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On some geometric properties for the combination of generalized Lommel–Wright function

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Abstract

The scope of our investigation is to study the geometric properties of the normalized form of the combination of generalized Lommel–Wright function $J_{\nu,\lambda}^{\mu,m}$ defined by $\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z) := \Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)2^{2\lambda+\nu}z^{1-(\nu/2)-\lambda}\mathcal{I}_{\nu,\lambda}^{\mu,m}(\sqrt{z})$, where $\mathcal{I}_{\nu,\lambda}^{\mu,m}(z) := (1-2\lambda-\nu)J_{\nu,\lambda}^{\mu,m}(z) + z(J_{\nu,\lambda}^{\mu,m}(z))'$ and

$$J_{\nu,\lambda}^{\mu,m}(z) = \left(\frac{z}{2}\right)^{2\lambda+\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma^m(n+\lambda+1)\Gamma(n\mu+\nu+\lambda+1)}\left(\frac{z}{2}\right)^{2n},$$

with $m \in \mathbb{N}$, $\mu > 0$ and $\lambda, \nu \in \mathbb{C}$, including starlikeness and convexity of order α ($0 \le \alpha < 1$) in the open unit disc using the two-sided inequality for the Fox–Wright functions that has been proved by Pogány and Srivastava in (Comput. Math. Appl. 57(1):127–140, 2009). Further, the orders of starlikeness and convexity are also evaluated using some classical tools. We then compare the orders of starlikeness and convexity given by both techniques to illustrate the efficacy of the approach. In addition, we proved that for some values of α, if $\lambda > -1$ then $\operatorname{Re}(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)/z) > \alpha$, $z \in \mathbb{U}$, and if $\lambda \ge (\sqrt{10}-6)/4$ then the function $(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z^2)/z)*\sin z$ is close-to-convex with respect to $1/2\log((1+z)/(1-z))$ where $*$ stands for the Hadamard product (or convolution) of two power series.

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1 Introduction and preliminaries

Special functions have attracted the particular attention of researchers due to their applications in many branches of mathematics and physics. Recently, special attention has further been devoted by several mathematicians to study the geometric properties of special functions like starlikeness, convexity, and close-to-convexity in the open unit disc. One can see the following papers in this direction: for hypergeometric functions ([6, 7] and [12, 13]), Bessel functions [1, 2], generalized Struve functions [8, 17], and Lommel

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functions [16]. Results of this type seem to be more and more visible in the developments of Geometric Function Theory. Their analyses are not only auxiliary tools but also provide powerful information to generalize to several types of functions and polynomials.

It is worthy of mention that generalized Lommel–Wright function have been constructed from concrete problems in mechanics, physics, engineering, and astronomy. Our main results are summarized in the following: we study some geometric properties for the normalized form of the combination of generalized Lommel–Wright function and its derivative, including starlikeness and convexity of order α ($0 \leq \alpha < 1$) in the open unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ using the two-sided inequality for the Fox–Wright functions that has been proved by Pogány and Srivastava in [11]. In addition, the orders of starlikeness and convexity are also established using some classical tools. We then compare the orders of starlikeness and convexity given by both techniques to illustrate the efficacy of the approach.

For all of what follows, let $\mathcal{H}$ denote the space of all analytic functions in $\mathbb{U}$. Denote by $\mathcal{A}$ the subclass of $\mathcal{H}$ of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad z \in \mathbb{U}, \tag{1.1}$$

and by $\mathcal{S}$ the subclass of all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$. If $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} B_n z^n, \quad z \in \mathbb{U},$$

then the *Hadamard product (or convolution)* of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} A_n B_n z^n = (g * f)(z), \quad z \in \mathbb{U}.$$

For $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is said to be in the class of *starlike functions of order α*, denoted by $\mathcal{S}^*(\alpha)$, if and only if $\mathrm{Re}(zf'(z)/f(z)) > \alpha$, for all $z \in \mathbb{U}$, and is said to be in the class of *convex functions of order α*, denoted by $\mathcal{K}(\alpha)$, if and only if $1 + \mathrm{Re}(zf''(z)/f'(z)) > \alpha$, for all $z \in \mathbb{U}$. Also, $f \in \mathcal{H}$ is said to be a *close-to-convex function* with respect to a fixed starlike function $g \in \mathcal{S}^*(0)$ (which need not be normalized), denoted by $\mathcal{C}_g$, if and only if $\mathrm{Re}(zf'(z)/g(z)) > 0, z \in \mathbb{U}$. It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$, and all these three subclasses of functions, namely $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, and $\mathcal{C}_g$, are subsets of $\mathcal{S}$. In addition, for $\alpha < 1$, let recall the class

$$\mathcal{P}(\alpha) := \{ p \in \mathcal{H} : p(0) = 1, \mathrm{Re}\, p(z) > \alpha, z \in \mathbb{U} \}.$$

The *generalized Lommel–Wright function* $J_{\nu,\lambda}^{\mu,m}(z)$ is defined by de Oteiza et al. in [3] as

$$J_{\nu,\lambda}^{\mu,m}(z) = \left(\frac{z}{2}\right)^{2\lambda+\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma^m(n+\lambda+1)\Gamma(n\mu+\nu+\lambda+1)} \left(\frac{z}{2}\right)^{2n}$$

$$= \left(\frac{z}{2}\right)^{2\lambda+\nu} {}_1\Psi_{m+1} \left[\begin{matrix} (1,1) \\ (\lambda+1,1),\ldots,(\lambda+1,1),(\lambda+\nu+1,\mu) \end{matrix} \;\middle|\; -\frac{z^2}{4} \right], \tag{1.2}$$

for $m \in \mathbb{N} := \{1, 2, \ldots\}$, $\mu > 0$ and $\lambda, \nu \in \mathbb{C}$ where ${}_p\Psi_q$ denotes the *Fox–Wright generalization of the hypergeometric function* which is defined by

$$
{}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \middle| z \right] =: {}_p\Psi_q \left[\begin{array}{c} (\mathbf{a}_p, \mathbf{A}_p) \\ (\mathbf{b}_q, \mathbf{B}_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \psi_n \frac{z^n}{n!}, \tag{1.3}
$$

with

$$
\psi_n = \frac{\Gamma(a_1 + A_1 n) \cdots \Gamma(a_p + A_p n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_q + B_q n)},
$$

for $a_i, b_j \in \mathbb{C}$ and $A_i, B_j \in \mathbb{R}^+$ ($i = 1, \ldots, p$, $j = 1, \ldots, q$). It is worth noting that the above series converges absolutely in the whole complex z-plane when $\Delta := \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1$, while if $\Delta = -1$, then the series (1.3) converges absolutely for $|z| < \rho$ and $|z| = \rho$ under the condition $\mathrm{Re}(\sigma) > 1/2$ where

$$
\rho = \left(\prod_{i=1}^{p} A_i^{-A_i} \right) \left(\prod_{j=1}^{q} B_j^{-B_j} \right), \qquad \sigma = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p - q}{2}.
$$

We refer the interested readers to [5] for more details concerning the Fox–Wright functions.

As a special case of the *generalized Lommel–Wright function* (1.2) obtained by taking $m = 1$ in (1.2), we obtain the following generalization of the *Bessel function* introduced by Pathak [10]:

$$
J_{\nu,\lambda}^{\mu}(z) := J_{\nu,\lambda}^{\mu,1}(z) = \left(\frac{z}{2} \right)^{2\lambda + \nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \lambda + 1)\Gamma(n\mu + \nu + \lambda + 1)} \left(\frac{z}{2} \right)^{2n},
$$

for $\mu > 0$ and $\lambda, \nu \in \mathbb{C}$. If $m = 1$, $\mu = 1$ and $\lambda = 1/2$, we obtain the *Struve function* defined by the power series expansion

$$
H_\nu(z) := J_{\nu,1/2}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + 3/2)\Gamma(n + \nu + 3/2)} \left(\frac{z}{2} \right)^{\nu + 2n}, \quad \nu \in \mathbb{C}.
$$

For $m = 1$, $\mu = 1$, and $\lambda = 0$ in (1.2), we get the *Bessel function* defined by

$$
J_\nu(z) := J_{\nu,1/2}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n + \nu + 1)} \left(\frac{z}{2} \right)^{\nu + 2n},
$$

where $z, \nu \in \mathbb{C}$, $z \neq 0$ and $\mathrm{Re}\,\nu > -1$.

We shall base our discussion upon the following definition:

Definition 1.1 Let $m \in \mathbb{N}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}^-$ with $\mathbb{Z}^- := \{-1, -2, -3, \ldots\}$ and $\nu, \mu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define the *normalized form of the combination of generalized Lommel–Wright function* by

$$
\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z) := \Gamma^m(\lambda + 1)\Gamma(\lambda + \nu + 1) 2^{2\lambda + \nu} z^{1 - (\nu/2) - \lambda} \mathcal{I}_{\nu,\lambda}^{\mu,m}(\sqrt{z}),
$$

where $\mathcal{I}_{\nu,\lambda}^{\mu,m}(z) := (1 - 2\lambda - \nu) J_{\nu,\lambda}^{\mu,m}(z) + z (J_{\nu,\lambda}^{\mu,m}(z))'$.

It can further be shown that

$$\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^{n+1}, \tag{1.4}$$

where $(a)_n$ represents the *Pochhammer symbol* defined by

$$(a)_n := \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

To establish our results, we need the following two lemma.

The first is a classical result of Fejér [4] and may be found in the next form in [14, Theorem 2.10]:

Lemma 1.1 *Assume* $A_1 = 1$ *and* $A_n \geq 0$ *for* $n \geq 2$ *such that* $(A_n)_{n \in \mathbb{N}}$ *is a convex decreasing sequence, that is,* $A_n - 2A_{n+1} + A_{n+2} \geq 0$ *and* $A_{n+1} - A_{n+2} \geq 0$*, for all* $n \in \mathbb{N}$*. Then,* $\operatorname{Re}(\sum_{n=1}^{\infty} A_n z^{n-1}) > 1/2$*,* $z \in \mathbb{U}$*.*

The next lemma is a special case of [9, Corollary 9] (see also [9, Theorem 10]) for the odd functions. It follows since the hypothesis ensures that f is close-to-convex function with respect to $\log \sqrt{(1+z)/(1-z)}$.

Lemma 1.2 *Suppose that the function* f *defined by* (1.1) *is an odd function (that is,* $A_{2n} = 0$ *for all* $n \in \mathbb{N}$*), such that* $1 \geq 3A_3 \geq \cdots \geq (2n+1)A_{2n+1} \geq 0$*, or* $1 \leq 3A_3 \leq \cdots \leq (2n+1)A_{2n+1} \leq 2$*. Then* $f \in \mathcal{S}$*.*

2　Main results

Theorem 2.1 *Suppose that* $\lambda > 0$ *and* $\nu, \mu \in \mathbb{N}$ *satisfy the inequalities*

$$\frac{1}{\Gamma(2\mu+\lambda+\nu+1)} > \frac{21}{20(\lambda+3)^m \Gamma(3\mu+\lambda+\nu+1)}, \tag{2.1}$$

$$\frac{21}{20(\lambda+3)^m \Gamma(\mu+\lambda+\nu+1)\Gamma(3\mu+\lambda+\nu+1)}$$
$$> \frac{5}{6(\lambda+2)^m \Gamma^2(2\mu+\lambda+\nu+1)}. \tag{2.2}$$

If

$$0 \leq \alpha$$
$$\leq 1 - \frac{3\Gamma(\lambda+\nu+1)}{4(\lambda+1)^m} \cdot \frac{(\lambda+1)^m (\lambda+\nu+1)^\mu - 1}{(\lambda+1)^m (\lambda+\nu+1)^\mu - 2}$$
$$\cdot \left[\frac{1}{\Gamma(\lambda+\nu+\mu+1)} - \frac{5(1-e)}{6(\lambda+2)^m \Gamma(\lambda+\nu+2\mu+1)} \right]$$
$$=: \widetilde{\alpha}_{\max}, \tag{2.3}$$

then $\mathfrak{J}_{\nu,\lambda}^{\mu,m} \in \mathcal{S}^*(\alpha)$*.*

Proof To obtain our result $\mathfrak{J}_{\nu,\lambda}^{\mu,m} \in \mathcal{S}^*(\alpha)$, it is sufficient to prove that

$$\left| \frac{z(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))'}{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{U}.$$

Using the theorem of the maximum of the module of an analytic function and the triangle inequality, a simple computation shows that

$$\left| \frac{z(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))'}{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)} - 1 \right| = \left| \frac{\sum_{n=1}^{\infty} \frac{(-1)^n n(2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^{n+1}}{z + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^{n+1}} \right|$$

$$= \frac{\left| \sum_{n=1}^{\infty} \frac{(-1)^n n(2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^n \right|}{\left| 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^n \right|}$$

$$< \frac{\sum_{n=1}^{\infty} \frac{n(2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}}}{1 - \sum_{n=1}^{\infty} \frac{2n+1}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}}} =: \frac{F_1}{F_2}, \quad z \in \mathbb{U}. \tag{2.4}$$

First, we see that

$$F_1 = \sum_{n=1}^{\infty} \frac{n(2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} \frac{(1)_n (2)_{2n}}{(1)_n (2)_{2n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} \frac{(1)_n (2)_{2n}}{(1)_{n-1} (1)_{2n}}, \quad n \in \mathbb{N}.$$

Using the *Legendre's formula*

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}} \Gamma(2z),$$

for $z = n + 1$ we get

$$\Gamma\left(n + \frac{3}{2}\right) = \frac{1 \cdot 3 \cdots (2n+1)}{2^{n+1}} \sqrt{\pi}, \quad n \in \mathbb{N}, \tag{2.5}$$

and also

$$(2)_{2n} = 4^n (1)_n \left(\frac{3}{2}\right)_n, \quad n \in \mathbb{N}. \tag{2.6}$$

From the identities (2.5) and (2.6), it follows that

$$\sum_{n=1}^{\infty} \frac{1}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} \frac{(1)_n (2)_{2n}}{(1)_{n-1} (1)_{2n}}$$

$$= \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\sqrt{\pi}}$$

$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+2)\Gamma(n+2)\Gamma(n+5/2)}{\Gamma^m(n+\lambda+2)\Gamma(n\mu+\mu+\lambda+\nu+1)\Gamma(n+1)\Gamma(2n+3)}$$

$$= \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\sqrt{\pi}}$$

$$\cdot {}_3\Psi_{m+2}\left[\begin{array}{c} (2,1),(2,1),(5/2,1) \\ (\lambda+2,1),\ldots,(\lambda+2,1),(\lambda+\nu+\mu+1,\mu),(3,2) \end{array}\middle| 1\right]. \tag{2.7}$$

On the other hand, according to Theorem 4 of [11], the next inequality holds for all ${}_p\Psi_q$ satisfying $\psi_1 > \psi_2$ and $\psi_1^2 < \psi_0\psi_2$:

$$\psi_0 e^{\psi_1\psi_0^{-1}|z|} \le {}_p\Psi_q\left[\begin{array}{c} (\mathbf{a}_p,\mathbf{A}_p) \\ (\mathbf{b}_q,\mathbf{B}_q) \end{array}\middle| z\right] \le \psi_0 - \left(1 - e^{|z|}\right)\psi_1, \quad z \in \mathbb{R}. \tag{2.8}$$

For the Fox–Wright function that appears in (2.7), we have

$$\psi_n = \frac{\Gamma(n+2)\Gamma(n+2)\Gamma(n+5/2)}{\Gamma^m(n+\lambda+2)\Gamma(n\mu+\mu+\lambda+\nu+1)\Gamma(2n+3)},$$

hence

$$\psi_0 = \frac{\Gamma(2)\Gamma(2)\Gamma(5/2)}{\Gamma^m(\lambda+2)\Gamma(\mu+\lambda+\nu+1)\Gamma(3)},$$

$$\psi_1 = \frac{\Gamma(3)\Gamma(3)\Gamma(7/2)}{\Gamma^m(\lambda+3)\Gamma(2\mu+\lambda+\nu+1)\Gamma(5)},$$

and

$$\psi_2 = \frac{\Gamma(4)\Gamma(4)\Gamma(9/2)}{\Gamma^m(\lambda+4)\Gamma(3\mu+\lambda+\nu+1)\Gamma(7)}.$$

It is easy to verify that the inequalities $\psi_1 > \psi_2$ and $\psi_1^2 < \psi_0\psi_2$ are equivalent to the assumptions (2.1) and (2.2), respectively. Hence, according to the right-hand side of the double inequality (2.8), from (2.7) we obtain

$$F_1 \le \frac{3\Gamma(\lambda+\nu+1)}{4(\lambda+1)^m}\left[\frac{1}{\Gamma(\lambda+\nu+\mu+1)} - \frac{5(1-e)}{6(\lambda+2)^m\Gamma(\lambda+\nu+2\mu+1)}\right]. \tag{2.9}$$

Moreover, since

$${}_3\Psi_{m+2}\left[\begin{array}{c} (2,1),(2,1),(5/2,1) \\ (\lambda+2,1),\ldots,(\lambda+2,1),(\lambda+\nu+\mu+1,\mu),(3,2) \end{array}\middle| 1\right] > 0$$

for $\lambda > 0$ and $\nu,\mu \in \mathbb{N}$, it follows that the right-hand side of the inequality (2.9) is positive.

Secondly, using the fact that $2n + 1 \le 4^n$, $n \in \mathbb{N}$, and under our assumptions

$$(\lambda+\nu+1)_{n\mu} \ge (\lambda+\nu+1)^{n\mu}, \tag{2.10}$$

we have

$$F_2 = 1 - \sum_{n=1}^{\infty} \frac{2n+1}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}}$$

$$\geq 1 - \sum_{n=1}^{\infty} \left(\frac{1}{(\lambda + 1)^m (\lambda + \nu + 1)^\mu} \right)^n$$

$$= \frac{(\lambda + 1)^m (\lambda + \nu + 1)^\mu - 2}{(\lambda + 1)^m (\lambda + \nu + 1)^\mu - 1} > 0, \quad \text{for } \lambda > 0 \text{ and } \nu, \mu \in \mathbb{N}. \tag{2.11}$$

Using (2.9) and (2.11), from the inequality (2.4) we deduce that

$$\left| \frac{z(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))'}{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)} - 1 \right| < \frac{F_1}{F_2}$$

$$\leq \frac{\frac{3\Gamma(\lambda + \nu + 1)}{4(\lambda + 1)^m} \left[\frac{1}{\Gamma(\lambda + \nu + \mu + 1)} - \frac{5(1-e)}{6(\lambda + 2)^m \Gamma(\lambda + \nu + 2\mu + 1)} \right]}{\frac{(\lambda + 1)^m (\lambda + \nu + 1)^\mu - 2}{(\lambda + 1)^m (\lambda + \nu + 1)^\mu - 1}}$$

$$\leq 1 - \alpha, \quad z \in \mathbb{U},$$

whenever the assumption (2.3) holds, and the theorem is completely proved. □

Theorem 2.2 *Suppose that $\lambda > 0$ and $\nu, \mu \in \mathbb{N}$ satisfy the inequalities*

$$\frac{1}{\Gamma(2\mu + \lambda + \nu + 1)} > \frac{7}{5(\lambda + 3)^m \Gamma(3\mu + \lambda + \nu + 1)}, \tag{2.12}$$

$$\frac{7}{5(\lambda + 3)^m \Gamma(\mu + \lambda + \nu + 1)\Gamma(3\mu + \lambda + \nu + 1)}$$

$$> \frac{5}{4(\lambda + 2)^m \Gamma^2(2\mu + \lambda + \nu + 1)}, \tag{2.13}$$

$$2(\lambda + 1)^m (\lambda + \nu + 1)^\mu > 5. \tag{2.14}$$

If

$$0 \leq \alpha$$

$$\leq 1 - \frac{(\lambda + 1)^m (\lambda + \nu + 1)^\mu - 1}{2(\lambda + 1)^m (\lambda + \nu + 1)^\mu - 5} \cdot \frac{3\Gamma(\lambda + \nu + 1)}{(\lambda + 1)^m}$$

$$\cdot \left[\frac{1}{\Gamma(\lambda + \nu + \mu + 1)} - \frac{5(1 - e)}{4(\lambda + 2)^m \Gamma(\lambda + \nu + 2\mu + 1)} \right]$$

$$=: \widetilde{\beta}_{\max}, \tag{2.15}$$

then $\mathfrak{J}_{\nu,\lambda}^{\mu,m} \in \mathcal{K}(\alpha)$.

Proof A sufficient condition for the function $\mathfrak{J}_{\nu,\lambda}^{\mu,m}$ to be convex of order α is to prove that

$$\left| \frac{z(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))''}{(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))'} \right| < 1 - \alpha, \quad z \in \mathbb{U}.$$

Like in the proof of the previous theorem, using the theorem of the maximum of the module of an analytic function and the triangle inequality, we find

$$\left| \frac{z(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))''}{(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))'} \right| < \frac{\sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}}}{1 - \sum_{n=1}^{\infty} \frac{(n+1)(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}}} =: \frac{G_1}{G_2}. \tag{2.16}$$

Like in the proof of the Theorem 2.1, from the identities (2.5) and (2.6), we have

$$\begin{aligned}
G_1 &= \sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}} \frac{(2)_n(1)_n(2)_{2n}}{(2)_n(1)_n(2)_{2n}} \\
&= \sum_{n=1}^{\infty} \frac{1}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}} \frac{(2)_n(1)_n(2)_{2n}}{(1)_n(1)_{n-1}(1)_{2n}} \\
&= \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\sqrt{\pi}} \\
&\quad \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+3)\Gamma(n+2)\Gamma(n+5/2)}{\Gamma^m(n+\lambda+2)\Gamma(n\mu+\mu+\lambda+\nu+1)\Gamma(n+1)\Gamma(2n+3)} \\
&= \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\sqrt{\pi}} \\
&\quad \cdot {}_3\Psi_{m+2}\left[\begin{matrix} (3,1),(2,1),(5/2,1) \\ (\lambda+2,1),\dots,(\lambda+2,1),(\lambda+\nu+\mu+1,\mu),(3,2) \end{matrix} \middle| 1 \right].
\end{aligned} \tag{2.17}$$

For the above Fox–Wright function (2.17), we have

$$\psi_n = \frac{\Gamma(n+3)\Gamma(n+2)\Gamma(n+5/2)}{\Gamma^m(n+\lambda+2)\Gamma(n\mu+\mu+\lambda+\nu+1)\Gamma(2n+3)},$$

hence

$$\psi_0 = \frac{\Gamma(3)\Gamma(2)\Gamma(5/2)}{\Gamma^m(\lambda+2)\Gamma(\mu+\lambda+\nu+1)\Gamma(3)},$$

$$\psi_1 = \frac{\Gamma(4)\Gamma(3)\Gamma(7/2)}{\Gamma^m(\lambda+3)\Gamma(2\mu+\lambda+\nu+1)\Gamma(5)},$$

and

$$\psi_2 = \frac{\Gamma(5)\Gamma(4)\Gamma(9/2)}{\Gamma^m(\lambda+4)\Gamma(3\mu+\lambda+\nu+1)\Gamma(7)}.$$

We could easily check that the inequalities $\psi_1 > \psi_2$ and $\psi_1^2 < \psi_0\psi_2$ are equivalent to the assumptions (2.12) and (2.13), respectively. Using again the right-hand side of the double inequality (2.8), from (2.17) we have

$$G_1 \leq \frac{3\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{2\Gamma^m(\lambda+2)}\left[\frac{1}{\Gamma(\lambda+\nu+\mu+1)} - \frac{5(1-e)}{4(\lambda+2)^m\Gamma(\lambda+\nu+2\mu+1)} \right]. \tag{2.18}$$

Also, since

$$
{}_3\Psi_{m+2}\left[\begin{array}{c}(3,1),(2,1),(5/2,1)\\(\lambda+2,1),\ldots,(\lambda+2,1),(\lambda+\nu+\mu+1,\mu),(3,2)\end{array}\middle|\,1\right]>0
$$

for $\lambda>0$ and $\nu,\mu\in\mathbb{N}$, it follows that the right-hand side of the inequality (2.18) is positive.

Furthermore, since $(n+1)(2n+1)\le(3/2)4^n$, $n\in\mathbb{N}$, using (2.10), we obtain

$$
\begin{aligned}
G_2 &= 1-\sum_{n=1}^{\infty}\frac{(n+1)(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}}\\
&\ge 1-\frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(\lambda+1)^m(\lambda+\nu+1)^\mu}\right)^n\\
&= 1-\frac{3}{2}\frac{1}{(\lambda+1)^m(\lambda+\nu+1)^\mu-1}\\
&= \frac{2(\lambda+1)^m(\lambda+\nu+1)^\mu-5}{2[(\lambda+1)^m(\lambda+\nu+1)^\mu-1]}>0,\quad\text{for }\lambda>0,\nu,\mu\in\mathbb{N},
\end{aligned}
\tag{2.19}
$$

if the assumption (2.14) holds.

Using (2.18) and (2.19), inequality (2.16) leads to

$$
\begin{aligned}
\left|\frac{z(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))''}{(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z))'}\right| &< \frac{G_1}{G_2}\\
&\le \frac{\frac{3\Gamma(\lambda+\nu+1)}{4(\lambda+1)^m}\left[\frac{1}{\Gamma(\lambda+\nu+\mu+1)}-\frac{5(1-e)}{4(\lambda+2)^m\Gamma(\lambda+\nu+2\mu+1)}\right]}{\frac{2(\lambda+1)^m(\lambda+\nu+1)^\mu-5}{2[(\lambda+1)^m(\lambda+\nu+1)^\mu-1]}}\\
&\le 1-\alpha,\quad z\in\mathbb{U},
\end{aligned}
$$

whenever the assumption (2.15) holds, and our result is proved. $\qquad\square$

Remark 2.1 We emphasize that the assumptions of Theorems 2.1 and 2.2 can be simultaneously fulfilled, as it is shown in the below Fig. 1 (the grey regions) obtained by using the MAPLE™ software.

Theorem 2.3 *Suppose that*

$$
0\le\alpha\le 1-\frac{3(\lambda+2)^m(\lambda+\nu+2)}{4(\lambda+1)^m(\lambda+\nu+1)[(\lambda+2)^m(\lambda+\nu+2)-1]-3(\lambda+2)^m(\lambda+\nu+2)}
$$
$$
=:\widehat{\alpha}_{\max},
\tag{2.20}
$$

with $\lambda>-1$ and $\nu,\mu\in\mathbb{N}$, which also satisfies

$$
\frac{4(\lambda+1)^m(\lambda+\nu+1)[(\lambda+2)^m(\lambda+\nu+2)-1]}{3(\lambda+2)^m(\lambda+\nu+2)}>1.
\tag{2.21}
$$

Then, $\mathfrak{J}_{\nu,\lambda}^{\mu,m}\in\mathcal{S}^*(\alpha)$.

(A) The region where the assumptions (2.1), (2.2) and (2.3) can be simultaneously fulfilled for $\mu = 2$, $\nu = 4$, with $\lambda \in [10, 80]$ and $m \in [2, 18]$

(B) The region where the assumptions (2.12), (2.13), (2.14) and (2.15) could be simultaneously fulfilled for $\mu = 2$, $\nu = 4$, with $\lambda \in [40, 80]$ and $m \in [2, 6]$

Figure 1 The figures for Remark 2.1
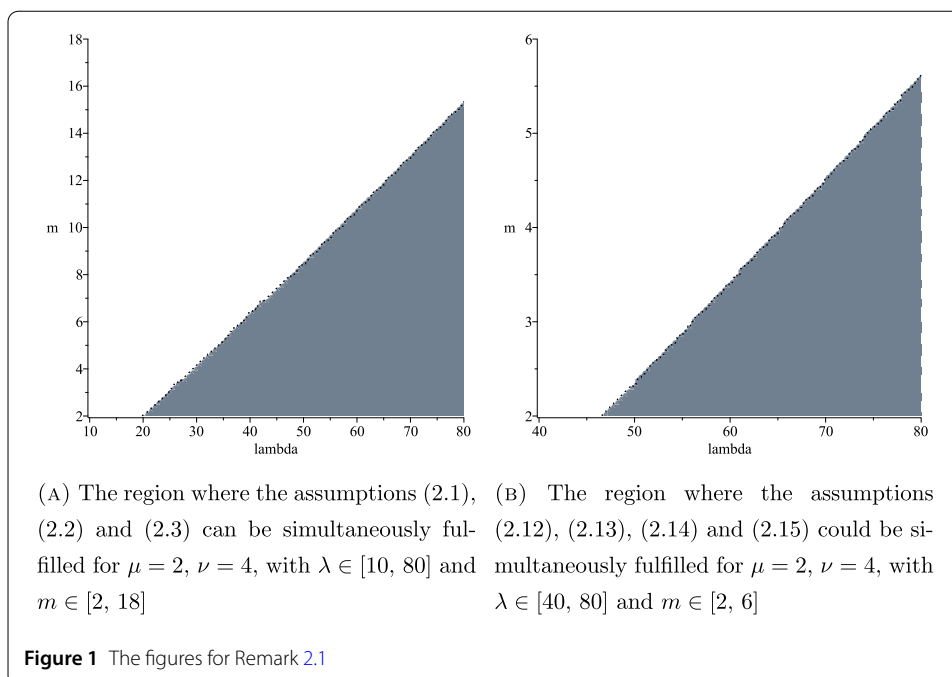
Proof It is well-known from [15, Theorem 1] that if f of the form (1.1) satisfies $\sum_{n=2}^{\infty}(n - \alpha)|A_n| \leq 1 - \alpha$, then $f \in \mathcal{S}^*(\alpha)$. Thus, according to (1.4), it is enough to prove that

$$H_1 := \sum_{n=2}^{\infty}(n - \alpha)\left|\frac{(-1)^{n-1}(2n-1)}{4^{n-1}[(\lambda+1)_{n-1}]^m(\lambda+\nu+1)_{(n-1)\mu}}\right| \leq 1 - \alpha.$$

Since $\lambda > -1$ and $\mu \in \mathbb{N}$, we have

$$H_1 = \sum_{n=2}^{\infty}\frac{(n-\alpha)(2n-1)}{4^{n-1}[(\lambda+1)_{n-1}]^m(\lambda+\nu+1)_{(n-1)\mu}}$$

$$= \sum_{n=1}^{\infty}\frac{n(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{2n+1}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}},$$

and using the fact that

$$(\lambda + \nu + 1)_{n\mu} \geq (\lambda + \nu + 1)_n, \quad \text{for all } n, \nu, \mu \in \mathbb{N},\tag{2.22}$$

it follows that

$$H_1 \leq \frac{1}{4(\lambda+1)^m(\lambda+\nu+1)}\sum_{n=1}^{\infty}\frac{n(2n+1)}{4^{n-1}[(\lambda+2)_{n-1}]^m(\lambda+\nu+2)_{n-1}}$$

$$+ \frac{1-\alpha}{4(\lambda+1)^m(\lambda+\nu+1)}\sum_{n=1}^{\infty}\frac{2n+1}{4^{n-1}[(\lambda+2)_{n-1}]^m(\lambda+\nu+2)_{n-1}}.\tag{2.23}$$

Moreover, it is easy to check that

$$n(2n+1) \leq 3 \cdot 4^{n-1}, \qquad 2n+1 \leq 3 \cdot 4^{n-1},$$

$$(\lambda + \nu + 2)_{n-1} \geq (\lambda + \nu + 2)^{n-1}, \qquad (\lambda + 2)_{n-1} \geq (\lambda + 2)^{n-1}$$

for all $n \in \mathbb{N}$. From (2.23), using the above inequalities together with the assumptions (2.20) and (2.21), as well as the facts that $\lambda > -1$ and $\nu \in \mathbb{N}$, we get

$$
\begin{aligned}
\mathrm{H}_1 &\leq \frac{3}{4(\lambda + 1)^m(\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{1}{(\lambda + 2)^{(n-1)m}(\lambda + \nu + 2)^{n-1}} \\
&\quad + \frac{3(1 - \alpha)}{4(\lambda + 1)^m(\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{1}{(\lambda + 2)^{(n-1)m}(\lambda + \nu + 2)^{n-1}} \\
&= \frac{3(2 - \alpha)}{4(\lambda + 1)^m(\lambda + \nu + 1)} \cdot \frac{(\lambda + 2)^m(\lambda + \nu + 2)}{(\lambda + 2)^m(\lambda + \nu + 2) - 1} \leq 1 - \alpha,
\end{aligned}
$$

and the proof is complete.   □

Theorem 2.4 *Suppose that*

$$0 \leq \alpha \leq 1 - \frac{\mathrm{T}_1}{\mathrm{T}_2} =: \widehat{\beta}_{\max}, \tag{2.24}$$

where

$$
\begin{aligned}
\mathrm{T}_1 &:= 8(\lambda + 2)^m(\lambda + \nu + 2)\big[2(\lambda + 2)^m(\lambda + \nu + 2) - 1\big], \\
\mathrm{T}_2 &:= 4(\lambda + 1)^m(\lambda + \nu + 1)\big[(\lambda + 2)^m(\lambda + \nu + 2) - 1\big] \cdot \big[2(\lambda + 2)^m(\lambda + \nu + 2) - 1\big] \\
&\quad - 3(\lambda + 2)^m(\lambda + \nu + 2) \cdot \big[2(\lambda + 2)^m(\lambda + \nu + 2) - 1\big] \\
&\quad - 6(\lambda + 2)^m(\lambda + \nu + 2) \cdot \big[(\lambda + 2)^m(\lambda + \nu + 2) - 1\big],
\end{aligned}
$$

with $\lambda > -1$ and $\nu, \mu \in \mathbb{N}$, which also satisfies

$$\mathrm{T}_2 > 0. \tag{2.25}$$

Then, $\mathfrak{I}_{\nu,\lambda}^{\mu,m} \in \mathcal{K}(\alpha)$.

Proof Like in the proof of the previous theorem, if f of the form (1.1) satisfies $\sum_{n=2}^{\infty}(n - \alpha)|A_n| \leq 1 - \alpha$, then $f \in \mathcal{S}^*(\alpha)$. From the Alexander duality relation, i.e., for f of the form (1.1), we have $f \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$. Thus, according to [15, Theorem 1], if f of the form (1.1) satisfies $\sum_{n=2}^{\infty} n(n - \alpha)|A_n| \leq 1 - \alpha$, then $f \in \mathcal{K}(\alpha)$ (see also [15, Corollary on p. 110]).

Therefore, it is sufficient to prove that

$$\mathrm{H}_2 := \sum_{n=2}^{\infty} n(n - \alpha) \left| \frac{(-1)^{n-1}(2n - 1)}{4^{n-1}[(\lambda + 1)_{n-1}]^m(\lambda + \nu + 1)_{(n-1)\mu}} \right| \leq 1 - \alpha. \tag{2.26}$$

Since $\lambda > -1$ and $\nu \in \mathbb{N}$, we have

$$\mathrm{H}_2 = \sum_{n=1}^{\infty} \frac{n^2(2n + 1)}{4^n[(\lambda + 1)_n]^m(\lambda + \nu + 1)_{n\mu}} + (2 - \alpha) \sum_{n=1}^{\infty} \frac{n(2n + 1)}{4^n[(\lambda + 1)_n]^m(\lambda + \nu + 1)_{n\mu}}$$

$$+ (1 - \alpha) \sum_{n=1}^{\infty} \frac{(2n + 1)}{4^n [(\lambda + 1)_n]^m (\lambda + \nu + 1)_{n\mu}},$$

and, using again inequality (2.22), we obtain

$$H_2 \leq \frac{1}{4(\lambda + 1)^m (\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{n^2(2n + 1)}{4^{n-1}[(\lambda + 2)_{n-1}]^m (\lambda + \nu + 2)_{n-1}}$$

$$+ \frac{2 - \alpha}{4(\lambda + 1)^m (\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{n(2n + 1)}{4^{n-1}[(\lambda + 2)_{n-1}]^m (\lambda + \nu + 2)_{n-1}}$$

$$+ \frac{1 - \alpha}{4(\lambda + 1)^m (\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{2n + 1}{4^{n-1}[(\lambda + 2)_{n-1}]^m (\lambda + \nu + 2)_{n-1}}.$$

Furthermore, a simple computation shows that the following inequalities hold:

$$n^2(2n + 1) \leq 5 \cdot 4^{n-1}, \qquad n(2n + 1) \leq 3 \cdot 4^{n-1}, \qquad 2n + 1 \leq 3 \cdot 2^{n-1},$$

$$(\lambda + \nu + 2)_{n-1} \geq (\lambda + \nu + 2)^{n-1}, \qquad (\lambda + 2)_{n-1} \geq (\lambda + 2)^{n-1}$$

for all $n \in \mathbb{N}$. Using the above inequalities, together with the assumption (2.25), we get

$$H_2 \leq \frac{5}{4(\lambda + 1)^m (\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{1}{(\lambda + 2)^{(n-1)m} (\lambda + \nu + 2)^{n-1}}$$

$$+ \frac{3(2 - \alpha)}{4(\lambda + 1)^m (\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{1}{(\lambda + 2)^{(n-1)m} (\lambda + \nu + 2)^{n-1}}$$

$$+ \frac{3(1 - \alpha)}{4(\lambda + 1)^m (\lambda + \nu + 1)} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}(\lambda + 2)^{(n-1)m} (\lambda + \nu + 2)^{n-1}}$$

$$= \frac{11 - 3\alpha}{4(\lambda + 1)^m (\lambda + \nu + 1)} \cdot \frac{(\lambda + 2)^m (\lambda + \nu + 2)}{(\lambda + 2)^m (\lambda + \nu + 2) - 1}$$

$$+ \frac{3(1 - \alpha)}{4(\lambda + 1)^m (\lambda + \nu + 1)} \cdot \frac{2(\lambda + 2)^m (\lambda + \nu + 2)}{2(\lambda + 2)^m (\lambda + \nu + 2) - 1}$$

$$= \frac{(\lambda + 2)^m (\lambda + \nu + 2)}{4(\lambda + 1)^m (\lambda + \nu + 1)} \left[\frac{11 - 3\alpha}{(\lambda + 2)^m (\lambda + \nu + 2) - 1} + \frac{6(1 - \alpha)}{2(\lambda + 2)^m (\lambda + \nu + 2) - 1} \right]$$

$$\leq 1 - \alpha,$$

whenever (2.24) and (2.25) hold. Thus, inequality (2.26) is satisfied and hence $\mathfrak{J}_{\nu,\lambda}^{\mu,m} \in \mathcal{K}(\alpha)$. $\qquad \square$

Theorem 2.5 *Suppose that*

$$0 \leq \alpha \leq 1 - \frac{3(\lambda + 2)^m (\lambda + \nu + 2)}{2(\lambda + 1)^m (\lambda + \nu + 1)[2(\lambda + 2)^m (\lambda + \nu + 2) - 1]}, \tag{2.27}$$

with $\lambda > -1$ and $\nu, \mu \in \mathbb{N}$. Then, $\frac{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)}{z} \in \mathcal{P}(\alpha)$.

Proof To obtain the required result, it is sufficient to prove that $|G(z) - 1| < 1$, $z \in \mathbb{U}$, where

$$G(z) := \frac{1}{1 - \alpha} \left(\frac{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)}{z} - \alpha \right), \quad z \in \mathbb{U}. \tag{2.28}$$

From the theorem of the maximum of the module of an analytic function, using the triangle inequality, inequality (2.22), and

$$2n + 1 \le 3 \cdot 2^{n-1}, \qquad (\lambda + \nu + 2)_{n-1} \ge (\lambda + \nu + 2)^{n-1}, \qquad (\lambda + 2)_{n-1} \ge (\lambda + 2)^{n-1},$$

we get

$$\begin{aligned}
\left| G(z) - 1 \right| &= \left| \frac{1}{1 - \alpha} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^n - \alpha \right] - 1 \right| \\
&= \frac{1}{1 - \alpha} \left| \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} z^n \right| \\
&< \frac{1}{1 - \alpha} \left| \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} e^{in\theta} \right| \\
&\le \frac{1}{1 - \alpha} \sum_{n=1}^{\infty} \frac{2n+1}{4^n [(\lambda+1)_n]^m (\lambda+\nu+1)_{n\mu}} \\
&\le \frac{1}{4(1-\alpha)(\lambda+1)^m(\lambda+\nu+1)} \sum_{n=1}^{\infty} \frac{2n+1}{4^{n-1}[(\lambda+2)_{n-1}]^m(\lambda+\nu+1)_n} \\
&\le \frac{3}{4(1-\alpha)(\lambda+1)^m(\lambda+\nu+1)} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}(\lambda+2)^{m(n-1)}(\lambda+\nu+2)^{n-1}} \\
&= \frac{3(\lambda+2)^m(\lambda+\nu+2)}{2(1-\alpha)(\lambda+1)^m(\lambda+\nu+1)[2(\lambda+2)^m(\lambda+\nu+2)-1]} \le 1, \quad z \in \mathbb{U},
\end{aligned}$$

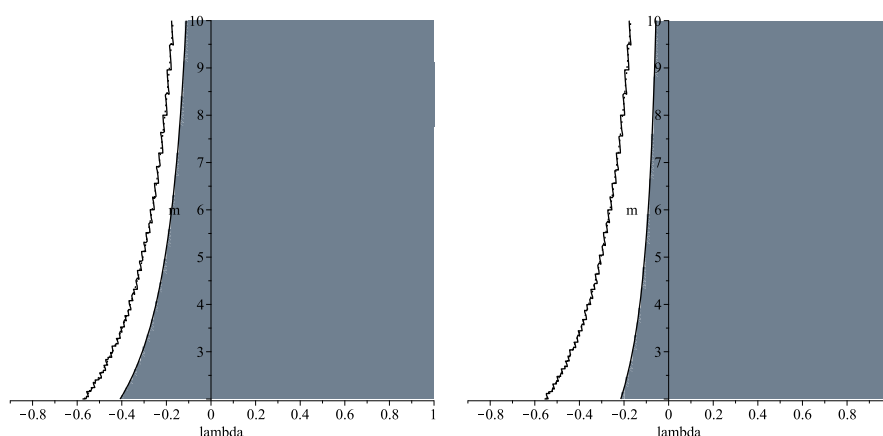with $\theta \in \mathbb{R}$, and the proof of our theorem is complete. $\qquad\square$

Remark 2.2 The assumptions of Theorems 2.3–2.5 can be simultaneously fulfilled, as it is shown in Fig. 2 (the grey regions) obtained by using the MAPLE™ software.

Theorem 2.6 *Suppose that* $m \in \mathbb{N}$, $\lambda \ge \frac{\sqrt{13}-3}{2} \simeq 0.302775\ldots$ *and* $\nu, \mu \in \mathbb{N}$. *Then,*

$$\mathrm{Re}\left(\left(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)\right)' * \frac{1}{1+z} \right) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

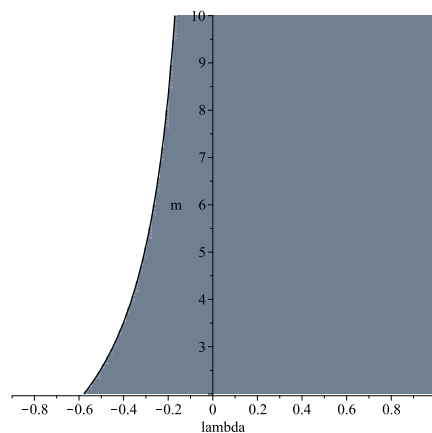*Proof* We have the following power series expansion:

$$\left(\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z)\right)' * \frac{1}{1+z} = \sum_{n=1}^{\infty} A_n z^{n-1}, \quad z \in \mathbb{U},$$

$$A_n = \frac{n(2n-1)}{4^{n-1}[(\lambda+1)_{n-1}]^m(\lambda+\nu+1)_{(n-1)\mu}}, \quad n \in \mathbb{N}.$$

(A) The region where the assumptions (2.20) and (2.21) can be simultaneously fulfilled for $\mu = 2$, $\nu = 4$, with $\lambda \in [-0.9, 1]$ and $m \in [2, 10]$

(B) The region where the assumptions (2.24) and (2.25) can be simultaneously fulfilled for $\mu = 2$, $\nu = 4$, with $\lambda \in [-0.9, 1]$ and $m \in [2, 10]$

(C) The region where the assumption (2.27) is fulfilled for $\mu = 2$, $\nu = 4$, with $\lambda \in [-0.9, 1]$ and $m \in [2, 10]$

Figure 2 The figures for Remark 2.2

We will use Lemma 1.1 to obtain the required conclusion. Thus, $A_1 = 1$, $A_n > 0$ for $n \geq 2$, and

$$A_n - A_{n+1} = \frac{1}{4^{n-1}[(\lambda + 1)_{n-1}]^m} \left[\frac{n(2n-1)}{(\lambda + \nu + 1)_{(n-1)\mu}} - \frac{(n+1)(2n+1)}{4(\lambda + n)^m (\lambda + \nu + 1)_{n\mu}} \right].$$

It is easy to check that

$$(\lambda + \nu + 1)_{n\mu} \geq \left(\lambda + \nu + 1 + (n-1)\mu \right)(\lambda + \nu + 1)_{(n-1)\mu}, \tag{2.29}$$

hence

$$A_n - A_{n+1} \geq \frac{1}{4^n [(\lambda + 1)_{n-1}]^m (\lambda + \nu + 1)_{(n-1)\mu}}$$

$$\cdot \left[4n(2n-1) - \frac{(n+1)(2n+1)}{(\lambda+n)^m(\lambda+\nu+1+(n-1)\mu)} \right]$$

$$= \frac{M(n)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{(n-1)\mu}(\lambda+\nu+1+(n-1)\mu)}, \tag{2.30}$$

where

$$M(n) := 4n(2n-1)(\lambda+n)^m\big(\lambda+\nu+1+(n-1)\mu\big) - (n+1)(2n+1)$$

$$= 4n(\lambda+n)^m\big[(\lambda+\nu+1)(2n-1) + \mu(2n-1)(n-1)\big]$$

$$- (2n-1)(n+2) - 3, \quad n \in \mathbb{N}.$$

First, to show that $A_n - A_{n+1} \geq 0$ for all $n \in \mathbb{N}$, we will prove that $M(n) \geq 0$, $n \in \mathbb{N}$.

Thus, since $m \in \mathbb{N}$, $\lambda \geq 0$ and $\nu, \mu \in \mathbb{N}$, we have $\lambda + \nu + 1 + (n-1)\mu > 0$, and it follows that

$$M(n) = (2n-1)\big[4n(\lambda+n)^m\big(\lambda+\nu+1+(n-1)\mu\big) - (n+2)\big] - 3$$

$$\geq (2n-1)\big[4n(\lambda+n)\big(\lambda+\nu+1+(n-1)\mu\big) - (n+2)\big] - 3 =: P(n). \tag{2.31}$$

For $n = 1$, we get $P(1) = 4(\lambda+1)(\lambda+\nu+1) - 6 \geq 4(\nu+1) - 6 > 0$ because $\nu \in \mathbb{N}$. For $n \geq 2$, since

$$P(n) = (2n-1)\big[4n(\lambda+n)\big(\lambda+\nu+1+(n-1)\mu\big) - (n+2)\big] - 3$$

and

$$4(\lambda+n)\big(\lambda+\nu+1+(n-1)\mu\big) - (n+2) \geq 4n(n+1) - (n+2) > 0, \quad n \in \mathbb{N},$$

it follows that

$$P(n) \geq 3\big[4n(\lambda+n)\big(\lambda+\nu+1+(n-1)\mu\big) - (n+2)\big] - 3 := 3Q(n), \tag{2.32}$$

where

$$Q(n) := 4n(\lambda+n)\big(\lambda+\nu+1+(n-1)\mu\big) - (n+3).$$

Under the assumptions of the theorem, we have

$$4n(\lambda+n) \geq 4n^2 > n+3, \quad \text{if } n \geq 2,$$

$$\lambda + \nu + 1 + (n-1)\mu > 1,$$

therefore $Q(n) > 0$ for all $n \geq 2$. Thus, according to inequalities (2.31) and (2.32), it follows that $M(n) > 0$, $n \in \mathbb{N}$, hence from (2.30) we conclude that $(A_n)_{n \in \mathbb{N}}$ is a decreasing sequence.

Secondly, we need to show that

$$A_n - 2A_{n+1} + A_{n+2} \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

Since $A_{n+2} > 0$ for all $n \in \mathbb{N}$, using inequality (2.29), we easily get

$$A_n - 2A_{n+1} + A_{n+2} > A_n - 2A_{n+1}$$

$$\geq \frac{1}{4^n[(\lambda + 1)_{n-1}]^m (\lambda + \nu + 1)_{(n-1)\mu}}$$

$$\cdot \left[4n(2n - 1) - \frac{2(n+1)(2n+1)}{(\lambda + n)^m (\lambda + \nu + 1 + (n-1)\mu)} \right]$$

$$= \frac{L^*(n)}{4^n[(\lambda + 1)_n]^m (\lambda + \nu + 1)_{(n-1)\mu}(\lambda + \nu + 1 + (n-1)\mu)}, \qquad (2.33)$$

where

$$L^*(n) := 4n(2n-1)(\lambda + n)^m\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+1)(2n+1)$$

$$= 4n(\lambda + n)^m\big[(\lambda + \nu + 1)(2n-1) + \mu(2n-1)(n-1)\big]$$

$$- 2(2n-1)(n+2) - 6.$$

Thus, to show that $A_n - 2A_{n+1} + A_{n+2} \geq 0$ for all $n \in \mathbb{N}$, it is sufficient prove that $L^*(n) \geq 0$ for all $n \in \mathbb{N}$.

Since $m \in \mathbb{N}$, $\lambda \geq 0$ and $\nu, \mu \in \mathbb{N}$, we have $\lambda + \nu + 1 + (n-1)\mu > 0$, and it follows that

$$L^*(n) = (2n-1)\big[4n(\lambda + n)^m\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+2)\big] - 6$$

$$\geq (2n-1)\big[4n(\lambda + n)\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+2)\big] - 6 =: P^*(n). \qquad (2.34)$$

For $n = 1$, we get

$$P^*(1) = 4(\lambda + 1)(\lambda + \nu + 1) - 12 \geq 4(\lambda + 1)(\lambda + \nu + 1) - 12$$

$$\geq 4(\lambda + 1)(\lambda + 2) - 12 > 0$$

whenever $\lambda \geq \frac{\sqrt{13} - 3}{2} \simeq 0.302775\dots$.

For $n \geq 2$, since

$$P^*(n) = (2n-1)\big[4n(\lambda + n)\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+2)\big] - 6$$

and

$$4(\lambda + n)\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+2) \geq 4n(n+1) - 2(n+2) > 0, \quad n \in \mathbb{N},$$

it follows that

$$P^*(n) \geq 3\big[4n(\lambda + n)\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+2)\big] - 6 =: 3Q^*(n), \qquad (2.35)$$

where

$$Q^*(n) := 4n(\lambda + n)\big(\lambda + \nu + 1 + (n-1)\mu\big) - 2(n+3).$$

Under the assumptions of the theorem, we have

$$4n(\lambda + n) \geq 4n^2 > 2(n + 3), \quad \text{if } n \geq 2,$$

$$\lambda + \nu + 1 + (n - 1)\mu > 1,$$

therefore $Q^*(n) > 0$ for all $n \geq 2$. Thus, according to inequalities (2.34) and (2.35), it follows that $L^*(n) > 0$, $n \in \mathbb{N}$, hence from (2.33) we conclude that $(A_n)_{n \in \mathbb{N}}$ is a convex sequence, which ends the proof. $\qquad \square$

Theorem 2.7 *Suppose that* $m \in \mathbb{N}$, $\lambda \geq \frac{\sqrt{10}-6}{4} \simeq -0.709430\ldots$, *and* $\nu, \mu \in \mathbb{N}$. *Then,* $\frac{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z^2)}{z} * \sin z$ *is a close-to-convex function with respect to* $\frac{1}{2} \log \frac{1+z}{1-z}$.

Proof From (1.4), since

$$\sin z = z + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C},$$

it follows that

$$\frac{\mathfrak{J}_{\nu,\lambda}^{\mu,m}(z^2)}{z} * \sin z = z + \sum_{n=1}^{\infty} A_{2n+1} z^{2n+1}, \quad z \in \mathbb{U},$$

where

$$A_{2n+1} = \frac{2n+1}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}(2n+1)!}, \quad n \in \mathbb{N}.$$

In order to use Lemma 1.2, we have $A_1 = 1$, $A_{2n+1} > 0$ for $n \geq 1$, and

$$(2n-1)A_{2n-1} - (2n+1)A_{2n+1}$$

$$= \frac{(2n-1)^2}{4^{n-1}[(\lambda+1)_{n-1}]^m(\lambda+\nu+1)_{(n-1)\mu}(2n-1)!} - \frac{(2n+1)^2}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}(2n+1)!}$$

$$= \frac{2n-1}{4^{n-1}[(\lambda+1)_{n-1}]^m(\lambda+\nu+1)_{(n-1)\mu}(2n-2)!} - \frac{2n+1}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}(2n)!}$$

$$\geq \frac{4(2n-1)(\lambda+n)^m[\lambda+\nu+1+(n-1)\mu](2n-1)(2n)-(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}(2n)!}$$

$$= \frac{8n(2n-1)^2(\lambda+n)^m[\lambda+\nu+1+(n-1)\mu]-(2n+1)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}(2n)!}$$

$$= \frac{W(n)}{4^n[(\lambda+1)_n]^m(\lambda+\nu+1)_{n\mu}(2n)!}, \quad n \in \mathbb{N},$$

where

$$W(n) := 8n(2n-1)^2(\lambda+n)^m[\lambda+\nu+1+(n-1)\mu] - (2n+1), \quad n \in \mathbb{N}.$$

It is easy to check that for $n \in \mathbb{N}$,

$$W(n) := 8n(2n-1)^2(\lambda+n)^m[\lambda+\nu+1+(n-1)\mu] - (2n+1)$$

Table 1 Comparison of the order of starlikeness given by Theorems 2.1 and 2.3

μ	ν	λ	m	$A := \widetilde{\alpha}_{\max}$ Theorem 2.1	$B := \widehat{\alpha}_{\max}$ Theorem 2.3	$A - B \leq 0$
1	1	20	3	0.99999630843873496490	0.99999631884528808850	true
1	1	40	3	0.99999974069866645409	0.99999974090403376158	true
1	1	75	3	0.99999997780611645794	0.99999997781140602240	true
2	6	30	3	0.99999998209380420596	0.99999931958308410376	false
3	6	60	3	0.99999999998948912236	0.99999995068295983574	false
4	6	80	3	0.99999999999997698708	0.99999998377865092704	false

Table 2 Comparison of the order of convexity given by Theorems 2.2 and 2.4

μ	ν	λ	m	$A := \widetilde{\beta}_{\max}$ Theorem 2.2	$B := \widehat{\beta}_{\max}$ Theorem 2.4	$A - B \leq 0$
1	1	90	2	0.99999803110440570718	0.99999737481228676390	false
1	1	60	5	0.99999999997135489658	0.99999999996180652878	false
2	4	90	6	0.99999999999999971037	0.99999999999996292697	false
2	4	60	2	0.99999990603320991858	0.99999173086604042377	false
1	3	70	5	0.99999999998876513662	0.99999999998502018216	false
1	3	50	4	0.99999999589402028998	0.99999999452536041351	false

$$\geq 8n(2n-1)^2(\lambda+n)\big[\lambda+\nu+1+(n-1)\mu\big]-(2n+1)$$

$$= (2n-1)\big\{8n(2n-1)(\lambda+n)\big[\lambda+\nu+1+(n-1)\mu\big]-1\big\}-2 =: W^*(n).$$

To show that $(2n-1)A_{2n-1}-(2n+1)A_{2n+1}\geq 0$ for all $n\in\mathbb{N}$, we will prove that $W^*(n)\geq 0$, $n\in\mathbb{N}$. For $n=1$, since $\nu\in\mathbb{N}$, we have

$$W^*(1) = 8(\lambda+1)(\lambda+\nu+1)-3 \geq 8(\lambda+1)(\lambda+2)-3 \geq 0,$$

whenever $\lambda \geq \frac{\sqrt{10}-6}{4} \simeq -0.709430\ldots$. For $n\geq 2$, since $\nu,\mu\in\mathbb{N}$, we get

$$W^*(n) \geq 3\big[48(\lambda+2)(\lambda+2+\mu)-1\big]-2 \geq 3\big[48(\lambda+2)(\lambda+3)-1\big]-2 > 0,$$

if $\lambda \geq \frac{\sqrt{10}-6}{4} \simeq -0.709430\ldots$.

Therefore, $(2n-1)A_{2n-1}-(2n+1)A_{2n+1}\geq 0$ for all $n\in\mathbb{N}$, and, according to Lemma 1.2, we get our result. $\qquad\square$

Remark 2.3 1. Theorems 2.1 and 2.3 give us sufficient conditions for the starlikeness of order α for the function $\mathfrak{J}_{\nu,\lambda}^{\mu,m}$. Like we see in the above table, for some values of the parameters the first theorem gives a better result than the second, and vice versa. See Table 1.

2. For the sufficient conditions of convexity of order α for the function $\mathfrak{J}_{\nu,\lambda}^{\mu,m}$ given by Theorems 2.2 and 2.4, we cannot formulate any opinion since in the above table, for some particular values of the parameters, we obtained better results by using Theorem 2.2. See Table 2.

3 Conclusions

In our present investigation we have studied some geometric properties of the normalized form of the combination of generalized Lommel–Wright function, including starlikeness and convexity of order α ($0 \leq \alpha < 1$), in the open unit disc using the two-sided inequality for the Fox–Wright functions that has been proved by Pogány and Srivastava in [11]. Further, we have also successfully considered conditions on λ, ν, μ, and m to evaluate the

orders of starlikeness and convexity using some classical tools. We have compared the orders of starlikeness and convexity given by both techniques to illustrate the efficacy of the approach.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors drafted and approved the final manuscript.

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