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# Some approximation properties of new $(p,q)$-analogue of Balázs–Szabados operators

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**Abstract**

In this paper, a new $(p,q)$-analogue of the Balázs–Szabados operators is defined. Moments up to the fourth order are calculated, and second order and fourth order central moments are estimated. Local approximation properties of the operators are examined and a Voronovskaja type theorem is given.

**Keywords:** $(p,q)$-calculus; Moments; Bernstein operators; Balázs–Szabados operators; $(p,q)$-Balázs–Szabados operators

## 1 Introduction

Bernstein type rational functions are defined and studied by Balázs in 1975 as follows (see [6]):

$$R_n(f;x) = \frac{1}{(1+a_nx)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) \binom{n}{k} (a_nx)^k \quad (n = 1,2,\ldots),$$

where $f$ is a real- and single-valued function defined on the interval $[0,\infty)$, $a_n$ and $b_n$ are real numbers which are appropriately selected and do not depend on $x$. Later in 1982 Balázs and Szabados together improved the estimate in [7] by choosing appropriate $a_n$ and $b_n$ under some restrictions for $f(x)$.

Several $q$-generalizations of Balázs–Szabados operators have been recently studied by Hamal and Sabancigil [14], Doğru [10], and Özkan [31]. Approximation properties of the $q$-Balázs–Szabados complex operators were studied by Mahmudov in [21] and by Ispir and Özkan in [16]. The Balázs–Szabados operator based on the $q$-integers were defined by Mahmudov in [21] as follows:

$$R_{n,q}(f,x) = \frac{1}{(1+a_nx)^n} \sum_{k=0}^{n} f\left(\frac{[k]_q}{b_n}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q (a_nx)^k \prod_{s=0}^{n-k-1} \left(1 + (1-q)[s]_q a_nx\right), \tag{1}$$

where $q > 0$, $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, $f : [0,\infty) \to \mathbb{R}$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$,

$$0 < \beta \le \frac{2}{3}, \qquad n \in \mathbb{N}, \quad \text{and} \quad x \ne -\frac{1}{a_n}.$$

On the other hand, the rapid rise of $(p,q)$-calculus has led to the discovery of new generalizations. Recently, Mursaleen et al. introduced and studied $(p,q)$-analogue of Bernstein operators, $(p,q)$-analogue of Bernstein–Stancu operators, Bernstein–Kantorovich operators based on $(p,q)$-calculus, $(p,q)$-Lorentz polynomials on a compact disc, Bleimann–Butzer–Hahn operators defined by $(p,q)$-integers and $(p,q)$-analogue of two parametric Stancu–Beta operators (see [24–30]). $(p,q)$-generalization of Szász–Mirakyan operators was studied by Acar (see [1]), Kantorovich modification of $(p,q)$-Bernstein operators was studied by Acar and Aral, see [2]. A generalization of $q$-Balázs–Szabados operators based on $(p,q)$-integers was studied by Özkan and İspir in [32].

In this paper, we study some approximation properties for the new $(p,q)$-analogue of Balázs–Szabados operators, and we prove a Voronovskaja type theorem. In [32], the authors gave the central moments without any estimations; in this paper we also give the estimation of central moments in detail. Before stating the results for these operators, we give some notations and definitions of $(p,q)$-calculus. For any $p>0$, $q>0$, nonnegative integer $n$, the $(p,q)$-integer of the number $n$ is defined as

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q} & \text{if } p \neq q \neq 1, \\ np^{n-1} & \text{if } p = q \neq 1, \\ [n]_q & \text{if } p = 1, \\ n & \text{if } p = q = 1. \end{cases}$$

It can be easily seen that $[n]_{p,q} = p^{n(n-1)/2}[n]_{\frac{q}{p}}$.

$(p,q)$-factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geq 1 \quad \text{and} \quad [0]_{p,q}! = 1,$$

and $(p,q)$-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n,$$

the formula of $(p,q)$-binomial expansion is defined by

$$\begin{aligned} (ax+by)_{p,q}^{n} &= \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k \\ &= (ax+by)(pax+qby)\left(p^2ax+q^2by\right)\cdots\left(p^{n-1}ax+q^{n-1}by\right), \end{aligned}$$

and

$$(x-y)_{p,q}^{n} = (x-y)(px-qy)\left(p^2x-q^2y\right)\left(p^3x-q^3y\right)\cdots\left(p^{n-1}x-q^{n-1}y\right).$$

From $(p,q)$-binomial expansion, we can see that

$$\sum_{k=0}^{n} p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n-k} = p^{n(n-1)/2}, \quad x \in [0,1]. \tag{2}$$

## 2 Operators and estimation of moments

**Definition 1** Let $0 < q < p \leq 1$, we introduce a new $(p,q)$-analogue of Balázs–Szabados operators by

$$R_{n,p,q}(f,x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2} f\left(\frac{p^{n-k}[k]_{p,q}}{b_n}\right)\left(\frac{a_n x}{1+a_n x}\right)^k \times \prod_{j=0}^{n-k-1}\left(p^j - q^j \frac{a_n x}{1+a_n x}\right),$$

where $a_n = [n]_{p,q}^{\beta-1}$, $b_n = [n]_{p,q}^{\beta}$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$, $x \geq 0$, $f$ is a real-valued function defined on $[0,\infty)$.

Note that in the case $p = 1$ these polynomials reduce to Mahmudov's $q$-Balázs–Szabados operator which is defined by (1). Also we consider the following two cases:

case (i) If $0 < p < q \leq 1$ or $1 \leq p < q < \infty$, then the positivity of the operators fails.

case (ii) If $1 \leq q < p < \infty$, then approximation by the new operators becomes difficult because if $p$ is large enough, then $R_{n,p,q}$ may diverge, so in this paper we study approximation properties of the operators for $0 < q < p \leq 1$.

Moments and central moments play an important role in the approximation theory. In the following lemma, we calculate the first five moments of our operators; in other words, we find formulas for $R_{n,p,q}(t^m,x)$ for $m = 0,1,2,3,4$.

**Lemma 2** *For all $n \in \mathbb{N}$, $x \in [0,\infty)$ and $0 < q < p \leq 1$, we have the following equalities*:

$$R_{n,p,q}(1,x) = 1, \tag{3}$$

$$R_{n,p,q}(t,x) = \frac{x}{1+a_n x}, \tag{4}$$

$$R_{n,p,q}\left(t^2,x\right) = \frac{p^{n-1}}{a_n b_n}\left(\frac{a_n x}{1+a_n x}\right) + \frac{q[n-1]_{p,q}}{a_n b_n}\left(\frac{a_n x}{1+a_n x}\right)^2, \tag{5}$$

$$R_{n,p,q}\left(t^3,x\right) = \frac{p^{2(n-1)}}{a_n b_n^2}\left(\frac{a_n x}{1+a_n x}\right) + \frac{(2+\frac{q}{p})p^{n-1}q[n-1]_{p,q}}{a_n b_n^2}\left(\frac{a_n x}{1+a_n x}\right)^2 + \frac{q^3[n-1]_{p,q}[n-2]_{p,q}}{a_n b_n^2}\left(\frac{a_n x}{1+a_n x}\right)^3, \tag{6}$$

$$\begin{aligned} R_{n,p,q}\left(t^4,x\right) = {} & \frac{p^{3(n-1)}}{a_n b_n^3}\left(\frac{a_n x}{1+a_n x}\right) + \frac{(3+\frac{3q}{p}+\frac{q^2}{p^2})p^{2(n-1)}q[n-1]_{p,q}}{a_n b_n^3}\left(\frac{a_n x}{1+a_n x}\right)^2 \\ & + \frac{(3+2\frac{q}{p^3}+\frac{q^2}{p^2})p^{n-1}q^3[n-1]_{p,q}[n-2]_{p,q}}{a_n b_n^3}\left(\frac{a_n x}{1+a_n x}\right)^3 \\ & + \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{a_n b_n^3}\left(\frac{a_n x}{1+a_n x}\right)^4. \end{aligned} \tag{7}$$

*Proof*

$$R_{n,p,q}(1,x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}\left(\frac{a_n x}{1+a_n x}\right)^k \prod_{j=0}^{n-k-1}\left(p^j - q^j \frac{a_n x}{1+a_n x}\right) = 1,$$

$$\begin{aligned}
R_{n,p,q}(t,x) &= \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2} \left(p^{n-k} \frac{[k]_{p,q}}{b_n}\right) \left(\frac{a_n x}{1+a_n x}\right)^k \\
&\quad \times \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \\
&= \frac{1}{a_n} \frac{1}{p^{(n-1)(n-2)/2}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+1} \\
&\quad \times \prod_{s=0}^{n-k-2} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \\
&= \frac{x}{1+a_n x}, \\
R_{n,p,q}\left(t^2,x\right) &= \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(p^{n-k} \frac{[k]_{p,q}}{b_n}\right)^2 \left(\frac{a_n x}{1+a_n x}\right)^k \\
&\quad \times \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right).
\end{aligned}$$

Now, by using $[k]_{p,q} = p^{k-1} + q[k-1]_{p,q}$, we get

$$\begin{aligned}
&R_{n,p,q}\left(t^2,x\right) \\
&= \frac{1}{a_n b_n} \frac{1}{p^{n(n-1)/2}} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}+2(n-k)} \left(p^{k-1} + q[k-1]_{p,q}\right) \left(\frac{a_n x}{1+a_n x}\right)^k \\
&\quad \times \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \\
&= \frac{p^{n-1}}{a_n b_n} \frac{1}{p^{(n-1)(n-2)/2}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+1} \\
&\quad \times \prod_{s=0}^{n-k-2} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \\
&\quad + \frac{q[n-1]_{p,q}}{a_n b_n} \frac{1}{p^{(n-2)(n-3)/2}} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+2} \\
&\quad \times \prod_{s=0}^{n-k-3} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \\
&= \frac{p^{n-1}}{a_n b_n} \left(\frac{a_n x}{1+a_n x}\right) + \frac{q[n-1]_{p,q}}{a_n b_n} \left(\frac{a_n x}{1+a_n x}\right)^2, \\
&R_{n,p,q}\left(t^3,x\right) \\
&= \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{p^{n-k}[k]_{p,q}}{b_n}\right)^3 \left(\frac{a_n x}{1+a_n x}\right)^k \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right)
\end{aligned}$$

$$= \frac{1}{a_n b_n^2} \frac{1}{p^{n(n-1)/2}} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}+3(n-k)} [k]_{p,q}^2 \left( \frac{a_n x}{1+a_n x} \right)^k \times \prod_{s=0}^{n-k-1} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right).$$

Again using the identity $[k]_{p,q} = p^{k-1} + q[k-1]_{p,q}$, we obtain

$$\begin{aligned} &R_{n,p,q}(t^3,x) \\ &= \frac{1}{a_n b_n^2} \frac{1}{p^{n(n-7)/2}} \sum_{k=1}^{n} \left\{ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-7)}{2}} \left(p^{k-1} + q[k-1]_{p,q}\right)^2 \right. \\ &\quad \left. \times \left( \frac{a_n x}{1+a_n x} \right)^k \prod_{s=0}^{n-k-1} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right) \right\} \\ &= \frac{p^{2(n-1)}}{a_n b_n^2} \frac{1}{p^{(n-1)(n-2)/2}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left( \frac{a_n x}{1+a_n x} \right)^{k+1} \prod_{s=0}^{n-k-2} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right) \\ &\quad + \frac{2p^{n-1} q[n-1]_{p,q}}{a_n b_n^2} \frac{1}{p^{(n-2)(n-3)/2}} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left( \frac{a_n x}{1+a_n x} \right)^{k+2} \\ &\quad \times \prod_{s=0}^{n-k-3} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right) \\ &\quad + \frac{p^{n-2} q^2 [n-1]_{p,q}}{a_n b_n^2} \frac{1}{p^{(n-2)(n-3)/2}} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left( \frac{a_n x}{1+a_n x} \right)^{k+2} \\ &\quad \times \prod_{s=0}^{n-k-3} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right) \\ &\quad + \frac{q^3 [n-1]_{p,q} [n-2]_{p,q}}{a_n b_n^2} \frac{1}{p^{(n-3)(n-4)/2}} \sum_{k=0}^{n-3} \begin{bmatrix} n-3 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left( \frac{a_n x}{1+a_n x} \right)^{k+3} \\ &\quad \times \prod_{s=0}^{n-k-4} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right). \end{aligned}$$

Then we obtain

$$\begin{aligned} R_{n,p,q}(t^3,x) &= \frac{p^{2(n-1)}}{a_n b_n^2} \left( \frac{a_n x}{1+a_n x} \right) + \frac{2p^{n-1} q[n-1]_{p,q} + p^{n-2} q^2 [n-1]_{p,q}}{a_n b_n^2} \left( \frac{a_n x}{1+a_n x} \right)^2 \\ &\quad + \frac{q^3 [n-1]_{p,q} [`n-2]_{p,q}}{a_n b_n^2} \left( \frac{a_n x}{1+a_n x} \right)^3, \\ R_{n,p,q}(t^4,x) &= \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left( \frac{p^{n-k}[k]_{p,q}}{b_n} \right)^4 \left( \frac{a_n x}{1+a_n x} \right)^k \\ &\quad \times \prod_{s=0}^{n-k-1} \left( p^s - q^s \frac{a_n x}{1+a_n x} \right) \end{aligned}$$

$$= \frac{1}{a_n b_n^3} \frac{1}{p^{n(n-1)/2}} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}+4(n-k)} [k]_{p,q}^3 \left(\frac{a_n x}{1+a_n x}\right)^k$$
$$\times \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right).$$

Using the identity $[k]_{p,q} = p^{k-1} + q[k-1]_{p,q}$, we have

$$R_{n,p,q}(t^4, x)$$
$$= \frac{1}{a_n b_n^3} \frac{1}{p^{n(n-9)/2}} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-9)}{2}} \Bigg\{ \left(p^{k-1} + q[k-1]_{p,q}\right)^3$$
$$\times \left(\frac{a_n x}{1+a_n x}\right)^k \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \Bigg\}$$
$$= \frac{p^{3(n-1)}}{a_n b_n^3} \frac{1}{p^{(n-1)(n-2)/2}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+1} \prod_{s=0}^{n-k-2} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right)$$
$$+ \frac{3p^{2(n-1)} q[n-1]_{p,q}}{a_n b_n^3} \frac{1}{p^{(n-2)(n-3)/2}} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+2}$$
$$\times \prod_{s=0}^{n-k-3} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right)$$
$$+ \frac{3p^{2n-3} q^2[n-1]_{p,q}}{a_n b_n^3} \frac{1}{p^{(n-2)(n-3)/2}} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+2}$$
$$\times \prod_{s=0}^{n-k-3} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right)$$
$$+ \frac{3p^{n-1} q^3[n-1]_{p,q}[n-2]_{p,q}}{a_n b_n^3} \frac{1}{p^{(n-3)(n-4)/2}} \sum_{k=0}^{n-3} \begin{bmatrix} n-3 \\ k \end{bmatrix}_{p,q} \Bigg\{ p^{\frac{k(k-1)}{2}} \left(\frac{a_n x}{1+a_n x}\right)^{k+3}$$
$$\times \prod_{s=0}^{n-k-4} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \Bigg\}$$
$$+ \frac{q^3[n-1]_{p,q}}{a_n b_n^3} \frac{1}{p^{n(n-9)/2}} \sum_{k=2}^{n} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{p,q} \Bigg\{ p^{\frac{k(k-9)}{2}} \left(p^{k-2} + q[k-2]_{p,q}\right)^2 \left(\frac{a_n x}{1+a_n x}\right)$$
$$\times \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{a_n x}{1+a_n x}\right) \Bigg\}.$$

By some calculations, we get

$$R_{n,p,q}(t^4, x)$$
$$= \frac{p^{3(n-1)}}{a_n b_n^3} \left(\frac{a_n x}{1+a_n x}\right) + \frac{(3p^{2(n-1)} + 3p^{2n-3} q + p^{2n-4} q^2) q[n-1]_{p,q}}{a_n b_n^3} \left(\frac{a_n x}{1+a_n x}\right)^2$$

$$+\frac{(3p^{n-1}+2p^{n-4}q+p^{n-3}q^2)q^3[n-1]_{p,q}[n-2]_{p,q}}{a_n b_n^3}\left(\frac{a_n x}{1+a_n x}\right)^3$$
$$+\frac{q^3[n-1]_{p,q}[‘n-2]_{p,q}[n-3]_{p,q}}{a_n b_n^3}\left(\frac{a_n x}{1+a_n x}\right)^4. \qquad \square$$

**Lemma 3** *For all $n \in \mathbb{N}$, $x \in [0,\infty)$, and $0 < q < p \leq 1$, we have the following central moments*:

$$R_{n,p,q}\big((t-x),x\big)=\frac{-a_n x^2}{1+a_n x}, \tag{8}$$

$$R_{n,p,q}\big((t-x)^2,x\big)=\frac{p}{b_n}^{n-1}\left(\frac{1}{a_n x+1}\right)x+\left\{\left(\frac{a_n x}{1+a_n x}\right)^2-\frac{p^{n-1}}{[n]_{p,q}}\frac{1}{(1+a_n x)^2}\right\}x^2, \tag{9}$$

$$\begin{aligned}
&R_{n,p,q}\big((t-x)^4,x\big)\\
&\quad=\left\{\frac{p}{b_n^3}^{3(n-1)}\frac{1}{(1+a_n x)}\right\}x\\
&\qquad+\left\{\frac{(3p^{2(n-1)}+3p^{2n-3}q+p^{2n-4}q^2)[n-1]_{p,q}}{[n]_{p,q}}\frac{1}{b_n^2(1+a_n x)^2}\right.\\
&\qquad\left.-\frac{4p^{2(n-1)}}{b_n^2(1+a_n x)}\right\}x^2+\left\{\frac{(3p^{n-1}+2p^{n-4}q+p^{n-3}q^2)q^3[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2 b_n(1+a_n x)^3}\right.\\
&\qquad\left.-\frac{4(2p^{n-1}+p^{n-2}q)q[n-1]_{p,q}}{[n]_{p,q}b_n(1+a_n x)^2}+\frac{6p^{n-1}}{b_n(1+a_n x)}\right\}x^3\\
&\qquad+\left\{\frac{q^3[n-1]_{p,q}[‘n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^3(1+a_n x)^4}-4\frac{q^3[n-1]_{p,q}[‘n-2]_{p,q}}{[n]_{p,q}^2(1+a_n x)^3}\right.\\
&\qquad\left.+6\frac{q[n-1]_{p,q}}{[n]_{p,q}(1+a_n x)^2}-\frac{4}{1+a_n x}+1\right\}x^4.
\end{aligned}\tag{10}$$

*Proof* The proof is done by using the linearity of the operators and the previous lemma. $\square$

**Lemma 4** *For all $n \in \mathbb{N}$ and $0 < q < p \leq 1$, we have the following estimations*:

$$\big(R_{n,p,q}\big((t-x),x\big)\big)^2 \leq x^2, \quad x \in [0,\infty), \tag{11}$$
$$R_{n,p,q}\big((t-x)^2,x\big) \leq D_1(1+x)^2, \quad x \in [0,\infty), \tag{12}$$
$$R_{n,p,q}\big((t-x)^4,x\big) \leq \frac{1}{b_n^2}D_2(1+x)^2, \quad x \in [0,\infty), \tag{13}$$

*where $D_1$* and *$D_2$ are positive constants.*

*Proof* First, we estimate $(R_{n,p,q}((t-x),x))^2$. For $x \in [0,\infty)$,

$$\begin{aligned}
\big(R_{n,p,q}\big((t-x),x\big)\big)^2 &= \big(R_{n,p,q}(t,x)-xR_{n,p,q}(1,x)\big)^2\\
&=\left(\frac{x}{1+a_n x}-x\right)^2=\left(\frac{-a_n x^2}{1+a_n x}\right)^2\\
&\leq x^2, \quad \text{since } \frac{a_n x}{1+a_n x}<1.
\end{aligned}$$

For the estimation of $R_{n,p,q}((t-x)^2,x)$, we use formula (9) which is given in the previous lemma. For $x \in [0,\infty)$, by using the facts that

$$\frac{1}{(1+a_n x)} \le 1, \qquad \frac{1}{(1+a_n x)^2} \le 1, \quad \text{and} \quad \frac{(a_n x)^2}{(1+a_n x)^2} \le 1,$$

we get

$$\begin{aligned} R_{n,p,q}\bigl((t-x)^2,x\bigr) &\le \frac{p^{n-1}}{b_n}x + \left\{1 - \frac{p^{n-1}}{[n]_{p,q}}\right\}x^2 \\ &\le C_1 \gamma_n(p,q)(1+x)^2, \end{aligned}$$

where $C_1 > 0$ and $\gamma_n(p,q) = \max\{\frac{p^{n-1}}{b_n}, 1 - \frac{p^{n-1}}{[n]_{p,q}}\}$. Since $\lim_{n\to\infty} \frac{p^{n-1}}{b_n} = 0$ and $\lim_{n\to\infty} \frac{p^{n-1}}{[n]_{p,q}} = 0$, then there exists a positive constant $D_1$ such that

$$R_{n,p,q}\bigl((t-x)^2,x\bigr) \le D_1(1+x)^2.$$

Now, for $x \in [0,\infty)$, we use similar calculations for the estimation of $R_{n,p,q}((t-x)^4,x)$, we use formula (10) which is given in the previous lemma. By using the inequalities

$$\begin{aligned} &\frac{1}{(1+a_n x)^i} \le 1 \quad \text{for } i \in \{1,2,3,4\}, \\ &p^{i(n-1)} \le 1 \quad \text{for } p \le 1, i \in \{1,2,3\}, \\ &p^{n-i} \le 1 \quad \text{for } p \le 1, i \in \{1,2,3,4\}, \end{aligned}$$

and the facts

$$\begin{aligned} &q[n-1]_{p,q} = [n]_{p,q} - p^{n-1}, \\ &q^2[n-2]_{p,q} = [n]_{p,q} - qp^{n-2} - p^{n-1}, \\ &q^3[n-3]_{p,q} = [n]_{p,q} - q^2p^{n-3} - qp^{n-2} - p^{n-1}, \end{aligned}$$

and by substituting these into formula (9) with some calculations, we obtain

$$\begin{aligned} R_{n,p,q}\bigl((t-x)^4,x\bigr) &\le \frac{1}{b_n^2}\varphi(p,q)x(1+x), \\ &\le \frac{1}{b_n^2}C_2\varphi(p,q)(1+x)^2, \end{aligned}$$

where $C_2 > 0$ and $\varphi(p,q) > 0$, so we get

$$R_{n,p,q}\bigl((t-x)^4,x\bigr) \le \frac{1}{b_n^2}D_2(1+x)^2, \quad \text{where } D_2 > 0. \qquad \square$$

*Remark* 1 To study the convergence results of the operators $R_{n,p,q}$, let $q = q_n$, $p = p_n$ be the sequences such that $0 < q_n < p_n \le 1$, if $q_n \to 1$ as $n \to \infty$, then by the sandwich theorem, $p_n \to 1$, which implies $\lim_{n\to\infty}[n]_{n,p_n,q_n} = \infty$. For example, if $0 < c < d$, we can choose $q_n = \frac{n}{n+d}$ and $p_n = \frac{n}{n+c}$ such that $0 < q_n < p_n \le 1$, it is obvious that $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n^n = e^{-d}$, and $\lim_{n\to\infty} p_n^n = e^{-c}$ so $\lim_{n\to\infty}[n]_{n,p_n,q_n} = \infty$.

**Lemma 5** *Assume that* $0 < q_n < p_n < 1$, $q_n \to 1$, *as* $n \to \infty$ *and* $0 < \beta < \frac{1}{2}$. *Then we have the following limits*:

$$\text{(i)} \quad \lim_{n\to\infty} b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x),x\big) = 0,$$

$$\text{(ii)} \quad \lim_{n\to\infty} b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x)^2,x\big) = x,$$

*where* $a_{n,p_n,q_n} = [n]_{p_n,q_n}^{\beta-1}$ *and* $b_{n,p_n,q_n} = [n]_{p_n,q_n}^{\beta}$.

*Proof* For the proof of this lemma, we use the formulas $R_{n,p_n,q_n}(t,x)$ and $R_{n,p_n,q_n}(t^2,x)$ given in Lemma 2. The first statement is clear

$$\begin{aligned}
\lim_{n\to\infty} b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x),x\big) &= \lim_{n\to\infty} b_{n,p_n,q_n}\big(R_{n,p_n,q_n}(t,x)-x\big) \\
&= \lim_{n\to\infty} b_{n,p_n,q_n}\left(\frac{-a_{n,p_n,q_n}x^2}{1+a_{n,p_n,q_n}x}\right) \\
&= \lim_{n\to\infty}\left(\frac{-[n]_{p_n,q_n}^{2\beta-1}x^2}{1+[n]_{p_n,q_n}^{\beta-1}x}\right) = 0.
\end{aligned}$$

For the second statement, we write

$$\begin{aligned}
&\lim_{n\to\infty} b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x)^2,x\big) \\
&\quad = \lim_{n\to\infty} b_{n,p_n,q_n}\big\{R_{n,p_n,q_n}\big(t^2,x\big) - x^2 - 2xR_{n,p_n,q_n}\big((t-x),x\big)\big\} \\
&\quad = \lim_{n\to\infty} b_{n,p_n,q_n}\left\{\frac{p_n^{n-1}}{b_{n,p_n,q_n}(1+a_{n,p_n,q_n}x)}\right\}x \\
&\qquad + \lim_{n\to\infty} b_{n,p_n,q_n}\left\{\left(\frac{a_{n,p_n,q_n}x}{1+a_{n,p_n,q_n}x}\right)^2 - \frac{p_n^{n-1}}{[n]_{p_n,q_n}}\frac{1}{(1+a_{n,p_n,q_n}x)^2}\right\}x^2.
\end{aligned}$$

Now, by substituting the following limits into the last equality

$$\lim_{n\to\infty}\left(\frac{p_n^{n-1}}{1+a_{n,p_n,q_n}x}\right) = 1, \qquad \lim_{n\to\infty}\frac{[n]_{p_n,q_n}^{3\beta-2}x^2}{(1+a_{n,p_n,q_n}x)^2} = 0,$$
$$\lim_{n\to\infty}\frac{p_n^{n-1}}{[n]_{p_n,q_n}}\frac{b_{n,p_n,q_n}}{(1+a_{n,p_n,q_n}x)^2} = 0,$$

we get

$$\lim_{n\to\infty} b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x)^2,x\big) = x,$$

which proves the lemma. □

## 3 Local approximation theorem

Here, in this section, the local approximation theorem for the new $(p,q)$-analogue of the Balázs–Szabados operators is established. Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions $f$ on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)}|f(x)|$.

We consider Peetre's K-functional

$$K_2(f,\delta) := \inf\bigl\{\|f-g\| + \delta\bigl\|g''\bigr\| : g \in C_B^2[0,\infty)\bigr\}, \quad \delta \geq 0,$$

where

$$C_B^2[0,\infty) := \bigl\{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\bigr\}.$$

An absolute constant $C_0 > 0$ exists from the known result given in [9] such that

$$K_2(f,\delta) \leq C_0\omega_2(f,\sqrt{\delta}), \tag{14}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0\leq h\leq\sqrt{\delta}} \sup_{x\pm h\in[0,\infty)} \bigl|f(x-h) - 2f(x) + f(x+h)\bigr|$$

is the second modulus of smoothness of $f \in C_B[0,\infty)$. Also we let

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \bigl|f(x+h) - f(x)\bigr|.$$

In the following theorem, we state the first main result of the local approximation for the operators $R_{n,p,q}(f,x)$.

**Theorem 6** *There exists an absolute constant $C > 0$ such that*

$$\bigl|R_{n,p,q}(f,x) - f(x)\bigr| \leq C\omega_2\bigl(f,\sqrt{\eta_n(x)}\bigr) + \omega\bigl(f,\theta_n(x)\bigr),$$

*where $f \in C_B[0,\infty)$, $0 \leq x < \infty$, $0 < q < p < 1$, and*

$$\eta_n(x) = \bigl\{D_1(1+x)^2 + x^2\bigr\}, \quad \textit{and} \quad \theta_n(x) = \frac{a_n x^2}{1+a_n x}.$$

*Proof* Let

$$R^*_{n,p,q}(f,x) = R_{n,p,q}(f,x) + f(x) - f\bigl(\zeta_n(x)\bigr),$$

where $f \in C_B[0,\infty)$, $\zeta_n(x) = \frac{x}{1+a_n x}$. From Taylor's formula we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)\,ds, \quad g \in C_B^2[0,\infty),$$

then we have

$$R^*_{n,p,q}(g,x) = g(x) + R_{n,p,q}\biggl(\int_x^t (t-s)g''(s)\,ds, x\biggr) - \int_x^{\zeta_n(x)} \bigl(\zeta_n(x)-s\bigr)g''(s)\,ds.$$

Hence

$$
\begin{aligned}
&\left|R^*_{n,p,q}(g,x)-g(x)\right| \\
&\quad\leq R_{n,p,q}\left(\left|\int_x^t |t-s|\left|g''(s)\right|\,ds\right|,x\right)+\left|\int_x^{\zeta_n(x)}\left|\zeta_n(x)-s\right|\left|g''(s)\right|\,ds\right| \qquad (15)\\
&\quad\leq \left\|g''\right\|R_{n,p,q}\left((t-x)^2,x\right)+\left\|g''\right\|\left(\zeta_n(x)-x\right)^2\\
&\quad= \left\|g''\right\|R_{n,p,q}\left((t-x)^2,x\right)+\left\|g''\right\|\left(R_{n,p,q}\left((t-x),x\right)\right)^2\\
&\quad\leq \left\|g''\right\|\left\{D_1(1+x)^2+x^2\right\}\\
&\quad= \left\|g''\right\|\eta_n(x). \qquad (16)
\end{aligned}
$$

Using (16) and the uniform boundedness of $R^*_{n,p,q}$, we get

$$
\begin{aligned}
\left|R_{n,p,q}(f,x)-f(x)\right| &\leq \left|R^*_{n,p,q}\left((f-g),x\right)\right|+\left|R^*_{n,p,q}(g,x)-g(x)\right|\\
&\quad+\left|f(x)-g(x)\right|+\left|f\left(\zeta_n(x)\right)-f(x)\right|\\
&\leq 4\|f-g\|+\left\|g''\right\|\eta_n(x)+\omega\left(f,\left|\zeta_n(x)-x\right|\right).
\end{aligned}
$$

If we take the infimum on the right-hand side overall $g\in C_B^2[0,\infty)$, we obtain

$$
\left|R_{n,p,q}(f,x)-f(x)\right|\leq 4K_2\left(f;\eta_n(x)\right)+\omega\left(f,\theta_n(x)\right),
$$

which together with (14) gives the proof of the theorem. □

**Corollary 7** *Let* $0<q_n<p_n\leq 1$, $q_n\to 1$ *as* $n\to\infty$. *Then, for each* $f\in C[0,\infty)$, *the sequence* $\{R_{n,p_n,q_n}(f,x)\}$ *converges to* $f$ *uniformly on* $[0,a]$, $a>0$.

Now, we give a Voronovskaja type theorem for the new $(p,q)$-analogue of the Balázs–Szabados operators.

**Theorem 8** *Assume that* $0<q_n<p_n\leq 1$, $q_n\to 1$ *as* $n\to\infty$, *and let* $0<\beta<\frac{1}{2}$. *For any* $f\in C_B^2[0,\infty)$, *the following equality holds*:

$$
\lim_{n\to\infty} b_{n,p_n,q_n}\left(R_{n,p_n,q_n}(f,x)-f(x)\right)=\frac{1}{2}xf''(x),
$$

*uniformly on* $[0,a]$.

*Proof* Let $f\in C_B^2[0,\infty)$ and $x\in[0,\infty)$ be fixed. By using Taylor's formula, we write

$$
f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \qquad (17)
$$

where the function $r(t,x)$ is the Peano form of the remainder, $r(t,x)\in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x)=0$. Applying $R_{n,p_n,q_n}$ to (17), we obtain

$$
b_{n,p_n,q_n}\left(R_{n,p_n,q_n}(f,x)-f(x)\right)
$$

$$= f'(x) b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x),x\big) + \frac{1}{2} f''(x) b_{n,p_n,q_n} R_{n,p_n,q_n}\big((t-x)^2,x\big) + b_{n,p_n,q_n} R_{n,p_n,q_n}\big(r(t,x)(t-x)^2,x\big).$$

By using the Cauchy–Schwartz inequality, we get

$$R_{n,p_n,q_n}\big(r(t,x)(t-x)^2,x\big) \le \sqrt{R_{n,p_n,q_n}\big(r^2(t,x),x\big)}\sqrt{R_{n,p_n,q_n}\big((t-x)^4,x\big)}. \tag{18}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x) \in C_B[0,\infty)$. Now, from Corollary 7, it follows that

$$\lim_{n\to\infty} R_{n,p_n,q_n}\big(r^2(t,x),x\big) = r^2(x,x) = 0, \tag{19}$$

uniformly for $x \in [0,a]$. Finally, from (18), (19), and Lemma 5, we get immediately

$$\lim_{n\to\infty} b_{n,p_n,q_n} R_{n,p_n,q_n}\big(r(t,x)(t-x)^2,x\big) = 0,$$

which completes the proof. □

**Theorem 9** *Let* $0 < q_n < p_n \le 1$, $q_n \to 1$ *as* $n \to \infty$, $\alpha \in (0,1]$ *and* $A$ *be any subset of the interval* $[0,\infty)$. *Then, if* $f \in C_B[0,\infty)$ *is locally* Lip$(\alpha)$, *i.e., the condition*

$$\big|f(y) - f(x)\big| \le L|y-x|^\alpha, \quad y \in A \text{ and } x \in [0,\infty) \tag{20}$$

*holds, then, for each* $x \in [0,\infty)$, *we have*

$$\big|R_{n,p_n,q_n}(f,x) - f(x)\big| \le L\big\{\lambda_n^{\frac{\alpha}{2}}(x) + 2\big(d(x,A)\big)^\alpha\big\},$$

*where* $L$ *is a constant depending on* $\alpha$ *and* $f$ *and* $d(x,A)$ *is the distance between* $x$ *and* $A$ *defined as*

$$d(x,A) = \inf\big\{|t-x| : t \in A\big\},$$

*where*

$$\lambda_n(x) = \big[D_1(1+x)^2\big].$$

*Proof* Let $\bar{A}$ be the closure of A in $[0,\infty)$. Then there exists a point $x_0 \in \bar{A}$ such that $|x - x_0| = d(x,A)$. By the triangle inequality

$$\big|f(t) - f(x)\big| \le \big|f(t) - f(x_0)\big| + \big|f(x) - f(x_0)\big|$$

and by (20), we get

$$\begin{aligned}\big|R_{n,p_n,q_n}(f,x) - f(x)\big| &\le R_{n,p_n,q_n}\big(\big|f(t) - f(x_0)\big|,x\big) + R_{n,p_n,q_n}\big(\big|f(x) - f(x_0)\big|,x\big)\\ &\le L\big\{R_{n,p_n,q_n}\big(|t-x_0|^\alpha,x\big) + |x-x_0|^\alpha\big\}\\ &\le L\big\{R_{n,p_n,q_n}\big(|t-x|^\alpha + |x-x_0|^\alpha,x\big) + |x-x_0|^\alpha\big\}\end{aligned}$$

$$\leq L\left\{R_{n,p_n,q_n}\left(|t-x|^{\alpha},x\right)+2|x-x_0|^{\alpha}\right\}.$$

Now, by using the H¨lder inequality with $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$, we get

$$\begin{aligned}\left|R_{n,p_n,q_n}(f,x)-f(x)\right| &\leq L\left\{\left(R_{n,p_n,q_n}\left(|t-x|^{\alpha p},x\right)\right)^{\frac{1}{p}}\left(R_{n,p_n,q_n}\left(1^q,x\right)\right)^{\frac{1}{q}}+2\left(d(x,A)\right)^{\alpha}\right\}\\ &= L\left\{\left(R_{n,p_n,q_n}\left(|t-x|^2,x\right)\right)^{\frac{\alpha}{2}}+2\left(d(x,A)\right)^{\alpha}\right\}\\ &\leq L\left\{\left[D_1(1+x)^2\right]^{\frac{\alpha}{2}}+2\left(d(x,A)\right)^{\alpha}\right\},\end{aligned}$$

and the proof is completed. □

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**Authors' contributions**
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