda^G(T^m x, T^m y, T^m z)} \end{array} \right\}.$$

Then T has a unique fixed point in X_{ω^G} and T^m is modular G-continuous at its fixed point u some positive integer, $m \geq 1$.

Proof By Corollary 3.11, T has a unique fixed point in X_{ω^G} and T^m is modular G-continuous its fixed point u some positive integer, $m \geq 1$. □

If we add some terms to Theorem 3.9 using condition (4) of Definition 2.1, we have the following result.

Theorem 3.13 *Let (X, ω^G) be a G-complete G-modular metric space satisfying a Δ_3-type condition, such that $C_2 \in (0, \frac{1}{4\rho})$, $\lambda > 0$, and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping for which the following condition holds*:

$$\begin{aligned}
&\omega_\lambda^G(Tx, Ty, Tz) \\
&\quad \leq \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x, y, z), \omega_\lambda^G(x, Tx, y), \omega_\lambda^G(x, Tx, Tx), \omega_\lambda^G(x, Tx, z), \\ \omega_\lambda^G(y, Ty, Ty), \omega_\lambda^G(y, Tz, Tz), \omega_\lambda^G\big(y, T^2x, Ty\big), \omega_\lambda^G(z, Tx, Tx), \\ \omega_\lambda^G(z, Tz, Tz), \omega_\lambda^G\big(z, T^2x, Tz\big), \omega_\lambda^G\big(Tx, T^2x, Ty\big), \omega_\lambda^G\big(Tx, T^2x, Tz\big) \end{array} \right\},
\end{aligned} \tag{3.76}$$

where $\rho \in (0, 1)$. Then T has a unique fixed point in X_{ω^G} and T is modular G-continuous at (say u).

Proof Let $x_0 \in X_{\omega^G}$ be arbitrary. We generate the sequence of iteration of T based on $x_0 \in X_{\omega^G}$ as follows:

$$\begin{aligned}
&Tx_0 = x_1 \\
&Tx_1 = x_2 \\
&\quad \vdots \\
&Tx_n = x_{n+1}
\end{aligned} \tag{3.77}$$

for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_0 is a fixed point of T. Now for all $n \in \mathbb{N}$, $x_{n+1} \neq x_n$ and $\lambda > 0$, take $x = x_n$ and $y = x_{n+1} = z$, then we have $\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) = \omega_\lambda^G(Tx, Ty, Tz) = \omega_\lambda^G(Tx_n, Tx_{n+1}, Tx_{n+1})$ so that

$$\begin{aligned}
&\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad = \omega_\lambda^G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\
&\quad \leq \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, Tx_n, x_{n+1}), \omega_\lambda^G(x_n, Tx_n, Tx_n), \\ \omega_\lambda^G(x_n, Tx_n, x_{n+1}), \omega_\lambda^G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \omega_\lambda^G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ \omega_\lambda^G\big(x_{n+1}, T^2x_n, Tx_{n+1}\big), \omega_\lambda^G(x_{n+1}, Tx_n, Tx_n), \omega_\lambda^G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ \omega_\lambda^G\big(x_{n+1}, T^2x_n, Tx_{n+1}\big), \omega_\lambda^G\big(Tx_n, T^2x_n, Tx_{n+1}\big), \omega_\lambda^G\big(Tx_n, T^2x_n, Tx_{n+1}\big) \end{array} \right\}
\end{aligned} \tag{3.78}$$

$$= \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \\ \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \end{array} \right\} \tag{3.79}$$

$$= \rho \max\bigl\{\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2})\bigr\}. \tag{3.80}$$

We have

$$\omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \le \rho\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}). \tag{3.81}$$

Therefore,

$$\begin{aligned} \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) &\le \rho\omega_\lambda^G(x_n, x_{n+1}, x_{n+1}) \\ &\le \rho^2\omega_\lambda^G(x_{n-1}, x_n, x_n) \\ &\le \rho^3\omega_\lambda^G(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\ \ \vdots \\ &\le \rho^n\omega_\lambda^G(x_0, x_1, x_1). \end{aligned} \tag{3.82}$$

But $\sum_{n\in\mathbb{N}} \rho^n < +\infty$. Now $\sum_{n\in\mathbb{N}} \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \le \omega_\lambda^G(x_0, x_1, x_1) \sum_{n\in\mathbb{N}} \rho^n < +\infty$ for all $\lambda > 0$. Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Observe that, for any arbitrary ϵ, using the rectangle inequality repeatedly and condition (2) of Proposition 2.2, we have

$$\begin{aligned} \omega_\lambda^G(x_n, x_m, x_m) &\le \omega_{\frac{\lambda}{m-n}}^G(x_n, x_{n+1}, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}^G(x_{n+1}, x_{n+2}, x_{n+2}) + \omega_{\frac{\lambda}{m-n}}^G(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \omega_{\frac{\lambda}{m-n}}^G(x_{n+3}, x_{n+4}, x_{n+4}) + \cdots + \omega_{\frac{\lambda}{m-n}}^G(x_{m-1}, x_m, x_m) \\ &\le \omega_{\frac{\lambda}{m}}^G(x_n, x_{n+1}, x_{n+1}) + \omega_{\frac{\lambda}{m}}^G(x_{n+1}, x_{n+2}, x_{n+2}) + \omega_{\frac{\lambda}{m}}^G(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \omega_{\frac{\lambda}{m}}^G(x_{n+3}, x_{n+4}, x_{n+4}) + \cdots + \omega_{\frac{\lambda}{m}}^G(x_{m-1}, x_m, x_m) \\ &\le \sum_{n=N} \omega_\lambda^G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &< \epsilon \end{aligned} \tag{3.83}$$

for all $m > n \ge N$ for some $N \in \mathbb{N}$. As ϵ is arbitrary, we have

$$\omega_\lambda^G(x_n, x_m, x_m) = 0 \quad \text{as } n, m \to \infty \quad \text{or} \quad \lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, x_m) = 0. \tag{3.84}$$

For $n, m, k \in \mathbb{N}$, condition (2) of Proposition 2.2 implies that

$$\omega_\lambda^G(x_n, x_m, x_k) \le \omega_{\frac{\lambda}{2}}^G(x_n, x_m, x_m) + \omega_{\frac{\lambda}{2}}^G(x_k, x_m, x_m), \tag{3.85}$$

so that on taking the limit of both sides of inequality (3.85) as $n, m, l \to \infty$ and by applying Definition 2.11 and Eq. (3.84), we get

$$\begin{aligned} \lim_{n,m,k\to\infty} \omega_\lambda^G(x_n, x_m, x_k) &\leq \lim_{n,m\to\infty} \omega_{\frac{\lambda}{2}}^G(x_n, x_m, x_m) + \lim_{k,m\to\infty} \omega_{\frac{\lambda}{2}}^G(x_k, x_m, x_m) \\ &\leq C_2 \lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, x_m) + C_2 \lim_{k,m\to\infty} \omega_\lambda^G(x_k, x_m, x_m) \\ &= C_2\Bigl(\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, x_m) + \lim_{k,m\to\infty} \omega_\lambda^G(x_k, x_m, x_m)\Bigr); \end{aligned} \tag{3.86}$$

thus, we have

$$\lim_{n,m,k\to\infty} \omega_\lambda^G(x_n, x_m, x_k) = 0 \quad \forall \lambda > 0. \tag{3.87}$$

By Equation (3.87), $\{x_n\}_{n\in\mathbb{N}}$ is a modular G-Cauchy sequence.

The completeness of (X, ω^G) implies that, for any $\lambda > 0$, $\lim_{n,m\to\infty} \omega_\lambda^G(x_n, x_m, u) = 0$, i.e. for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim_{n\to\infty} x_n \to u$. Suppose, if possible, that $Tu \neq u$, i.e. for $\lambda > 0$, $\omega_\lambda^G(u, Tu, Tu) > 0$ then from inequality (3.76), with $x = x_n$, $y = u = z$, then we have

$$\omega_\lambda^G(x_{n+1}, Tu, Tu) = \omega_\lambda^G(Tx_n, Tu, Tu) \leq \rho K_\lambda(x_n, u, u) \quad \text{for all } x_n, u, \in X_{\omega^G}, \lambda > 0, \tag{3.88}$$

so that

$$\begin{aligned} &K_\lambda(x_n, u, u) \\ &= \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, u, u), \omega_\lambda^G(x_n, Tx_n, u), \omega_\lambda^G(x_n, Tx_n, Tx_n), \omega_\lambda^G(x_n, Tx_n, u), \\ \omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G\bigl(u, T^2x_n, Tu\bigr), \omega_\lambda^G(u, Tx_n, Tx_n), \\ \omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G\bigl(u, T^2x_n, Tu\bigr), \omega_\lambda^G\bigl(Tx_n, T^2x_n, Tu\bigr), \omega_\lambda^G\bigl(Tx_n, T^2x_n, Tu\bigr) \end{array} \right\} \end{aligned} \tag{3.89}$$

$$= \rho \max \left\{ \begin{array}{l} \omega_\lambda^G(x_n, u, u), \omega_\lambda^G(x_n, x_{n+1}, u), \omega_\lambda^G(x_n, x_{n+1}, x_{n+1}), \omega_\lambda^G(x_n, x_{n+1}, u), \\ \omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G(u, x_{n+2}, Tu), \omega_\lambda^G(u, x_{n+1}, x_{n+1}), \\ \omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G(u, x_{n+2}, Tu), \omega_\lambda^G(x_{n+1}, x_{n+2}, Tu), \omega_\lambda^G(x_{n+1}, x_{n+2}, Tu) \end{array} \right\}, \tag{3.90}$$

as $n \to \infty$, we get

$$\omega_\lambda^G(u, Tu, Tu) \leq \rho \max\bigl\{\omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G(u, u, Tu)\bigr\}. \tag{3.91}$$

Now we consider the following cases.

Case 1. If $\max\{\omega_\lambda^G(u, Tu, Tu), \omega_\lambda^G(u, u, Tu)\} = \omega_\lambda^G(u, Tu, Tu)$, then inequality (3.91) becomes

$$\omega_\lambda^G(u, Tu, Tu) \leq \rho \omega_\lambda^G(u, Tu, Tu), \tag{3.92}$$

which implies that

$$(1-\rho)\omega_\lambda^G(u,Tu,Tu)\le 0, \quad \rho\in(0,1). \tag{3.93}$$

Hence for all $\lambda>0$, $Tu=u$.

Case 2. If $\max\{\omega_\lambda^G(u,Tu,Tu),\omega_\lambda^G(u,u,Tu)\}=\omega_\lambda^G(u,u,Tu)$, then, by condition (4) of Proposition 2.2, inequality (3.91) becomes

$$\begin{aligned}\omega_\lambda^G(u,Tu,Tu)&\le\rho\omega_\lambda^G(u,u,Tu)\\&\le 2\rho\omega_{\frac{\lambda}{2}}^G(u,Tu,Tu)\\&\le 2C_2\rho\omega_\lambda^G(u,Tu,Tu),\quad C_2\in\left(0,\frac{1}{2\rho}\right),\end{aligned} \tag{3.94}$$

which implies that

$$(1-2\rho C_2)\omega_\lambda^G(u,Tu,Tu)\le 0, \tag{3.95}$$

for all $\lambda>0$ and $C_2\in(0,\frac{1}{2\rho})$, which is a contradiction. Hence, $Tu=u$. Following Theorem 3.1 carefully, T has a unique fixed point in X_{ω^G} and T is modular G-continuous at say u. □

Remark 3.7 Let (X,ω^G) be a G-complete G-modular metric space satisfying a Δ_3-type condition, such that $C_2\in(0,\frac{1}{2\rho})$, $\lambda>0$, and let $T:X_{\omega^G}\to X_{\omega^G}$ be a mapping for which the following condition holds:

$$\begin{aligned}&\omega_\lambda^G(Tx,Ty,Tz)\\&\quad\le\rho\max\left\{\begin{array}{l}\omega_\lambda^G(x,y,z),\omega_\lambda^G(x,Tx,y),\omega_\lambda^G(x,Tx,Tx),\omega_\lambda^G(x,Tx,z),\\ \omega_\lambda^G(y,Ty,Ty),\omega_\lambda^G(y,Tz,Tz),\omega_\lambda^G\big(y,T^2x,Ty\big),\omega_\lambda^G(z,Tx,Tx),\\ \omega_\lambda^G(z,Tz,Tz),\omega_\lambda^G\big(z,T^2x,Tz\big),\omega_\lambda^G\big(Tx,T^2x,Ty\big),\omega_\lambda^G\big(Tx,T^2x,Tz\big)\end{array}\right\} &(3.96)\\&\quad=\alpha\omega_\lambda^G(x,Tx,Tx)+\beta\omega_\lambda^G(y,Ty,Ty)+\gamma\omega_\lambda^G(z,Tz,Tz) &(3.97)\end{aligned}$$

for $\alpha+\beta+\gamma\in(0,1)$. Then T has a unique fixed point in X_{ω^G} and T is modular G-continuous at u. In fact this is Theorem 3.2 of [14].

Remark 3.8 Let (X,ω^G) be a G-complete G-modular metric space satisfying a Δ_3-type condition, such that $C_2\in(0,\frac{1}{2\rho})$, $\lambda>0$, and let $T:X_{\omega^G}\to X_{\omega^G}$ be a mapping for which the following condition holds:

$$\begin{aligned}&\omega_\lambda^G(Tx,Ty,Tz)\\&\quad\le\rho\max\left\{\begin{array}{l}\omega_\lambda^G(x,y,z),\omega_\lambda^G(x,Tx,y),\omega_\lambda^G(x,Tx,Tx),\omega_\lambda^G(x,Tx,z),\\ \omega_\lambda^G(y,Ty,Ty),\omega_\lambda^G(y,Tz,Tz),\omega_\lambda^G\big(y,T^2x,Ty\big),\omega_\lambda^G(z,Tx,Tx),\\ \omega_\lambda^G(z,Tz,Tz),\omega_\lambda^G\big(z,T^2x,Tz\big),\omega_\lambda^G\big(Tx,T^2x,Ty\big),\omega_\lambda^G\big(Tx,T^2x,Tz\big)\end{array}\right\}\end{aligned} \tag{3.98}$$

$$= \alpha\omega_\lambda^G(x,Tx,Tx) + \beta\omega_\lambda^G(y,Ty,Ty) + \gamma\omega_\lambda^G(z,Tz,Tz) + \delta\omega_\lambda^G(x,y,z), \tag{3.99}$$

for $\alpha + \beta + \gamma + \delta \in (0,1)$. Then T has a unique fixed point in X_{ω^G} and T is modular G-continuous at u. This is Theorem 3.3 of [14].

Corollary 3.14 *Let (X,ω^G) be a G-complete modular G-metric space satisfying a Δ_3-type condition, such that $C_2 \in (0,\frac{1}{2\rho})$, $\lambda > 0$, $m = 1,2,\dots,k$, and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping for which the following condition holds*:

$$\begin{aligned}
&\omega_\lambda^G\bigl(T^m x, T^m y, T^m z\bigr) \\
&\quad\le \rho \max\left\{\begin{aligned}
&\omega_\lambda^G(x,y,z), \omega_\lambda^G\bigl(x,T^m x,y\bigr), \omega_\lambda^G\bigl(x,T^m x,T^m x\bigr), \\
&\omega_\lambda^G\bigl(x,T^m x,z\bigr), \omega_\lambda^G\bigl(y,T^m y,T^m y\bigr), \omega_\lambda^G\bigl(y,T^m z,T^m z\bigr), \\
&\omega_\lambda^G\bigl(y,T^m x,T^m y\bigr), \omega_\lambda^G\bigl(z,T^m x,T^m x\bigr), \omega_\lambda^G\bigl(z,T^m z,T^m z\bigr), \\
&\omega_\lambda^G\bigl(z,T^m x,T^m z\bigr), \omega_\lambda^G\bigl(T^m x,T^m x,T^m y\bigr), \omega_\lambda^G\bigl(T^m x,T^m x,T^m z\bigr)
\end{aligned}\right\},
\end{aligned} \tag{3.100}$$

where $\rho \in (0,1)$. Then T has a unique fixed point in X_{ω^G} and T is modular G-continuous at its fixed point (*say u*).

Proof $T^m u$ has a unique fixed point u say, that is $T^m u = u$. Now we see that $Tu = T(T^m u) = T^{m+1}u = T^m(Tu)$. But Tu is another fixed point for $T^m u$ and by uniqueness of limit, $Tu = u$. By Theorem 3.13, $T^m u$ is modular G-continuous at u, i.e. $T^m x_n = u = T^m u$. □

Remark 3.9 Let (X,ω^G) be a G-complete modular G-metric space satisfying a Δ_3-type condition, such that $C_2 \in (0,\frac{1}{2\rho})$ and there is some positive integer, $m \ge 1$, $\lambda > 0$, and let $T : X_{\omega^G} \to X_{\omega^G}$ be a mapping such that

$$\begin{aligned}
&\omega_\lambda^G\bigl(T^m x, T^m y, T^m z\bigr) \\
&\quad\le \rho \max\left\{\begin{aligned}
&\omega_\lambda^G(x,y,z), \omega_\lambda^G\bigl(x,T^m x,y\bigr), \omega_\lambda^G\bigl(x,T^m x,T^m x\bigr), \\
&\omega_\lambda^G\bigl(x,T^m x,z\bigr), \omega_\lambda^G\bigl(y,T^m y,T^m y\bigr), \omega_\lambda^G\bigl(y,T^m z,T^m z\bigr), \\
&\omega_\lambda^G\bigl(y,T^m x,T^m y\bigr), \omega_\lambda^G\bigl(z,T^m x,T^m x\bigr), \omega_\lambda^G\bigl(z,T^m z,T^m z\bigr), \\
&\omega_\lambda^G\bigl(z,T^m x,T^m z\bigr), \omega_\lambda^G\bigl(T^m x,T^m x,T^m y\bigr), \omega_\lambda^G\bigl(T^m x,T^m x,T^m z\bigr)
\end{aligned}\right\}
\end{aligned} \tag{3.101}$$

$$\begin{aligned}
&= \alpha\omega_\lambda^G\bigl(x,T^m x,T^m x\bigr) + \beta\omega_\lambda^G\bigl(y,T^m y,T^m y\bigr) \\
&\quad + \gamma\omega_\lambda^G\bigl(z,T^m z,T^m z\bigr) + \delta\omega_\lambda^G(x,y,z),
\end{aligned} \tag{3.102}$$

for $\alpha + \beta + \gamma + \delta \in (0,1)$. Then T has a unique fixed point in X_{ω^G} and T is modular G-continuous at its fixed point u. This is a variant form of Theorem 3.3 in [14].

4 Fixed point theorems of weakly compatible mappings in modular *G*-metric space

In this section we will briefly investigate fixed point theorems of compatible mappings in modular G-metric space.

Theorem 4.1 ([15]) *Let (X,ω^G) be a modular G-metric space and $g,h:X\to X$ be two self-mappings satisfying either*

$$\omega_\lambda^G(gx,gy,gz)\le\phi\bigl(\max\bigl\{\omega_\lambda^G(hx,gx,gx),\omega_\lambda^G(hy,gy,gy),\omega_\lambda^G(hz,gz,gz)\bigr\}\bigr),\tag{4.1}$$

or

$$\omega_\lambda^G(gx,gy,gz)\le\phi\bigl(\max\bigl\{\omega_\lambda^G(hx,hx,gx),\omega_\lambda^G(hy,hy,gy),\omega_\lambda^G(hz,hz,gz)\bigr\}\bigr),\tag{4.2}$$

for all $x,y,x\in X_{\omega^G}$ and $\lambda>0$. If the range of h contains the range of g and $h(X_{\omega^G})$ is a complete subspace of X, then g and h has a unique point of coincidence in X_ω. Furthermore, if g, h are weakly compatible, then g and h have a unique common fixed point.

Theorem 4.2 *Let (X,ω^G) be a modular G-metric space satisfying a Δ_3-type condition, $C_2+C_4\in(0,\frac{1}{k})$ and $T,S:X_{\omega^G}\to X_{\omega^G}$ be two self-mappings for which $T(X_{\omega^G})\subseteq S(X_{\omega^G})$, where $S(X_{\omega^G})$ is a G-complete subspace of X_{ω^G}. Suppose that, for all $\lambda>0$, the following condition holds*:

$$\omega_\lambda^G(Tx,Ty,Tz)\le k\max\left\{\begin{array}{l}\omega_\lambda^G(Sx,Sy,Sz),\omega_\lambda^G(Tx,Tx,Sz),\\ \omega_\lambda^G(Ty,Ty,Sy),\omega_\lambda^G(Sy,Tz,Sz),\\ \omega_\lambda^G(Sy,Tx,Sy),\omega_\lambda^G(Sz,Tz,Tz),\\ \omega_\lambda^G(Tz,Sx,Tz),\omega_\lambda^G(Tx,Sx,Sy)\end{array}\right\},\tag{4.3}$$

where $k<1$, for all $x,y,z\in X_{\omega^G}$. Then T, S has a unique coincidence point in X_{ω^G}. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X_{ω^G}.

Proof Let $x_0\in X_{\omega^G}$ be an arbitrary point. Since $T(X_{\omega^G})\subseteq S(X_{\omega^G})$, there exists $x_1\in X_{\omega^G}$ such that $Tx_0=Sx_1$. Continuing in this way we have $Tx_n=Sx_{n+1}$, for $n\in\mathbb{N}$. So for any $\lambda>0$, $\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n)\le\omega_\lambda^G(Tx_n,Tx_{n-1},Tx_{n-1})$. From inequality (4.3), we have

$$\begin{aligned}&\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n)\\ &\quad\le k\max\left\{\begin{array}{l}\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1}),\omega_\lambda^G(Tx_n,Tx_n,Sx_{n-1}),\\ \omega_\lambda^G(Tx_{n-1},Tx_{n-1},Sx_{n-1}),\omega_\lambda^G(Sx_{n-1},Tx_{n-1},Sx_{n-1}),\\ \omega_\lambda^G(Sx_{n-1},Tx_n,Sx_{n-1}),\omega_\lambda^G(Sx_{n-1},Tx_{n-1},Tx_{n-1}),\\ \omega_\lambda^G(Tx_{n-1},Sx_n,Tx_{n-1}),\omega_\lambda^G(Tx_n,Sx_n,Sx_{n-1})\end{array}\right\}\qquad(4.4)\\ &\quad=k\max\left\{\begin{array}{l}\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1}),\omega_\lambda^G(Sx_{n+1},Sx_{n+1},Sx_{n-1}),\\ \omega_\lambda^G(Sx_n,Sx_n,Sx_{n-1}),\omega_\lambda^G(Sx_{n-1},Sx_n,Sx_{n-1}),\\ \omega_\lambda^G(Sx_{n-1},Sx_{n+1},Sx_{n-1}),\omega_\lambda^G(Sx_{n-1},Sx_n,Sx_n),\\ \omega_\lambda^G(Sx_n,Sx_n,Sx_n),\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_{n-1})\end{array}\right\},\qquad(4.5)\end{aligned}$$

using condition (4) of Definition 2.2 which we simplify as

$$\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \le k \max \left\{ \begin{array}{l} \omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1}), \omega_\lambda^G(Sx_{n+1},Sx_{n+1},Sx_{n-1}), \\ \omega_\lambda^G(Sx_n,Sx_n,Sx_{n-1}), \omega_\lambda^G(Sx_{n+1},Sx_{n-1},Sx_{n-1}), \\ \omega_\lambda^G(Sx_{n+1},Sx_n,Sx_{n-1}) \end{array} \right\}, \tag{4.6}$$

here we have the following cases.

Case 1. If the right hand side of inequality (4.6) is $k\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1})$, then we have

$$\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \le k\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1}), \quad k \in (0,1). \tag{4.7}$$

Case 2. If the right hand side of inequality (4.6) is $k\omega_\lambda^G(Sx_{n+1},Sx_{n+1},Sx_{n-1})$, then we have

$$\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \le k\omega_\lambda^G(Sx_{n+1},Sx_{n+1},Sx_{n-1}), \tag{4.8}$$

using conditions (6), (3) of Proposition 2.2 and Definition 2.11, we have

$$\begin{aligned} \omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) &\le k\omega_\lambda^G(Sx_{n+1},Sx_{n+1},Sx_{n-1}) \\ &\le k\Big\{\omega_{\frac{\lambda}{2}}^G(Sx_{n+1},Sx_n,Sx_n) + \omega_{\frac{\lambda}{4}}^G(Sx_{n+1},Sx_n,Sx_n) \\ &\quad + \omega_{\frac{\lambda}{4}}^G(Sx_n,Sx_n,Sx_{n-1})\Big\} \\ &\le k\Big\{C_2\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) + C_4\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \\ &\quad + C_4\omega_\lambda^G(Sx_n,Sx_n,Sx_{n-1})\Big\} \\ &\le k\Big\{(C_2+C_4)\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) + 2C_4\omega_{\frac{\lambda}{2}}^G(Sx_n,Sx_{n-1},Sx_{n-1})\Big\} \\ &\le k\Big\{(C_2+C_4)\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) + 2C_2C_4\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1})\Big\}. \end{aligned}$$

Hence

$$\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \le p\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1}), \tag{4.9}$$

where $p = \frac{2kC_2C_4}{1-k(C_2+C_4)} < 1$, and $C_2 + C_4 \in (0, \frac{1}{k})$.

Case 3. If the right hand side of inequality (4.6) is $k\omega_\lambda^G(Sx_n,Sx_n,Sx_{n-1})$, then we have

$$\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \le k\omega_\lambda^G(Sx_n,Sx_n,Sx_{n-1}), \tag{4.10}$$

using condition (3) of Proposition 2.2 and Definition 2.11, we have

$$\begin{aligned} \omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) &\le k\omega_\lambda^G(Sx_n,Sx_n,Sx_{n-1}) \\ &\le 2k\omega_{\frac{\lambda}{2}}^G(Sx_n,Sx_{n-1},Sx_{n-1}) \\ &\le 2kC_2\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1}), \end{aligned}$$

hence

$$\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) \le d\omega_{\lambda}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1}), \tag{4.11}$$

where $d = 2kC_2 < 1$ and $C_2 \in (0, \frac{1}{k})$.

Case 4. If the right hand side of inequality (4.6) is $k\omega_{\lambda}^{G}(Sx_{n+1}, Sx_{n-1}, Sx_{n-1})$, then we have

$$\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) \le k\omega_{\lambda}^{G}(Sx_{n+1}, Sx_{n-1}, Sx_{n-1}), \tag{4.12}$$

using condition (5) of Definition 2.1 and Definition 2.11, we have

$$\begin{aligned} \omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) &\le k\omega_{\lambda}^{G}(Sx_{n+1}, Sx_{n-1}, Sx_{n-1}) \\ &\le k\bigl(\omega_{\frac{\lambda}{2}}^{G}(Sx_{n+1}, Sx_n, Sx_n) + \omega_{\frac{\lambda}{2}}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1})\bigr) \\ &\le kC_2\bigl(\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) + \omega_{\lambda}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1})\bigr), \end{aligned}$$

so that

$$\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) \le b\omega_{\lambda}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1}), \tag{4.13}$$

where $b = \frac{kC_2}{1-kC_2} < 1$ and $C_2 \in (0, \frac{1}{k})$.

Case 5. If the right hand side of inequality (4.6) is $k\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_{n-1})$, using condition (1) of Definitions 2.1, 2.11 and condition (6), (3) of Proposition 2.2, then we have

$$\begin{aligned} \omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) &\le k\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_{n-1}) \\ &\le k\bigl\{\omega_{\frac{\lambda}{2}}^{G}(Sx_{n+1}, Sx_n, Sx_n) + \omega_{\frac{\lambda}{4}}^{G}(Sx_n, Sx_n, Sx_n) \\ &\quad + \omega_{\frac{\lambda}{4}}^{G}(Sx_n, Sx_n, Sx_{n-1})\bigr\} \\ &= k\bigl\{\omega_{\frac{\lambda}{2}}^{G}(Sx_{n+1}, Sx_n, Sx_n) + \omega_{\frac{\lambda}{4}}^{G}(Sx_n, Sx_n, Sx_{n-1})\bigr\} \\ &\le k\bigl\{C_2\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) + C_4\omega_{\lambda}^{G}(Sx_n, Sx_n, Sx_{n-1})\bigr\} \\ &\le k\bigl\{C_2\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) + 2C_4\omega_{\frac{\lambda}{2}}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1})\bigr\} \\ &\le k\bigl\{C_2\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) + 2C_2C_4\omega_{\lambda}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1})\bigr\}, \end{aligned}$$

which implies that

$$\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) \le \frac{2kC_2C_4}{1-kC_2}\omega_{\lambda}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1}).$$

Therefore,

$$\omega_{\lambda}^{G}(Sx_{n+1}, Sx_n, Sx_n) \le r\omega_{\lambda}^{G}(Sx_n, Sx_{n-1}, Sx_{n-1}), \tag{4.14}$$

where $r = \frac{2kC_2C_4}{1-kC_2} < 1$ and $C_2 \in (0, \frac{1}{k})$.

So take $\eta = \max\{k,p,d,b,r\} < 1$, so we have

$$\begin{aligned}\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) &\le \eta\omega_\lambda^G(Sx_n,Sx_{n-1},Sx_{n-1})\\ &\vdots\\ &\le \eta^n\omega_\lambda^G(Sx_0,Sx_1,Sx_1).\end{aligned}\tag{4.15}$$

But $\sum_{n\in\mathbb{N}}\eta^n < +\infty$. Now $\sum_{n\in\mathbb{N}}\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n) \le \omega_\lambda^G(Sx_0,Sx_1,Sx_1)\sum_{n\in\mathbb{N}}\eta^n < +\infty$ for all $\lambda > 0$. Suppose that $m,n\in\mathbb{N}$ and $m > n \in \mathbb{N}$. Observe that, for any arbitrary ϵ, using the rectangle inequality repeatedly and condition (2) of Proposition 2.2, we have

$$\begin{aligned}\omega_\lambda^G(Sx_n,Sx_m,Sx_m) &\le \omega^G_{\frac{\lambda}{m-n}}(Sx_n,Sx_{n+1},Sx_{n+1}) + \omega^G_{\frac{\lambda}{m-n}}(Sx_{n+1},Sx_{n+2},Sx_{n+2})\\ &\quad+\omega^G_{\frac{\lambda}{m-n}}(Sx_{n+2},Sx_{n+3},Sx_{n+3})\\ &\quad+\omega^G_{\frac{\lambda}{m-n}}(Sx_{n+3},Sx_{n+4},Sx_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{m-n}}(Sx_{m-1},Sx_m,Sx_m)\\ &\le \omega^G_{\frac{\lambda}{m}}(Sx_n,Sx_{n+1},Sx_{n+1}) + \omega^G_{\frac{\lambda}{m}}(Sx_{n+1},Sx_{n+2},Sx_{n+2})\\ &\quad+\omega^G_{\frac{\lambda}{m}}(Sx_{n+2},Sx_{n+3},Sx_{n+3})\\ &\quad+\omega^G_{\frac{\lambda}{m}}(Sx_{n+3},Sx_{n+4},Sx_{n+4}) + \cdots + \omega^G_{\frac{\lambda}{m}}(Sx_{m-1},Sx_m,Sx_m)\\ &\le \sum_{n=N}\omega_\lambda^G(Sx_{n+1},Sx_n,Sx_n)\\ &< \epsilon\end{aligned}\tag{4.16}$$

for all $m > n \ge N$ for some $N\in\mathbb{N}$. As ϵ is arbitrary, we have

$$\omega_\lambda^G(Sx_n,Sx_m,Sx_m) = 0 \quad \text{as } n,m\to\infty \quad \text{or} \quad \lim_{n,m\to\infty}\omega_\lambda^G(Sx_n,Sx_m,Sx_m) = 0.\tag{4.17}$$

For $n,m,k\in\mathbb{N}$, condition (2) of Proposition 2.2 implies that

$$\omega_\lambda^G(Sx_n,Sx_m,Sx_k) \le \omega^G_{\frac{\lambda}{2}}(Sx_n,Sx_m,Sx_m) + \omega^G_{\frac{\lambda}{2}}(Sx_k,Sx_m,Sx_m),\tag{4.18}$$

so that on taking the limit of both sides of inequality (4.18) as $n,m,l\to\infty$ and applying Definition 2.11 and Eq. (4.17), we get

$$\begin{aligned}&\lim_{n,m,k\to\infty}\omega_\lambda^G(Sx_n,Sx_m,Sx_k)\\ &\quad\le \lim_{n,m\to\infty}\omega^G_{\frac{\lambda}{2}}(Sx_n,Sx_m,Sx_m) + \lim_{k,m\to\infty}\omega^G_{\frac{\lambda}{2}}(Sx_k,Sx_m,Sx_m)\\ &\quad\le C_2\lim_{n,m\to\infty}\omega_\lambda^G(Sx_n,Sx_m,Sx_m) + C_2\lim_{k,m\to\infty}\omega_\lambda^G(Sx_k,Sx_m,Sx_m)\\ &\quad= C_2\Bigl(\lim_{n,m\to\infty}\omega_\lambda^G(Sx_n,Sx_m,Sx_m) + \lim_{k,m\to\infty}\omega_\lambda^G(Sx_k,Sx_m,Sx_m)\Bigr),\end{aligned}\tag{4.19}$$

hence, we have

$$\lim_{n,m,k\to\infty}\omega_\lambda^G(Sx_n,Sx_m,Sx_k) = 0.\tag{4.20}$$

By Eq. (4.20), the sequence $\{Sx_n\}_{n\in\mathbb{N}}$ is a modular G-Cauchy sequence in $S(X_\omega)$. Because of completeness of $S(X_\omega)$, $Sx_n \to v$ and there exists $u \in X_\omega$ such that $Su = v$. Hence, inequality (4.3) becomes $\omega_\lambda^G(Sx_n, Tu, Tu) = \omega_\lambda^G(Tx_{n-1}, Tu, Tu)$, so

$$\omega_\lambda^G(Sx_n, Tu, Tu) \leq k \max \left\{ \begin{array}{l} \omega_\lambda^G(Sx_{n-1}, Su, Su), \omega_\lambda^G(Tx_{n-1}, Tx_{n-1}, Su), \\ \omega_\lambda^G(Tu, Tu, Su), \omega_\lambda^G(Su, Tu, Su), \\ \omega_\lambda^G(Su, Tx_{n-1}, Su), \omega_\lambda^G(Su, Tu, Tu), \\ \omega_\lambda^G(Tu, Sx_{n-1}, Tu), \omega_\lambda^G(Tx_{n-1}, Sx_{n-1}, Su) \end{array} \right\}, \tag{4.21}$$

so that

$$\omega_\lambda^G(Sx_n, Tu, Tu) \leq k \max \left\{ \begin{array}{l} \omega_\lambda^G(Sx_{n-1}, Su, Su), \omega_\lambda^G(Sx_n, Sx_n, Su), \\ \omega_\lambda^G(Tu, Tu, Su), \omega_\lambda^G(Su, Tu, Su), \\ \omega_\lambda^G(Su, Sx_n, Su), \omega_\lambda^G(Su, Tu, Tu), \\ \omega_\lambda^G(Tu, Sx_{n-1}, Tu), \omega_\lambda^G(Sx_n, Sx_{n-1}, Su) \end{array} \right\}, \tag{4.22}$$

as $n \to \infty$, and $Su = v$ we obtain

$$\omega_\lambda^G(v, Tu, Tu) \leq k \max\bigl\{\omega_\lambda^G(v, Tu, Tu), \omega_\lambda^G(v, v, Tu)\bigr\}, \quad \forall \lambda > 0. \tag{4.23}$$

We consider the following cases.

Case 1. If $k \max\{\omega_\lambda^G(v, Tu, Tu), \omega_\lambda^G(v, v, Tu)\} = \omega_\lambda^G(v, Tu, Tu)$, then inequality (4.23) becomes

$$\omega_\lambda^G(v, Tu, Tu) \leq k\omega_\lambda^G(v, Tu, Tu), \tag{4.24}$$

which implies that

$$(1 - k)\omega_\lambda^G(v, Tu, Tu) \leq 0 \quad \forall \lambda > 0, k \in (0, 1), \tag{4.25}$$

hence $Tu = v = Su$.

Case 2. If $k \max\{\omega_\lambda^G(v, Tu, Tu), \omega_\lambda^G(v, v, Tu)\} = \omega_\lambda^G(v, v, Tu)$, then inequality (4.23) becomes

$$\omega_\lambda^G(v, Tu, Tu) \leq k\omega_\lambda^G(v, v, Tu), \tag{4.26}$$

using condition (4) of Proposition 2.2, and Definition 2.11, we get

$$\begin{aligned} \omega_\lambda^G(v, Tu, Tu) &\leq k\bigl(\omega_{\frac{\lambda}{2}}^G(v, Tu, Tu) + \omega_{\frac{\lambda}{2}}^G(Tu, v, Tu)\bigr) \\ &\leq k\bigl(C_2\omega_\lambda^G(v, Tu, Tu) + C_2\omega_\lambda^G(v, Tu, Tu)\bigr) \\ &= 2kC_2\omega_\lambda^G(v, Tu, Tu), \end{aligned} \tag{4.27}$$

which implies that

$$(1-2kC_2)\omega_\lambda^G(v,Tu,Tu)\leq 0, \tag{4.28}$$

for some $C_2 \in (0,\frac{1}{k})$, hence $\omega_\lambda^G(v,Tu,Tu)\leq 0$, this implies that $Tu = v = Su$, which shows that v is a point of coincidence of T and S. Now suppose that v is not a point of coincidence of T and S, then there exists another point of coincidence (say v^*) of T and S such that $Tu^* = v^* = Su^*$. For $\lambda > 0$, $\omega_\lambda^G(Su,Su^*,Su^*) = \omega_\lambda^G(Tu,Tu^*,Tu^*)$, so from inequality (4.3), we obtain

$$\omega_\lambda^G(Su,Su^*,Su^*)\leq k\max\left\{\begin{array}{l}\omega_\lambda^G(Su,Su^*,Su^*),\omega_\lambda^G(Tu,Tu,Su^*),\\ \omega_\lambda^G(Tu^*,Tu^*,Su^*),\omega_\lambda^G(Su^*,Tu^*,Su^*),\\ \omega_\lambda^G(Su^*,Tu,Su^*),\omega_\lambda^G(Su^*,Tu^*,Tu^*),\\ \omega_\lambda^G(Tu^*,Su,Tu^*),\omega_\lambda^G(Tu,Su,Su^*)\end{array}\right\}, \tag{4.29}$$

hence

$$\omega_\lambda^G(Su,Su^*,Su^*)\leq k\max\bigl\{\omega_\lambda^G(Su,Su^*,Su^*),\omega_\lambda^G(Su,Su,Su^*)\bigr\}. \tag{4.30}$$

If $k\max\{\omega_\lambda^G(Su,Su^*,Su^*),\omega_\lambda^G(Su,Su,Su^*)\} = k\omega_\lambda^G(Su,Su^*,Su^*)$, then inequality (4.30) becomes

$$(1-k)\omega_\lambda^G(Su,Su^*,Su^*)\leq 0,\quad k\in(0,1), \tag{4.31}$$

which implies that $Su = Su^*$.

Again, if $k\max\{\omega_\lambda^G(Su,Su^*,Su^*),\omega_\lambda^G(Su,Su,Su^*)\} = k\omega_\lambda^G(Su,Su,Su^*)$, then using condition (4) of Proposition 2.2, and Definition 2.11, we get

$$\begin{aligned}\omega_\lambda^G(Su,Su^*,Su^*) &\leq k\bigl(\omega_{\frac{\lambda}{2}}^G(Su,Su^*,Su^*)+\omega_{\frac{\lambda}{2}}^G(Su^*,Su,Su^*)\bigr)\\ &\leq k\bigl(C_2\omega_\lambda^G(Su,Su^*,Su^*)+C_2\omega_\lambda^G(Su,Su^*,Su^*)\bigr)\\ &=2kC_2\omega_\lambda^G(Su,Su^*,Su^*),\end{aligned} \tag{4.32}$$

which implies that

$$(1-2kC_2)\omega_\lambda^G(Su,Su^*,Su^*)\leq 0, \tag{4.33}$$

for some $C_2 \in (0,\frac{1}{k})$, hence $\omega_\lambda^G(Su,Su^*,Su^*)\leq 0$. Therefore, $Su = Su^*$, which shows that v is a unique point of coincidence of T and S. By Proposition 2.1, v is a common unique point of T and S. □

We give some examples here.

Example 4.1 Let $X = [0,1]\cup\{\infty\}$, and for $\lambda > 0$, define $\omega_\lambda^G : X\times X\times X\to[0,\infty)$ by $\omega_\lambda^G(x,y,z) = \frac{1}{\lambda}\max\{|x-y|,|y-z|,|x-z|\}$ and $T,S : X_{\omega^G}\to X_{\omega^G}$ such that $Tx = \frac{1}{2}$ and

$Sx = \frac{1}{2} - x$. Evidently, the range of T contains the range of S, $R(S) \subseteq R(T)$ and $S(X_{\omega^G})$ is a complete subspace of X_{ω^G}, also T and S has a unique point of coincidence at $x = 0$ in X. Furthermore, T and S are weakly compatible. But $\omega_\lambda^G(Tx, Ty, Tz) = 0$, so T satisfies inequality (4.3).

Example 4.2 Let $X = \mathbb{R}^r \cup \{\infty\}$ for $r \geq 1$. Define $G : \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^r \to [0,\infty)$ by $G(x,y,z) = \frac{1}{3}\max\{|x-y|, |y-z|, |x-z|\}$ for all distinct $x,y,z \in \mathbb{R}^r$ and 0 for $x = y = z$, where $|x - y| = (\sum_{k=1}^r (x_k - y_k)^2)^{\frac{1}{2}}$, $x = (x_1, x_2, \dots, x_r)$, $y = (y_1, y_2, \dots, y_r)$ and $z = (z_1, z_2, \dots, z_r)$. For any $\lambda > 0$, let $\omega_\lambda^G(x,y,z) = \frac{G(x,y,z)}{3\lambda}$ for all $x,y,z \in \mathbb{R}^r$. Suppose that two mappings $T, S : \mathbb{R}^r \to \mathbb{R}^r$ defined by $Tx = \frac{x}{6}$ and $Sx = \frac{x}{3}$. Indeed observe that $R(S) \subseteq R(T)$ and $S(X_{\omega^G})$ is a complete subspace of X, $T(S(0)) = S(T(0))$ also T and S has a unique point of coincidence at $x = 0$ in X, i.e. $u = T(0) = S(0) = 0$, so that the common unique fixed point of T and S is 0. Now

$$\begin{aligned}
&\omega_\lambda^G(Tx, Ty, Tz) \\
&\quad = \frac{1}{6\lambda}\max\left\{\left(\sum_{k=1}^r (x_k - y_k)^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^r (y_k - z_k)^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^r (x_k - z_k)^2\right)^{\frac{1}{2}}\right\} \\
&\quad = \frac{1}{2}\left\{\frac{1}{3\lambda}\max\left\{\left(\sum_{k=1}^r (x_k - y_k)^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^r (y_k - z_k)^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^r (x_k - z_k)^2\right)^{\frac{1}{2}}\right\}\right\} \\
&\quad = \frac{1}{2}\omega_\lambda^G(Sx, Sy, Sz),
\end{aligned} \tag{4.34}$$

so that

$$\omega_\lambda^G(Tx, Ty, Tz) \leq \frac{1}{2}\max\left\{\begin{array}{l} \omega_\lambda^G(Sx, Sy, Sz), \omega_\lambda^G(Tx, Tx, Sz), \\ \omega_\lambda^G(Ty, Ty, Sy), \omega_\lambda^G(Sy, Tz, Sz), \\ \omega_\lambda^G(Sy, Tx, Sy), \omega_\lambda^G(Sz, Tz, Tz), \\ \omega_\lambda^G(Tz, Sx, Tz), \omega_\lambda^G(Tx, Sx, Sy) \end{array}\right\}, \tag{4.35}$$

hence all the conditions of Theorem 4.2 are satisfied. Thus T and S has a unique point of coincidence at $x = 0$ in X.

Corollary 4.3 *Let (X, ω^G) be a modular G-metric space satisfying a Δ_3-type condition and $T, S : X_{\omega^G} \to X_{\omega^G}$ be two self-mappings for which $T(X_{\omega^G}) \subseteq S(X_{\omega^G})$, where $S(X_{\omega^G})$ is a G-complete subspace of X_{ω^G}. Suppose that, for all $\lambda > 0$, the following condition holds*:

$$\omega_\lambda^G(Tx, Ty, Tz) \leq k\omega_\lambda^G(Sx, Sy, Sz), \tag{4.36}$$

where $k < 1$, for all $x, y, z \in X_{\omega^G}$. Then T, S has a unique coincidence point in X_{ω^G}. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X_{ω^G}.

Proof Let $x_0 \in X_{\omega^G}$ be an arbitrary point. Since $T(X_{\omega^G}) \subseteq S(X_{\omega^G})$, there exists $x_1 \in X_{\omega^G}$ such that $Tx_0 = Sx_1$. Continuing in this way we have $Tx_n = Sx_{n+1}$, for $n \in \mathbb{N}$. So for any

$\lambda > 0$, $\omega_\lambda^G(Sx_{n+1}, Sx_n, Sx_n) \le \omega_\lambda^G(Tx_n, Tx_{n-1}, Tx_{n-1})$. From inequality (4.36), we have

$$\omega_\lambda^G(Sx_{n+1}, Sx_n, Sx_n) \le k\omega_\lambda^G(Sx_n, Sx_{n-1}, Sx_{n-1}), \quad k \in (0,1). \tag{4.37}$$

Theorem 4.2 tells us that T and S have a unique common fixed point in X_{ω^G}. □

Corollary 4.4 *Let (X,ω^G) be a modular G-metric space satisfying a Δ_3-type condition, such that $C_2 + C_4 \in (0, \frac{1}{k})$ and $T, S : X_{\omega^G} \to X_{\omega^G}$ be two self-mappings for which $T^m(X_{\omega^G}) \subseteq S^m(X_{\omega^G})$, where $S^m(X_{\omega^G})$ is a G-complete subspace of X_{ω^G}. Suppose that, for all $\lambda > 0$, the following condition holds*:

$$\omega_\lambda^G\bigl(T^m x, T^m y, T^m z\bigr) \le k \max \left\{ \begin{array}{l} \omega_\lambda^G\bigl(S^m x, S^m y, S^m z\bigr), \omega_\lambda^G\bigl(T^m x, T^m x, S^m z\bigr), \\ \omega_\lambda^G\bigl(T^m y, T^m y, S^m y\bigr), \omega_\lambda^G\bigl(S^m y, T^m z, S^m z\bigr), \\ \omega_\lambda^G\bigl(S^m y, T^m x, S^m y\bigr), \omega_\lambda^G\bigl(S^m z, T^m z, T^m z\bigr), \\ \omega_\lambda^G\bigl(T^m z, S^m x, T^m z\bigr), \omega_\lambda^G\bigl(T^m x, S^m x, S^m y\bigr) \end{array} \right\}, \tag{4.38}$$

where $k < 1$ for all $x, y, z \in X_{\omega^G}$. Then T, S has a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X_{ω^G} for some positive integer $m \ge 1$.

Proof By Theorem 4.2, $T^m u = S^m u = v$. Hence v is a common unique point of T^m and S^m. □

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