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# Some bounds of the generalized $\mu$-scrambling indices of primitive digraphs with $d$ loops

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**Abstract**
In 2010, Huang and Liu introduced a useful parameter called the generalized $\mu$-scrambling indices of a primitive digraph. In this paper, we give some bounds for $\mu$-scrambling indices of some primitive digraphs with $d$ loops and the digraphs attained the sharp upper bounds are provided.

**MSC:** 05C20; 05C50

**Keywords:** Generalized $\mu$-scrambling index; Primitive digraph; Bound; Scrambling index

## 1 Introduction

For the research on the competition index, $m$-competition index, the scrambling index and the generalized $\mu$-scrambling index, please refer to [1–3, 5, 6, 8, 9] and [7, 11], respectively. Cho et al. [6] defined the $m$-step competition graph of a digraph which is an extension of a competition graph. In 2009, Akelbek and Kirkland [2] defined and studied the scrambling index of a primitive digraph and provided an upper bound on the scrambling index of a primitive digraph. The $m$-competition index of a primitive digraph was introduced by Kim [8]. Kim investigated the $m$-competition index of a primitive digraph and gave an upper bound for the $m$-competition indices of primitive digraphs. In 2010, Huang and Liu [7] gave the definition of the generalized $\mu$-scrambling indices for a primitive digraph which are a generalization of the scrambling index and $m$-competition index and they provided some bounds for the generalized $\mu$-scrambling indices of some primitive digraphs. In this paper, we give some bounds for $\mu$-scrambling indices of some primitive digraphs.

The outline of this paper is as follows: Some notation and notions used throughout this paper are introduced in Sect. 2. In Sect. 3, we study the generalized $\mu$-scrambling indices of the primitive digraphs with $d$ loops.

## 2 Definitions and terminology

In this section, we introduce some definitions,notations which are needed to use in the presentations and proofs of our main results in this paper.

A *digraph* $D$ consists of a nonempty set $V = V(D)$ and an arc set $E = E(D)$. In $D$, loops are permitted but multiple arcs are not. A *path* $P = x \to y$ is a sequence of edges $\{(x, v_1), (v_1, v_2), \ldots, (v_{k-1}, y)\}$ in which all vertices are distinct. A *cycle* $C$ is a closed path with the first and the last vertices coincided. A *walk* from $x$ to $y$ is a sequence of arcs: $e_1, e_2, \ldots, e_k$ such that the terminal vertex of $e_i$ is the same as the initial vertex of $e_{i+1}$ for $i = 1, 2, \ldots, k-1$, denoted by $W = x \to y$. The *length* of a walk or cycle is the number of arcs. A walk $W = x \to y$ of length $k$ is denoted by $x \xrightarrow{k} y$. A cycle of length $l$ is denoted by $C_l$. The *girth* of $D$ which has at least one cycle, is the length of a shortest cycle in $D$.

A digraph $D$ is *primitive* with a walk of length $k$ from each vertex $x$ to each vertex $y$ (not necessarily distinct). The digraph $D$ is primitive if and only if $D$ is strongly connected and the greatest common divisor of the lengths of its cycles is 1 (see [4]). For a positive integer $s$, the *sth power* of $D$, denoted by $D^{(s)}$, is the digraph on the same vertex set $V(D)$ and with an arc from $i$ to $j$ if and only if $i \xrightarrow{s} j$ in $D$. The *scrambling index* $k(D)$ of a primitive digraph $D$ is the smallest positive integer $k$ such that, for every pair of vertices $u$ and $v$, there exists a vertex $w$ such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in $D$ (see [2]).

Let $D$ be a digraph with vertex set $V$ and let $k$ be a positive integer. A vertex $w$ of $D$ is a *k-step* common prey for $u$ and $v$ if $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$. The *k-step m-competition graph* of $D$ has the same vertex set of $D$ and an edge between vertices $u$ and $v$ if and only if there are at least $m$ distinct vertices $v_1, \ldots, v_m$ in $D$ such that $u \xrightarrow{k} v_i$ and $v \xrightarrow{k} v_i$ for $i = 1, 2, \ldots, m$ (see [6]). The *m-competition index* $c(D, m)$ of a primitive digraph $D$ is the smallest positive integer $k$ such that, for every pair of vertices $u$ and $v$, there are $m$ distinct vertices $v_1, \ldots, v_m$ in $D$ such that $u \xrightarrow{k} v_i$ and $v \xrightarrow{k} v_i$ for $i = 1, 2, \ldots, m$ (see [2]). That is to say, the $m$-competition index of $D$ is the smallest positive integer $k$ such that the $k$-step $m$-competition graph is complete.

Let $P_n$ denote the set of all primitive digraphs of order $n$.

**Definition 2.1** ([7]) Let $D \in P_n$, and $\lambda$, $\mu$ be integers with $1 \le \lambda, \mu \le n$. For $X \subseteq V(D)$, let $k_X^{(\mu)}$ be the smallest positive integer $m$ such that there exist $\mu$ vertices $w_1, w_2, \ldots, w_\mu$ of $D$ such that $x \xrightarrow{m} w_i$ $(i = 1, 2, \ldots, \mu)$ in $D$ for every vertex $x$ of $X$. Then

$$h(D, \lambda, \mu) := \min\{k_X^{(\mu)} \mid X \subseteq V(D) \text{ and } |X| = \lambda\} \quad \text{and}$$
$$k(D, \lambda, \mu) := \max\{k_X^{(\mu)} \mid X \subseteq V(D) \text{ and } |X| = \lambda\}$$

are called the $\lambda$th lower and upper $\mu$-scrambling indices of $D$, respectively. For convenience, let $k_X(D) := k_X^{(1)}(D)$, $h(D, \lambda) := h(D, \lambda, 1)$ and $k(D, \lambda) := k(D, \lambda, 1)$.

Since $k(D, 2) = k(D)$, in [7] Huang and Liu called $h(D, \lambda, \mu)$ and $k(D, \lambda, \mu)$ the generalized $\mu$-scrambling indices, $h(D, \lambda)$ and $k(D, \lambda)$ the generalized scrambling indices of $D$ in $P_n$. As $k(D, 2, m) = c(D, m)$, the generalized $\mu$-scrambling indices are also generalizations of the $m$-competition index.

## 3 Generalized $\mu$-scrambling indices

In [7], Huang and Liu investigated generalized scrambling indices of the primitive digraphs with $d$ loops. In this section, we study the generalized $\mu$-scrambling indices of the primitive digraphs with $d$ loops.

For a vertex subset $X \subseteq V(D)$, define $R_t^D(X)$ to be the set of vertices in $D$ reachable from some vertices in $X$ via a walk of length $t$.

Let $d$ be an integer with $1 \leq d \leq n$ and let $P_n(d)$ be the class of primitive digraphs with $n$ vertices and $d$ loops. Let $L_{n,d}$ $(1 \leq d \leq n)$ be the digraph with vertex set $V(L_{n,d}) = \{1, 2, \ldots, n\}$ and arc set

$$E(L_{n,d}) = \big\{(i, i+1) | 1 \leq i \leq n-1\big\} \cup \big\{(n, 1)\big\} \cup \big\{(i, i) | n-d+1 \leq i \leq n\big\}.$$

**Theorem 3.1** *Let $D \in P_n(d)$ and $1 \leq \lambda, \mu \leq n$.*

$$h(D, \lambda, \mu) \leq \begin{cases} \lambda + \mu - 2, & \lambda + \mu < n+1, \\ n-1, & d \geq \lambda, \lambda + \mu \geq n+1, \\ n-1, & d < \lambda, n+1 \leq \lambda + \mu \leq n+d, \\ \lambda + \mu - d - 1, & d < \lambda, \lambda + \mu > n+d, \end{cases}$$

*and the bound can be attained by the digraph $L_{n,d}$.*

*Proof* Since $D \in P_n(d)$, there exists a loop vertex $u$ such that there is a set $Y$ of $\lambda - 1$ vertices whose distances to $u$ are at most $\lambda - 1$. If $\lambda + \mu < n+1$, let $X = Y \cup \{u\}$. Then $|X| = \lambda$. Since $D$ is strongly connected and $u$ is a loop vertex, the minimum number of vertices that can be reached from $u$ at $(\mu - 1)$-step in $D$ is $\mu$. Therefore, $|\bigcap_{x \in X} R^D_{\lambda+\mu-2}(\{x\})| \geq \mu$, which implies that $h(D, \lambda, \mu) \leq \lambda + \mu - 2$.

If $d \geq \lambda$ and $\lambda + \mu \geq n+1$, let $X$ be a vertex set which contains $\lambda$ loop vertices. Since each vertex in $X$ is a loop vertex, we have $R^D_{n-1}(X) = V(D)$. Therefore, $|R^D_{n-1}(X)| = |V(D)| = n \geq \mu$, which implies that $h(D, \lambda, \mu) \leq \lambda + \mu - 2$.

If $d < \lambda$, let $Z$ be the vertex set of $d$ loop vertices and $X_i \subseteq (V(D) \setminus Z)$ be the vertex set of $x_i$ vertices whose shortest distance to vertices of $Z$ is $i$, where $1 \leq i \leq \lambda - d$. Assume $\sum_{i=1}^{r} x_i \leq \lambda - d < \sum_{i=1}^{r+1} x_i$, where $1 \leq r \leq \lambda - d$. Let $X = Z \cup X_1 \cup \cdots \cup X_r \cup \bar{X}_{r+1}$, where $\bar{X}_{r+1} \subseteq X_{r+1}$ contains $\bar{x}_{r+1}$ vertices and $\sum_{i=1}^{r} x_i + \bar{x}_{r+1} = \lambda - d$. Then $|X| = \lambda$.

If $d < \lambda$ and $n+1 \leq \lambda + \mu \leq n+d$, since $R^D_{n-1}(Z) = V(D)$ and $R^D_{n-1}(X_1 \cap \cdots \cap X_r \cap \bar{X}_{r+1})$ contains at least $n - \sum_{i=1}^{r} x_i - \bar{x}_{r+1} = n - \lambda + d$ vertices, we have $|R^D_{n-1}(X)| \geq n - \lambda + d \geq \mu$. Therefore, $h(D, \lambda, \mu) \leq \lambda + \mu - 2$.

If $d < \lambda$ and $\lambda + \mu > n+d$, let $k = \lambda + \mu - n - d$. Notice that $R^D_{\lambda+\mu-d-1}(X_i) = V(D)$, for $1 \leq i \leq k$. If $k \geq r+1$, then $|R^D_{\lambda+\mu-d-1}(X)| = |V(D)| = n \geq \mu$.

If $k < r+1$, then $R^D_{\lambda+\mu-d-1}(X) = \bigcap_{i=k+1}^{r} [R^D_{\lambda+\mu-d-1}(X_i)] \cap R^D_{\lambda+\mu-d-1}(\bar{X}_{r+1})$. Since $\bigcap_{i=k+1}^{r} [R^D_{\lambda+\mu-d-1}(X_i)] \cap R^D_{\lambda+\mu-d-1}(\bar{X}_{r+1})$ contains at least $n - (\sum_{i=k+1}^{r} x_i + \bar{x}_{r+1})$ vertices, we have $|R^D_{\lambda+\mu-d-1}(X)| \geq n - (\lambda - d - \sum_{i=1}^{k} x_i) \geq n - \lambda + d + k = n - \lambda + d + (\lambda + \mu - n - d) = \mu$. We thus arrive at $h(D, \lambda, \mu) \leq \lambda + \mu - 2$.

On the other hand, consider the digraph $L_{n,d}$. Lex $X$ be a vertex set with $\lambda$ vertices. If $\lambda + \mu < n+1$, since $R^{L_{n,d}}_{\lambda+\mu-3}(i) = \{i, \ldots, n, \ldots, \lambda + \mu + i - n - 3\}$, for $n - d + 1 \leq i \leq n$ and $R^{L_{n,d}}_{\lambda+\mu-3}(i) = \{n - d + 1, \ldots, n, \ldots, \lambda + \mu + i - n - 3\}$, for $1 \leq i \leq n - d + 1$, we obtain $|\bigcap_{x \in X} R^{L_{n,d}}_{\lambda+\mu-3}(\{x\})| \leq \mu - 1$.

Noticing that $R^{L_{n,d}}_{n-2}(i) = \{i, \ldots, n, \ldots, i-2\}$, for $n - d + 1 \leq i \leq n$, and $R^{L_{n,d}}_{n-2}(i) = \{n - d + 1, \ldots, n, \ldots, i - 2\}$, for $1 \leq i < n - d + 1$. If $d \geq \lambda$ and $\lambda + \mu \geq n+1$, then, for any vertex $x \in X$, $R^{L_{n,d}}_{n-2}(\{x\})$ contains at most $n-2$ vertices. Therefore, $\bigcap_{x \in X} R^{L_{n,d}}_{n-2}(\{x\})$ contains at most $n-\lambda-1$ vertices. As $\lambda+\mu\geq n+1$, we have $|\bigcap_{x\in X} R_{n-2}^{L_{n,d}}(\{x\})|\leq \mu-2$. Consequently, we obtain $h(D,\lambda,\mu)=\lambda+\mu-2$.

If $d<\lambda$ and $n+1\leq\lambda+\mu\leq n+d$, we have for any set $X$, there is at least one vertex $x\in X$ such that $R_{n-2}^{L_{n,d}}(\{x\})$ contains at most $n-3$ vertices. Thus, $\bigcap_{x\in X} R_{n-2}^{L_{n,d}}(\{x\})$ contains at most $n-\lambda-2$ vertices. Since $\lambda+\mu\geq n+1$, we obtain $|\bigcap_{x\in X} R_{n-2}^{L_{n,d}}(\{x\})|\leq\mu-3$. Therefore, $h(D,\lambda,\mu)=\lambda+\mu-2$.

If $d<\lambda$ and $\lambda+\mu>n+d$, we have $R_{\lambda+\mu-d-2}^{L_{n,d}}(i)=V(L_{n,d})$, for $2n+2-\lambda-\mu\leq i\leq n$, and $R_{\lambda+\mu-d-2}^{L_{n,d}}(i)=\{n-d+1,\dots,n,\dots,\lambda+\mu+i-n-d-2\}$, for $1\leq i<2n+2-\lambda-\mu$. Thus, for any vertex set $X$ of $\lambda$ vertices, there is a set $Y\subseteq X$ of at least $n+1-\mu$ vertices, such that, for any vertex $y\in Y$, $R_{\lambda+\mu-d-2}^{L_{n,d}}(\{y\})$ contains at most $\lambda+\mu+y-n-2$ vertices, where $1\leq y<2n+2-\lambda-\mu$. It follows that $\bigcap_{y\in Y} R_{\lambda+\mu-d-2}^{L_{n,d}}(\{y\})$ containing at most $\mu-1$ vertices. Therefore, $|\bigcap_{x\in X} R_{\lambda+\mu-d-2}^{L_{n,d}}(\{x\})|\leq|\bigcap_{y\in Y} R_{\lambda+\mu-d-2}^{L_{n,d}}(\{y\})|\leq\mu-1$. It follows that $h(D,\lambda,\mu)=\lambda+\mu-2$. This completes the proof. □

**Lemma 3.2** ([10]) *Let $D\in P_n(d)$ and $\emptyset\neq X\subseteq V(D)$. Then, for nonnegative integers $i, j, t, k$, we have $R_i^D(X)=R_{i-j}^D(R_j^D(X))$ for $i\geq j$, and $|\bigcup_{t=0}^k R_t^D(X)|\geq\min\{|X|+k,n\}$.*

**Theorem 3.3** *Let $D\in P_n(d)$ and $1\leq\lambda,\mu\leq n$. Then*

$$k(D,\lambda,\mu)\leq\begin{cases} n-\lceil\frac{d-\mu+1}{\lambda}\rceil, & \mu\leq d,\\ n+\mu-d-1, & \mu>d,\end{cases}$$

*and the bound can be attained by the digraph $L_{n,d}$.*

*Proof* Let $X\subseteq V(D)$ be a vertex set of any $\lambda$ vertices. Set $X=\{v_1,v_2,\dots,v_\lambda\}$.

*Case 1.* If $\mu\leq d$.

For any vertex $v_i\in X$, since $D\in P_n(d)\subseteq P_n$, by Lemma 3.2,

$$\left|\bigcup_{t=0}^{n-\lceil\frac{d-\mu+1}{\lambda}\rceil} R_t^D(\{v_i\})\right|\geq n-\left\lceil\frac{d-\mu+1}{\lambda}\right\rceil+1,$$

where $i=1,2,\dots,\lambda$. Let $\frac{d-\mu+1}{\lambda}=k'+a$ where $k'$ is a nonnegative integer and $0\leq a<1$. Therefore, if $0<a<1$,

$$\left|\bigcap_{i=1}^{\lambda}\left[\bigcup_{t=0}^{n-k'-1} R_t^D(\{v_i\})\right]\right|\geq\lambda(n-k')-n(\lambda-1)\geq n-d+\mu+\lambda a-1.$$

Since $\lambda a\geq 1$,

$$\left|\bigcap_{i=1}^{\lambda}\left[\bigcup_{t=0}^{n-k'-1} R_t^D(\{v_i\})\right]\right|\geq n-d+\mu.$$

If $a=0$,

$$\left|\bigcap_{i=1}^{\lambda}\left[\bigcup_{t=0}^{n-k'-1} R_t^D(\{v_i\})\right]\right|\geq\lambda(n-k'+1)-n(\lambda-1)\geq n-d+\mu+\lambda-1.$$

Since $\lambda \geq 1$, we have

$$\left|\bigcap_{i=1}^{\lambda}\left[\bigcup_{t=0}^{n-k'-1} R_t^D(\{v_i\})\right]\right| \geq n-d+\mu.$$

It follows that there are at least $\mu$ loop vertices $u_1, u_2, \ldots, u_\mu$ such that $u_i \in \bigcap_{i=1}^{\lambda}[\bigcup_{t=0}^{n-k'-1} R_t^D(\{v_i\})]$, where $i = 1, 2, \ldots, \mu$. That is to say,

$$\left|\bigcap_{i=1}^{\lambda} R_{n-k'-1}^D(\{v_i\})\right| \geq \mu.$$

Therefore, $k(D,\lambda,\mu) \leq n - \lceil \frac{d-\mu+1}{\lambda} \rceil$.

Consider the digraph $L_{n,d}$. Let

$$t = n - \left\lceil \frac{d-\mu+1}{\lambda} \right\rceil \quad \text{and} \quad k^* = \left\lceil \frac{d-\mu+1}{\lambda} \right\rceil.$$

Then we consider the following two subcases.

*Subcase 1.* If $\lambda \leq d+1$.

When $\lambda = 1$, then $t = n-d+\mu-1$. Noting that

$$R_{t-1}^{L_{n,d}}(\{1\}) = \{n-d+1, n-d+2, \ldots, n-d+\mu-1\},$$

then

$$\left|R_{t-1}^{L_{n,d}}(\{1\})\right| = \mu - 1.$$

When $\lambda = 2$, as

$$R_{t-1}^{L_{n,d}}(\{1\}) = \{n-d+1, \ldots, n-k^*\}$$

and

$$R_{t-1}^{L_{n,d}}(\{n-k^*-\mu+2\}) = \{n-k^*-\mu+2, \ldots, n, 1, \ldots, n-2k^*-\mu+1\},$$

we have

$$\left|R_{t-1}^{L_{n,d}}(\{1\}) \cap R_{t-1}^{L_{n,d}}(\{n-k^*-\mu+2\})\right| = \left|\{n-k^*-\mu+2, \ldots, n-1\}\right| = \mu - 1.$$

When $\lambda = 3$, if $n-2k^*-\mu+1 < n-d+1$, then $k^* = 1$, $t = n-1$. Let

$$X = \{1, n-k^*-\mu+2, u\} \subseteq V(L_{n,d}),$$

where $u \in V(L_{n,d}) \setminus \{1, n-k^*-\mu+2\}$. Then $|X| = 3$. Since

$$R_{t-1}^{L_{n,d}}(\{1\}) = \{n-d+1, \ldots, n-k^*\}$$

and

$$R_{t-1}^{L_{n,d}}\left(\left\{n-k^*-\mu+2\right\}\right)=\left\{n-k^*-\mu+2,\dots,n,1,\dots,n-2k^*-\mu+1\right\},$$

we have

$$R_{t-1}^{L_{n,d}}\left(\{1\}\right)\cap R_{t-1}^{L_{n,d}}\left(\left\{n-k^*-\mu+2\right\}\right)=\left\{n-k^*-\mu+2,\dots,n-k^*\right\}.$$

Thus,

$$\left|\bigcap_{x\in X}R_{t-1}^{L_{n,d}}\left(\{x\}\right)\right|\le\left|R_{t-1}^{L_{n,d}}\left(\{1\}\right)\cap R_{t-1}^{L_{n,d}}\left(\left\{n-k^*-\mu+2\right\}\right)\right|=\mu-1.$$

If $n-2k^*-\mu+1\ge n-d+1$, then $n-2k^*-\mu+1<n-d+k^*+1<n-k^*-\mu+2$. Let

$$X=\left\{1,n-k^*-\mu+2,n-d+k^*\right\}\subseteq V(L_{n,d}),$$

where $u\in V(L_{n,d})\setminus\{1,n-k^*-\mu+2\}$. Then $|X|=3$. Since

$$R_{t-1}^{L_{n,d}}\left(\{1\}\right)=\left\{n-d+1,\dots,n-k^*\right\},$$
$$R_{t-1}^{L_{n,d}}\left(\left\{n-d+k^*+1\right\}\right)=\left\{n-d+k^*+1,\dots,n,1,\dots,n-d\right\}$$

and

$$R_{t-1}^{L_{n,d}}\left(\left\{n-k^*-\mu+2\right\}\right)=\left\{n-k^*-\mu+2,\dots,n,1,\dots,n-2k^*-\mu+1\right\},$$

we have

$$\bigcap_{x\in X}R_{t-1}^{L_{n,d}}\left(\{x\}\right)=\left\{n-k^*-\mu+2,\dots,n-k^*\right\},$$

which implies

$$\left|\bigcap_{x\in X}R_{t-1}^{L_{n,d}}\left(\{x\}\right)\right|=\mu-1.$$

When $\lambda=4$, if $n-2k^*-\mu+1<n-d+1$, then $k^*=1$ and $t=n-1$. Let

$$X=\{1,n-\mu+1,u_1,u_2\}\subseteq V(L_{n,d}),$$

where $u_1,u_2\in V(L_{n,d})\setminus\{1,n-\mu+1\}$. Then $|X|=4$. Since

$$R_{t-1}^{L_{n,d}}\left(\{1\}\right)=\{n-d+1,\dots,n-\mu+1,\dots n-1\}$$

and

$$R_{t-1}^{L_{n,d}}\left(\{n-\mu+1\}\right)=\{n-\mu+1,\dots,n,1,\dots,n-\mu-1\},$$

we have

$$\left|R_{t-1}^{L_{n,d}}\left(\{1\}\right)\cap R_{t-1}^{L_{n,d}}\left(\{n-\mu+1\}\right)\right| = \left|\{n-\mu+1,\dots,n-1\}\right| = \mu-1.$$

This implies

$$|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}\left(\{x\}\right) \le \mu-1.$$

If $n-d+k^*+1 > n-2k^*-\mu+1 \ge n-d+1$, letting

$$X = \left\{1, n-k^*-\mu+2, n-d+k^*+1, w\right\} \subseteq V(L_{n,d}),$$

where $w \in V(L_{n,d}) \setminus \{1, n-k^*-\mu+2, n-d+k^*+1\}$, we have $|X| = 4$. Since

$$\begin{aligned}
&R_{t-1}^{L_{n,d}}\left(\{1\}\right) = \left\{n-d+1,\dots,n-k^*\right\},\\
&R_{t-1}^{L_{n,d}}\left(\left\{n-d+k^*+1\right\}\right) = \left\{n-d+k^*+1,\dots,n,1,\dots,n-d\right\},\\
&R_{t-1}^{L_{n,d}}\left(\left\{n-\mu-k^*+2\right\}\right) = \left\{n-\mu-k^*+2,\dots,n,1,\dots,n-2k^*-\mu+1\right\},
\end{aligned}$$

we have

$$R_{t-1}^{L_{n,d}}\left(\{1\}\right)\cap R_{t-1}^{L_{n,d}}\left(\left\{n-\mu-k^*+2\right\}\right)\cap R_{t-1}^{L_{n,d}}\left(\left\{n-d+k^*+1\right\}\right) = \left\{n-k^*-\mu+2,\dots,n-k^*\right\}.$$

Thus,

$$\left|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}\left(\{x\}\right)\right| \le \left|R_{t-1}^{L_{n,d}}\left(\{1\}\right)\cap R_{t-1}^{L_{n,d}}\left(\left\{n-\mu-k^*+2\right\}\right)\cap R_{t-1}^{L_{n,d}}\left(\left\{n-d+k^*+1\right\}\right)\right| \le \mu-1.$$

If $n-d+k^*+1 \le n-2k^*-\mu+1$, letting

$$X = \left\{1, n-d+k^*+1, n-k^*-\mu+2, n-2k^*-\mu+2\right\} \subseteq V(L_{n,d}),$$

we have $|X| = 4$. Since

$$\begin{aligned}
&R_{t-1}^{L_{n,d}}\left(\{1\}\right) = \left\{n-d+1,\dots,n-k^*\right\},\\
&R_{t-1}^{L_{n,d}}\left(\left\{n-d+k^*+1\right\}\right) = \left\{n-d+k^*+1,\dots,n,1,\dots,n-d\right\},\\
&R_{t-1}^{L_{n,d}}\left(\left\{n-k^*-\mu+2\right\}\right) = \left\{n-\mu-k^*+2,\dots,n,1,\dots,n-2k^*-\mu+1\right\},\\
&R_{t-1}^{L_{n,d}}\left(\left\{n-2k^*-\mu+2\right\}\right) = \left\{n-2k^*-\mu+2,\dots,n,1,\dots,n-3k^*-\mu+1\right\},
\end{aligned}$$

we have

$$\left|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}\left(\{x\}\right)\right| = \left|\left\{n-k^*-\mu+2, n-k^*-\mu+3,\dots,n-k^*\right\}\right| = \mu-1.$$

When $\lambda \geq 5$, if $n-2k^*-\mu+1 < n-d+1$, letting

$$X = \{1, n-k^*-\mu+2, v_1, \dots, v_{\lambda-2}\} \subseteq V(L_{n,d}),$$

where $v_i \in V(L_{n,d}) \setminus \{1, n-k^*-\mu+2\}$ for $i = 1, 2, \dots, \lambda-2$, then $|X| = \lambda$. Since

$$R_{t-1}^{L_{n,d}}(\{1\}) = \{n-d+1, \dots, n-k^*\}$$

and

$$R_{t-1}^{L_{n,d}}(\{n-k^*-\mu+2\}) = \{n-\mu-k^*+2, \dots, n, 1, \dots, n-2k^*-\mu+1\},$$

we have

$$(\bigcap_{x\in X} R_{t-1}^{L_{n,d}}(\{x\}) \subseteq (R_{t-1}^{L_{n,d}}(\{1\}) \cap R_{t-1}^{L_{n,d}}(\{n-\mu-k^*+2\})) = \{n-k^*-\mu+2, \dots, n-k^*\},$$

which implies that

$$\left|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}(\{x\})\right| \leq \mu-1.$$

If $n-d+k^*+1 > n-2k^*-\mu+1 \geq n-d+1$, letting

$$X = \{1, n-d+k^*+1, n-k^*-\mu+2, v_1, \dots, v_{\lambda-3}\} \subseteq V(L_{n,d}),$$

where $v_i \in V(L_{n,d}) \setminus \{1, n-d+k^*+1, n-k^*-\mu+2\}$, for $i = 1, 2, \dots, \lambda-3$, then $|X| = \lambda$. As

$$R_{t-1}^{L_{n,d}}(\{1\}) = \{n-d+1, \dots, n-k^*\},$$
$$R_{t-1}^{L_{n,d}}(\{n-d+k^*+1\}) = \{n-d+k^*+1, \dots, n, 1, \dots, n-d\},$$

and

$$R_{t-1}^{L_{n,d}}(\{n-k^*-\mu+2\}) = \{n-\mu-k^*+2, \dots, n, 1, \dots, n-2k^*-\mu+1\},$$

we have

$$R_{t-1}^{L_{n,d}}(\{1\}) \cap R_{t-1}^{L_{n,d}}(\{n-\mu-k^*+2\}) \cap R_{t-1}^{L_{n,d}}(\{n-d+k^*+1\}) = \{n-k^*-\mu+2, \dots, n-k^*\}.$$

Thus,

$$\left|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}(\{x\})\right| \leq \left|R_{t-1}^{L_{n,d}}(\{1\}) \cap R_{t-1}^{L_{n,d}}(\{n-\mu-k^*+2\}) \cap R_{t-1}^{L_{n,d}}(\{n-d+k^*+1\})\right| = \mu-1.$$

If $n-(r+1)k^*-\mu+1 < n-d+k^*+1 \leq n-rk^*-\mu+1$ and $2 \leq r \leq \lambda-3$, letting

$$Y = \{1, n-d+k^*+1, n-k^*-\mu+2, n-2k^*-\mu+2, \dots, n-rk^*-\mu+2\} \subseteq V(L_{n,d}),$$

and

$$X = \bigl(Y \cup \{v_1, \ldots, v_{\lambda-r-2}\}\bigr) \subseteq V(L_{n,d}),$$

where $v_i \in V(L_{n,d}) \setminus \{1, n-d+k^*+1, n-k^*-\mu+2, n-2k^*-\mu+2, \ldots, n-rk^*-\mu+2\}$ for $i = 1, 2, \ldots, \lambda-r-2$, then $|X| = \lambda$. Since

$$R_{t-1}^{L_{n,d}}\bigl(\{1\}\bigr) = \bigl\{n-d+1, \ldots, n-k^*\bigr\},$$
$$R_{t-1}^{L_{n,d}}\bigl(\bigl\{n-d+k^*+1\bigr\}\bigr) = \bigl\{n-d+k^*+1, \ldots, n, 1, \ldots, n-d\bigr\},$$

and for $1 \le i \le r$,

$$R_{t-1}^{L_{n,d}}\bigl(\bigl\{n-\mu-ik^*+2\bigr\}\bigr) = \bigl\{n-\mu-ik^*+2, \ldots, n, 1, \ldots, n-(i+1)k^*-\mu+1\bigr\},$$

we have

$$\bigcap_{x\in Y} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr) = \bigl\{n-k^*-\mu+2, \ldots, n-k^*\bigr\}.$$

Thus,

$$\left|\bigcap_{x\in Y} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr)\right| \le \left|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr)\right| = \mu-1.$$

If $n-d+k^*+1 \le n-(\lambda-2)k^*-\mu+1$, letting

$$X = \bigl\{1, n-d+k^*+1, n-k^*-\mu+2, n-2k^*-\mu+2, \ldots, n-(\lambda-2)k^*-\mu+2\bigr\} \subseteq V(L_{n,d}),$$

we have $|X| = \lambda$. Since

$$R_{t-1}^{L_{n,d}}\bigl(\{1\}\bigr) = \bigl\{n-d+1, \ldots, n-k^*\bigr\},$$
$$R_{t-1}^{L_{n,d}}\bigl(\bigl\{n-d+k^*+1\bigr\}\bigr) = \bigl\{n-d+k^*+1, \ldots, n, 1, \ldots, n-d\bigr\},$$

for $1 \le i \le \lambda-2$

$$R_{t-1}^{L_{n,d}}\bigl(\bigl\{n-\mu-ik^*+2\bigr\}\bigr) = \bigl\{n-\mu-ik^*+2, \ldots, n, 1, \ldots, n-(i+1)k^*-\mu+1\bigr\},$$

and $n-d+k^*+1 > n-(\lambda-1)k^*-\mu+1$, we have

$$\bigcap_{x\in Y} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr) = \bigl\{n-k^*-\mu+2, \ldots, n-k^*\bigr\}.$$

Therefore,

$$\left|\bigcap_{y\in Y} R_{t-1}^{L_{n,d}}\bigl(\{y\}\bigr)\right| \le \left|\bigcap_{x\in X} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr)\right| = \mu-1.$$

From the above, we have $k(L_{n,d},\lambda,\mu) \geq n - \lceil\frac{d-\mu+1}{\lambda}\rceil$, it follows that $k(L_{n,d},\lambda,\mu) = n - \lceil\frac{d-\mu+1}{\lambda}\rceil$.

*Subcase 2.* If $\lambda > d+1$.

If $\lambda > d+1$, then $t = n-1$. It is easy to see that

$$\begin{aligned} R_{t-1}^{L_{n,d}}(\{1\}) &= \{n-d+1, n-d+2, \ldots, n-1\}, \\ R_{t-1}^{L_{n,d}}(\{2\}) &= \{n-d+1, n-d+2, \ldots, n\}, \end{aligned}$$

for $3 \leq i \leq \lambda - d$,

$$R_{t-1}^{L_{n,d}}(\{\lambda-d\}) = \{n-d+1, \ldots, n, 1, \ldots, \lambda-d-2\}$$

and for $i = n-d+1, \ldots, n$,

$$R_{t-1}^{L_{n,d}}(\{i\}) = \{i, \ldots, n, 1, \ldots, i-2\}.$$

Let

$$X_1 = \{1, \ldots, \lambda-d, n-d+1\}, \qquad X_2 = \{n-d+2, \ldots, n\} \quad \text{and} \quad X = X_1 \cup X_2.$$

Then $|X| = \lambda$. Since

$$\bigcap_{x \in X_1} R_{t-1}^{L_{n,d}}(\{x\}) = \{n-d+1, n-d+2, \ldots, n-1\}$$

and

$$\bigcap_{x \in X_2} R_{t-1}^{L_{n,d}}(\{x\}) = \{n, 1, \ldots, n-d-1\},$$

we have $\bigcap_{x \in X} R_{t-1}^{L_{n,d}}(\{x\}) = \phi$. Therefore,

$$k(L_{n,d},\lambda,\mu) \geq n - \left\lceil \frac{d-\mu+1}{\lambda} \right\rceil,$$

it follows that

$$k(L_{n,d},\lambda,\mu) = n - \left\lceil \frac{d-\mu+1}{\lambda} \right\rceil.$$

*Case 2.* If $\mu > d+1$.

Let $X \subseteq V(D)$ be a vertex set of any $\lambda$ vertices. Set $X = \{v_1, v_2, \ldots, v_\lambda\}$. For any vertex $v_i \in X$, since $D \in P_n(d) \subseteq P_n$, by Lemma 3.2,

$$\left| \bigcup_{t=0}^{n-1} R_t^D(\{v_i\}) \right| \geq n - 1 + 1 = n,$$

where $i = 1, 2, \dots, \lambda$. Therefore, each loop vertex $u_i \in \bigcup_{t=0}^{n-1} R_t^D(\{v_i\})$, where $i = 1, 2, \dots, d$. Then

$$\{u_1, u_2, \dots, u_d\} \subseteq \left(\bigcap_{i=1}^{\lambda} R_{n-1}^D\bigl(\{v_i\}\bigr)\right).$$

Since $u_1, u_2, \dots, u_d$ are loop vertices, there are at least $\mu - d$ vertices $w_1, w_2, \dots, w_{\mu-d}$ and $w_i \notin \{u_1, u_2, \dots, u_d\}$ such that

$$\{w_1, w_2, \dots, w_{\mu-d}\} \subseteq R_{\mu-d}^D\bigl(\{u_1, u_2, \dots, u_d\}\bigr),$$

where $i = 1, 2, \dots, \mu - d$. It follows that

$$\left|\bigcap_{i=1}^{\lambda} R_{n+\mu-d+1}^D\bigl(\{v_i\}\bigr)\right| \geq d + \mu - d = \mu.$$

We thus arrive at

$$k(D, \lambda, \mu) \leq n + \mu - d - 1.$$

Next we consider the digraph $L_{n,d}$. Let $X \subseteq L_{n,d}$ be a vertex set of $\lambda$ vertices and set $X = \{1, 2, \dots, \lambda\}$. Let $t = n + \mu - d - 1$. Since for $i = 2, \dots, n - d$,

$$R_{t-1}^{L_{n,d}}\bigl(\{1\}\bigr) = \{n - d + 1, \dots, n, 1, \dots, \mu - d - 1\} \subseteq R_{t-1}^{L_{n,d}}\bigl(\{i\}\bigr),$$

and for $j = n - d + 1, \dots, n$,

$$R_{t-1}^{L_{n,d}}\bigl(\{j\}\bigr) = \{1, \dots, n\},$$

we have

$$\bigcap_{x \in X} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr) = \{n - d + 1, \dots, n, 1, \dots, \mu - d - 1\},$$

which implies that

$$\left|\bigcap_{x \in X} R_{t-1}^{L_{n,d}}\bigl(\{x\}\bigr)\right| = \mu - 1.$$

Therefore, $k(L_{n,d}, \lambda, \mu) \geq n + \mu - d - 1$. It follows that

$$k(L_{n,d}, \lambda, \mu) = n + \mu - d - 1.$$

Combining the proofs of Cases 1 and 2, the theorem follows as expected. □

**Theorem 3.4** *Let $D \in P_n$ with girth $s$. Then*

$$k(D, \lambda, \mu) \leq \begin{cases} n - s + (n - 1 - \lfloor \frac{n-\mu}{\lambda} \rfloor)s, & \lambda \leq s, \\ n - s + (n - 1 - \lfloor \frac{n-\mu}{s} \rfloor)s, & \lambda > s. \end{cases}$$

*Proof* Let $C_s$ be a directed cycle of length $s$ in $D^{(s)}$. Consider the digraph $D^{(s)}$. Choose any $r$ vertices $w_1, w_2, \ldots, w_r$ of $C_s$. Let $\frac{n-\mu}{r} = k + \frac{b}{r}$ where $0 \leq b < r-1$. In $D^{(s)}$, since $w_i$ is a loop vertex, $|R^{D^s}_{n-k-1}\{w_i\}| \geq n-k = n - \frac{n-\mu}{r} + \frac{b}{r}$ where $i = 1, 2, \ldots, r$. Therefore

$$\left|\bigcap_{i=1}^{r} R^{D^s}_{n-k-1}\{w_i\}\right| \geq r\left(n - \frac{n-\mu}{r} + \frac{b}{r}\right) - (r-1)n = \mu + b \geq \mu.$$

It follows that

$$\left|\bigcap_{i=1}^{r} R^{D}_{(n-k-1)s}\{w_i\}\right| \geq \mu.$$

For any $\lambda$ vertices $v_1, v_2, \ldots, v_\lambda \in V(D)$, there is a walk of length $n-s$ from $v_i$ to a vertex $u_i$ of $C_s$ where $i = 1, 2, \ldots, \lambda$. If $\lambda \leq s$, then $|u_1, u_2, \ldots, u_\lambda| \leq \lambda$ and if $\lambda > s$, then $|u_1, u_2, \ldots, u_\lambda| \leq s$. Hence,

$$k(D, \lambda, \mu) \leq \begin{cases} n - s + (n - 1 - \lfloor \frac{n-\mu}{\lambda} \rfloor)s, & \lambda \leq s, \\ n - s + (n - 1 - \lfloor \frac{n-\mu}{s} \rfloor)s, & \lambda > s. \end{cases}$$ □

## 4 Conclusions

In this paper, we studied $\mu$-scrambling indices of primitive digraphs and gave some bounds for the $\lambda$th lower and upper $\mu$-scrambling indices of primitive digraphs with $d$ loops. However, the digraphs attaining the sharp upper bounds are not determined completely. For a general given primitive digraph, its $\mu$-scrambling indices are not given. It would be nice to settle these problems in further research.

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**Authors' contributions**
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