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Generalized fuzzy GV-Hausdorff distance in GFGV-fractal spaces with application in integral equation

Alireza Alihajimohammad[1] and Reza Saadati[2*]

*Correspondence:
rsaadati@iust.ac.ir; rsaadati@eml.cc
[2]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
Full list of author information is available at the end of the article

Abstract

We propose a method for constructing a generalized fuzzy Hausdorff distance on the set of the nonempty compact subsets of a given generalized fuzzy metric space in the sence of George–Veeramni and Tian–Ha–Tian. Next, we define the generalized fuzzy fractal spaces. Morever, we obtain a fixed point theorem of a class of generalized fuzzy contractions and present an application in integranl equation.

MSC: 54H25; 54A40; 28A80; 47H10

Keywords: GFGV-metric spaces; GFGV-Hausdorff distance; Fixed points; Fuzzy fractal space

1 Introduction

George and Veeramani [1, 2] have rectified in absorbing manner the structure of Menger space and introduced a first countable Hausdorff topology on it, which is very popular in contemporary research.

We present and discuss an appropriate concept for the generalized fuzzy Hausdorff distance of a given generalized fuzzy GV-metric space on the set of its nonempty compact subsets. As an application, we use the concept of contraction on a generalized fuzzy GV-metric space to define a new concept of the generalized fuzzy fractal spaces and prove an interesting fixed point theorem in these spaces. In this paper, we present some results to extend and get uncertain models of recent results discused in [3–5] and [6].

2 Basic notions and preliminaries

In this paper, we denote $\mathbb{I} = [0,1]$, $\mathbb{I}^\circ = (0,1)$, $\mathbb{J} = [0,\infty)$, and $\mathbb{J}^\circ = (0,\infty)$.

Definition 2.1 ([7–9]) A mapping $*: \mathbb{I}^2 \to \mathbb{I}$ is called a continuous t-norm (CTN) if

(i) $*$ is associative and commutative;

(ii) $*$ is continuous;

(iii) $\jmath * 1 = \jmath$ for all $\jmath \in \mathbb{I}$;

(iv) $\jmath * \imath \leq \jmath' * \imath'$ whenever $\jmath \leq \jmath'$ and $\imath \leq \imath'$, $\imath, \jmath, \imath', \jmath' \in \mathbb{I}$.

Some examples of CTNs are $\imath *_P \jmath = \imath \cdot \jmath$ and $\imath *_M \jmath = \min\{\imath, \jmath\}$. Note that $\imath * \jmath \le \imath *_M \jmath$ for $\imath, \jmath \in \mathbb{I}$.

Assume that for every $ß \in \mathbb{I}^\circ$, there exists $l \in \mathbb{I}^\circ$ (which does not depend on ℓ) such that the following inequality holds:

$$\overbrace{(1-l) * \cdots * (1-l)}^{\ell} > 1 - ß \quad \text{for each } \ell \in \{2, 3, \ldots\}. \tag{2.1}$$

In this case, we say the CTN $*$ has the (D) property (CTND); for example, $*_M$ is CTND. Now we consider a generalized fuzzy metric space in the George–Veeramani sense of (GFGVM-space); for details and results on fuzzy metric spaces introduced by George and Veeramani, we refer to [2, 10–13], and [14].

Definition 2.2 ([15]) The triple $(S, G, *)$ is a GFGVM-space if $S \neq \emptyset$, $*$ is a CTN, and G is a fuzzy set on $S \times S \times S \times \mathbb{J}^\circ$ such that for all $w, x, y, z \in S$ and σ, δ in $\mathbb{J}^\circ$, we have:

(GGVFM-1) $G_{x,y,z}(\sigma) \in \mathbb{J}^\circ$;

(GGVFM-2) $G_{x,y,z}(\sigma) = 1$ iff $x = y = z$;

(GGVFM-3) $G_{x,y,y}(\sigma) \ge G_{x,y,z}(\sigma)$ for $x \neq y$;

(GGVFM-4) $G_{x,y,z}(\sigma) = G_{x,z,y}(\sigma) = G_{y,x,z}(\sigma) = \cdots$;

(GGVFM-5) $G_{x,y,z}(\cdot) : \mathbb{J}^\circ \to \mathbb{I}^\circ$ is continuous;

(GGVFM-6) $G_{x,y,w}(\sigma + \delta) \ge G_{x,z,z}(\sigma) * G_{z,y,w}(\delta)$.

Tian et al. [15] proved that in a GFGVM-space $(S, G, *)$, we have that $G_{x,y,y}(\cdot)$ is increasing for all $x, y \in S$. For $\varepsilon \in \mathbb{I}^\circ$, $\alpha \in \mathbb{J}^\circ$, and $x_0 \in S$, the set $B_G(x_0, \varepsilon, \alpha) = \{y \in S : G_{x_0,y,y}(\alpha) > 1 - \varepsilon, G_{x_0,x_0,y}(\alpha) > 1 - \varepsilon\}$ is called the neighborhood with center x_0 and radius ε. Consider

$$\tau = \bigl\{X \subset S : \forall x \in X, \exists \alpha \in \mathbb{J}^\circ, \varepsilon \in \mathbb{I}^\circ \text{ such that } B_G(x, \varepsilon, \alpha) \subset X\bigr\}.$$

Then τ is the topology induced by GFGVM G on S. Moreover, the local base $\{B_G(u, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, \ldots\}$ at u leads to the first countability of τ. Also, they proved that every GFGVM-space is Hausdorff.

Now we consider Cauchy sequences and completeness in GFGVM-spaces $(S, G, *)$ (see [15]).

(1) A sequence $\{a_n\}$ in S converges to a point $a \in S$ if for any $\alpha \in \mathbb{J}^\circ$ and $\varepsilon \in \mathbb{I}^\circ$, there is an integer $N_{\alpha,\varepsilon} \in \mathbb{J}^\circ$ such that $a_n \in B_G(a, \varepsilon, \alpha)$ whenever $n > N_{\alpha,\varepsilon}$.

(2) A sequence $\{a_n\}$ in S is called a GFGV-Cauchy sequence (GFGVCS) if for any $\alpha \in \mathbb{J}^\circ$ and $\varepsilon \in \mathbb{I}^\circ$, there is an integer $N_{\alpha,\varepsilon} > 0$ such that $G_{a_n,a_m,a_p}(\alpha) > 1 - \varepsilon$ whenever $m, n, p > N_{\alpha,\varepsilon}$.

(3) A GFGVM-space is said to be GFGV-complete if every GFGVCS is convergent in it.

Let $\{a_n\}$ be a sequence in S. Then the following statements are equivalent:

(i) The sequence $\{a_n\}$ in S converges to a point $a \in S$;

(ii) As $n \to \infty$, $G_{a_n,a_n,a}(\alpha) \to 1$ for each $\alpha \in \mathbb{J}^\circ$;

(iii) As $n \to \infty$, $G_{a_n,a,a}(\alpha) \to 1$ for each $\alpha \in \mathbb{J}^\circ$.

Also, the following statements are equivalent:

(i) A sequence $\{a_n\}$ in S is a GFGVCS;

(ii) For any $\alpha \in \mathbb{J}^\circ$ and $\varepsilon \in \mathbb{I}^\circ$, there is an integer $N'_{\alpha,\varepsilon} > 0$ such that $G_{a_n,a_m,a_m}(\alpha) > 1 - \varepsilon$ whenever $n, m > N'_{\alpha,\varepsilon}$.

Proposition 2.3 ([15]) *Consider the GFGVM-space $(S,G,*)$. Then the function G is continuous on $S \times S \times S \times \mathbb{J}^\circ$.*

Let (S,g) be a G-metric space (for more detail, we refer to [16–19] and [20]). Let G^g be the function defined on $S \times S \times S \times \mathbb{J}^\circ$ by

$$G^g_{x,y,z}(\alpha) = \frac{\alpha}{\alpha + g(x,y,z)}$$

for all $x,y,z \in S$ and $\alpha \in \mathbb{J}^\circ$. Then both $(S,G^g,*_P)$ and $(S,G^g,*_M)$ are GFGVM-spaces (standard GFGVM-spaces). Consider the GFGVM-space $(S,G,*)$. We denote by $\mathfrak{T}_0(S)$, $\mathfrak{F}_0(S)$, and $\mathfrak{R}_0(S)$ the sets of nonempty subsets, of nonempty finite subsets, and of nonempty compact subsets of (S,τ_G), respectively.

Let X and Y be two (nonempty) subsets of a GFGVM-space $(S,G,*)$. For $s \in S$ and $\alpha > 0$, we define $G_{s,X,Y}(\alpha) := \sup\{G_{s,x,y}(\alpha) : x \in X, y \in Y\}$.

Lemma 2.4 *Let $(S,G,*)$ be a GFGVM-space. Then, for all $s \in S$, $X,Y \in \mathfrak{R}_0(S)$, and $\alpha \in \mathbb{J}^\circ$, there are $x_0 \in X$ and $y_0 \in Y$ such that*

$$G_{s,X,Y}(\alpha) = G_{s,x_0,y_0}(\alpha).$$

Proof Let $s \in S$, $X,Y \in \mathfrak{R}_0(S)$, and $\alpha > 0$. By Proposition 2.3 the functions $x \mapsto G_{s,x,y}(\alpha)$ and $y \mapsto G_{s,x,y}(\alpha)$ are continuous. By the compactness of X and Y, there exist $x_0 \in X$ and $y_0 \in Y$ such that

$$\sup_{x\in X, y\in Y} G_{s,x,y}(\alpha) = G_{s,x_0,y_0}(\alpha),$$

and thus

$$G_{s,X,Y}(\alpha) = G_{s,x_0,y_0}(\alpha). \qquad \square$$

Lemma 2.5 *Let $(S,G,*)$ be a GFGVM-space. Then, for all $s \in S$ and $X,Y \in \mathfrak{R}_0(S)$, the function $\alpha \mapsto G_{s,X,Y}(\alpha)$ is continuous on $\mathbb{J}^\circ$.*

Proof From the equality

$$G_{s,X,Y}(\alpha) = \sup_{x\in X, y\in Y} G_{s,x,y}(\alpha)$$

and the continuity of the function $\alpha \mapsto G_{s,x,y}(\alpha)$ for all $x \in X$ and $y \in Y$ on $\mathbb{J}^\circ$, we get the lower semicontinuity of $\alpha \mapsto G_{s,X,Y}(\alpha)$ on $\mathbb{J}^\circ$.

It suffices to show that $\alpha \mapsto G_{s,X,Y}(\alpha)$ is upper semicontinuous on $\mathbb{J}^\circ$. Consider $\alpha \in \mathbb{J}^\circ$ and a sequence $\{\alpha_n\}_n$ in $\mathbb{J}^\circ$ converging to α. Lemma 2.4 implies that or every $n \in \mathbb{N}$, we can find $x_n \in X$ and $y_n \in Y$ such that $G_{s,X,Y}(\alpha) = G_{s,x_n,y_n}(\alpha_n)$. Since $X,Y \in \mathfrak{R}_0(S)$, we can find subsequences $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and $\{y_{n_k}\}_k$ of $\{y_n\}_n$ and two points $x_0 \in X$ and $y_0 \in Y$ such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$ in $(S,G,*)$. Hence $\lim_k G_{s,x_{n_k},y_{n_k}}(\alpha_{n_k}) = G_{s,x_0,y_0}(\alpha)$. Using

Proposition 2.3, we get

$$\lim_k G_{s,X,Y}(\alpha_{n_k}) = G_{s,x_0,y_0}(\alpha) \\ \leq G_{s,X,Y}(\alpha).$$

Consequently, $\alpha \mapsto G_{s,X,Y}(\alpha)$ is upper semicontinuous on $\mathbb{J}^\circ$. □

Lemma 2.6 *Consider the GFGVM-space $(S, G, *)$. Then for all $X \in \mathfrak{R}_0(S)$, $Y, Z \in \mathfrak{T}_0(S)$, and $\alpha \in \mathbb{J}^\circ$, we can find $x_0 \in X$ such that*

$$\inf_{x\in X} G_{x,Y,Z}(\alpha) = G_{x_0,Y,Z}(\alpha).$$

Proof Put $\delta = \inf_{x\in X} G_{x,Y,Z}(\alpha)$. Then we can find a sequence $\{x_n\}_n$ in X such that $\delta + \frac{1}{n} > G_{x_n,Y,Z}(\alpha)$ for all $n \in \mathbb{N}$. Since $X \in \mathfrak{R}_0(S)$, we can find a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and $x_0 \in X$ such that $x_{n_k} \to x_0$ in $(S, G, *)$.

Let $y \in Y$ and $z \in Z$. By Proposition 2.3, $\lim_k G_{x_{n_k},y,z}(\alpha) = G_{x_0,y,z}(\alpha)$. Since $\delta + \frac{1}{n_k} > G_{x_{n_k},y,z}(\alpha)$ for each $k \in \mathbb{N}$, we get $\delta \geq G_{x_0,y,z}(\alpha)$. Hence $\delta = G_{x_0,Y,Z}(\alpha)$. □

Now Lemmas 2.5 and 2.6 imply that the following result.

Corollary 2.7 *Consider the GFGVM-space $(S, G, *)$. Let $X, Y, Z \in \mathfrak{R}_0(S)$ and $\alpha \in \mathbb{J}^\circ$. Then we can find $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ such that*

$$\inf_{x\in X} G_{x,Y,Z}(\alpha) = G_{x_0,y_0,z_0}(\alpha).$$

Proposition 2.8 *Consider the GFGVM-space $(S, G, *)$. Then for all $X, Y, Z \in \mathfrak{R}_0(S)$, the function $\alpha \mapsto \inf_{x\in X} G_{x,Y,Z}$ is continuous on $\mathbb{J}^\circ$.*

Proof The continuity of $\alpha \mapsto G_{x,Y,Z}(\alpha)$ on $\mathbb{J}^\circ$ follows from Lemma 2.5, which implies the upper semicontinuity of $\alpha \mapsto \inf_{x\in X} G_{x,Y,Z}(\alpha)$ on $\mathbb{J}^\circ$.

It suffices to show that $\alpha \mapsto \inf_{x\in X} G_{x,Y,Z}(\alpha)$ is lower semicontinuous on $\mathbb{J}^\circ$. Consider $\alpha \in \mathbb{J}^\circ$ and a sequence $\{\alpha_n\}_n$ in $\mathbb{J}^\circ$ converging to α. Lemma 2.6 implies that we can find $x_n \in X$ such that $G_{x_n,Y,Z}(\alpha_n) = \inf_{x\in X} G_{x,Y,Z}(\alpha_n)$. Since $X \in \mathfrak{R}_0(S)$, we can find a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and $x_0 \in X$ such that $x_{n_k} \to x_0$ in $(S, G, *)$. Then Lemma 2.4 implies that we can find $y_0 \in Y$ and $z_0 \in Z$ such that $G_{x_0,y_0,z_0}(\alpha) = G_{x_0,Y,Z}(\alpha)$, and then $\lim_k G_{x_{n_k},y_0,z_0}(\alpha_{n_k}) = G_{x_0,y_0,z_0}(\alpha)$ by Proposition 2.3. Then for $\varepsilon \in \mathbb{I}^\circ$, we can find $k_0 \in \mathbb{N}$ such that $G_{x_0,y_0,z_0}(\alpha) < \varepsilon + G_{x_{n_k},y_0,z_0}(\alpha)$ for every $k \geq k_0$. So

$$\inf_{x\in X} G_{x,Y,Z}(\alpha) \leq G_{x_0,y_0,z_0}(\alpha) \\ < \varepsilon + G_{x_{n_k},Y,Z}(\alpha_{n_k}) \\ = \varepsilon + \inf_{x\in X} G_{x,Y,Z}(\alpha_{n_k})$$

for every $k \geq k_0$. Hence $\alpha \mapsto \inf_{x\in X} G_{x,Y,Z}(\alpha)$ is lower semicontinuous on $\mathbb{J}^\circ$. □

Remark 2.9 Proposition 2.8 implies the continuity of $\alpha \mapsto \inf_{y\in Y} G_{X,y,Z}(\alpha)$ and $\alpha \mapsto \inf_{z\in Z} G_{X,Y,z}(\alpha)$ on $\mathbb{J}^\circ$ for $X, Y, Z \in \mathfrak{R}_0(S)$.

3 GFGV-Hausdorff distance on $\mathfrak{R}_0(S)$

Consider the GFGVM-space $(S, G, *)$. We define the function H_G on $\mathfrak{R}_0(S) \times \mathfrak{R}_0(S) \times \mathfrak{R}_0(S) \times \mathbb{J}^\circ$ by

$$H_G(X, Y, Z, \alpha) = \min\Big\{\inf_{x\in X} G_{x,Y,Z}(\alpha), \inf_{y\in Y} G_{X,y,Z}(\alpha), \inf_{z\in Z} G_{X,Y,z}(\alpha)\Big\}$$

for $X, Y, Z \in \mathfrak{R}_0(S)$ and $\alpha \in \mathbb{J}^\circ$.

Lemma 3.1 *Consider the GFGVM-space $(S, G, *)$. Let $x \in S$, $Y, Z \in \mathfrak{R}_0(S)$, $W \in \mathfrak{T}_0(S)$, and $\alpha, \beta \in \mathbb{J}^\circ$. Then*

$$G_{x,Y,W}(\alpha + \beta) \geq G_{x,Z,Z}(\alpha) * G_{z_x,Y,W}(\beta),$$

where $z_x \in Z$ satisfies $G_{x,Z,Z}(\alpha) = G_{x,z_x,z_x}(\alpha)$.

Proof By Lemma 2.4, for $z_x \in Z$, we have $G_{x,Z,Z}(\alpha) = G_{x,z_x,z_x}(\alpha)$. Now, for all $y \in Y$ and $w \in W$, we have

$$\begin{aligned} G_{x,Y,W}(\alpha + \beta) &\geq G_{x,y,w}(\alpha + \beta) \\ &\geq G_{x,z_x,z_x}(\alpha) * G_{z_x,y,w}(\beta). \end{aligned}$$

Then by the continuity of CTN $*$ we get

$$G_{x,Y,W}(\alpha + \beta) \geq G_{x,Z,Z}(\alpha) * G_{z_x,Y,W}(\beta). \qquad \square$$

Theorem 3.2 *Consider the GFGVM-space $(S, G, *)$. Then $(\mathfrak{R}_0(S), H_G, *)$ is a GFGVM-space.*

Proof Let $X, Y, Z, W \in \mathfrak{R}_0(S)$ and $\alpha, \beta \in \mathbb{J}^\circ$. By Lemma 2.6 there exist $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ such that

$$\inf_{x\in X} G_{x,Y,Z} = G_{x_0,Y,Z},$$

$$\inf_{y\in Y} G_{X,y,Z} = G_{X,y_0,Z},$$

and

$$\inf_{z\in Z} G_{X,Y,z} = G_{X,Y,z_0}.$$

Then $H_G(X, Y, Z, \alpha) > 0$. Furthermore, it is obvious that $X = Y = Z$ if and only if $H_G(X, Y, Z, \alpha) = 1$, and so we get $H_G(X, Y, Z, \alpha) = H_G(X, Z, Y, \alpha) = H_G(Y, X, Z, \alpha) = \cdots$.

Now by Lemma 3.1 and the continuity of CTN $*$ we have

$$\inf_{x\in X} G_{x,Y,W}(\alpha + \beta) \geq \inf_{x\in X} G_{x,Z,Z}(\alpha) * \inf_{x\in X} G_{z_x,Y,W}(\beta).$$

Since $\{z_x : x \in X\} \subseteq Z$, we have $\inf_{x\in X} G_{z_x,Y,W}(\beta) \geq \inf_{z\in Z} G_{z,Y,W}(\beta)$. Then

$$\inf_{x\in X} G_{x,Y,W}(\alpha+\beta) \geq \inf_{x\in X} G_{x,Z,Z}(\alpha) * \inf_{z\in Z} G_{z,Y,W}(\beta).$$

In the same way, we obtain

$$\inf_{y\in Y} G_{x,y,W}(\alpha+\beta) \geq \inf_{y\in Y} G_{y,Z,Z}(\alpha) * \inf_{z\in Z} G_{z,X,W}(\beta)$$

and

$$\inf_{w\in W} G_{x,Y,w}(\alpha+\beta) \geq \inf_{w\in W} G_{w,Z,Z}(\alpha) * \inf_{z\in Z} G_{z,X,W}(\beta).$$

Then it easily follows that

$$H_G(X,Y,W,\alpha+\beta) \geq H_G(X,Z,Z,\alpha) * H_G(Z,Y,W,\beta).$$

By Proposition 2.8 and Remark 2.9 we conclude that $\alpha \mapsto H_G(X,Y,Z,\alpha)$, is continuous on $\mathbb{J}^\circ$. Then $(\mathfrak{R}_0(S), H_G, *)$ is a GFGVM-space. □

We call the GFGVM $(H_G, *)$ the GFGV-Hausdorff distance on $\mathfrak{R}_0(S)$.

Proposition 3.3 *The GFGV-Hausdorff distance $(H_{G^g}, *_P)$ of the standard GFGVM $(G^g, *_P)$ coincides with the standard GFGVM $(G^{h_g}, *_P)$ of the Hausdorff distance*

$$h_g(X,Y,Z) := \max\left\{\sup_{x\in X} g(x,Y,Z), \sup_{y\in Y} g(X,y,Z), \sup_{z\in Z} g(X,Y,z)\right\}$$

on $\mathfrak{R}_0(S)$.

Proof Let $X, Y, Z \in \mathfrak{R}_0(S)$ and $\alpha \in \mathbb{J}^\circ$, and let

$$G^g_{x,Y,Z}(\alpha) = \frac{\alpha}{\alpha + g(x,Y,Z)}$$

for $x \in X$. Now we have

$$\inf_{x\in X} G^g_{x,Y,Z}(\alpha) = \inf_{x\in X}\left(\frac{\alpha}{\alpha + g(x,Y,Z)}\right) = \frac{\alpha}{\alpha + \sup_{x\in X} g(x,Y,Z)}.$$

Similarly, we obtain

$$\inf_{y\in Y} G^g_{X,y,Z}(\alpha) = \inf_{y\in Y}\left(\frac{\alpha}{\alpha + g(X,y,Z)}\right) = \frac{\alpha}{\alpha + \sup_{y\in Y} g(X,y,Z)},$$

$$\inf_{z\in Z} G^g_{X,Y,z}(\alpha) = \inf_{z\in Z}\left(\frac{\alpha}{\alpha + g(X,Y,z)}\right) = \frac{\alpha}{\alpha + \sup_{z\in Z} g(X,Y,z)}.$$

Therefore $H_{G^g}(X,Y,Z,\alpha) = G^{h_g}_{X,Y,Z}(\alpha)$. □

Now we present some examples to support our idea.

Example 3.4 Consider the discrete G-metric g on S (see [18]) with $|S| \geq 3$. Let X, Y, and Z be nonempty finite subsets of S such that $X \neq Y(Y \neq Z$ or $X \neq Z)$. Then $h_g(X,Y,Z) = 1$, and so, by Proposition 3.3, we get $H_{G_g}(X,Y,Z,\alpha) = \frac{\alpha}{\alpha+1}$ for all $\alpha \in \mathbb{J}^\circ$.

Example 3.5 Let g be the Euclidean G-metric on $\mathbb{R}$ (see [18]), and let $X = [x_1,x_2]$, $Y = [y_1,y_2]$, and $Z = [z_1,z_2]$ be compact intervals. Then

$$h_g(X,Y,Z) = \max\bigl\{|x_1 - y_1|, |x_2 - y_2|, |x_1 - z_1|, |x_2 - z_2|, |y_1 - z_1|, |y_2 - z_2|\bigr\}.$$

Now by Proposition 3.3 we get

$$H_{G^g}(X,Y,Z,\alpha) = \frac{\alpha}{\alpha + h_g(X,Y,Z)}$$

for all $\alpha \in \mathbb{J}^\circ$.

4 GFGVCS

Now we assume that all $*$ are CTND.

Lemma 4.1 *Consider the GFGVM-space $(S,G,*)$. For each $\mu \in \mathbb{I}^\circ$, define the function*

$$E_{\mu,G}(x,y,z) = \inf\bigl\{\alpha > 0, G_{x,y,z}(\alpha) > 1 - \mu\bigr\}$$

for $x,y,z \in S$. Then:

(i) *For any $\lambda \in \mathbb{I}^\circ$, we can find $\mu \in \mathbb{I}^\circ$ such that*

$$E_{\lambda,G}(s_0,s_m,s_m) \leq \sum_{i=0}^{m-1} E_{\mu,G}(s_i,s_{i+1},s_{i+1}),$$

$$E_{\lambda,G}(s_0,s_0,s_m) \leq \sum_{i=0}^{m-1} E_{\mu,G}(s_i,s_i,s_{i+1}) \quad \textit{for all } s_0,s_1,\dots,s_m \in S.$$

(ii) *Let $\{s_n\}_n$ be a convergent sequence in $(S,G,*)$. Then we have $E_{\lambda,G}(s,s_n,s_n) \to 0$ and $E_{\lambda,G}(s_n,s,s) \to 0$ and vice versa.*

Also, if $\{s_n\}$ is a GFGVCS in $(S,G,*)$, then it is a GFGVCS with $E_{\lambda,G}$ and vice versa.

Proof (i) For every $\lambda \in \mathbb{I}^\circ$, we can find $\mu \in \mathbb{I}^\circ$ such that

$$\overbrace{(1-\mu)*,\dots,*(1-\mu)}^{m} > 1 - \lambda.$$

For any given $m \in Z^+$, we put

$$E_{\mu,G}(s_i,s_{i+1},s_{i+1}) = \alpha_i, \quad i = 0,1,2,\dots,m-1.$$

It is obvious that for every $\varepsilon > 0$,

$$E_{\mu,G}(s_i,s_{i+1},s_{i+1}) < \alpha_i + \frac{\varepsilon}{m}.$$

For $i = 0, 1, \dots, m-1$, we have $G_{s_i,s_{i+1},s_{i+1}}(\alpha_i + \frac{\varepsilon}{m}) > 1 - \mu$, and so

$$\begin{aligned}
&G_{s_0,s_m,s_m}(\alpha_0 + \alpha_1 + \cdots + \alpha_{m-1} + \varepsilon) \\
&\quad \geq \overbrace{G_{s_0,s_1,s_1}\left(\alpha_0 + \frac{\varepsilon}{m}\right) * \dots * G_{s_{m-1},s_m,s_m}\left(\alpha_{m-1} + \frac{\varepsilon}{m}\right)}^{m} \\
&\quad \geq \overbrace{(1-\mu)*, \dots, *(1-\mu)}^{m} \\
&\quad > 1 - \lambda.
\end{aligned}$$

Then

$$E_{\lambda,G}(s_0, s_m, s_m) \leq \alpha_0 + \alpha_1 + \cdots + \alpha_{m-1} + \varepsilon,$$

and so

$$E_{\lambda,G}(s_0, s_m, s_m) \leq \sum_{i=0}^{m-1} E_{\mu,G}(s_i, s_i, s_i) + \varepsilon. \tag{4.1}$$

Taking the limit in (4.1) as $\varepsilon \downarrow 0$, we get

$$E_{\lambda,G}(s_0, s_m, s_m) \leq \sum_{i=0}^{m-1} E_{\mu,G}(s_i, s_i, s_i) \tag{4.2}$$

for all $s_0, s_1, \dots, s_m \in S$.

Similarly, we get

$$E_{\lambda,G}(s_0, s_0, s_m) \leq \sum_{i=0}^{m-1} E_{\mu,G}(s_i, s_i, s_{i+1})$$

for all $s_0, s_1, \dots, s_m \in S$.

(ii) We have

$$G_{s,s_n,s_n}(\eta) > 1 - \lambda \quad \Longleftrightarrow \quad E_{\lambda,G}(s, s_n, s_n) < \eta$$

for every $\eta > 0$.

Similarly,

$$G_{s_n,s,s}(\eta) > 1 - \lambda \quad \Longleftrightarrow \quad E_{\lambda,G}(s_n, s, s) < \eta$$

for every $\eta \in \mathbb{J}^\circ$. □

Lemma 4.2 *Consider the GFGVM-space* $(S, G, *)$. *If*

$$G_{x,y,z}(\alpha) = C \tag{4.3}$$

for all $x, y, z \in S$ *and* $\alpha \in \mathbb{J}^\circ$, *then* $C = 1$.

Proof Taking $x = y = z$ in (4.3), we get $C = 1$. □

Consider the class of mappings $\phi : \mathbb{J}^\circ \to \mathbb{J}^\circ$ that are onto, strictly increasing, and $\phi(\alpha) < \alpha$ for all $\alpha \in \mathbb{J}^\circ$.

Lemma 4.3 *Consider the GFGVM-space $(S, G, *)$. Then*

$$\inf\{\phi^n(\alpha) > 0 : G_{x,y,z}(\alpha) > 1 - \lambda\} \le \phi^n(\inf\{\alpha > 0 : G_{x,y,z}(\alpha) > 1 - \lambda\})$$

for all $x, y, z \in S$, $\lambda \in \mathbb{I}^\circ$, and $n \in \mathbb{N}$.

Proof Fix $\alpha \in \mathbb{J}^\circ$ with $G_{x,y,z}(\alpha) > 1 - \lambda$. Then $\phi^n(\alpha) \in \mathbb{J}^\circ$. Also,

$$\phi^n(\alpha) > \inf\{\phi^n(\beta) > 0 : G_{x,y,z}(\beta) > 1 - \lambda\},$$

and so we have

$$\alpha \ge (\phi^n)^{-1}(\inf\{\phi^n(\beta) > 0 : G_{x,y,z}(\beta) > 1 - \lambda\}).$$

Then

$$\inf\{\alpha > 0 : G_{x,y,z}(\alpha) > 1 - \lambda\} \ge (\phi^n)^{-1}(\inf\{\phi^n(\beta) > 0 : G_{x,y,z}(\beta) > 1 - \lambda\}),$$

and we conclude that

$$\inf\{\phi^n(\alpha) > 0 : G_{x,y,z}(\alpha) > 1 - \lambda\} \le \phi^n(\inf\{\alpha > 0 : G_{x,y,z}(\alpha) > 1 - \lambda\}).$$ □

Lemma 4.4 *Consider the GFGVM-space $(S, G, *)$. Suppose that $\{s_n\} \subseteq S$ satisfies*

$$G_{s_n,s_{n+1},s_{n+1}}(\phi^n(\alpha)) \ge G_{s_0,s_1,s_1}(\alpha) \quad \text{for all } \alpha \in \mathbb{J}^\circ.$$

Then $\{s_n\}$ is a GFGVCS.

Proof Using Lemma 4.3, we get

$$\begin{aligned}
&\mathrm{E}_{\mu,G}(s_n, s_{n+1}, s_{n+1}) \\
&\quad = \inf\{\phi^n(\alpha) > 0 : G_{s_n,s_{n+1},s_{n+1}}(\phi^n(\alpha)) > 1 - \mu\} \\
&\quad \le \inf\{\phi^n(\alpha) > 0 : G_{s_0,s_1,s_1}(\alpha) > 1 - \mu\} \\
&\quad \le \phi^n(\inf\{\alpha > 0 : G_{s_0,s_1,s_1}(\alpha) > 1 - \mu\}) \\
&\quad = \phi^n(\mathrm{E}_{\mu,G}(s_0, s_1, s_1))
\end{aligned}$$

for every $\mu \in \mathbb{I}^\circ$.

For every $\lambda \in \mathbb{I}^\circ$, there exists $\theta \in \mathbb{I}^\circ$ such that

$$\begin{aligned}
&\mathrm{E}_{\lambda,G}(s_n, s_m, s_m) \\
&\quad \le \mathrm{E}_{\theta,G}(s_{m-1}, s_m, s_m) + \mathrm{E}_{\theta,G}(s_{m-2}, s_{m-1}, s_{m-1}) + \cdots
\end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&+ \mathrm{E}_{\theta,G}(s_n, s_{n+1}, s_{n+1}) \\
&\leq \sum_{i=n}^{m-1} \phi^i\bigl(E_{\theta,G}(s_0, s_1, s_1)\bigr) \\
&\to 0
\end{aligned}
$$

as $m, n \to \infty$. By Lemma 4.1, $\{s_n\}$ is a GFGVCS. □

5 GFGV-fractal spaces

Hutchinson [21] considered the concept of fractal theory by studying the iterated function system (IFS). This subject was generalized by Barnsley [22], Bisht [6], Imdad [23], and Ri [5].

Definition 5.1 Consider the GFGVM-space $(S, G, *)$. A mapping $\Omega : S \to S$ is said to be a GFGV-ϕ-contractive mapping if

$$
G_{\Omega(x),\Omega(y),\Omega(z)}\bigl(\phi(\alpha)\bigr) \geq G_{x,y,z}(\alpha)
$$

for all $x, y, z \in S$ and $\alpha \in \mathbb{J}^\circ$.

Definition 5.2 A GFGV iterated function system (GFGVIFS) is a finite set of GFGV-ϕ-contractions $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ $(m \geq 2)$ defined on a complete GFGVM-space $(S, G, *)$.

Consider the given GFGVIFS, if there is a unique nonempty compact set Γ of the complete GFGVM-space $(S, G, *)$ such that $\Gamma = \bigcup_{i=1}^m \Omega_i(\Gamma)$ in which Γ is a fractal set called the attractor of the respective GFGVIFS. The related attractor GFGVIFS is called a GFGV-fractal space.

Lemma 5.3 *Consider the GFGVM-space $(S, G, *)$. Assume that $\Omega : S \to S$ is a mapping such that*

$$
G_{\Omega(x),\Omega(y),\Omega(z)}\bigl(\phi(\alpha)\bigr) \geq G_{x,y,z}(\alpha) \tag{5.1}
$$

for all $x, y, z \in S$ and $\alpha \in \mathbb{J}^\circ$. Then the sequence $\{\Omega^n(x)\}_{n=1}^{+\infty}$ is GFGVCS.

Proof We use induction. In (5.1), taking $y = z = \Omega(x)$, we get

$$
G_{\Omega(x),\Omega^2(x),\Omega^2(x)}\bigl(\phi(\alpha)\bigr) \geq G_{x,\Omega(x),\Omega(x)}(\alpha).
$$

Let $G_{\Omega^n(x),\Omega^{n+1}(x),\Omega^{n+1}(x)}(\phi^n(\alpha)) \geq G_{x,\Omega(x),\Omega(x)}(\alpha)$. Then

$$
\begin{aligned}
&G_{\Omega^{n+1}(x),\Omega^{n+2}(x),\Omega^{n+2}(x)}\bigl(\phi^{n+1}(\alpha)\bigr) \\
&= G_{\Omega(\Omega^n(x)),\Omega(\Omega^{n+1}(x)),\Omega(\Omega^{n+1}(x))}\bigl(\phi\bigl(\phi^n(\alpha)\bigr)\bigr) \\
&\geq G_{\Omega^n(x),\Omega^{n+1}(x),\Omega^{n+1}(x)}\bigl(\phi^n(\alpha)\bigr) \\
&\geq G_{x,\Omega(x),\Omega(x)}(\alpha).
\end{aligned}
$$

Put $\{s_n\}_{n=1}^{+\infty} = \{\Omega^n(x)\}_{n=1}^{+\infty}$. Then $\{s_n\}$ is a sequence that satisfies the conditions of Lemma 4.4. Therefore

$$G_{s_n,s_{n+1},s_{n+1}}\big(\phi^n(\alpha)\big) \ge G_{s_0,s_1,s_1}(\alpha),$$

and hence $\{s_n\}_{n=1}^{+\infty} = \{\Omega^n(x)\}_{n=1}^{+\infty}$ is a GFGVCS. □

Lemma 5.4 *Consider the GFGVM-space $(S,G,*)$ and GFGVF-ϕ-contractive map Ω such that*

$$G_{\Omega(x),\Omega(y),\Omega(z)}\big(\phi(\alpha)\big) \ge G_{x,y,z}(\alpha) \tag{5.2}$$

for all $x,y,z \in S$ and $\alpha \in \mathbb{J}^\circ$. Then Ω has a unique fixed point δ in S.

Proof Lemma 5.3 and (5.2) imply that the sequence $\{\Omega^n(x)\}_{n=1}^{+\infty}$ is GFGVCS for each $x \in S$ and $\lim_{n\to+\infty} \Omega^n(x) = \delta \in S$.

Letting $x_0 = x$ and $x_n = \Omega^n(x)$ for each $n \ge 1$, since $\lim_{n\to+\infty} \Omega^n(x) = \delta$, we have $\lim_{n\to+\infty} G_{x_n,\delta,\delta}(\alpha) = 1$ for each $\alpha \in \mathbb{J}^\circ$.

On the other hand, we have

$$G_{\Omega(\delta),x_{n+1},x_{n+1}}\big(\phi(\alpha)\big) \ge G_{\delta,x_n,x_n}(\alpha)$$

for each $n \in \mathbb{N}$ and each $\alpha > 0$. Then

$$G_{\Omega(\delta),\delta,\delta}\big(\phi(\alpha)\big) = \lim_{n\to+\infty} G_{\Omega(\delta),x_{n+1},x_{n+1}}\big(\phi(\alpha)\big) \ge \lim_{n\to+\infty} G_{\delta,x_n,x_n}(\alpha) = 1$$

for each $\alpha > 0$. Therefore $\delta = \Omega(\delta)$, that is, δ is a fixed point of Ω.

Now we prove that δ is the unique fixed point of Ω. If σ is another fixed point of Ω, then for any $\alpha \in \mathbb{J}^\circ$,

$$G_{\delta,\delta,\sigma}(\alpha) = G_{\Omega(\delta),\Omega(\delta),\Omega(\sigma)}(\alpha) \le G_{\Omega(\delta),\Omega(\delta),\Omega(\sigma)}\big(\phi(\alpha)\big).$$

On the other hand, since $G_{x,y,y}(\alpha)$ is nondecreasing and $\phi(\alpha) < \alpha$, we have

$$G_{\Omega(\delta),\Omega(\delta),\Omega(\sigma)}\big(\phi(\alpha)\big) \le G_{\Omega(\delta),\Omega(\delta),\Omega(\sigma)}(\alpha) = G_{\delta,\delta,\sigma}(\alpha).$$

Hence $G_{\delta,\delta,\sigma}(\alpha) = C$ for all $\alpha \in \mathbb{J}^\circ$. From Lemma 4.2 we get $C = 1$. Therefore $\delta = \sigma$, that is, δ is a unique fixed point of Ω. □

Now we present an example illustrating our results; for more applications, we refer to [11, 17, 24–29].

Example 5.5 Let $S = C(\mathbb{I})$ be the set of all continuous functions defined on $\mathbb{I}$. Define G on $S \times S \times S \times \mathbb{J}^0$ by

$$G_{u,v,w}(\alpha) = \inf_{\delta\in\mathbb{I}}\left(\frac{\alpha}{\alpha + |u(\delta) - v(\delta)| + |v(\delta) - w(\delta)| + |w(\delta) - u(\delta)|}\right)$$

for $u, v, w \in S$ and $\alpha \in \mathbb{J}^0$. Then

$$G_{u,v,w}(\alpha) = \frac{\alpha}{\alpha + \sup_{\delta\in\mathbb{I}} |u(\delta) - v(\delta)| + \sup_{\delta\in\mathbb{I}} |v(\delta) - w(\delta)| + \sup_{\delta\in\mathbb{I}} |w(\delta) - u(\delta)|}.$$

We denote

$$g(u, v, w) = \sup_{\delta\in\mathbb{I}}\bigl|u(\delta) - v(\delta)\bigr| + \sup_{\delta\in\mathbb{I}}\bigl|v(\delta) - w(\delta)\bigr| + \sup_{\delta\in\mathbb{I}}\bigl|w(\delta) - u(\delta)\bigr|.$$

It is obvious that (S, g) is a complete G-metric space [16, 27]. Then $(S, G, *_M)$ is a GFGVM-space.

Let $\phi(\alpha) : \mathbb{J} \to \mathbb{J}$ be defined as $\phi(\alpha) = \frac{\alpha}{\alpha+1}$.

Consider the following integral equation:

$$\Omega\bigl(u(\delta)\bigr) = \int_0^1 p(\delta, \sigma)f\bigl(\sigma, u(\sigma)\bigr)\, d\sigma, \quad \sigma \in \mathbb{I}. \tag{5.3}$$

Suppose that the following conditions are satisfied:

(i) $p : \mathbb{I} \times \mathbb{I} \to \mathbb{R}^+$ is continuous.

(ii) $f : \mathbb{I} \times \mathbb{R} \to \mathbb{R}^+$ is continuous.

(iii) There exists a constant $\lambda > 0$ such that

$$\bigl|f(\delta, u) - f(\delta, v)\bigr| \le \lambda|u - v|$$

for all $\delta \in \mathbb{I}$ and $u, v \in \mathbb{R}$.

(iv) $\lambda\|p\|_\infty \le \frac{1}{\alpha+1}$, where

$$\|p\|_\infty = \sup\bigl\{p(\delta, \sigma) : \delta, \sigma \in \mathbb{I}\bigr\}.$$

Then, under conditions (i)–(iv), integral (5.3) has a unique solution in $C(\mathbb{I})$.

Proof First, consider $\Omega : S \to S$. It is clear that Ω is well defined (i.e., for $u \in S$, we have $\Omega(u) \in S$). Then we have

$$\begin{aligned}
&G_{\Omega(u(\delta)),\Omega(v(\delta)),\Omega(w(\delta))}\bigl(\phi(\alpha)\bigr)\\
&\quad = \inf_{\delta\in\mathbb{I}}\left(\frac{\phi(\alpha)}{\phi(\alpha) + |\Omega(u(\delta)) - \Omega(v(\delta))| + |\Omega(v(\delta)) - \Omega(w(\delta))| + |\Omega(w(\delta)) - \Omega(u(\delta))|}\right)\\
&\quad = \bigl(\phi(\alpha)\bigr)\Big/\Bigl(\phi(\alpha) + \sup_{\delta\in\mathbb{I}}\bigl|\Omega\bigl(u(\delta)\bigr) - \Omega\bigl(v(\delta)\bigr)\bigr|\\
&\qquad + \sup_{\delta\in\mathbb{I}}\bigl|\Omega\bigl(v(\delta)\bigr) - \Omega\bigl(w(\delta)\bigr)\bigr| + \sup_{\delta\in\mathbb{I}}|\Omega\bigl(w(\delta)\bigr) - \Omega_(u(\delta)\bigr)|)\\
&\quad = \bigl(\phi(\alpha)\bigr)\Big/\Bigl(\phi(\alpha) + \Bigl(\sup_{\delta\in\mathbb{I}}\Bigl|\int_0^1 p(\delta, \sigma)f\bigl(\sigma, u(\sigma)\bigr)\, d\sigma - \int_0^1 p(\delta, \sigma)f\bigl(\sigma, v(\sigma)\bigr)\, d\sigma\Bigr|\\
&\qquad + \sup_{\delta\in\mathbb{I}}\Bigl|\int_0^1 p(\delta, \sigma)f\bigl(\sigma, v(\sigma)\bigr)\, d\sigma - \int_0^1 p(\delta, \sigma)f\bigl(\sigma, w(\sigma)\bigr)\, d\sigma\Bigr|\\
&\qquad + \sup_{\delta\in\mathbb{I}}\Bigl|\int_0^1 p(\delta, \sigma)f\bigl(\sigma, w(\sigma)\bigr)\, d\sigma - \int_0^1 p(\delta, \sigma)f\bigl(\sigma, u(\sigma)\bigr)\, d\sigma\Bigr|\Bigr)\Bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \big(\phi(\alpha)\big) \Big/ \bigg(\phi(\alpha) + \bigg(\sup_{\delta\in\mathbb{I}}\bigg|\int_0^1 p(\delta,\sigma)\big(f\big(\sigma,u(\sigma)\big) - f\big(\sigma,v(\sigma)\big)\big)\,d\sigma\bigg| \\
&\quad + \sup_{\delta\in\mathbb{I}}\bigg|\int_0^1 p(\delta,\sigma)\big(f\big(\sigma,v(\sigma)\big) - f\big(\sigma,w(\sigma)\big)\big)\,d\sigma\bigg| \\
&\quad + \sup_{\delta\in\mathbb{I}}\bigg|\int_0^1 p(\delta,\sigma)\big(f\big(\sigma,w(\sigma)\big) - f\big(\sigma,u(\sigma)\big)\big)\,d\sigma\bigg|\bigg)\bigg) \\
&\geq \big(\phi(\alpha)\big) \Big/ \bigg(\phi(\alpha) + \bigg(\lambda \sup_{\delta\in\mathbb{I}} \int_0^1 p(\delta,\sigma)\big|u(\sigma) - v(\sigma)\big|\,d\sigma \\
&\quad + \lambda \sup_{\delta\in\mathbb{I}} \int_0^1 p(\delta,\sigma)\big|v(\sigma) - w(\sigma)\big|\,d\sigma \\
&\quad + \lambda \sup_{\delta\in\mathbb{I}} \int_0^1 p(\delta,\sigma)\big|w(\sigma) - u(\sigma)\big|\,d\sigma\bigg)\bigg).
\end{aligned}
\tag{5.4}
$$

Using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\int_0^1 p(\delta,\sigma)\big|u(\sigma) - v(\sigma)\big|\,d\sigma &\leq \bigg(\int_0^1 p^2(\delta,\sigma)\,d\sigma\bigg)^{\frac{1}{2}} \Big(\int_0^1 \big(\big|u(\sigma) - v(\sigma)\big|^2\,d\sigma\Big)^{\frac{1}{2}} \\
&\leq \|p\|_\infty \sup_{\delta\in\mathbb{I}}\big|u(\delta) - v(\delta)\big|.
\end{aligned}
\tag{5.5}
$$

In the same way, we have

$$
\int_0^1 p(\delta,\sigma)\big|v(\sigma) - w(\sigma)\big|\,d\sigma \leq \|p\|_\infty \sup_{\delta\in\mathbb{I}}\big|v(\delta) - w(\delta)\big| \tag{5.6}
$$

and

$$
\int_0^1 p(\delta,\sigma)\big|w(\sigma) - u(\sigma)\big|\,d\sigma \leq \|p\|_\infty \sup_{\delta\in\mathbb{I}}\big|w(\delta) - u(\delta)\big|. \tag{5.7}
$$

Replacing (5.5), (5.6), and (5.7) in (5.4), we obtain that

$$
\begin{aligned}
&G_{\Omega(u),\Omega(v),\Omega(w)}\big(\phi(\alpha)\big) \\
&\quad\geq \big(\phi(\alpha)\big) \Big/ \bigg(\phi(\alpha) + \bigg(\lambda \sup_{\delta\in\mathbb{I}} \int_0^1 p(\delta,\sigma)\big|u(\sigma) - v(\sigma)\big|\,d\sigma \\
&\qquad + \lambda \sup_{\delta\in\mathbb{I}} \int_0^1 p(\delta,\sigma)\big|v(\sigma) - w(\sigma)\big|\,d\sigma \\
&\qquad + \lambda \sup_{\delta\in\mathbb{I}} \int_0^1 p(\delta,\sigma)\big|w(\sigma) - u(\sigma)\big|\,d\sigma\bigg)\bigg) \\
&\quad\geq \frac{\phi(\alpha)}{\phi(\alpha) + \lambda\|p\|_\infty(\sup_{\delta\in\mathbb{I}}|u(\delta) - v(\delta)| + \sup_{\delta\in\mathbb{I}}|v(\delta) - w(\delta)| + \sup_{\delta\in\mathbb{I}}|w(\delta) - u(\delta)|)} \\
&\quad= \frac{\phi(\alpha)}{\phi(\alpha) + \lambda\|p\|_\infty g(u,v,w)} \\
&\quad\geq \frac{\frac{\alpha}{\alpha+1}}{\frac{\alpha}{\alpha+1} + \frac{1}{\alpha+1}g(u,v,w)}
\end{aligned}
$$

$$= \frac{\alpha}{\alpha + g(u,v,w)}$$
$$= G_{u,v,w}(\alpha).$$

By Lemma 5.4 Ω has a unique fixed point, that is, $\Omega(u) = u$, and so u is the unique solution of equation (5.3). □

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Authors' contributions
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Author details
[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran. [2]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran.

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