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Local existence–uniqueness and monotone iterative approximation of positive solutions for p-Laplacian differential equations involving tempered fractional derivatives

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Abstract
In this paper, we are concerned with a kind of tempered fractional differential equation Riemann–Stieltjes integral boundary value problems with p-Laplacian operators. By means of the sum-type mixed monotone operators fixed point theorem based on the cone P_h, we obtain not only the local existence with a unique positive solution, but also construct two successively monotone iterative sequences for approximating the unique positive solution. Finally, we present an example to illustrate our main results.

MSC: 34B18; 34A08; 35J05

Keywords: Tempered fractional derivative; p-Laplacian operator; Sum-type mixed monotone operator; Existence and uniqueness; Riemann–Stieltjes integral boundary value conditions

1 Introduction

In this paper, we are concerned with local existence–uniqueness of the following nonlinear tempered fractional differential equation involving p-Laplacian operator:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t))) = f(t,u(t),u(t)) + g(t,u(t)), \quad 0 \le t \le 1, \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)\,dA(s), \\ \varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u))(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u))(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s))]\,dA(s), \end{cases} \tag{1.1}$$

where $2 < \alpha \le 3$, $0 < \beta_2 < \beta_1 < \alpha - 1$, $1 < \alpha - \gamma < 2$, $a \in C(0,1)$, φ_p is the p-Laplacian operator with $p = 2$, and ${}_0^R\mathbb{D}_t^{\alpha,\lambda}u$, ${}_0^R\mathbb{D}_t^{\gamma,\lambda}u$, and ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}u$ $(i = 1,2)$ are the empered fractional derivatives defined by

$${}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) = e^{-\lambda t}{}_0^R D_t^\alpha\bigl(e^{\lambda t}u(t)\bigr), \quad \lambda \ge 0.$$

Here ${}_0^R D_t^{\alpha}$ is the standard Riemann–Liouville fractional derivative defined by

$$ {}_0^R D_t^{\alpha} u(t) = \frac{d^n}{dt^n}\big({}_0 I_t^{n-\alpha} u(t)\big), $$

where ${}_0 I_t^{\beta}$ is the β-order fractional integral operator defined by

$$ {}_0 I_t^{\beta} \psi = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \psi(s)\, ds, $$

A is a function of bounded variation, and $\int_0^1 a(s) {}_0^R\mathbb{D}_t^{\beta_2} u(s)\, dA(s)$ is the Riemann–Stieltjes integral with respect to A. By using the sum-type mixed monotone fixed point theorem based on cone P_h we investigate the existence–uniqueness and monotone iteration of positive solutions for p-Laplacian differential systems with tempered fractional derivatives (1.1).

In the past decades, fractional calculus and all kinds of fractional differential equations have been proved to be powerful tools in the modeling of various phenomena in a great deal of fields of science and engineering, such as chemical physics, fluid mechanics, heat conduction, control theory, economics, and so on; see, for example, [1–4]. Abdullah and Zeynep [5] investigated the generalized fractional integral inequalities for continuous random variables and obtained new generalized integral inequalities for the generalized dispersion and the generalized fractional variance functions of a continuous random variable having the probability density function. Muhammad et al. [6] considered one of the important classes of Caputo fractional-order evolution equations by using fixed point theorems of Banach and Krasnoselskii type, obtained the existence and uniqueness of the solution, and studied Ulam–Hyer-type stability of the numerical solution.

In fact, a standard Riemann–Liouville (or Caputo) fractional derivative is a convolution with power law, so does fractional integration, and the difference between the two fractional derivatives only lies in the order of derivation and integration. Based on the definition of classical fractional derivative, the tempered fractional derivative multiplies the power law kernel by exponential factor, and various differential equation models based on tempered fractional derivative open up a new possibility for robust mathematical modeling of anomalous phenomena and complex multiscale problems; we refer the readers to [7–10]. In [11], we studied two kinds of tempered fractional differential systems involving the following Riemann–Stieltjes integral boundary value conditions:

$$ \begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t) + f(t,u(t),u(t)) + g(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma_1,\lambda} u(0) = {}_0^R\mathbb{D}_t^{\gamma_2,\lambda} u(0) = \cdots = {}_0^R\mathbb{D}_t^{\gamma_{n-2},\lambda} u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda} u(1) = \int_0^{\eta} b(s) {}_0^R\mathbb{D}_t^{\beta_2,\lambda} u(s)\, dA(s) + \int_0^1 a(s) {}_0^R\mathbb{D}_t^{\beta_3,\lambda} u(s)\, dA(s) \end{cases} \tag{1.2} $$

and

$$ \begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t) + \psi(t,u(t)) = c, \quad t \in (0,1), \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma_1,\lambda} u(0) = {}_0^R\mathbb{D}_t^{\gamma_2,\lambda} u(0) = \cdots = {}_0^R\mathbb{D}_t^{\gamma_{n-2},\lambda} u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda} u(1) = \int_0^{\eta} b(s) {}_0^R\mathbb{D}_t^{\beta_2,\lambda} u(s)\, dA(s) + \int_0^1 a(s) {}_0^R\mathbb{D}_t^{\beta_3,\lambda} u(s)\, dA(s), \end{cases} \tag{1.3} $$

where ${}_0^R\mathbb{D}_t^{\alpha,\lambda}u$, ${}_0^R\mathbb{D}_t^{\gamma_k,\lambda}u$ ($k = 1,2,\dots,n-2$), and ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}u$ ($i = 1,2,3$) are the tempered fractional derivatives. By using a class of sum-type mixed monotone operators fixed point theorems and increasing φ-(h,σ)-concave operators fixed point theorems, respectively, we constructed sufficient conditions to guarantee the existence–uniqueness of positive solutions for Riemann–Stieltjes integral boundary value problems (1.2) and (1.3), respectively.

It is well known that the p-Laplacian operator is used in analyzing various complex problems in physics, mechanics, and the related fields of mathematical modeling; see [12–14]. In [12], for studying the turbulent flow in a kind of porous media, Leibenson introduced the p-Laplacian differential equation

$$\bigl(\varphi_p\bigl(u'(t)\bigr)\bigr)' = f\bigl(t,u(t),u'(t)\bigr), \quad t \in (0,1), \tag{1.4}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. Motivated by Leibenson's work, Ren, Li, and Zhang [15] studied the existence of maximum and minimum solutions for the nonlocal p-Laplacian fractional differential system

$$\begin{cases} -D_t^{\beta_1}(\varphi_{p_1}(-D_t^{\alpha_1}x_1))(t) = f_1(x_1(t),x_2(t)), \\ -D_t^{\beta_2}(\varphi_{p_2}(-D_t^{\alpha_2}x_2))(t) = f_2(x_1(t),x_2(t)), \\ x_1(0) = 0, \qquad D_t^{\alpha_1}x_1(0) = D_t^{\alpha_1}x_1(1) = 0, \qquad x_1(1) = \int_0^1 x_1(t)\,dA_1(t), \\ x_2(0) = 0, \qquad D_t^{\alpha_2}x_2(0) = D_t^{\alpha_2}x_2(1) = 0, \qquad x_2(1) = \int_0^1 x_2(t)\,dA_2(t), \end{cases} \tag{1.5}$$

where φ_{p_i} denotes the p-Laplacian operator, $D_t^{\alpha_i}$, $D_t^{\beta_i}$ are the standard Riemann–Liouville derivatives with $1 < \alpha_i$, $\beta_i < 2$, $\int_0^1 x_i(t)\,dA_i(t)$ denotes the Riemann–Stieltjes integral, and A_i is a function of bounded variation. By employing the cone theory and monotone iterative technique, some new existence results on maximal and minimal solutions were established. Furthermore, the estimation of the bounds of maximum and minimum solutions was derived.

In [16], we investigated the existence of multiple positive solutions for the following p-Laplacian fractional differential equations with two-point boundary values:

$$\begin{cases} {}_0^RD_t^{\alpha}(\varphi_p({}_0^RD_t^{\alpha}u(t))) = f(t,u(t),{}_0^RD_t^{\alpha}u(t)), \quad 0 \le t \le 1; \\ u^{(i)}(0) = 0, \qquad [\varphi_p({}_0^RD_t^{\alpha}u)]^{(i)}(0) = 0, \quad i = 0,1,2,\dots,n-2; \\ [{}_0^RD_t^{\beta}u(t)]_{t=1} = 0, \quad 0 < \beta \le \alpha - 1; \\ [{}_0^RD_t^{\beta}(\varphi_p({}_0^RD_t^{\alpha}u(t)))]_{t=1} = 0; \end{cases} \tag{1.6}$$

where $n-1 < \alpha \le n$, ${}_0^RD_t^{\alpha}$ is the standard Riemann–Liouville fractional derivative, and φ_p is the p-Laplacian operator. By employing the functional-type cone expansion–compression fixed point theorem and Leggett–Williams fixed point theorem, we obtained the existence of multiple positive solutions for p-Laplacian differential systems (1.6).

To study more boundary value problems for complex fractional differential equations, we combine the Riemann–Stieltjes integral boundary value conditions with p-Laplacian operators, where the nonlinear terms are sum-type nonlinear terms in (1.1). Comparing with the previous references, this paper has the following characteristics. Firstly, the tempered fractional derivative ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is more general than the standard Riemann–Liouville fractional derivative ${}_0^R D_t^\alpha$, for instance, for $\lambda = 0$, it is clear that ${}_0^R \mathbb{D}_t^{\alpha,\lambda}$ is equivalent to ${}_0^R D_t^\alpha$. Secondly, Riemann–Stieltjes integral boundary conditions are more general and cover the common integral boundary conditions as particular cases. Finally, comparing with p-Laplacian differential systems (1.6), the integral operator in this paper need not be completely continuous or compact. Furthermore, we not only obtain the local existence with a unique positive solution, but also construct a Cauchy sequence to approximate the unique positive solution.

The organization of the paper is as follows. In Sect. 2, we list some concepts, symbols, definitions, and lemmas in abstract Banach spaces, which need to be used in the subsequent proof process. In Sect. 3, by employing the sum-type mixed monotone operators fixed point theorem based on cone P_h we show that the existence–uniqueness and monotone iteration of positive solutions of the two-point boundary value problems for the p-Laplacian differential equation (1.1). In Sect. 4, we present an example to demonstrate our main results.

2 Preliminaries

A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following conditions:

(I_1) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$;

(I_2) $x \in P$, $-x \in P \Rightarrow x = \theta$.

In addition, let $(E, \|\cdot\|)$ be a real Banach space that is partially ordered by a cone $P \subset E$, that is, $y - x \in P$ implies that $x \leq y$. If $x \leq y$ and $x \neq y$, then we write $x < y$ or $y > x$. We denote the zero element of E by θ. If for all $x, y \in E$, there exists $M > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq \|y\|$, then the cone P is called normal; in this case, M is the infimum of such constants and is called the normality constant of P.

Furthermore, for $h > \theta$, denote $P_h = \{x \in E \mid x \sim h\}$, where $\sim$ is an equivalence relation, that is, for all $x, y \in E$, $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \geq y \geq \mu x$.

Definition 2.1 ([17]) $A : P \to P$ is said to be subhomogeneous if

$$A(tx) \geq tAx, \quad \forall t \in (0,1), x \in P.$$

Definition 2.2 ([18]) An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, that is, u_i, v_i $(i = 1, 2) \in P$, $u_1 < u_2$, $v_1 > v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.3 ([12]) For $p > 1$, the p-Laplacian operator is given by

$$\varphi_p(x) = |x|^{p-2}x \quad \text{and} \quad \varphi_p^{-1} = \varphi_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.1 ([11]) *Let* $h(t) \in C[0,1] \cap L^1[0,1]$, $\alpha > 0$. *Then*

$${}_0I_t^\alpha {}_0^R D_t^\alpha h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, 3, \ldots, n$ $(n = [\alpha] + 1)$.

Lemma 2.2 ([16])

(1) *If* $u \in L^1(0,1)$, $\alpha > \beta > 0$, *then*

$$ {}_0I_t^{\alpha}\,{}_0I_t^{\beta}u(t) = {}_0I_t^{\alpha+\beta}u(t), \qquad {}_0^RD_t^{\beta}\,{}_0I_t^{\alpha}u(t) = {}_0I_t^{\alpha-\beta}u(t), \qquad {}_0^RD_t^{\beta}\,{}_0I_t^{\beta}u(t) = u(t); $$

(2) *If* $\rho > 0$, $\mu > 0$, *then*

$$ {}_0^RD_t^{\rho}t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)}t^{\mu-\rho-1}. $$

Lemma 2.3 *Given* $g \in C(0,1)$, *the unique solution of*

$$ \begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) + g(t) = 0, \quad 2 < \alpha \leq 3, t \in (0,1), \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^{\eta} a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)\,dA(s), \end{cases} \tag{2.1} $$

is

$$ u(t) = \int_0^1 G(t,s)g(s)\,ds, \quad t \in [0,1], \tag{2.2} $$

where

$$ G(t,s) = G_1(t,s) + \frac{t^{\alpha-1}e^{-\lambda t}}{\Delta\Gamma(\alpha-\beta_2)}\int_0^{\eta} a(t)G_2(t,s)\,dA(t) \tag{2.3} $$

with

$$ \begin{aligned} &\Delta = \frac{e^{-\lambda}}{\Gamma(\alpha-\beta_1)} - \frac{\delta}{\Gamma(\alpha-\beta_2)}, \quad \delta = \int_0^{\eta} e^{-\lambda s}s^{\alpha-\beta_2-1}a(s)\,dA(s), \\ &G_1(t,s) = \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}\begin{cases} (1-s)^{\alpha-\beta_1-1}t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-\beta_1-1}t^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ &G_2(t,s) = \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}\begin{cases} (1-s)^{\alpha-\beta_1-1}t^{\alpha-\beta_2-1} - (t-s)^{\alpha-\beta_2-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-\beta_1-1}t^{\alpha-\beta_2-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} $$

Proof For system (2.1), by means of Lemma 2.1 we have

$$ e^{\lambda t}u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\lambda s}g(s)\,ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}. \tag{2.4} $$

From $u(0) = 0$ we get $c_3 = 0$, and hence

$$ u(t) = -e^{-\lambda t}{}_0I_t^{\alpha}e^{\lambda t}g(t) + c_1e^{-\lambda t}t^{\alpha-1} + c_2e^{-\lambda t}t^{\alpha-2}. \tag{2.5} $$

By using the tempered fractional-order derivative operator ${}_0^R\mathbb{D}_t^{\gamma,\lambda}$ on both sides of (2.5), we obtain

$$\begin{aligned}
{}_0^R\mathbb{D}_t^{\gamma,\lambda}u(t) &= {}_0^R\mathbb{D}_t^{\gamma,\lambda}\big(-e^{-\lambda t}{}_0I_t^{\alpha}\big(e^{\lambda t}g(t)\big) + c_1e^{-\lambda t}t^{\alpha-1} + c_2e^{-\lambda t}t^{\alpha-2}\big)\\
&= e^{-\lambda t}{}_0^RD_t^{\gamma}\big(-{}_0I_t^{\alpha}\big(e^{\lambda t}g(t)\big) + c_1t^{\alpha-1} + c_2t^{\alpha-2}\big)\\
&= -e^{-\lambda t}{}_0I_t^{\alpha-\gamma}\big(e^{\lambda t}g(t)\big) + c_1e^{-\lambda t}{}_0^RD_t^{\gamma}t^{\alpha-1} + c_2e^{-\lambda t}{}_0^RD_t^{\gamma}t^{\alpha-2}\\
&= -\int_0^t \frac{(t-s)^{\alpha-\gamma-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\gamma)}g(s)\,ds + c_1\frac{\Gamma(\alpha)e^{-\lambda t}}{\Gamma(\alpha-\gamma)}t^{\alpha-1-\gamma}\\
&\quad + c_2\frac{\Gamma(\alpha-1)e^{-\lambda t}}{\Gamma(\alpha-1-\gamma_1)}t^{\alpha-2-\gamma}.
\end{aligned}$$

From ${}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0$ and $1 < \alpha - \gamma \leq 2$ we know that $c_2 = 0$. Hence equation (2.5) can be reduced to

$$u(t) = -e^{-\lambda t}\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda s}}{\Gamma(\alpha)}g(s)\,ds + c_1e^{-\lambda t}t^{\alpha-1}. \tag{2.6}$$

Once again, applying the tempered fractional derivative operator ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}$ on the both sides of (2.6), we have

$$\begin{aligned}
{}_0^R\mathbb{D}_t^{\beta_i,\lambda}u(t) &= -{}_0^R\mathbb{D}_t^{\beta_i,\lambda}\big(e^{-\lambda t}{}_0I_t^{\alpha}\big(e^{\lambda t}g(t)\big)\big) + c_1{}_0^R\mathbb{D}_t^{\beta_i,\lambda}\big(e^{-\lambda t}t^{\alpha-1}\big)\\
&= -e^{-\lambda t}{}_0^RD_t^{\beta_i}\big({}_0I_t^{\alpha}\big(e^{\lambda t}g(t)\big)\big) + c_1e^{-\lambda t}{}_0^RD_t^{\beta_i}\big(t^{\alpha-1}\big)\\
&= -e^{-\lambda t}{}_0I_t^{\alpha-\beta_i}\big(e^{\lambda t}g(t)\big) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}e^{-\lambda t}t^{\alpha-1-\beta_i}\\
&= -\int_0^t \frac{(t-s)^{\alpha-\beta_i-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_i)}g(s)\,ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}e^{-\lambda t}t^{\alpha-1-\beta_i}.
\end{aligned} \tag{2.7}$$

From (2.7) it is clear that

$$\begin{cases}
{}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \frac{-1}{\Gamma(\alpha-\beta_1)}\int_0^1(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}g(s)\,ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_1)}e^{-\lambda},\\
{}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(t) = \frac{-1}{\Gamma(\alpha-\beta_2)}\int_0^t(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}g(s)\,ds + c_1\frac{\Gamma(\alpha)e^{-\lambda t}}{\Gamma(\alpha-\beta_2)}t^{\alpha-1-\beta_2}.
\end{cases} \tag{2.8}$$

Substituting (2.8) into ${}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)\,dA(s)$, we obtain

$$\begin{aligned}
c_1 &= \big[\Gamma(\alpha)\Delta\big]^{-1}\bigg\{\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha-\beta_1)}g(s)\,ds\\
&\quad - \int_0^\eta a(t)\,dA(t)\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}g(s)\,ds\bigg\}.
\end{aligned} \tag{2.9}$$

Finally, combining (2.9) with (2.6), we obtain

$$
\begin{aligned}
u(t) &= -e^{-\lambda t}\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda s}}{\Gamma(\alpha)}g(s)\,ds + \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)\Delta}\left\{\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{-\lambda}}{\Gamma(\alpha-\beta_1)}e^{\lambda s}g(s)\,ds \right.\\
&\quad \left. - \int_0^\eta a(t)\,dA(t)\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}g(s)\,ds\right\}\\
&= -\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda(s-t)}}{\Gamma(\alpha)}g(s)\,ds + \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta_1-1}e^{\lambda s}g(s)\,ds\\
&\quad + \frac{e^{-\lambda t}t^{\alpha-1}\delta}{\Gamma(\alpha)\Gamma(\alpha-\beta_2)\Delta}\int_0^1 (1-s)^{\alpha-\beta_1-1}e^{\lambda s}g(s)\,ds\\
&\quad - \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha-\beta_2)\Delta}\int_0^\eta a(t)\,dA(t)\int_0^t (t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}g(s)\,ds\\
&= \int_0^1 G_1(t,s)g(s)\,ds + \frac{t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha-\beta_2)\Delta}\int_0^1 g(s)\,ds\int_0^\eta G_2(t,s)a(t)\,dA(t)\\
&= \int_0^1 G(t,s)g(s)\,ds,
\end{aligned}
$$

where $G(t,s)$ is the Green function of system (2.1). The proof is complete. □

Lemma 2.4 *Suppose that*

$$(H)\quad \Gamma(\alpha-\beta_1)e^{\lambda}\delta < \Gamma(\alpha-\beta_2).$$

Then for all $(t,s)\in[0,1]\times[0,1]$, $G(t,s)$, $G_1(t,s)$, *and* $G_2(t,s)$ *satisfy*

(A_1) $G(t,s)$, $G_1(t,s)$, *and* $G_2(t,s)$ *are all continuous in* $(t,s)\in[0,1]\times[0,1]$;

(A_2) $G_i(t,s)\geq 0$ $(i=1,2)$, *and* $G(t,s)\geq 0$;

(A_3) $\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1} \leq G_1(t,s) \leq \frac{e^{\lambda s}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1}$;

(A_4) $\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-\beta_2-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-\beta_2-1} \leq G_2(t,s) \leq \frac{e^{\lambda s}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-\beta_2-1}$;

(A_5) $m(s)e^{-\lambda t}t^{\alpha-1} \leq G(t,s) \leq M(s)e^{-\lambda t}t^{\alpha-1}$, *where*

$$M(s) = \left[\frac{1}{\Gamma(\alpha)} + \frac{\delta}{\Delta\Gamma(\alpha)\Gamma(\alpha-\beta_2)}\right]e^{\lambda s}(1-s)^{\alpha-\beta_1-1}$$

and

$$m(s) = \frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)} + \frac{\delta e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-\beta_2-1}]}{\Delta\Gamma(\alpha)\Gamma(\alpha-\beta_2)}.$$

Proof Firstly, for $(t,s)\in[0,1]\times[0,1]$, it is obvious that $G(t,s)$ and $G_i(t,s)$ $(i=1,2)$ are continuous.

Secondly, for $G_i(t,s)$ $(i=1,2)$ in (A_3) and (A_4), it is evident that the right sides of the inequalities hold, so we only need to prove the left sides of the inequalities. If $0\leq s\leq t\leq 1$, then we easily see that $0\leq t-s\leq t-ts=(1-s)t$, and thus $(t-s)^{\alpha-1}\leq(1-s)^{\alpha-1}t^{\alpha-1}$. So we get

$$G_1(t,s) \geq \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-\beta_1-1} - (1-s)^{\alpha-1}t^{\alpha-1}\right]$$

$$= \frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1}.$$

If $0 \le t \le s \le 1$, then, evidently, $G_1(t,s) \ge \frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1}$.

Furthermore, from $(1-s)^{\alpha-\beta_1-1} > (1-s)^{\alpha-1}$ we get $G_1(t,s) \ge 0$ for all $(t,s) \in [0,1]\times[0,1]$. In the same way, we can get that $G_2(t,s) \ge 0$ and inequality (A_4) holds.

Finally, from (A_3) and (A_4) we can get that $m(s)e^{-\lambda t}t^{\alpha-1} \le G(t,s) \le M(s)e^{-\lambda t}t^{\alpha-1}$. In addition, from condition (H) we can deduce that $\Delta > 0$. Combining $(1-s)^{\alpha-\beta_1-1} > (1-s)^{\alpha-1}$ with $\Delta > 0$, we obtain $m(s) \ge 0$, that is, $G(s,t) \ge 0$ for all $(t,s) \in [0,1]\times[0,1]$. The proof is complete. □

Lemma 2.5 ([17]) *Let $\xi \in (0,1)$, let $A : P\times P \to P$ be a mixed monotone operator satisfying*

$$A\bigl(tx, t^{-1}y\bigr) \ge t^{\xi}A(x,y), \quad \forall t \in (0,1), x,y \in P. \tag{2.10}$$

Let $B : P \to P$ be an increasing subhomogeneous operator. Assume that

(I) *there exists $h_0 \in P_h$ such that $A(h_0,h_0) \in P_h$ and $Bh_0 \in P_h$;*

(II) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \ge \delta_0 Bx$, $\forall x,y \in P$;*

Then

(1) $A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;

(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \le u_0 < v_0, \qquad u_0 \le A(u_0,v_0) + Bu_0 \le A(v_0,u_0) + B(v_0) \le v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution x^* in P_h;*

(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1},y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1},x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3 Main results

Lemma 3.1 *For $\widetilde{g} \in C[0,1]$, the p-Laplacian tempered fractional differential system*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t))) = \widetilde{g}(t), \quad 2 < \alpha \le 3, 0 \le t \le 1, \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0, \\ \varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0)) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0))) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)\,dA(s), \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(1))) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s))]\,dA(s), \end{cases} \tag{3.1}$$

has a unique integral formal solution

$$u(t) = \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)\widetilde{g}(\tau)\,d\tau\biggr)\,ds, \tag{3.2}$$

where $G(t,s)$ is given in (2.3).

Proof Firstly, applying the fractional integral operator ${}_0I_t^\alpha$ on both sides of the first equation of integral boundary value problems (3.1), we have

$$\begin{aligned} e^{\lambda t}\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) &= {}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1t^{\alpha-1} + d_2t^{\alpha-2} + d_3t^{\alpha-3} \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\lambda s}\widetilde{g}(s)\,ds + d_1t^{\alpha-1} + d_2t^{\alpha-2} + d_3t^{\alpha-3}. \end{aligned}$$

From $\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)(0) = 0$ we can deduce that $d_3 = 0$. So

$$\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) = e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}t^{\alpha-1} + d_2e^{-\lambda t}t^{\alpha-2}. \tag{3.3}$$

Furthermore, applying the tempered fractional derivative operator ${}_0^R\mathbb{D}_t^{\gamma,\lambda}$ on both sides of (3.3), we have

$$\begin{aligned} {}_0^R\mathbb{D}_t^{\gamma,\lambda}\big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\big) &= {}_0^R\mathbb{D}_t^{\gamma,\lambda}\big(e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}t^{\alpha-1} + d_2e^{-\lambda t}t^{\alpha-2}\big) \\ &= e^{-\lambda t}{}_0I_t^{\alpha-\gamma}\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}{}_0^RD_t^\gamma t^{\alpha-1} + d_2e^{-\lambda t}{}_0^RD_t^\gamma t^{\alpha-2} \\ &= \int_0^t \frac{(t-s)^{\alpha-\gamma-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\gamma)}\widetilde{g}(s)\,ds + d_1\frac{\Gamma(\alpha)e^{-\lambda t}}{\Gamma(\alpha-\gamma)}t^{\alpha-1-\gamma} \\ &\quad + d_2\frac{\Gamma(\alpha-1)e^{-\lambda t}}{\Gamma(\alpha-1-\gamma_1)}t^{\alpha-2-\gamma}. \end{aligned}$$

From ${}_0^R\mathbb{D}_t^{\gamma,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u))(0) = 0$ and $1 < \alpha - \gamma < 2$ we deduce that $d_2 = 0$, that is,

$$\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) = e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}t^{\alpha-1}. \tag{3.4}$$

Secondly, applying the tempered fractional derivative operator ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}$ $(i = 1,2)$ on both sides of (3.4), we get

$$\begin{aligned} {}_0^R\mathbb{D}_t^{\beta_i,\lambda}\big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\big) &= {}_0^R\mathbb{D}_t^{\beta_i,\lambda}\big(e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big)\big) + d_1{}_0^R\mathbb{D}_t^{\beta_i,\lambda}\big(e^{-\lambda t}t^{\alpha-1}\big) \\ &= e^{-\lambda t}{}_0I_t^{\alpha-\beta_i}\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}{}_0^RD_t^{\beta_i}\big(t^{\alpha-1}\big) \\ &= \int_0^t \frac{(t-s)^{\alpha-\beta_i-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_i)}\widetilde{g}(s)\,ds + d_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}e^{-\lambda t}t^{\alpha-1-\beta_i}. \end{aligned} \tag{3.5}$$

From (3.5) it is clear that

$$\begin{cases} {}_0^R\mathbb{D}_t^{\beta_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(1))) = \int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha-\beta_1)}\widetilde{g}(s)\,ds + d_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_1)}e^{-\lambda}, \\ {}_0^R\mathbb{D}_t^{\beta_2,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t))) = \int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}\widetilde{g}(s)\,ds + d_1\frac{\Gamma(\alpha)e^{-\lambda t}}{\Gamma(\alpha-\beta_2)}t^{\alpha-1-\beta_2}. \end{cases} \tag{3.6}$$

Combining (3.6) with the Riemann–Stieltjes integral boundary value condition ${}_0^R\mathbb{D}_t^{\beta_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u))(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s))]\,dA(s)$, we obtain

$$\begin{aligned} d_1 = \frac{-1}{\Gamma(\alpha)\Delta}\bigg\{&\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha-\beta_1)}\widetilde{g}(s)\,ds \\ &- \int_0^\eta a(t)\,dA(t)\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}\widetilde{g}(s)\,ds\bigg\}. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.4), we obtain

$$\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) = -\int_0^1 G(t,s)\widetilde{g}(s)\,ds. \tag{3.8}$$

Furthermore, applying the p-Laplacian operator φ_q on both sides of (3.8), we get

$${}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) + \varphi_q\left(\int_0^1 G(t,s)\widetilde{g}(s)\,ds\right) = 0. \tag{3.9}$$

Finally, setting $g(t) :\triangleq \varphi_q(\int_0^1 G(t,s)\widetilde{g}(s)\,ds)$, we easily see that p-Laplacian fractional differential system (3.1) is equivalent to the following fraction differential equation integral boundary value problem:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) + g(t) = 0, & t \in (0,1), 2 < \alpha \leq 3; \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)\,dA(s). \end{cases} \tag{3.10}$$

By means of Lemma 2.3 we get that the tempered fractional differential system with p-Laplacian operator (3.10) has a unique integral solution

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)g(s)\,ds \\ &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\widetilde{g}(\tau)\,d\tau\right)ds. \end{aligned}$$

This completes the proof. □

From Lemma 3.1 we can deduce that Riemann–Stieltjes integral boundary value problem with p-Laplacian operator (1.1) is equivalent to the integral formulation

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,u(\tau),u(\tau)\big) + g\big(\tau,u(\tau)\big)\big]\,d\tau\right)ds. \tag{3.11}$$

For the convenience of further research, we define the operator T by

$$T(u,v)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,u(\tau),v(\tau)\big) + g\big(\tau,u(\tau)\big)\big]\,d\tau\right)ds. \tag{3.12}$$

Theorem 3.1 *Suppose that condition* (H) *holds,* $a(t) : [0,1] \to R^+$, $f(t,u,v) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, *and* $g(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *are all continuous functions with* $g(t,u) \not\equiv 0$, *and the following conditions are satisfied*:

(H_1) *for fixed* $t \in [0,1]$, $f(t,u,v)$ *is increasing in* $u \in [0,+\infty)$ *and decreasing in* $v \in [0,+\infty)$. *In addition, for all* $\gamma \in (0,1)$ *and* $u,v \in [0,+\infty)$, *there exists a constant* $\xi \in (0,1)$ *such that*

$$f\big(t,\gamma u,\gamma^{-1}v\big) \geq \varphi_p^{\xi}(\gamma)f(t,u,v); \tag{3.13}$$

(H_2) *for fixed* $t \in [0,1]$, $g(t,u)$ *is increasing in* $u \in [0,+\infty)$, *and for all* $t \in [0,1]$, $u \in [0,+\infty)$, *and* $\gamma \in (0,1)$,

$$g(t,\gamma u) \geq \varphi_p(\gamma) g(t,u); \tag{3.14}$$

(H_3) *for all* $u,v \in [0,+\infty)$, *there exists a constant* $\delta_0 > 0$ *such that*

$$f(t,u,v) \geq \varphi_p(\delta_0) g(t,u), \quad t \in [0,1]. \tag{3.15}$$

Then we have:

(I) *the tempered fractional differential equation Riemann–Stieltjes integral boundary value problem involving the p-Laplacian operator* (1.1) *has a unique positive solution* $u^* \in P_h$, *where* $h(t) = e^{-\lambda t} t^{\alpha-1}$;

(II) *for all* $t \in [0,1]$, *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leq u_0 < v_0$ *and*

$$u_0(t) \leq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,u_0(\tau),v_0(\tau)\big) + g\big(\tau,u_0(\tau)\big)\big]\,d\tau\right) ds,$$
$$v_0(t) \geq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,v_0(\tau),u_0(\tau)\big) + g\big(\tau,v_0(\tau)\big)\big]\,d\tau\right) ds;$$

(III) *for any initial values* $x_0, y_0 \in P_h$, *making the successive sequences*

$$x_n = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,x_{n-1}(\tau),y_{n-1}(\tau)\big) + g\big(\tau,x_{n-1}(\tau)\big)\big]\,d\tau\right) ds,$$
$$y_n = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,y_{n-1}(\tau),x_{n-1}(\tau)\big) + g\big(\tau,y_{n-1}(\tau)\big)\big]\,d\tau\right) ds,$$
$$n = 0,1,2,\ldots,$$

we obtain $x_n \to u^*$ *and* $y_n \to u^*$ *as* $n \to \infty$.

Proof To begin with, we define two operators $A : P \times P \to E$ and $B : P \to E$ by

$$A(u,v)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau) f\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right) ds, \tag{3.16}$$

$$B(u)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau) g\big(\tau,u(\tau)\big)\,d\tau\right) ds. \tag{3.17}$$

From $p = 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ we easily see that $q = 2$. Evidently, we have $T(u,v) = A(u,v) + B(u)$. In addition, u^* is a solution of the Riemann–Stieltjes integral boundary value problem (1.1) if and only if $T(u^*,u^*) = u^*$. From Lemma 2.4 we get $A : P \times P \to P$ and $B : P \to P$. Furthermore, it follows from (H_1) and (H_2) that A is a mixed monotone operator and B is an increasing operator. For all $\gamma \in (0,1)$ and $u,v \in P$, from (3.13) we obtain

$$\begin{aligned} A\big(\gamma u,\gamma^{-1}v\big)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau) f\big(\tau,\gamma u(\tau),\gamma v(\tau)\big)\,d\tau\right) ds \\ &\geq \int_0^1 G(t,s)\varphi_q\left(\varphi_p^{\xi}(\gamma)\int_0^1 G(s,\tau) f\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right) ds \\ &= \gamma^{\xi} A(u,v)(t). \end{aligned} \tag{3.18}$$

Hence the mixed monotone operator A satisfies condition (2.10) in Lemma 2.5. In addition, for all $\gamma \in (0,1)$ and $u \in P$, from (3.14) we have

$$\begin{aligned} B(\gamma u)(t) &= \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)g\bigl(\tau,\gamma u(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq \varphi_q\bigl(\varphi_p(\gamma)\bigr)\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)g\bigl(\tau,u(\tau)\bigr)\,d\tau\biggr)\,ds \\ &= \gamma B(u)(t). \end{aligned} \tag{3.19}$$

So B is a subhomogeneous operator.

Next, we show that $A(h,h) \in P_h$ and $Bh \in P_h$. From Lemma 2.4 we have

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 M(\tau)e^{-\lambda s}s^{\alpha-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \int_0^1 M(s)e^{-\lambda t}t^{\alpha-1}\varphi_q\biggl(\int_0^1 M(\tau)e^{-\lambda s}s^{\alpha-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \biggl(\int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\biggl(\int_0^1 M(\tau)f(\tau,h_{\max},0)\,d\tau\biggr)\,ds\biggr)e^{-\lambda t}t^{\alpha-1} \end{aligned}$$

and

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 m(\tau)e^{-\lambda s}s^{\alpha-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq \int_0^1 m(s)e^{-\lambda t}t^{\alpha-1}\varphi_q\biggl(\int_0^1 m(\tau)e^{-\lambda s}s^{\alpha-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\geq \biggl(\int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\biggl(\int_0^1 m(\tau)f(\tau,0,h_{\max})\,d\tau\biggr)\,ds\biggr)e^{-\lambda t}t^{\alpha-1}, \end{aligned}$$

where $h_{\max} = \max\{h(t): t \in [0,1]\}$. Letting

$$\begin{aligned} L_1 &\triangleq \int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\biggl(\int_0^1 M(\tau)f(\tau,h_{\max},0)\,d\tau\biggr)\,ds, \\ l_1 &\triangleq \int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\biggl(\int_0^1 m(\tau)f(\tau,0,h_{\max})\,d\tau\biggr)\,ds. \end{aligned}$$

It is clear that $L_1 > l_1 > 0$. Hence $l_1 h(t) \leq A(h,h) \leq L_1 h(t)$, that is, $A(h,h) \in P_h$. Similarly, for the subhomogeneous operator B, from Lemma 2.4 we get

$$\begin{aligned} B(h)(t) &= \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)g\bigl(\tau,h(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\leq \biggl(\int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\biggl(\int_0^1 M(\tau)g(\tau,h_{\max})\,d\tau\biggr)\,ds\biggr)e^{-\lambda t}t^{\alpha-1} \end{aligned}$$

and

$$\begin{aligned} B(h)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g\big(\tau,h(\tau)\big)\,d\tau\right)ds \\ &\geq \left(\int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 m(\tau)g(\tau,0)\,d\tau\right)ds\right)e^{-\lambda t}t^{\alpha-1}. \end{aligned}$$

Letting

$$\begin{aligned} L_2 &\triangleq \int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 M(\tau)g(\tau,h_{\max})\,d\tau\right)ds, \\ l_2 &\triangleq \int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 m(\tau)g(\tau,0)\,d\tau\right)ds. \end{aligned}$$

From $L_2 > l_2 > 0$ and $l_2 h(t) \leq B(h) \leq L_2 h(t)$ we get $Bh \in P_h$. Since $h \in P_h$, letting $h_0 = h$, we get that condition (I_1) in Lemma 2.5 holds.

Finally, for all $u, v \in P$, from (3.15) we have

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\ &\geq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 \varphi_p(\delta_0)G(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\right)ds \\ &= \delta_0(Bu)(t), \end{aligned} \tag{3.20}$$

that is, $A(u,v) \geq \delta_0 Bu$. All the conditions in Lemma 2.5 are satisfied. So the conclusion in Theorem 3.1 follows from Lemma 2.5. □

Corollary 3.1 *Assume that condition (H) holds and*

- (H_1') *$a(t) : [0,1] \to R^+$ and $f(t,u,v) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are all continuous functions;*
- (H_2') *for fixed $t \in [0,1]$, $f(t,u,v)$ is increasing in $u \in [0,+\infty)$ and decreasing in $v \in [0,+\infty)$;*
- (H_3') *for all $u, v \in [0,+\infty)$ and $\gamma \in (0,1)$, there exists a constant $\xi \in (0,1)$ such that*

$$f\big(t,\gamma u,\gamma^{-1}v\big) \geq \varphi_p^{\xi}(\gamma)f(t,u,v), \quad t \in [0,1]. \tag{3.21}$$

Then we have:

- (I) *the p-Laplacian differential equation Riemann–Stieltjes integral boundary value problem*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t))) = f(t,u(t),u(t)), \quad 0 \leq t \leq 1, \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)\,dA(s), \\ \varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0)) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0))) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(1))) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s))]\,dA(s), \end{cases}$$

  has a unique positive solution $u^* \in P_h$, where $h(t) = e^{-\lambda t}t^{\alpha-1}$;

(*II*) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \le u_0 < v_0$ *and*

$$u_0(t) \le \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,u_0(\tau),v_0(\tau)\big)\,d\tau\right)ds,$$
$$v_0(t) \ge \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,v_0(\tau),u_0(\tau)\big)\,d\tau\right)ds;$$

(*III*) *for any initial values* $x_0, y_0 \in P_h$, *making the successive sequences*

$$x_n = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,x_{n-1}(\tau),y_{n-1}(\tau)\big)\,d\tau\right)ds,$$
$$y_n = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,y_{n-1}(\tau),x_{n-1}(\tau)\big)\,d\tau\right)ds,$$
$$n = 0,1,2,\ldots,$$

we obtain $x_n \to u^*$ *and* $y_n \to u^*$ *as* $n \to \infty$.

Proof Setting $g(t,u(t)) \equiv 0$, by means of Theorem 3.1 we get the conclusions. □

4 Applications

Example 1 We consider the following tempered fractional differential systems involving the p-Laplacian operator:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{5}{2},1}(\varphi_p({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(t))) = F(t,u(t)), \quad 0 \le t \le 1; \\ u(0) = {}_0^R\mathbb{D}_t^{\frac{3}{4},1}u(0) = 0; \\ {}_0^R\mathbb{D}_t^{1,1}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\frac{5}{8},1}u(s)\,dA(s); \\ \varphi_p({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(0)) = {}_0^R\mathbb{D}_t^{\frac{3}{4},1}(\varphi_p({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(0))) = 0; \\ {}_0^R\mathbb{D}_t^{1,1}(\varphi_p({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(1))) = \int_0^1 {}_0^R\mathbb{D}_t^{\frac{5}{8},1}[\varphi_p({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(s))]\,dA(s); \end{cases} \tag{4.1}$$

where $F(t,u(t)) = f(t,u(t),u(t)) + g(t,u(t))$, $A(t) = \frac{t}{2}$, and $p = 2$, in which $f(t,u,v) = (1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}}$, $g(t,u) = (1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}u^{\frac{1}{3}}$. In addition, for any $t \in (0,1)$, $u > 0$ and $v > 0$.

Let us check that all the conditions in Theorem 3.1 are satisfied. Evidently, $\alpha = \frac{5}{2}$, $\beta_1 = 1$, $\beta_2 = \frac{5}{8}$, $\gamma = \frac{3}{4}$, $a(t) \equiv 1$, $\lambda = 1 > 0$, and $\eta = 1$ in system (4.1).

(1) From $\delta = \int_0^\eta e^{-\lambda s}s^{\alpha-\beta_2-1}a(s)\,dA(s) = 0.1432$ we can get $\Gamma(\alpha-\beta_1)e^\lambda\delta = 0.345 < 0.9534 = \Gamma(\alpha-\beta_2)$; clearly, condition (H) is satisfied.

(2) From the expressions of f and g it is evident that $f(t,u,v) : (0,1) \times R^+ \times R^+ \to R^+$ and $g(t,u) : (0,1) \times R^+ \to R^+$ are continuous. Furthermore, $f(t,u,v)$ is increasing in u for fixed $t \in (0,1)$ and $v \in R^+$, decreasing in v for fixed $t \in (0,1)$ and $u \in R^+$, and, in addition, for fixed $t \in (0,1)$, $g(t,u)$ is increasing in u.

(3) For any $\gamma \in (0,1)$, $t \in (0,1)$, and $u, v > 0$, taking $\xi = \frac{1}{2} \in (0,1)$, we have

$$\begin{aligned} f\left(t, \gamma u, \gamma^{-1} v\right) &= (1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} (\gamma u)^{\frac{1}{3}} + \left(\gamma^{-1} v\right)^{-\frac{1}{5}} \\ &\geq \gamma^{\frac{1}{2}} \left[(1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} u^{\frac{1}{3}} + v^{-\frac{1}{5}}\right] \\ &\geq \gamma \left[(1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} u^{\frac{1}{3}} + v^{-\frac{1}{5}}\right] \\ &= \varphi_p^{\xi}(\gamma) f(t,u,v) \end{aligned}$$

and

$$\begin{aligned} g(t, \gamma u) &= (1-t)^{-\frac{1}{8}} t^{-\frac{1}{6}} (\gamma u)^{\frac{1}{3}} \\ &\geq \gamma^2 \left[(1-t)^{-\frac{1}{8}} t^{-\frac{1}{6}} u^{\frac{1}{3}}\right] \\ &= \varphi_p(\gamma) g(t,u). \end{aligned}$$

(4) Taking $\delta_0 = \frac{1}{2}$, for all $t \in (0,1)$ and $u, v \in [0, +\infty)$, we have

$$\begin{aligned} f(t,u,v) &= (1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} u^{\frac{1}{3}} + v^{-\frac{1}{5}} \\ &\geq \frac{1}{4} \left[(1-t)^{-\frac{1}{8}} t^{-\frac{1}{6}} u^{\frac{1}{3}}\right] \\ &= \varphi_p(\delta_0) g(t,u). \end{aligned}$$

From the above conclusions, obviously, Theorem 3.1 implies that the tempered fractional differential equation integral boundary value problem (4.1) has a unique positive solution $u^* \in P_h$, where $h(t) = e^{-t} t^{\frac{3}{2}}$.

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