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Statistical convergence in probabilistic generalized metric spaces w.r.t. strong topology

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Abstract

In this paper, the concept of probabilistic g-metric space with degree l, which is a generalization of probabilistic G-metric space, is introduced. Then, by endowing strong topology, the definition of l-dimensional asymptotic density of a subset A of $\mathbb{N}^l$ is used to introduce a statistically convergent and Cauchy sequence and to study some basic facts.

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1 Introduction

The theory of probabilistic metric space (PM-space) as a generalization of ordinary metric space was introduced by Menger in [12]. In this space, distribution functions are considered as the distance of a pair of points in statistics rather than deterministic.

The concept of the generalized metric space (briefly G-metric space) was introduced by Mustafa and Sims in 2006 [16]. Then, in 2014, Zhou et al. [26] generalized the notion of PM-space to the G-metric spaces and defined the probabilistic generalized metric space which is denoted by PGM-space.

In [3], Choi et al. proposed a generalization of G-metric space named g-metric space with degree l, in which the distance function with degrees $l = 1, 2$ is equivalent to ordinary and G-metric, respectively.

The idea of statistical convergence was first introduced by Steinhaus [25] for real sequences and developed by Fast [7], then reintroduced by Shoenberg [22]. Many authors, such as [4, 6, 8, 9, 17, 21], have discussed and developed this concept. The theory of statistical convergence has many applications in various fields such as approximation theory [5], finitely additive set functions [4], trigonometric series [27], and locally convex spaces [11].

In 2008, Sencimen and Pehlivan [24] introduced the concepts of statistically convergent sequence and statistically Cauchy sequence in the probabilistic metric space endowed with strong topology.

The purpose of this paper is to develop a concept to generalize the probabilistic G-metric space to the probabilistic g-metric space with degree l. Here, the notation of the generalized space is still referred as PGM-space. The l-dimensional asymptotic density of a subset A of $\mathbb{N}^l$ defined previously by the author in [1] is used to introduce the statistically convergent and Cauchy sequences with respect to strong topology, and some basic facts are studied. Note that in this definition $l = 1$ and $l = 2$ values coincide exactly with the statistical convergence in PM-space and PGM-space (related to G-metric), respectively. Thus, the definitions and the obtained results show that this study is more comprehensive.

2 Preliminaries

In this section, some basic definitions and results related to PM-space, PGM-space, and statistical convergence are presented and discussed. First, recall the definition of triangular norm (t-norm) as follows.

Definition 2.1 ([23]) A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if T satisfies the following conditions:

(i) T is commutative and associative, i.e., $T(a,b) = T(b,a)$ and $T(a,T(b,c)) = T(T(a,b),c)$ for all $a,b,c \in [0,1]$;

(ii) T is continuous;

(iii) $T(a,1) = a$ for all $a \in [0,1]$;

(iv) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

A distribution function F is a map from extended reals $\mathbb{R}_\infty := \mathbb{R} \cup \{-\infty,\infty\}$ into $[0,1]$ such that it is nondecreasing, left-continuous at every real number, and $F(-\infty) = 0$ and $F(\infty) = 1$. The set of all distribution functions is denoted by Δ and $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Definition 2.2 ([23]) A Menger probabilistic metric space (PM-space) is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm. and F is a mapping from $X \times X \to \Delta^+$ satisfying the following conditions:

($F_{(x,y)}$ denotes the value of F at the pair (x,y))

(i) $F_{(x,y)}(t) = 1$ for all $x,y \in X$ and $t > 0$ if and only if $x = y$;

(ii) $F_{(x,y)}(t) = F_{(y,x)}(t)$;

(iii) $F_{(x,y)}(t+s) \geq T(F_{(x,z)}(t), F_{(z,y)}(s))$ for all $x,y,z \in X$ and $t,s \geq 0$.

Definition 2.3 ([16]) Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying:

1) $G(x,y,z) = 0$ if $x = y = z$;

2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;

3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;

4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);

5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then the pair (X,G) is called G-metric space.

The following definition is a developing of PM-space on G-metric.

Definition 2.4 ([26]) A *Menger probabilistic G-metric space* (PGM-space) is a triple (X,G^*,T), where X is a nonempty set, T is a continuous t-norm, and G^* is a mapping from $X \times X \times X$ into Δ^+, satisfying the following conditions:

(i) $G^*_{(x,y,z)}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
(ii) $G^*_{(x,x,y)}(t) \geq G^*_{(x,y,z)}(t)$ for all $x, y \in X$ with $z \neq y$ and $t > 0$;
(iii) $G^*_{(x,y,z)}(t) = G^*_{(x,z,y)}(t) = G^*_{(y,x,z)}(t) = \cdots$ (symmetry in all three variables);
(iv) $G^*_{(x,y,z)}(t+s) \geq T(G^*_{(x,a,a)}(t), G^*_{(a,y,z)}(s))$ for all $x, y, z, a \in X$ and $s, t \geq 0$.

Definition 2.5 ([26]) Let (X, G^*, T) be a *PGM*-space and $x_0 \in X$. For $\epsilon > 0$ and $0 < \delta < 1$, the (ϵ, δ)-neighborhood of x_0 is defined as follows:

$$N_{x_0}(\epsilon, \delta) = \{ y \in X : G^*_{(x_0,y,y)}(\epsilon) > 1 - \delta, G^*_{(y,x_0,x_0)}(\epsilon) > 1 - \delta \}.$$

Definition 2.6 ([26])
(i) A sequence $\{x_n\}$ in a *PGM*-space (X, G^*, T) is said to be *convergent* to a point $x \in X$ if, for every $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_x(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$.
(ii) A sequence $\{x_n\}$ in a *PGM*-space (X, G^*, T) is called a *Cauchy* sequence if, for every $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{(x_m,x_n,x_l)}(\epsilon) > 1 - \delta$ whenever $m, n, l > M_{\epsilon,\delta}$.
(iii) A *PGM*-space (X, G^*, T) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

In the following, some basic concepts of statistical convergence are discussed.

Definition 2.7 ([7]) Let $A \subset \mathbb{N}$ and $A(n) = \{k \in A; k \leq n\}$. Then the *asymptotic density* of A is defined as follows:

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}.$$

For a subset A of $\mathbb{N}$, if $\delta(A) = 1$, then it is said to be *statistically dense*. It is clear that $\delta(\mathbb{N} - A) = 1 - \delta(A)$.

Definition 2.8 ([7]) A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be *statistically convergent* to a point x in $\mathbb{R}$ if, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \{ k \leq n : |x_k - x| \geq \epsilon \} \right| = 0.$$

For more information about statistical convergence, the references [2, 4, 7–10, 13–15, 18–20] can be addressed.

3 Main results

In this section the main definitions and results are introduced and discussed. First of all, consider the following definition which is a generalization of a G-metric space to an l-dimensional case, where $l \in \mathbb{N}$.

Definition 3.1 ([3]) Let X be a nonempty set. A function $g : X^{l+1} \longrightarrow \mathbb{R}_+$ is called a g-metric with degree l on X if it satisfies the following conditions:
g1) $g(x_0, x_1, \ldots, x_l) = 0$ if and only if $x_0 = x_1 = \cdots = x_l$,
g2) $g(x_0, x_1, \ldots, x_l) = g(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, \ldots, x_{\sigma(l)})$ for permutation σ on $\{0, 1, \ldots, l\}$,

g3) $g(x_0, x_1, \ldots, x_l) \le g(y_0, y_1, \ldots, y_l)$ for all $(x_0, x_1, \ldots, x_l), (y_0, y_1, \ldots, y_l) \in X^{l+1}$ with $\{x_i : i = 0, 1, \ldots, l\} \subseteq \{y_i : i = 0, 1, \ldots, l\}$,

g4) For all $x_0, x_1, \ldots, x_s, y_0, y_1, \ldots, y_t, w \in X$ with $s + t + 1 = l$,

$$g(x_0, x_1, \ldots, x_s, y_0, y_1, \ldots, y_t) \le g(x_0, x_1, \ldots, x_s, w, w, \ldots, w) + g(y_0, y_1, \ldots, y_t, w, w, \ldots, w).$$

The pair (X, g) is called a g-metric space. It is noteworthy that, if $l = 1$ (resp. $l = 2$), then it is equivalent to an ordinary metric space (resp. G-metric space).

Definition 3.2 ([3]) Let (X, g) be a g-metric space, $x \in X$ be a point, and $\{x_k\} \subseteq X$ be a sequence.

1) $\{x_k\}$ g-converges to x if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$i_1, \ldots, i_l \ge N \quad \Longrightarrow \quad g(x, x_1, \ldots, x_l) < \epsilon.$$

2) $\{x_k\}$ is said to be g-Cauchy if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$i_0, i_1, \ldots, i_l \ge N \quad \Longrightarrow \quad g(x_{i_0}, x_{i_1}, \ldots, x_{i_l}) < \epsilon.$$

3) (X, g) is complete if every g-Cauchy sequence in (X, g) is g-convergent in (X, g).

Now, by equipping Definition 2.4 with g-metric, we introduce the following definition that is a generalization.

Definition 3.3 A Menger probabilistic g-metric space (still is denoted as PGM-space) is a triple (X, F, T), where X is a nonempty set, T is a continuous t-norm, and F is a mapping from X^{l+1} into Δ^+, satisfying the following conditions:

(i) $F_{(x_0,x_1,\ldots,x_l)}(t) = 1$ for all $x_0, x_1, \ldots, x_l \in X$ and $t > 0$ if and only if $x_0 = x_1 = \cdots = x_l$;

(ii) $F_{(x_0,x_1,\ldots,x_l)}(t) \ge F_{(y_0,y_1,\ldots,y_l)}(t)$ for all $(x_0, x_1, \ldots, x_l), (y_0, y_1, \ldots, y_l) \in X^{l+1}$ with $\{x_i : i = 0, 1, \ldots, l\} \subseteq \{y_i : i = 0, 1, \ldots, l\}$;

(iii) $F_{(x_0,x_1,\ldots,x_l)}(t) = F_{(x_{\sigma(0)},x_{\sigma(1)},\ldots,\ldots,x_{\sigma(l)})}(t)$ for permutation σ on $\{0, 1, \ldots, l\}$;

(iv) For all $x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n, w \in X$ with $m + n + 1 = l$,

$$F_{(x_0,x_1,\ldots,x_m,y_0,y_1,\ldots,y_n)}(t + s) \ge T\bigl(F_{(x_0,x_1,\ldots,x_m,w,w,\ldots,w)}(t), F_{(y_0,y_1,\ldots,y_n,w,w,\ldots,w)}(s)\bigr).$$

In the following, according to the generalization of asymptotic density given in [1], statistically convergent and Cauchy sequences in a PGM-space are introduced.

Definition 3.4 Let (X, F, T) be a PGM-space. For any $\epsilon > 0$, $0 < \delta < 1$ and $x \in X$, the strong (ϵ, δ)-vicinity of x is defined by the subset $M_x(\epsilon, \delta)$ of X^l as follows:

$$M_x(\epsilon, \delta) = \bigl\{(x_1, x_2, \ldots, x_l) \in X^l; F_{(x,x_1,x_2,\ldots,x_l)}(\epsilon) > 1 - \delta\bigr\}.$$

Next, we generalize the concept of asymptotic density of a set in an l-dimensional case.

Definition 3.5 Let $K \subset \mathbb{N}^l$, the *l-dimensional asymptotic density* of K is defined by

$$\delta_l(K) = \lim_{n\to\infty} \frac{l!}{n^l} \bigl|\bigl\{(i_1, i_2, \ldots, i_l) \in K; i_1, i_2, \ldots, i_l \le n\bigr\}\bigr|.$$

Definition 3.6 Let (X,F,T) be a PGM-space.

(i) A sequence $\{x_n\}$ in X is statistically convergent to a point x in X w.r.t. strong topology if, for any $\epsilon > 0$ and $0 < \delta < 1$,

$$\delta_l\big(\big\{(i_1,i_2,\ldots,i_l) \in \mathbb{N}^l : F_{(x_{i_1},x_{i_2},\ldots,x_{i_l},x)}(\epsilon) \leq 1-\delta\big\}\big) = 0,$$

and is denoted by $x_n \overset{st}{\longrightarrow} x$ or $st-\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is said to be statistically Cauchy w.r.t. strong topology if, for all $\epsilon > 0$ and $0 < \delta < 1$, there exists $i_\epsilon \in \mathbb{N}$ such that

$$\delta_l\big(\big\{(i_1,i_2,\ldots,i_l) \in \mathbb{N}^l : F_{(x_{i_1},x_{i_2},\ldots,x_{i_l},x_{i_\epsilon})}(\epsilon) \leq 1-\delta\big\}\big) = 0.$$

We can restate part (i) of the above definition as follows:

(i') $x_n \overset{st}{\longrightarrow} x$ if and only if, for any $\epsilon > 0$ and $0 < \delta < 1$,

$$\delta_l\big(\big\{(i_1,i_2,\ldots,i_l) \in \mathbb{N}^l : (x_{i_1},x_{i_2},\ldots,x_{i_l}) \notin M_x(\epsilon,\delta)\big\}\big) = 0.$$

The following theorem shows that if a sequence is statistically convergent to a point in X, then that point is unique.

Theorem 3.7 *Let $\{x_n\}$ be a sequence in a PGM-space (X,F,T) such that $x_n \overset{st}{\longrightarrow} x$ and $x_n \overset{st}{\longrightarrow} y$, then $x = y$.*

Proof Let $\epsilon > 0$ and $0 < \delta < 1$, by the continuity of T, there exists $0 < \delta_0 < 1$ such that

$$T(1-\delta_0, 1-\delta_0) > 1-\delta.$$

Set

$$A(\epsilon,\delta) := \left\{(i_1,i_2,\ldots,i_l) \in \mathbb{N}^l : F_{(x_{i_1},x_{i_2},\ldots,x_{i_l},x)}\left(\frac{\epsilon}{2}\right) \leq 1-\delta_0\right\},$$
$$B(\epsilon,\delta) := \left\{(i_1,i_2,\ldots,i_l) \in \mathbb{N}^l : F_{(x_{i_1},x_{i_2},\ldots,x_{i_l},y)}\left(\frac{\epsilon}{2}\right) \leq 1-\delta_0\right\},$$

and

$$C(\epsilon,\delta) := A(\epsilon,\delta) \cup B(\epsilon,\delta).$$

Since $x_n \overset{st}{\longrightarrow} x$ and $x_n \overset{st}{\longrightarrow} y$, so $\delta_l(A(\epsilon,\delta)) = \delta_l(B(\epsilon,\delta)) = 0$ and hence $\delta_l(C(\epsilon,\delta)) = 0$, therefore $\delta_l(C^c(\epsilon,\delta)) = 1$. Suppose $(i_1,i_2,\ldots,i_l) \in C^c(\epsilon,\delta)$, then by parts (ii) of Definition 3.3 and (iv) of Definition 2.1 we have

$$\begin{aligned} F_{(x,y,y,\ldots,y)}(\epsilon) &\geq T\left(F_{(x_{i_1},x_{i_1},\ldots,x_{i_1},x)}\left(\frac{\epsilon}{2}\right), F_{(x_{i_1},y,y,\ldots,y)}\left(\frac{\epsilon}{2}\right)\right) \\ &\geq T\left(F_{(x_{i_1},x_{i_2},\ldots,x_{i_l},x)}\left(\frac{\epsilon}{2}\right), F_{(x_{i_1},x_{i_2},\ldots,x_{i_l},y)}\left(\frac{\epsilon}{2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&> T(1-\delta_0, 1-\delta_0)\\
&> 1-\delta.
\end{aligned}
$$

Since $\delta > 0$ is arbitrary, we conclude that $F_{(x,y,y,\ldots,y)}(\epsilon) = 1$, and therefore $x = y$. □

Theorem 3.8 *Every convergent sequence in a PGM-space is statistically convergent.*

Proof Let $\{x_n\}$ be a sequence in the PGM-space (X, F, T) that converges to a point $x \in X$. For $\epsilon > 0$ and $0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that, for all $i_1, i_2, \ldots, i_l \geq n_0$,

$$F_{(x_{i_1}, x_{i_2}, \ldots, x_{i_l}, x)}(\epsilon) > 1 - \delta.$$

Set

$$A(n) := \left\{(i_1, i_2, \ldots, i_l) \in \mathbb{N}^l : i_1, i_2, \ldots, i_l \leq n, F_{(x_{i_1}, x_{i_2}, \ldots, x_{i_l}, x)}(\epsilon) > 1 - \delta\right\},$$

then

$$\left|A(n)\right| \geq \binom{n - n_0}{l}$$

and

$$\lim_{n\to\infty} \frac{l!|A(n)|}{n^l} \geq \lim_{n\to\infty} \frac{l!}{n^l}\binom{n - n_0}{l} = 1,$$

so

$$st - \lim_{n\to\infty} x_n = x.$$

□

Example 3.9 shows that the converse of the above theorem is not valid.

Example 3.9 Let $X = \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be a G-metric on $\mathbb{R}$ defined by

$$G(x, y, z) = |x - y| + |x - z| + |y - z|.$$

($T = \min$) Define a function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ as follows:

$$F_{(x,y,z)}(t) = \begin{cases} H(t), & x = y = z, \\ \mathcal{D}(\frac{t}{G(x,y,z)}), & \text{otherwise}, \end{cases}$$

where $H(t)$ and $\mathcal{D}(t)$ are distribution functions as follows:

$$H(t) = \begin{cases} 0, & t \leq 0, \\ t, & t > 0. \end{cases}, \qquad \mathcal{D} = \begin{cases} 0, & t \leq 0, \\ 1 - e^{-t}, & t > 0. \end{cases}$$

Now, consider the following sequence in $\mathbb{R}$:

$$x_n = \begin{cases} n, & n \text{ is square}, \\ 1, & \text{otherwise}. \end{cases}$$

It is clear that $\{x_n\}$ statistically converges to 1 but it is not convergent normally.

Definition 3.10 A set $A = \{n_k : k \in \mathbb{N}\}$ is said to be *statistically dense* in $\mathbb{N}$ if the set

$$A(n) = \{(i_1, i_2, \dots, i_l) \in A^l, i_1, i_2, \dots, i_l \le n\}$$

has asymptotic density 1, i.e.,

$$\delta_l(A) = \lim_{n\to\infty} \frac{l!|A(n)|}{n^l} = 1.$$

Theorem 3.11 *Let $\{x_n\}$ be a sequence in the PGM-space (X, F, T). Then the following are equivalent*:

(i) *$\{x_n\}$ statistically converges to a point $x \in X$.*

(ii) *There is a sequence $\{y_n\}$ in X such that $x_n = y_n$ for almost all n, and $\{y_n\}$ converges to x.*

(iii) *There is a statistically dense subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is convergent.*

(iv) *There is a statistically dense subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is statistically convergent.*

Proof $(i \Longrightarrow ii)$ Let $\{x_n\}$ be a sequence that converges to x, so

$$\begin{aligned} &\delta_l\bigl(\bigl\{(i_1, i_2, \dots, i_l) \in \mathbb{N}^l : F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x)}(\epsilon) > 1 - \delta\bigr\}\bigr) \\ &\quad = \lim_{n\to\infty} \frac{l!}{n^l} \bigl|\bigl\{(i_1, i_2, \dots, i_l) \in \mathbb{N}^l; i_1, i_2, \dots, i_l \le n, F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x)}(\epsilon) > 1 - \delta\bigr\}\bigr| = 1. \end{aligned}$$

For each $k \in \mathbb{N}$, we can choose an increasing sequence $\{n_k\}$ such that, for every $n > n_k$,

$$\frac{l!}{n^l}\left|\left\{(i_1, i_2, \dots, i_l) \in \mathbb{N}^l; i_1, i_2, \dots, i_l \le n, F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x)}(\epsilon) > 1 - \frac{1}{2^k}\right\}\right| > 1 - \frac{1}{2^k}.$$

Define the sequence $\{y_n\}$ as follows:

$$y_m = \begin{cases} x_m, & 1 \le m \le n_1, \\ x_m, & n_k < m \le n_{k+1}, i_1, i_2, \dots, i_{l-1} \le n_{k+1}, F_{(x_{i_1}, x_{i_2}, \dots, x_{i_{l-1}}, x_m, x)}(\epsilon) > 1 - \frac{1}{2^k}, \\ x, & \text{otherwise}. \end{cases}$$

Choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \delta$. It is clear that $\{y_m\}$ converges to x. Fix $n \in \mathbb{N}$, for $n_k < n \le n_{k+1}$, we have

$$\begin{aligned} &\bigl\{(i_1, i_2, \dots, i_l) \in \mathbb{N}^l; i_1, i_2, \dots, i_l \le n; x_{i_j} \ne y_{i_j}\bigr\} \\ &\quad \subseteq \bigl\{(i_1, i_2, \dots, i_l) \in \mathbb{N}^l; i_1, i_2, \dots, i_l \le n\bigr\} \\ &\qquad - \left\{(i_1, i_2, \dots, i_l) \in \mathbb{N}^l; i_1, i_2, \dots, i_l \le n_k, F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x)}(\epsilon) > 1 - \frac{1}{2^k}\right\}. \end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n; x_{i_j}\ne y_{i_j}\right\}\right| \\
&\quad\le 1-\lim_{n\to\infty}\frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n_k, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon)>1-\frac{1}{2^k}\right\}\right| \\
&\quad< \frac{1}{2^k}<\delta,
\end{aligned}$$

so

$$\begin{aligned}
&\delta_l\left(\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n; x_{i_j}\ne y_{i_j}\right\}\right) \\
&\quad=\lim_{n\to\infty}\frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n; x_{i_j}\ne y_{i_j}\right\}\right|=0.
\end{aligned}$$

($ii \Longrightarrow iii$) Let $\{y_n\}$ be a convergent sequence in X and $A=\{n\in\mathbb{N}: y_n\ne x_n\}$. We have $\delta_l(A)=1$, so the sequence $\{y_n\}$ is a statistical dense subsequence of $\{x_n\}$ that is convergent.

($iii \Longrightarrow iv$) It is obvious from Theorem 3.8.

($iv \Longrightarrow i$) Let $\{x_{n_k}\}$ be a statistically dense subsequence of $\{x_n\}$ that is statistically convergent to a point $x\in X$. Set $A=\{n_k: k\in\mathbb{N}\}$, so $\delta_l(A)=1$. For $\epsilon>$ and $0<\delta<1$,

$$\begin{aligned}
&\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon)>1-\delta\right\} \\
&\quad\supseteq\left\{(i_1,i_2,\dots,i_l)\in\mathbb{A}^l; i_1,i_2,\dots,i_l\le n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon)>1-\delta\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon)>1-\delta\right\}\right| \\
&\quad\ge\lim_{n\to\infty}\frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{A}^l; i_1,i_2,\dots,i_l\le n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon)>1-\delta\right\}\right|=1.
\end{aligned}$$

So,

$$\delta_l\left(\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l\le n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon)>1-\delta\right\}\right)=1.$$

Therefore $\{x_n\}$ statistically converges to x. □

The following corollary is a direct consequence of the above theorem.

Corollary 3.12 *Every statistically convergent sequence in a PGM-space has a convergent subsequence.*

Theorem 3.13 *Every statistically convergent sequence in a PGM-space is statistically Cauchy.*

Proof Suppose that $\{x_n\}$ is a sequence that statistically converges to a point x. Let $\epsilon>0$ and $0<\delta<1$. Since T is continuous, there are $0<\delta_1<1$ and $0<\delta_2<1$ such that $T(1-$

$\delta_1, 1-\delta_2) > 1-\delta$. On the other hand, there exists i_ϵ such that

$$F_{(x_{i_\epsilon},x,x,\dots,x)}\left(\frac{\epsilon}{2}\right) > 1-\delta_1.$$

Since

$$F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon) \geq T\left(F_{(x_{i_\epsilon},x,\dots,x)}\left(\frac{\epsilon}{2}\right), F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x_{i_\epsilon})}\left(\frac{\epsilon}{2}\right)\right),$$

so

$$\begin{aligned}&\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l \leq n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x_{i_\epsilon})}\left(\frac{\epsilon}{2}\right) > 1-\delta_2\right\}\\ &\quad\subseteq \left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l \leq n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon) > 1-\delta\right\}.\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{n\to\infty}\frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l \leq n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x_{i_\epsilon})}\left(\frac{\epsilon}{2}\right) > 1-\delta_2\right\}\right|\\ &\quad\leq \lim_{n\to\infty}\frac{l!}{n^l}\left|\left\{(i_1,i_2,\dots,i_l)\in\mathbb{N}^l; i_1,i_2,\dots,i_l \leq n, F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x)}(\epsilon) > 1-\delta\right\}\right|.\end{aligned}$$

Since $\{x_n\}$ is statistically convergent, so the right-hand side of the previous inequality is zero. Therefore it shows that the sequence $\{x_n\}$ is statistically Cauchy. □

Definition 3.14 Let (X,F,T) be a *PGM*-space. If every statistically Cauchy sequence is statistically convergent, then (X,F,T) is said to be *statistically complete*.

Corollary 3.15 *Every statistically complete PGM-space is complete.*

Proof Let (X,F,T) be a statistically complete *PGM*-space. Suppose that $\{x_n\}$ is a Cauchy sequence in (X,F,T), so it is a statistically Cauchy sequence. Since X is statistically complete, so $\{x_n\}$ is statistically convergent. By Corollary 3.12, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $x\in X$. By the continuity of T, for $0<\delta<1$, there exist $0<\delta_1,\delta_2,\delta_3,\delta_4<1$ such that

$$\begin{cases} T(1-\delta_1, 1-\delta_2) > 1-\delta,\\ T(1-\delta_3, 1-\delta_4) > 1-\delta_1.\end{cases}$$

Let $\delta_5 := \max\{\delta_2,\delta_3\}$, then we have

$$T\big(T(1-\delta_5, 1-\delta_4), 1-\delta_5\big) > 1-\delta.$$

For $\epsilon>0$, since $\{x_n\}$ is Cauchy, then there exist $N_1\in\mathbb{N}$ and $x_{i_\epsilon}\in\{x_n\}$ such that, for all $i_1,i_2,\dots,i_l \geq N_1$,

$$F_{(x_{i_1},x_{i_2},\dots,x_{i_l},x_{i_\epsilon})}\left(\frac{\epsilon}{4}\right) > 1-\delta_5,$$

and since $x_{n_k} \longrightarrow x$, there exists $N_2 \geq N_1$ such that, for $i_{n_1}, i_{n_2}, \dots, i_{n_l} \geq N_2$,

$$F_{(x_{i_{n_1}}, x_{i_{n_2}}, \dots, x_{i_{n_l}}, x)}\left(\frac{\epsilon}{4}\right) > 1 - \delta_5.$$

For $i_1, i_2, \dots, i_l, i_{n_1}, i_{n_2}, \dots, i_{n_l} \geq N_2$, we have

$$\begin{aligned}
&F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x)}(\epsilon) \\
&\quad \geq T\left(F_{(x_{i_\epsilon}, x, x, \dots, x)}\left(\frac{\epsilon}{2}\right), F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x_{i_\epsilon})}\left(\frac{\epsilon}{2}\right)\right) \\
&\quad \geq T\left(T\left(F_{(x_{i_\epsilon}, x_{i_{n_1}}, x_{i_{n_1}}, \dots, x_{i_{n_1}})}\left(\frac{\epsilon}{4}\right), F_{(x_{i_{n_1}}, x, x, \dots, x)}\left(\frac{\epsilon}{4}\right)\right), F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x_{i_\epsilon})}\left(\frac{\epsilon}{2}\right)\right) \\
&\quad \geq T\left(T\left(F_{(x_{i_\epsilon}, x_{i_{n_1}}, x_{i_{n_2}}, \dots, x_{i_{n_l}})}\left(\frac{\epsilon}{4}\right), F_{(x_{i_{n_1}}, x_{i_{n_1}}, x_{i_{n_2}}, \dots, x_{i_{n_l}})}\left(\frac{\epsilon}{4}\right)\right), F_{(x_{i_1}, x_{i_2}, \dots, x_{i_l}, x_{i_\epsilon})}\left(\frac{\epsilon}{4}\right)\right) \\
&\quad > T\big(T(1 - \delta_5, 1 - \delta_4), 1 - \delta_5\big) \\
&\quad > 1 - \delta.
\end{aligned}$$

The third inequality arises from part (*ii*) of Definition 3.3 and the nondecreasing property of F. So, $\{x_n\}$ is convergent and therefore (X, F, T) is complete. □

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References
1. Abazari, R.: Statistical convergence in g-metric spaces. arxiv:2103.05527v1 (2021)
2. Alotaibi, A., Mursaleen, M.: Statistical convergence in random paranormed space. J. Comput. Anal. Appl. **17**(2), 297–304 (2014)
3. Choi, H., Kim, S., Yang, S.: Structure for g-metric spaces and related fixed point theorem. arxiv:1804.03651v1 (2018)
4. Connor, J., Kline, J.: On statistical limit points and the consistency of statistical convergence. J. Math. Anal. Appl. **197**, 392–399 (1996)
5. Duman, O., Khan, M.K., Orhan, C.: *A*-Statistical convergence of approximating operators. Math. Inequal. Appl. **6**, 689–699 (2003)
6. Erdös, P., Tenenbaum, G.: Sur les densities de certeines suites d'entiers. Proc. Lond. Math. Soc. **3**(59), 417–438 (1989)
7. Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)
8. Fridy, J.A.: On statistical convergence. Analysis **5**, 301–313 (1985)
9. Fridy, J.A., Miller, H.I.: A matrix characterization of statistical convergence. Analysis **11**, 59–66 (1991)
10. Jena, B.B., Paikray, S.K., Dutta, H.: On various new concepts of statistical convergence for sequences of random variables via deferred Cesaro mean. J. Math. Anal. Appl. **487**, 123950 (2020)
11. Maddox, I.J.: Statistical convergence in a locally convex space. Math. Proc. Camb. Philos. Soc. **104**, 141–145 (1988)
12. Menger, K.: Statistical metrics. Proc. Natl. Acad. Sci. USA **28**, 535–537 (1942)

13. Mursaleen, M., Danish Lohani, Q.M.: Statistical limit superior and limit inferior in probabilistic normed spaces. Filomat **25**(3), 55–67 (2011)
14. Mursaleen, M., Mohiuddine, S.A.: On ideal convergence of double sequences in probabilistic normed spaces. Math. Rep. **12**(62), 359–371 (2010)
15. Mursaleen, M., Mohiuddine, S.A.: On ideal convergence in probabilistic normed spaces. Math. Slovaca **62**(1), 49–62 (2012)
16. Mustafa, Z., Sims, B.: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**(2), 289–297 (2006)
17. Pehlivan, S., Güncan, A., Mamedov, A.: On statistical convergent sequences of real numbers. Math. Slovaca **30**, 139–150 (1980)
18. Raj, K., Anand, R.: On statistical convergence in generalized lacunary sequence spaces. Bul. Acad. Ştiinţe Repub. Mold. Mat. **2**, 17–29 (2018)
19. Raj, K., Jamwal, S.: Applications of statistical convergence to n-normed spaces. Adv. Pure Appl. Math. **7**(3), 197–204 (2016)
20. Raj, K., Pandoh, S., Devi, M.: Orlicz b-metric spaces of infinite matrix and their Kothe–Toeplitz duals. J. Interdiscip. Math. **21**(7–8), 1587–1595 (2018)
21. Salat, T.: Statistical cluster points of sequences in finite dimensional spaces. Czechoslov. Math. J. **54**, 95–102 (2004)
22. Schoenberg, I.J.: The integrability of certain functions and related summability methods. Am. Math. Mon. **66**, 361–375 (1959)
23. Schweizer, B.: Probabilistic Metric Space. Elsevier, New York (1983)
24. Sencimen, C., Pehlivan, S.: Strong statistical convergence in probabilistic metric spaces. Stoch. Anal. Appl. **26**, 651–664 (2008)
25. Steinhaus, H.: Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. **2**, 73–74 (1951)
26. Zhou, C., Wang, S., Ciric, L., Alsulami, S.: Generalized probabilistic metric spaces and fixed point theorems. Fixed Point Theory Appl. **2014**, 91 (2014)
27. Zygmund, A.: Trigonometric Series. Cambridge University Press, Cambridge (1979)