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# On two kinds of the reverse half-discrete Mulholland-type inequalities involving higher-order derivative function

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**Abstract**

By means of the weight functions, Hermite–Hadamard's inequality, and the techniques of real analysis, a new more accurate reverse half-discrete Mulholland-type inequality involving one higher-order derivative function is given. The equivalent statements of the best possible constant factor related to a few parameters, the equivalent forms, and several particular inequalities are provided. Another kind of the reverses is also considered.

**MSC:** 26D15; 26D10; 26A42

**Keywords:** Weight function; Hermite–Hadamard's inequality; Half-discrete Mulholland-type inequality; Higher-order derivative function; Parameter; Best possible constant factor

## 1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following Hardy–Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m+n} a_m b_n < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1}$$

Replacing $\frac{1}{m+n}$ with $\frac{1}{m+n-1}$ in (1), we have a more accurate form of (1) (cf. [1], Theorem 323). We still have the following Mulholland's inequality with the same best possible constant factor of (1) (cf. [1], Theorem 343):

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{\ln mn} a_m b_n < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=2}^{\infty} m^{p-1} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=2}^{\infty} n^{q-1} b_n^q \right)^{\frac{1}{q}}. \tag{2}$$

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In 2006, by means of Euler–Maclaurin's summation formula, Krnic et al. [2] provided an extension of (1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{3}$$

where $\lambda_i \in (0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda\in(0,4]$, and the constant factor $B(\lambda_1,\lambda_2)$ is best possible, $B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\,dt$ $(u,v>0)$ is the beta function. For $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, inequality (3) reduces to the published result in [3].

In 2019, by using (2) and Abel's summation by parts formula, Adiyasuren et al. [4] published a Hardy–Hilbert-type inequality with the kernel as $\frac{1}{(m+n)^{\lambda}}$ involving partial sums. Inequalities (1)–(3) with their reverses play an important role in analysis and its applications (cf. [5–15]).

In 1934, Hardy et al. [1] also published a half-discrete Hilbert-type inequality in Theorem 351: If $K(t)$ $(t>0)$ is decreasing, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s)=\int_0^{\infty}K(t)t^{s-1}\,dt<\infty$, $a_n\geq 0$, such that $0<\sum_{n=1}^{\infty}a_n^p<\infty$, then

$$\int_0^{\infty} x^{p-2}\left(\sum_{n=1}^{\infty}K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty}a_n^p. \tag{4}$$

Some new extensions of (4) were given in [16–20].

In 2016, by the use of the techniques of real analysis, Hong et al. [21] gave an equivalent statement of the best possible constant factor related to several parameters in the general form of (1). The other similar results were provided by [22–29]. Recently, Yang et al. [30] gave a new result in a reverse half-discrete Hilbert-type inequality.

In this paper, following the way of [4, 21], by means of the weight functions, Hermite–Hadamard's inequality, and the techniques of real analysis, a new more accurate reverse half-discrete Mulholland-type inequality with the kernel as $\frac{1}{[x+\ln^{\alpha}(n-\xi)]^{\lambda+m}}$ involving one higher-order derivative function is given. The equivalent statements of the best possible constant factor related to a few parameters, the equivalent forms, and several particular inequalities are provided. Another kind of the reverses is also considered.

## 2 Some lemmas

In what follows, we suppose that $p<1$ $(p\neq 0)$, $\frac{1}{p}+\frac{1}{q}=1$, $\mathrm{N}:=\{1,2,\ldots\}$, $\mathrm{m}\in\mathrm{N}\cup\{0\}$, $\alpha\in(0,1]$, $\xi\in[0,\frac{1}{2}]$, $\lambda\in(0,\infty)$, $\lambda_1\in(0,\lambda)$, $\lambda_2\in(0,\lambda)\cap(0,\frac{1}{\alpha}]$,

$$k_{\lambda}(\lambda_i):=B(\lambda_i,\lambda-\lambda_i)\quad(i=1,2),$$

$\hat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. We also assume that $f(x)=f^{(0)}(x)$ is a continuous derivative function of m-order unless finite points in $R_+:=(0,\infty)$, $f^{(k-1)}(y)\geq 0$, $f^{(k-1)}(0+)=0$, $f^{(k-1)}(x)=o(e^{tx})$ $(t>0;\ x\to\infty)$ $(k=1,\ldots,m)$, and for $f^{(m)}(x)$, $a_n\geq 0$, $0<\int_0^{\infty}x^{p(1-\hat{\lambda}_1)-1}\times(f^{(m)}(x))^p\,dx<\infty$, and

$$0<\sum_{n=2}^{\infty}\left[\ln(n-\xi)\right]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q<\infty.$$

**Lemma 1** *Define the following weight function*:

$$\varpi_\lambda(\lambda_2,x) := \alpha x^{\lambda-\lambda_2}\sum_{n=2}^{\infty}\frac{[\ln(n-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(n-\xi)]^\lambda(n-\xi)} \quad (x\in \mathrm{R}_+). \tag{5}$$

*We have the following inequalities*:

$$0 < k_\lambda(\lambda_2)\left[1-O\left(\frac{1}{x^{\lambda_2}}\right)\right] < \varpi_\lambda(\lambda_2,x) < k_\lambda(\lambda_2) \quad (x\in \mathrm{R}_+), \tag{6}$$

*where we define* $O(\frac{1}{x^{\lambda_2}}) := \frac{1}{k_\lambda(\lambda_2)}\int_0^{\frac{\ln^\alpha(2-\xi)}{x}}\frac{v^{\lambda_2-1}}{(1+v)^\lambda}\,dv$ *satisfying* $0 < O(\frac{1}{x^{\lambda_2}}) < \frac{\ln^{\alpha\lambda_2}(2-\xi)}{k_\lambda(\lambda_2)x^{\lambda_2}}$.

*Proof* For fixed $x\in \mathrm{R}_+$, the function $g(t) := \frac{[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)}$ is strictly decreasing and strictly convex in $(\frac{3}{2},\infty)$. In fact, for $\alpha\in(0,1]$, $\lambda_2\in(0,\frac{1}{\alpha}]$, $\xi\in[0,\frac{1}{2}]$, $t\in(\frac{3}{2},\infty)$,

$$\begin{aligned}
\frac{d}{dt}g(t) &= -\frac{(1-\alpha\lambda_2)[\ln(t-\xi)]^{\alpha\lambda_2-2}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)^2} - \frac{\lambda\alpha[\ln(t-\xi)]^{\alpha\lambda_2+\alpha-2}}{[x+\ln^\alpha(t-\xi)]^{\lambda+1}(t-\xi)^2} \\
&\quad - \frac{[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)^2} < 0, \\
\frac{d^2}{dt^2}g(t) &= \frac{(1-\alpha\lambda_2)(2-\alpha\lambda_2)[\ln(t-\xi)]^{\alpha\lambda_2-3}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)^3} + \frac{\lambda\alpha(3-2\alpha\lambda_2-\alpha)[\ln(t-\xi)]^{\alpha\lambda_2+\alpha-3}}{[x+\ln^\alpha(t-\xi)]^{\lambda+1}(t-\xi)^3} \\
&\quad + \frac{3(1-\alpha\lambda_2)[\ln(t-\xi)]^{\alpha\lambda_2-2}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)^3} + \frac{\lambda\alpha^2(\lambda+1)[\ln(t-\xi)]^{\alpha\lambda_2+2\alpha-3}}{[x+\ln^\alpha(t-\xi)]^{\lambda+2}(t-\xi)^3} \\
&\quad + \frac{3\lambda\alpha[\ln(t-\xi)]^{\alpha\lambda_2+\alpha-2}}{[x+\ln^\alpha(t-\xi)]^{\lambda+1}(t-\xi)^3} + \frac{2[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)^3} > 0.
\end{aligned}$$

By the decreasingness property of series and Hermite–Hadamard's inequality (cf. [31]), we have

$$\begin{aligned}
\int_2^\infty\frac{[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)}\,dt &< \sum_{n=2}^{\infty}\frac{[\ln(n-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(n-\xi)]^\lambda(n-\xi)} \\
&< \int_{\frac{3}{2}}^\infty\frac{[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)}\,dt.
\end{aligned} \tag{7}$$

Setting $v=\frac{\ln^\alpha(t-\xi)}{x}$ $(\frac{1}{t-\xi}\,dt=\frac{1}{\alpha}x^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}-1}\,dv)$, for $\frac{3}{2}-\xi\geq 1$, we obtain

$$\begin{aligned}
&\int_{\frac{3}{2}}^\infty\frac{[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)}\,dt \\
&\quad = \frac{1}{\alpha x^\lambda}\int_{\frac{\ln^\alpha(\frac{3}{2}-\xi)}{x}}^\infty\frac{(xv)^{\frac{1}{\alpha}(\alpha\lambda_2-1)}}{(1+v)^\lambda}x^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}-1}\,dv \leq \frac{1}{\alpha x^{\lambda-\lambda_2}}\int_0^\infty\frac{v^{\lambda_2-1}\,dv}{(1+v)^\lambda} = \frac{1}{\alpha x^{\lambda-\lambda_2}}k_\lambda(\lambda_2).
\end{aligned}$$

By (5) and (7), we have

$$\varpi_\lambda(\lambda_2,x) < \alpha x^{\lambda-\lambda_2}\frac{1}{\alpha x^{\lambda-\lambda_2}}k_\lambda(\lambda_2) = k_\lambda(\lambda_2).$$

On the other hand, in the same way, we find

$$\begin{aligned}\varpi_\lambda(\lambda_2,x) &> \alpha x^{\lambda-\lambda_2}\int_2^\infty \frac{[\ln(t-\xi)]^{\alpha\lambda_2-1}}{[x+\ln^\alpha(t-\xi)]^\lambda(t-\xi)}\,dt\\ &= \int_0^\infty \frac{v^{\lambda_2-1}\,dv}{(1+v)^\lambda} - \int_0^{\frac{\ln^\alpha(2-\xi)}{x}} \frac{v^{\lambda_2-1}\,dv}{(1+v)^\lambda} = k_\lambda(\lambda_2)\biggl[1-O\biggl(\frac{1}{x^{\lambda_2}}\biggr)\biggr],\end{aligned}$$

where $O(\frac{1}{x^{\lambda_2}}) = \frac{1}{k_\lambda(\lambda_2)}\int_0^{\frac{\ln^\alpha(2-\xi)}{x}} \frac{v^{\lambda_2-1}}{(1+v)^\lambda}\,dv$ satisfying

$$0 < O\biggl(\frac{1}{x^{\lambda_2}}\biggr) < \frac{1}{k_\lambda(\lambda_2)}\int_0^{\frac{\ln^\alpha(2-\xi)}{x}} v^{\lambda_2-1}\,dv = \frac{\ln^{\alpha\lambda_2}(2-\xi)}{k_\lambda(\lambda_2)x^{\lambda_2}} \quad (x>0).$$

Hence, inequalities (6) follow.

The lemma is proved. □

**Lemma 2** *For* $p<0$ $(0<q<1)$, *we have the following reverse Mulholland-type inequality*:

$$\begin{aligned}I_0 &:= \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x+\ln^\alpha(n-\xi)]^\lambda} f^{(m)}(x)\,dx > \biggl(\frac{1}{\alpha}k_\lambda(\lambda_2)\biggr)^{\frac{1}{p}}\bigl(k_\lambda(\lambda_1)\bigr)^{\frac{1}{q}}\\ &\quad\times \biggl[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\biggr]^{\frac{1}{p}}\Biggl\{\sum_{n=2}^\infty \bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\Biggr\}^{\frac{1}{q}}.\end{aligned} \tag{8}$$

*Proof* For $\alpha>0$, setting $v = x/\ln^\alpha(n-\xi)$, we can obtain another weight function:

$$\begin{aligned}\omega_\lambda(\lambda_1,n) &:= \bigl[\ln(n-\xi)\bigr]^{\alpha(\lambda-\lambda_1)}\int_0^\infty \frac{x^{\lambda_1-1}\,dx}{[x+\ln^\alpha(n-\xi)]^\lambda}\\ &= \int_0^\infty \frac{v^{\lambda_1-1}\,dv}{(v+1)^\lambda} = k_\lambda(\lambda_1) \quad \bigl(n\in \mathrm{N}\backslash\{1\}\bigr).\end{aligned} \tag{9}$$

By reverse Hölder's inequality (cf. [31]), we have

$$\begin{aligned}I_0 &= \int_0^\infty \sum_{n=2}^\infty \frac{1}{[x+\ln^\alpha(n-\xi)]^\lambda}\biggl\{\frac{x^{(1-\lambda_1)/q}(n-\xi)^{-1/p}}{[\ln(n-\xi)]^{(1-\alpha\lambda_2)/p}}f^{(m)}(x)\biggr\}\biggl\{\frac{[\ln(n-\xi)]^{(1-\alpha\lambda_2)/p}}{x^{(1-\lambda_1)/q}(n-\xi)^{-1/p}}a_n\biggr\}\,dx\\ &\geq \Biggl\{\int_0^\infty \sum_{n=2}^\infty \frac{1}{[x+\ln^\alpha(n-\xi)]^\lambda}\frac{x^{(1-\lambda_1)(p-1)}(n-\xi)^{-1}}{[\ln(n-\xi)]^{1-\alpha\lambda_2}}\bigl(f^{(m)}(x)\bigr)^p\,dx\Biggr\}^{\frac{1}{p}}\\ &\quad\times\Biggl\{\sum_{n=2}^\infty \int_0^\infty \frac{1}{[x+\ln^\alpha(n-\xi)]^\lambda}\frac{[\ln(n-\xi)]^{(1-\alpha\lambda_2)(q-1)}a_n^q}{x^{1-\lambda_1}(n-\xi)^{1-q}}\,dx\Biggr\}^{\frac{1}{q}}\\ &= \biggl\{\frac{1}{\alpha}\int_0^\infty \varpi_\lambda(\lambda_2,x)x^{p(1-\hat{\lambda}_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\biggr\}^{\frac{1}{p}}\\ &\quad\times\Biggl\{\sum_{n=2}^\infty \omega_\lambda(\lambda_1,n)\bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\Biggr\}^{\frac{1}{q}}.\end{aligned} \tag{10}$$

We show that (10) does not keep the form of equality. Otherwise (cf. [31]), there exist constants $A$ and $B$ such that both of them are not zero and

$$A\frac{x^{(1-\lambda_1)(p-1)}(n-\xi)^{-1}}{[\ln(n-\xi)]^{1-\alpha\lambda_2}}\left(f^{(m)}(x)\right)^p = B\frac{[\ln(n-\xi)]^{(1-\alpha\lambda_2)(q-1)}a_n^q}{x^{1-\lambda_1}(n-\xi)^{1-q}} \quad \text{a.e. in } \mathrm{R}_+ \times \mathrm{N}\backslash\{1\}.$$

Assuming that $A \neq 0$, there exists $n \in \mathrm{N}\backslash\{1\}$ satisfying

$$x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p = \frac{B}{A}\left[\ln(n-\xi)\right]^{q(1-\alpha\lambda_2)}(n-\xi)^{-q}a_n^q\frac{1}{x^{1+(\lambda-\lambda_1-\lambda_2)}} \quad \text{a.e. in } \mathrm{R}_+,$$

which contradicts the fact that $0 < \int_0^\infty x^{p(1-\hat{\lambda}_1)-1}(f^{(m)}(x))^p\,dx < \infty$, since $\int_0^\infty \frac{1}{x^{1+(\lambda-\lambda_1-\lambda_2)}}\,dx = \infty$. Then, by (6) and (9), we have (8).

The lemma is proved. □

**Lemma 3** *For $t > 0$, we have the following expression*:

$$\int_0^\infty e^{-tx}f(x)\,dx = \frac{1}{t^m}\int_0^\infty e^{-tx}f^{(m)}(x)\,dx. \tag{11}$$

*Proof* For $m = 0$, in view of $f^{(0)}(x) = f(x)$, (11) is valid. For $m \in \mathrm{N}$, since $f^{(k-1)}(0+) = 0$ ($k = 1,\ldots,m$), integration by parts, we find

$$\begin{aligned}\int_0^\infty e^{-tx}f^{(k)}(x)\,dx &= \int_0^\infty e^{-tx}\,df^{(k-1)}(x)\\ &= e^{-tx}f^{(k-1)}(x)|_0^\infty - \int_0^\infty f^{(k-1)}(x)\,de^{-tx}\\ &= \lim_{x\to\infty}\frac{f^{(k-1)}(x)}{e^{tx}} + t\int_0^\infty e^{-tx}f^{(k-1)}(x)\,dx.\end{aligned}$$

In view of

$$f^{(k-1)}(x) = o\left(e^{tx}\right) \quad (t > 0, x \to \infty; k = 1,\ldots,m),$$

it follows that $\lim_{x\to\infty}\frac{f^{(k-1)}(x)}{e^{tx}} = 0$, and then

$$\int_0^\infty e^{-tx}f^{(k-1)}(x)\,dx = \frac{1}{t}\int_0^\infty e^{-tx}f^{(k)}(x)\,dx.$$

By substitution of $k = 1,\ldots,m$ in the above expression, (11) follows.

The lemma is proved. □

## 3 Main results

**Theorem 1** *For $p < 0$ ($0 < q < 1$), we have the following more accurate reverse half-discrete Mulholland-type inequality involving one higher-order derivative function*:

$$I := \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}}f(x)\,dx > \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$

$$\times \left[\int_0^\infty x^{p(1-\overleftarrow{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \left\{\sum_{n=2}^{\infty}\left[\ln(n-\xi)\right]^{q(1-\alpha\overleftarrow{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}. \quad (12)$$

*In particular, for* $\lambda_1 + \lambda_2 = \lambda$, *we have*

$$0 < \int_0^\infty x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p dx < \infty, \qquad 0 < \sum_{n=2}^{\infty}\left[\ln(n-\xi)\right]^{q(1-\alpha\lambda_2)-1}(n-\xi)^{q-1}a_n^q < \infty,$$

*and the following inequality*:

$$\begin{aligned} I &= \int_0^\infty \sum_{n=2}^{\infty} \frac{a_n}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}} f(x)\,dx \\ &> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1,\lambda_2) \\ &\quad \times \left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \left\{\sum_{n=2}^{\infty}\left[\ln(n-\xi)\right]^{q(1-\alpha\lambda_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \quad (13)$$

*Proof* Since we have

$$\frac{1}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}} = \frac{1}{\Gamma(\lambda+m)} \int_0^\infty t^{\lambda+m-1} e^{-[x+\ln^\alpha(n-\xi)]t}\,dt,$$

by the Lebesgue term by term integration theorem (cf. [32]) and (11), we find

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty \sum_{n=2}^{\infty} a_n f(x) \int_0^\infty t^{\lambda+m-1} e^{-[x+\ln^\alpha(n-\xi)]t}\,dt\,dx \\ &= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty t^{\lambda+m-1} \left(\int_0^\infty e^{-xt} f(x)\,dx\right) \sum_{n=2}^{\infty} e^{-t\ln^\alpha(n-\xi)} a_n\,dt \\ &= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty t^{\lambda+m-1} \left(t^{-m}\int_0^\infty e^{-xt} f^{(m)}(x)\,dx\right) \sum_{n=2}^{\infty} e^{-t\ln^\alpha(n-\xi)} a_n\,dt \\ &= \frac{1}{\Gamma(\lambda+m)} \int_0^\infty \sum_{n=2}^{\infty} a_n f^{(m)}(x) \left\{\int_0^\infty t^{\lambda-1} e^{-[x+\ln^\alpha(n-\xi)]t}\,dt\right\} dx \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \int_0^\infty \sum_{n=2}^{\infty} \frac{a_n}{[x+\ln^\alpha(n-\xi)]^{\lambda}} f^{(m)}(x)\,dx = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} I_0. \end{aligned}$$

Then, by (8), we have (12).

The theorem is proved. □

**Theorem 2** *For* $p < 0$ $(0 < q < 1)$, *if* $\lambda_1 + \lambda_2 = \lambda$, *then the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} \left(\frac{1}{\alpha} k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$

*in* (12) *is best possible. On the other hand, if we add the condition that* $\lambda - \lambda_1 \leq \frac{1}{\alpha}$, *the same constant factor in* (12) *is best possible, then* $\lambda_1 + \lambda_2 = \lambda$.

*Proof* In the following, for $m = 0$, $\varepsilon \geq 0$, we define $\prod_{i=0}^{m-1}(\lambda_1 + i - \frac{\varepsilon}{p}) = 1$.

*If* $\lambda_1 + \lambda_2 = \lambda$, then (12) reduces to (13). For any $0 < \varepsilon < q\lambda_2$, we set

$$\tilde{f}^{(0)}(x) = \tilde{f}(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1}, & x \geq 1 \end{cases},$$

$$\tilde{a}_n := \bigl[\ln(n-\xi)\bigr]^{\alpha(\lambda_2 - \frac{\varepsilon}{q})-1}(n-\xi)^{-1} \quad \bigl(n \in \mathrm{N}\backslash\{1\}\bigr),.$$

and find

$$\tilde{f}^{(k)}(x) = \begin{cases} 0, \quad 0 < x < 1, \\ \prod_{i=0}^{k-1}(\lambda_1 + i - \frac{\varepsilon}{p})x^{\lambda_1 + m - k - \frac{\varepsilon}{p} - 1}, \quad x > 1 \end{cases} \quad (k = 0, \dots, m)$$

satisfying $\tilde{f}^{(k)}(0+) = 0, \tilde{f}^{(k)}(x) = o(e^{tx})$ $(t > 0, x \to \infty; k = 0, \dots, m-1)$.

If there exists a constant $M(\geq \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1, \lambda_2))$ such that (13) is valid when we replace $\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1, \lambda_2)$ with $M$, then in particular we have

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \sum_{n=2}^\infty \frac{\tilde{a}_n \tilde{f}(x)}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}}\,dx \\ &> M\biggl[\int_0^\infty x^{p(1-\lambda_1)-1}\bigl(\tilde{f}^{(m)}(x)\bigr)^p\,dx\biggr]^{\frac{1}{p}} \Biggl\{\sum_{n=2}^\infty \bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\lambda_2)-1}(n-\xi)^{q-1}\tilde{a}_n^q\Biggr\}^{\frac{1}{q}}. \end{aligned} \tag{14}$$

By the decreasingness property of series, we find

$$\begin{aligned} \tilde{I} &> M\prod_{i=0}^{m-1}\biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr)\biggl[\int_1^\infty x^{p(1-\lambda_1)-1}x^{p(\lambda_1-1)-\varepsilon}\,dx\biggr]^{\frac{1}{p}} \\ &\quad \times \Biggl\{\sum_{n=2}^\infty \bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\lambda_2)-1}(n-\xi)^{q-1}\bigl[\ln(n-\xi)\bigr]^{q\alpha\lambda_2-\alpha\varepsilon-q}(n-\xi)^{-q}\Biggr\}^{\frac{1}{q}} \\ &= M\prod_{i=0}^{m-1}\biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr)\biggl(\int_1^\infty x^{-\varepsilon-1}\,dx\biggr)^{\frac{1}{p}}\Biggl\{\sum_{n=2}^\infty \bigl[\ln(n-\xi)\bigr]^{-\alpha\varepsilon-1}(n-\xi)^{-1}\Biggr\}^{\frac{1}{q}} \\ &> M\prod_{i=0}^{m-1}\biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr)\biggl(\int_1^\infty x^{-\varepsilon-1}\,dx\biggr)^{\frac{1}{p}}\biggl\{\int_2^\infty \bigl[\ln(y-\xi)\bigr]^{-\alpha\varepsilon-1}(y-\xi)^{-1}\,dy\biggr\}^{\frac{1}{q}} \\ &= \frac{M}{\varepsilon}\prod_{i=0}^{m-1}\biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr)\biggl\{\frac{1}{\alpha}\bigl[\ln(2-\xi)\bigr]^{-\alpha\varepsilon}\biggr\}^{\frac{1}{q}}. \end{aligned}$$

Replacing $\lambda$ with $\lambda + m$, setting $\tilde{\lambda}_2 := \lambda_2 - \frac{\varepsilon}{q} \in (0, \lambda + m) \cap (0, \frac{1}{\alpha}]$, $\tilde{\lambda}_1 := \lambda_1 + m + \frac{\varepsilon}{q} \in (0, \lambda + m)$ in (5), by (6), we have

$$\tilde{I} = \int_1^\infty \Biggl\{x^{\lambda_1 + m + \frac{\varepsilon}{q}}\sum_{n=2}^\infty \frac{[\ln(n-\xi)]^{\alpha(\lambda_2 - \frac{\varepsilon}{q})-1}}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}(n-\xi)}\Biggr\}x^{-\varepsilon-1}\,dx$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\int_1^\infty \varpi_{\lambda+m}(\tilde{\lambda}_2, x)x^{-\varepsilon-1}\,dx \\
&< \frac{1}{\alpha}\int_1^\infty k_{\lambda+m}(\tilde{\lambda}_2)x^{-\varepsilon-1}\,dx = \frac{1}{\varepsilon\alpha}k_{\lambda+m}(\tilde{\lambda}_2).
\end{aligned}
$$

Based on the above results, we have

$$
\frac{1}{\alpha}k_{\lambda+m}(\tilde{\lambda}_2) > \varepsilon\tilde{I} > M\prod_{i=0}^{m-1}\left(\lambda_1 + i - \frac{\varepsilon}{p}\right)\left\{\frac{1}{\alpha}\left[\ln(2-\xi)\right]^{-\alpha\varepsilon}\right\}^{\frac{1}{q}}.
$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, it follows that

$$
\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2) = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\alpha^{1/p}\Gamma(\lambda+m)} = \frac{B(\lambda_1+m,\lambda_2)}{\alpha^{1/p}\prod_{i=0}^{m-1}(\lambda_1+i)} \geq M.
$$

Hence, $M = \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)$ is the best possible constant factor in (13).

On the other hand, for $\hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, $\lambda - \lambda_1 \leq \frac{1}{\alpha}$, we find

$$
\begin{aligned}
&\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda, \\
&0 < \hat{\lambda}_1, \qquad \hat{\lambda}_2 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \qquad \hat{\lambda}_2 \leq \frac{1/\alpha}{p} + \frac{1/\alpha}{q} = \frac{1}{\alpha},
\end{aligned}
$$

and $\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\hat{\lambda}_1,\hat{\lambda}_2) \in \mathrm{R}_+$. Substituting $\hat{\lambda}_i = \lambda_i$ $(i=1,2)$ in (13), we still have

$$
\begin{aligned}
I &= \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}}f(x)\,dx \\
&> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\hat{\lambda}_1,\hat{\lambda}_2) \\
&\quad\times\left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\left\{\sum_{n=2}^\infty\left[\ln(n-\xi)\right]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{15}
$$

By reverse Hölder's inequality (cf. [31]), we still have

$$
\begin{aligned}
B(\hat{\lambda}_1,\hat{\lambda}_2) &= k_\lambda\left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}\right) \\
&= \int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du = \int_0^\infty \frac{1}{(1+u)^\lambda}\left(u^{\frac{\lambda-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1-1}{q}}\right)du \\
&\geq \left[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\lambda-\lambda_2-1}\,du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\lambda_1-1}\,du\right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty \frac{1}{(1+v)^\lambda}v^{\lambda_2-1}\,dv\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\lambda_1-1}\,du\right]^{\frac{1}{q}} \\
&= \left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}.
\end{aligned} \tag{16}
$$

In view of $\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ being the best possible constant factor in (12), by (15), we have the following inequality:

$$\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\big(k_\lambda(\lambda_2)\big)^{\frac{1}{p}}\big(k_\lambda(\lambda_1)\big)^{\frac{1}{q}} \geq \frac{\Gamma(\lambda)}{\alpha^{1.p}\Gamma(\lambda+m)}B(\hat{\lambda}_1,\hat{\lambda}_2)(\in \mathrm{R}_+),$$

namely, $B(\hat{\lambda}_1,\hat{\lambda}_2) \leq (k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$, and then (16) keeps the form of equality.

We observe that (16) keeps the form of equality if and only if there exist constants $A$ and $B$ (cf. [31]) such that they are not both zero and $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ *a.e.* in $\mathrm{R}_+$. Assuming that $A \neq 0$, we find $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in $\mathrm{R}_+$, namely, $\lambda - \lambda_2 - \lambda_1 = 0$. Hence we have $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. □

## 4 Equivalent forms and some particular inequalities

**Theorem 3** *For $p < 0$ $(0 < q < 1)$, we have the following reverse half-discrete Mulholland-type inequality equivalent to* (12):

$$\begin{aligned} J &:= \left\{\sum_{n=2}^{\infty}\big[\ln(n-\xi)\big]^{p\alpha\hat{\lambda}_2-1}(n-\xi)^{-1}\left[\int_0^\infty \frac{f(x)}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}}\,dx\right]^p\right\}^{\frac{1}{p}} \\ &> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\big(k_\lambda(\lambda_2)\big)^{\frac{1}{p}}\big(k_\lambda(\lambda_1)\big)^{\frac{1}{q}}\left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\big(f^{(m)}(x)\big)^p\,dx\right]^{\frac{1}{p}}. \end{aligned} \tag{17}$$

*In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have the following inequality equivalent to* (13):

$$\begin{aligned} &\left\{\sum_{n=2}^{\infty}\big[\ln(n-\xi)\big]^{p\alpha\lambda_2-1}(n-\xi)^{-1}\left[\int_0^\infty \frac{f(x)}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}}\,dx\right]^p\right\}^{\frac{1}{p}} \\ &> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}\big(f^{(m)}(x)\big)^p\,dx\right]^{\frac{1}{p}}. \end{aligned} \tag{18}$$

*Proof* Suppose that (17) is valid. By reverse Hölder's inequality, we have

$$\begin{aligned} I &= \sum_{n=2}^{\infty}\left\{\big[\ln(n-\xi)\big]^{\alpha\hat{\lambda}_2-\frac{1}{p}}(n-\xi)^{-\frac{1}{p}}\int_0^\infty \frac{f(x)\,dx}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}}\right\} \\ &\quad\times\left\{\big[\ln(n-\xi)\big]^{-\alpha\hat{\lambda}_2+\frac{1}{p}}(n-\xi)^{\frac{1}{p}}a_n\right\} \\ &\geq J\left\{\sum_{n=2}^{\infty}\big[\ln(n-\xi)\big]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{19}$$

Then, by (17), we have (12).

On the other hand, assuming that (12) is valid, we set

$$a_n := \big[\ln(n-\xi)\big]^{p\alpha\lambda_2-1}(n-\xi)^{-1}\left[\int_0^\infty \frac{f(x)}{[x+\ln^\alpha(n-\xi)]^{\lambda+m}}\,dx\right]^{p-1}, \quad n \in \mathrm{N}\backslash\{1\}.$$

If $J = \infty$, then (17) is naturally valid; if $J = 0$, then it is impossible that makes (17) valid, i.e., $J > 0$. Suppose that $0 < J < \infty$. By (12), we have

$$\begin{aligned}
\infty &> \sum_{n=2}^{\infty} \bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q = J^p = I \\
&> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\bigl(k_\lambda(\lambda_2)\bigr)^{\frac{1}{p}}\bigl(k_\lambda(\lambda_1)\bigr)^{\frac{1}{q}}\left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\bigl(f^{(m)}(x)\bigr)\,dx\right]^{\frac{1}{p}} J^{p-1} > 0, \\
J &= \left\{\sum_{n=2}^{\infty} \bigl[\ln(n-\xi)\bigr]^{q(1-\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{p}} \\
&> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\bigl(k_\lambda(\lambda_2)\bigr)^{\frac{1}{p}}\bigl(k_\lambda(\lambda_1)\bigr)^{\frac{1}{q}}\left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\bigl(f^{(m)}(x)\bigr)\,dx\right]^{\frac{1}{p}},
\end{aligned}$$

namely, (17) follows, which is equivalent to (12).

The theorem is proved. □

**Theorem 4** *For $p < 0$ $(0 < q < 1)$, if $\lambda_1 + \lambda_2 = \lambda$, then the constant factor*

$$\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\bigl(k_\lambda(\lambda_2)\bigr)^{\frac{1}{p}}\bigl(k_\lambda(\lambda_1)\bigr)^{\frac{1}{q}}$$

*in (17) is best possible. On the other hand, if we add the condition that $\lambda - \lambda_1 \le \frac{1}{\alpha}$, the same constant factor in (17) is best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.*

*Proof* If $\lambda_1 + \lambda_2 = \lambda$, then by Theorem 2, the constant factor

$$\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\bigl(k_\lambda(\lambda_2)\bigr)^{\frac{1}{p}}\bigl(k_\lambda(\lambda_1)\bigr)^{\frac{1}{q}}$$

in (12) is best possible. By (19), the constant factor in (17) is still best possible. Otherwise, we would reach a contradiction that the constant factor in (12) is not best possible.

On the other hand, if the constant factor in (17) is best possible, then, by the equivalency of (17) and (12), in view of $J^p = I$ (see the proof of Theorem 3), we still can show that the constant factor in (12) is best possible. By the assumption and Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. □

*Remark* 1 (i) For $\alpha = 1$ in (13) and (18), we have the following equivalent inequalities:

$$\begin{aligned}
&\int_0^\infty \sum_{n=2}^{\infty} \frac{a_n}{\ln^{\lambda+m} e^x(n-\xi)} f(x)\,dx \\
&\quad > \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)} B(\lambda_1,\lambda_2) \\
&\qquad \times \left[\int_0^\infty x^{p(1-\lambda_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty}\bigl[\ln(n-\xi)\bigr]^{q(1-\lambda_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}, \qquad (20)
\end{aligned}$$

$$\left\{\sum_{n=2}^{\infty}\bigl[\ln(n-\xi)\bigr]^{p\lambda_2-1}(n-\xi)^{-1}\left[\int_0^{\infty}\frac{f(x)}{\ln^{\lambda+m}e^{x}(n-\xi)}\,dx\right]^p\right\}^{\frac{1}{p}}$$
$$>\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^{\infty}x^{p(1-\lambda_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}}. \tag{21}$$

(ii) For $\xi = 0$ in (13) and (18), we have the following equivalent inequalities:

$$\int_0^{\infty}\sum_{n=2}^{\infty}\frac{a_n}{(x+\ln^{\alpha}n)^{\lambda+m}}f(x)\,dx$$
$$>\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^{\infty}x^{p(1-\lambda_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}(\ln n)^{q(1-\alpha\lambda_2)-1}n^{q-1}a_n^q\right]^{\frac{1}{q}}, \tag{22}$$

$$\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{p\alpha\lambda_2-1}}{n}\left[\int_0^{\infty}\frac{f(x)}{(x+\ln^{\alpha}n)^{\lambda+m}}\,dx\right]^p\right\}^{\frac{1}{p}}$$
$$>\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^{\infty}x^{p(1-\lambda_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}}. \tag{23}$$

Hence,(13) (resp. (18)) is a more accurate form of (22) (resp. (23)).

(ii) For $\xi = \frac{1}{2}$ in (13) and (18), we have the following equivalent inequalities:

$$\int_0^{\infty}\sum_{n=2}^{\infty}\frac{a_n}{[x+\ln^{\alpha}(n-\frac{1}{2})]^{\lambda+m}}f(x)\,dx$$
$$>\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^{\infty}x^{p(1-\lambda_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=2}^{\infty}\left[\ln\left(n-\frac{1}{2}\right)\right]^{q(1-\alpha\lambda_2)-1}\left(n-\frac{1}{2}\right)^{q-1}a_n^q\right\}^{\frac{1}{q}}, \tag{24}$$

$$\left\{\sum_{n=2}^{\infty}\left[\ln\left(n-\frac{1}{2}\right)\right]^{p\alpha\lambda_2-1}\left(n-\frac{1}{2}\right)^{-1}\left[\int_0^{\infty}\frac{f(x)}{[x+\ln^{\alpha}(n-\frac{1}{2})]^{\lambda+m}}\,dx\right]^p\right\}^{\frac{1}{p}}$$
$$>\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^{\infty}x^{p(1-\lambda_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}}. \tag{25}$$

The constant factors in the above inequalities are all best possible.

## 5 Another kind of reverses

Similar to Lemma 2, we have the following.

**Lemma 4** *For* $0 < p < 1$ $(q < 0)$, *we have the following reverse Mulholland-type inequality*:

$$I_0 = \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n-\xi)]^\lambda} f^{(m)}(x)\,dx > \left(\frac{1}{\alpha}k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$
$$\times \left\{\int_0^\infty \left[1 - O\left(\frac{1}{x^{\lambda_2}}\right)\right] x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=2}^\infty \left[\ln(n-\xi)\right]^{q(1-\alpha\hat{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}. \tag{26}$$

**Theorem 5** *For* $0 < p < 1$ $(q < 0)$, *we have the following equivalent reverse inequalities*:

$$I = \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}} f(x)\,dx > \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$
$$\times \left[\int_0^\infty \left(1 - O\left(\frac{1}{x^{\lambda_2}}\right)\right) x^{p(1-\overleftarrow{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=2}^\infty \left[\ln(n-\xi)\right]^{q(1-\alpha\overleftarrow{\lambda}_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}, \tag{27}$$

$$J = \left\{\sum_{n=2}^\infty \left[\ln(n-\xi)\right]^{p\alpha\hat{\lambda}_2-1}(n-\xi)^{-1}\left[\int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}}\,dx\right]^p\right\}^{\frac{1}{p}}$$
$$> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}\left(k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(k_\lambda(\lambda_1)\right)^{\frac{1}{q}}$$
$$\times \left[\int_0^\infty \left(1 - O\left(\frac{1}{x^{\lambda_2}}\right)\right) x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}. \tag{28}$$

*In particular, for* $\lambda_1 + \lambda_2 = \lambda$, *we have the following equivalent reverse inequalities*:

$$\int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}} f(x)\,dx$$
$$> \frac{\Gamma(\lambda)}{\alpha^{1p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)$$
$$\times \left[\int_0^\infty \left(1 - O\left(\frac{1}{x^{\lambda_2}}\right)\right) x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=2}^\infty \left[\ln(n-\xi)\right]^{q(1-\alpha\lambda_2)-1}(n-\xi)^{q-1}a_n^q\right\}^{\frac{1}{q}}, \tag{29}$$

$$\left\{\sum_{n=2}^\infty \left[\ln(n-\xi)\right]^{p\alpha\lambda_2-1}(n-\xi)^{-1}\left[\int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}}\,dx\right]^p\right\}^{\frac{1}{p}}$$
$$> \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)\left[\int_0^\infty \left(1 - O\left(\frac{1}{x^{\lambda_2}}\right)\right) x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}, \tag{30}$$

*where the constant factor* $\frac{\Gamma(\lambda)}{\alpha^{1p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)$ *is best possible.*

*Proof* We only prove that the constant factor in (29) is best possible. The others are omitted.

For any $0 < \varepsilon < p\lambda_1$, we set

$$\tilde{f}^{(0)}(x) = \tilde{f}(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1}, & x \geq 1 \end{cases},$$
$$\tilde{a}_n := \bigl[\ln(n-\xi)\bigr]^{\alpha(\lambda_2 - \frac{\varepsilon}{q}) - 1}(n-\xi)^{-1} \quad \bigl(n \in \mathrm{N}\backslash\{1\}\bigr),$$

and find

$$\tilde{f}^{(k)}(x) = \begin{cases} 0, \quad 0 < x < 1, \\ \prod_{i=0}^{k-1}(\lambda_1 + i - \frac{\varepsilon}{p}) x^{\lambda_1 + m - k - \frac{\varepsilon}{p} - 1}, \quad x > 1 \end{cases} \qquad (k = 0, \ldots, m)$$

satisfying $\tilde{f}^{(k)}(0+) = 0, \tilde{f}^{(k)}(x) = o(e^{tx})$ $(t > 0, x \to \infty; k = 0, \ldots, m-1)$.

If there exists a constant $M(\geq \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1, \lambda_2))$ such that (29) is valid when we replace $\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)} B(\lambda_1, \lambda_2)$ with $M$, then in particular we have

$$\begin{aligned} \tilde{I} &= \int_0^\infty \sum_{n=2}^\infty \frac{\tilde{a}_n \tilde{f}(x)}{[x + \ln^\alpha(n-\xi)]^{\lambda+m}}\, dx \\ &> M \biggl\{ \int_0^\infty \biggl(1 - O\biggl(\frac{1}{x^{\lambda_2}}\biggr)\biggr) x^{p(1-\lambda_1)-1} \bigl(\tilde{f}^{(m)}(x)\bigr)^p \, dx \biggr\}^{\frac{1}{p}} \cdot \\ &\quad \times \Biggl\{ \sum_{n=2}^\infty \bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\lambda_2)-1} (n-\xi)^{q-1} \tilde{a}_n^q \Biggr\}^{\frac{1}{q}}. \end{aligned} \tag{31}$$

By the decreasingness property of series, we find

$$\begin{aligned} \tilde{I} &> M \prod_{i=0}^{m-1} \biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr) \biggl[ \int_1^\infty \biggl(1 - O\biggl(\frac{1}{x^{\lambda_2}}\biggr)\biggr) x^{p(1-\lambda_1)-1} x^{p(\lambda_1 - 1) - \varepsilon} \, dx \biggr]^{\frac{1}{p}} \\ &\quad \times \Biggl\{ \sum_{n=2}^\infty \bigl[\ln(n-\xi)\bigr]^{q(1-\alpha\lambda_2)-1} (n-\xi)^{q-1} \bigl[\ln(n-\xi)\bigr]^{q\alpha\lambda_2 - \alpha\varepsilon - q} (n-\xi)^{-q} \Biggr\}^{\frac{1}{q}} \\ &= M \prod_{i=0}^{m-1} \biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr) \biggl[ \int_1^\infty x^{-\varepsilon-1}\, dx - \int_1^\infty O\biggl(\frac{1}{x^{\lambda_2 + \varepsilon + 1}}\biggr) dx \biggr]^{\frac{1}{p}} \\ &\quad \times \Biggl\{ \bigl[\ln(2-\xi)\bigr]^{-\alpha\varepsilon - 1} (2-\xi)^{-1} + \sum_{n=3}^\infty \bigl[\ln(n-\xi)\bigr]^{-\alpha\varepsilon-1} (n-\xi)^{-1} \Biggr\}^{\frac{1}{q}} \\ &> M \prod_{i=0}^{m-1} \biggl(\lambda_1 + i - \frac{\varepsilon}{p}\biggr) \biggl(\frac{1}{\varepsilon} - O(1)\biggr)^{\frac{1}{p}} \\ &\quad \times \biggl\{ \bigl[\ln(2-\xi)\bigr]^{-\alpha\varepsilon-1} (2-\xi)^{-1} + \int_2^\infty \bigl[\ln(y-\xi)\bigr]^{-\alpha\varepsilon-1} (y-\xi)^{-1}\, dy \biggr\}^{\frac{1}{q}} \end{aligned}$$

$$= \frac{M}{\varepsilon}\prod_{i=0}^{m-1}\left(\lambda_1 + i - \frac{\varepsilon}{p}\right)\left(1 - \varepsilon O(1)\right)^{\frac{1}{p}}$$
$$\times \left\{\varepsilon\left[\ln(2-\xi)\right]^{-\alpha\varepsilon-1}(2-\xi)^{-1} + \frac{1}{\alpha}\left[\ln(2-\xi)\right]^{-\alpha\varepsilon}\right\}^{\frac{1}{q}}.$$

Replacing $\lambda$ with $\lambda + m$, setting $\tilde{\lambda}_2 := \lambda_2 + \frac{\varepsilon}{p} \in (0, \lambda + m)$, $\tilde{\lambda}_1 := \lambda_1 + m - \frac{\varepsilon}{p} \in (0, \lambda + m)$ in (9), we have

$$\tilde{I} = \sum_{n=2}^{\infty}\left\{\left[\ln(n-\xi)\right]^{\alpha(\lambda_2+\frac{\varepsilon}{p})}\int_1^{\infty}\frac{x^{\lambda_1+m-\frac{\varepsilon}{p}-1}}{[x+\ln^{\alpha}(n-\xi)]^{\lambda+m}}\,dx\right\}\left[\ln(n-\xi)\right]^{-\alpha\varepsilon-1}(n-\xi)^{-1}$$
$$\le \sum_{n=2}^{\infty}\left\{\left[\ln(n-\xi)\right]^{\alpha\tilde{\lambda}_2}\int_0^{\infty}\frac{x^{\tilde{\lambda}_1-1}}{[x+\ln^{\alpha}(n-\xi)]^{\lambda+m}}\,dx\right\}\left[\ln(n-\xi)\right]^{-\alpha\varepsilon-1}(n-\xi)^{-1}$$
$$= \sum_{n=2}^{\infty}\omega_{\lambda+m}(\tilde{\lambda}_1, n)\left[\ln(n-\xi)\right]^{-\alpha\varepsilon-1}(n-\xi)^{-1}$$
$$= k_{\lambda+m}(\tilde{\lambda}_1)\left\{\left[\ln(2-\xi)\right]^{-\alpha\varepsilon-1}(2-\xi)^{-1} + \sum_{n=3}^{\infty}\left[\ln(n-\xi)\right]^{-\alpha\varepsilon-1}(n-\xi)^{-1}\right\}$$
$$< k_{\lambda+m}(\tilde{\lambda}_1)\left\{\left[\ln(2-\xi)\right]^{-\alpha\varepsilon-1}(2-\xi)^{-1} + \int_2^{\infty}\left[\ln(y-\xi)\right]^{-\alpha\varepsilon-1}(y-\xi)^{-1}\,dy\right\}$$
$$= \frac{1}{\varepsilon\alpha}k_{\lambda+m}(\tilde{\lambda}_1)\left\{\varepsilon\alpha\left[\ln(2-\xi)\right]^{-\alpha\varepsilon-1}(2-\xi)^{-1} + \left[\ln(2-\xi)\right]^{-\alpha\varepsilon}\right\}.$$

Based on the above results, we have

$$\frac{1}{\alpha}k_{\lambda+m}\left(\lambda_1 + m - \frac{\varepsilon}{p}\right)\left\{\varepsilon\alpha\left[\ln(2-\xi)\right]^{-\alpha\varepsilon-1}(2-\xi)^{-1} + \left[\ln(2-\xi)\right]^{-\alpha\varepsilon}\right\}$$
$$> \varepsilon\tilde{I}$$
$$> M\prod_{i=0}^{m-1}\left(\lambda_1 + i - \frac{\varepsilon}{p}\right)\left(1 - \varepsilon O(1)\right)^{\frac{1}{p}}\left\{\varepsilon\left[\ln(2-\xi)\right]^{-\alpha\varepsilon-1}(2-\xi)^{-1} + \frac{1}{\alpha}\left[\ln(2-\xi)\right]^{-\alpha\varepsilon}\right\}^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, it follows that

$$\frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2) = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\alpha^{1/p}\Gamma(\lambda+m)} = \frac{B(\lambda_1+m,\lambda_2)}{\alpha^{1/p}\prod_{i=0}^{m-1}(\lambda_1+i)} \ge M.$$

Hence, $M = \frac{\Gamma(\lambda)}{\alpha^{1/p}\Gamma(\lambda+m)}B(\lambda_1,\lambda_2)$ is the best possible constant factor in (29).

The theorem is proved. □

In the same way of proving Theorem 4, we have the following.

**Theorem 6** *For $0 < p < 1$ $(q < 0)$, if we add the condition that $\lambda - \lambda_1 \le \frac{1}{\alpha}$, the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}k_{\lambda}(\lambda_2)\right)^{\frac{1}{p}}\left(k_{\lambda}(\lambda_1)\right)^{\frac{1}{q}}$$

*in (27) (or (28)) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.*

## 6 Conclusions

In this paper, following the way of [4, 21], by means of the weight functions, Hermite–Hadamard's inequality, and the techniques of real analysis, a new more accurate reverse half-discrete Mulholland-type inequality with the kernel as $\frac{1}{[x+\ln^{\alpha}(n-\xi)]^{\lambda+m}}$ involving one higher-order derivative function is given (for $p < 0$, $0 < q < 1$) in Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters, the equivalent forms, and several particular inequalities are provided in Theorems 2–4 and Remark 1. Another kind of the reverses is also considered (for $0 < p < 1$, $q < 0$) in Theorems 5–6. The lemmas and theorems provide an extensive account of this type of inequalities.

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**Availability of data and materials**
The data used to support the findings of this study are included within the article.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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