Stephen et al. *Journal of Inequalities and Applications* (2021) 2021:125
https://doi.org/10.1186/s13660-021-02661-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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# On fixed points of rational contractions in generalized parametric metric and fuzzy metric spaces

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**Abstract**

We introduce the notion of generalized parametric metric spaces along with the study of its various properties. Further, we prove some new fixed point theorems for $(\alpha, \psi)$-rational-type contractive mappings in generalized parametric metric spaces. As a consequence, we deduce fixed point theorems for $(\alpha, \psi)$-rational-type contractive mappings in partially ordered rectangular generalized fuzzy metric spaces.

**MSC:** 47H10; 54H25

**Keywords:** Fixed point; $\alpha$-admissible; $(\alpha, \psi)$-rational contraction; Generalized parametric metric spaces; Fuzzy metric spaces

## 1 Introduction

Hussain et al. [1] gave the definition of parametric metric spaces. They also studied the existence of fixed points for mappings under different contractions in such spaces. A generalization of parametric metric spaces, parametric $b$-metric spaces, was given by Hussain et al. [2]. Another extension of parametric metric spaces to three dimensions, parametric $S$-metric spaces, was introduced by Nihal et al. [3]. Also, Priyobarta et al. [4] introduced the notion of parametric A-metric spaces. Branciari [5] introduced generalized metric spaces. Suzuki [6] and others have pointed out that the topology of a generalized metric space has some drawbacks as a generalized metric need not be continuous, need not have a compatible topology, and in a generalized metric space, a convergent sequence may be a non-Cauchy sequence. Also, a generalized metric is not Hausdrof,f and a limit with respect yo it is not unique. Various forms of parametric metric spaces can be found in [7–18] and references therein. Also, there many applications in the literature [19–25].

First, we recall the following definitions.

**Definition 1.1** ([1]) Consider a set $\Omega \neq \phi$. A function $\mathcal{P}m : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$ is called a parametric metric on $\Omega$ if

(i) $\mathcal{P}m(\zeta, \eta, x) = 0$ for all $x > 0$ implies $\zeta = \eta$;
(ii) $\mathcal{P}m(\zeta, \eta, x) = \mathcal{P}(\eta, \zeta, x)$ for all $x > 0$;

(iii) $\mathcal{P}m(\zeta,\eta,x) \leq \mathcal{P}(\zeta,\mu,x) + \mathcal{P}(\mu,\eta,x)$ for all $\zeta,\eta,\mu \in \Omega$ and $x > 0$.
The pair $(\Omega,\mathcal{P}m)$ is said to be a parametric metric space.

**Definition 1.2** ([5]) Consider a set $\Omega \neq \phi$. A function $d : \Omega \times \Omega \to [0,+\infty)$ is called a generalized metric on $\Omega$ if

(i) $d(\zeta,\eta) = 0$ implies $\zeta = \eta$;
(ii) $d(\zeta,\eta) = d(\eta,\zeta)$;
(iii) $d(\zeta,\eta) \leq d(\zeta,\mu) + d(\mu,\lambda) + d(\lambda,\eta)$

for all distinct $\mu,\lambda \in \Omega - \{\zeta,\eta\}$. The pair $(\Omega,d)$ is said to be a generalized metric space.

Now we introduce generalized parametric metric spaces.

**Definition 1.3** Consider a set $\Omega \neq \phi$. A function $\mathcal{P}m : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$ is called a generalized parametric metric on $\Omega$ if

(i) $\mathcal{P}m(\zeta,\eta,x) = 0$ for all $x > 0$ implies $\zeta = \eta$;
(ii) $\mathcal{P}m(\zeta,\eta,x) = \mathcal{P}m(\eta,\zeta,x)$ for all $x > 0$;
(iii) $\mathcal{P}m(\zeta,\eta,x) \leq \mathcal{P}m(\zeta,\mu,x) + \mathcal{P}m(\mu,\lambda,x) + \mathcal{P}m(\lambda,\eta,x)$ for all distinct $\mu,\lambda \in \Omega - \{\zeta,\eta\}$.

The pair $(\Omega,\mathcal{P}m)$ is said to be a generalized parametric metric space.

**Definition 1.4** Consider a sequence $\{\zeta_n\}$ in a generalized parametric metric space $(\Omega,\mathcal{P}m)$.

1. $\{\zeta_n\}$ is called a convergent sequence converging to $\zeta \in \Omega$ and expressed as $\lim_{n\to\infty}\zeta_n = \zeta$ if $\lim_{n\to\infty}\mathcal{P}m(\zeta_n,\zeta,x) = 0$ for all $x > 0$.
2. $\{\zeta_n\}$ is called a Cauchy sequence in $\Omega$ if $\lim_{n\to\infty}\mathcal{P}m(\zeta_n,\zeta_m,x) = 0$ for all $x > 0$.
3. $(\Omega,\mathcal{P}m)$ is said to be complete if every Cauchy sequence in it is convergent.

**Definition 1.5** Let $C$ be a self-mapping in a generalized parametric metric space $(\Omega,\mathcal{P}m)$. If for every sequence $\{\zeta_n\}$ in $\Omega$ satisfying $\zeta_n \to \zeta$ as $n \to \infty$, $C(\zeta_n) \to C(\zeta)$, then we say that $C$ is a continuous mapping at $\zeta$ in $\Omega$.

Following the definition of $\alpha$-admissibility introduced in [26] and [27], we give the corresponding definition for generalized parametric metric space.

**Definition 1.6** Suppose that $\Omega \neq \phi$, and let $\alpha : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$. A mapping $C : \Omega \to \Omega$ is called an $\alpha$-admissible mapping if $\alpha(\zeta,\eta,x) \geq 1$ gives $\alpha(C\zeta,C\eta,x) \geq 1$ for all $\zeta,\eta \in \Omega$ and $x > 0$.

**Definition 1.7** Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $\alpha : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$. Then $\Omega$ is called an $\alpha$-regular generalized parametric metric space if for any sequence $\{\zeta_n\}$ in $\Omega$ such that $\zeta_n \to \zeta$ and $\alpha(\zeta_n,\zeta_{n+1},x) \geq 1$, there is a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $\alpha(\zeta_{n_k},\zeta,x) \geq 1$ for all $k \in \mathbb{N}$ and $x > 0$.

**Proposition 1.8** *Let $\{\zeta_n\}$ be a Cauchy sequence in a generalized parametric metric space $(\Omega,\mathcal{P}m)$ and $\lim_{n\to\infty}\mathcal{P}m(\zeta_n,a,x) = 0$ for all $a \in \Omega$. Then $\lim_{n\to\infty}\mathcal{P}m(\zeta_n,b,x) = \mathcal{P}m(a,b,x)$ for all $b \in \Omega$ and $x > 0$. Particularly, sequence $\{\zeta_n\}$ does not converge to $b$ if $b \neq a$.*

We denote by $F(C)$ the set of fixed points of mapping $C$.

## 2 Main results

$(\alpha, \psi)$-rational type contractive mappings were used by Salimi et al. [28] and Hamid et al. [29], to prove some fixed point theorems. Here we present their concept in generalized parametric metric spaces. The mapping $\psi$ is defined as before.

Let $\Psi$ be a collection of mappings $\psi : [0,+\infty) \to [0,+\infty)$ such that

(i) $\psi$ is strictly increasing and upper semicontinuous;

(ii) for all $t > 0$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$;

(iii) $\psi(t) < t$ for all $t > 0$.

**Definition 2.1** Let $(\Omega, \mathcal{P}m)$ be a generalized parametric metric space, and let $\alpha : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$. A mapping let $C : \Omega \to \Omega$ is called an $(\alpha, \psi)$-rational contractive mapping of type-*I* if for all $\zeta, \eta \in \Omega$ and $\psi \in \Psi$,

$$\alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) \le \psi\Bigl(\prod(\zeta,\eta,x)\Bigr), \quad x > 0, \tag{2.1}$$

where

$$\prod(\zeta,\eta,x) = \max\biggl\{\mathcal{P}m(\zeta,\eta,x), \mathcal{P}m(\zeta,C\zeta,x), \mathcal{P}m(\eta,C\eta,x), \frac{\mathcal{P}m(\zeta,C\zeta,x)\mathcal{P}m(\eta,C\eta,x)}{1+\mathcal{P}m(\zeta,\eta,x)}, \frac{\mathcal{P}m(\zeta,C\zeta,x)\mathcal{P}m(\eta,C\eta,x)}{1+\mathcal{P}m(C\zeta,C\eta,x)}\biggr\}.$$

Next, we prove a theorem that generalizes the results in [28, 29].

**Theorem 2.2** *Let $(\Omega, \mathcal{P}m)$ be a complete generalized parametric metric space, and let $\alpha : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$. Let $C : \Omega \to \Omega$ be an $\alpha$-admissible mapping satisfying*

(i) *there exists $\zeta_0 \in \Omega$ satisfying $\alpha(\zeta_0, C\zeta_0, x) \ge 1$ and $\alpha(\zeta_0, C^2\zeta_0, x) \ge 1$;*

(ii) *$C$ is an $(\alpha, \psi)$-rational contractive mapping of type-I.*

(iii) *$C$ is continuous, or $\Omega$ is $\alpha$-regular.*

*Then there is a fixed point $\zeta^* \in \Omega$ of $C$, and $\{C^n\zeta_0\}$ converges to $\zeta^*$. Further, if for all $\zeta, \eta \in F(C)$ and $x > 0$, we have $\alpha(\zeta,\eta,x) \ge 1$, then the fixed point of $C$ in $\Omega$ is unique.*

*Proof* Let $\zeta_0 \in \Omega$ satisfy $\alpha(\zeta_0, C\zeta_0, x) \ge 1$ and $\alpha(\zeta_0, C^2\zeta_0, x) \ge 1$. Let us construct the sequence $\{\zeta_n\}$ in $\Omega$ by $\zeta_n = C^n\zeta_0 = C\zeta_{n-1}$ for $n \in \mathbb{N}$. If $\zeta_{n_0} = \zeta_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $\zeta_{n_0}$ is a fixed point of $C$. Thus suppose that $\zeta_n \ne \zeta_{n+1}$ for all $n \in \mathbb{N}$.

As $C$ is $\alpha$-admissible, $\alpha(\zeta_0, C\zeta_0, x) = \alpha(\zeta_0, \zeta_1, x) \ge 1 \Rightarrow \alpha(C\zeta_0, C\zeta_1, x) = \alpha(\zeta_1, \zeta_2, x) \ge 1$, and thus $\alpha(C\zeta_1, C\zeta_2, x) = \alpha(\zeta_2, \zeta_3, x) \ge 1, \ldots$. So by induction we have $\alpha(\zeta_n, \zeta_{n+1}, x) \ge 1$ for all $n \ge 0$.

Similarly, for $\alpha(\zeta_0, C^2\zeta_0, x) \ge 1$, we have $\alpha(\zeta_0, \zeta_2, x) = \alpha(\zeta_0, C^2\zeta_0, x) \ge 1$, $\alpha(C\zeta_0, C\zeta_2, x) = \alpha(\zeta_1, \zeta_3, x) \ge 1$. By induction we get $\alpha(\zeta_n, \zeta_{n+2}, x) \ge 1$ for all $n \ge 0$. By (2.1) with $\zeta = \zeta_n$ and $\eta = \zeta_{n+1}$ we get

$$\begin{aligned}\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x) &\le \mathcal{P}m(C\zeta_n, C\zeta_{n+1},x)\\ &\le \alpha(\zeta_n,\zeta_{n+1},x)\mathcal{P}m(C\zeta_n,C\zeta_{n+1},x)\\ &\le \psi\Bigl(\prod(\zeta_n,\zeta_{n+1},x)\Bigr),\end{aligned}$$

where

$$\begin{aligned}\prod(\zeta_n,\zeta_{n+1},x) &= \max\Big\{\mathcal{P}m(\zeta_n,\zeta_{n+1},x),\mathcal{P}m(\zeta_n,C\zeta_n,x),\mathcal{P}m(\zeta_{n+1},C\zeta_{n+1},x),\\ &\quad \frac{\mathcal{P}m(\zeta_n,C\zeta_n,x)\mathcal{P}m(\zeta_{n+1},C\zeta_{n+1},x)}{1+\mathcal{P}m(\zeta_n,\zeta_{n+1},x)},\frac{\mathcal{P}m(\zeta_n,C\zeta_n,x)\mathcal{P}(\zeta_{n+1},C\zeta_{n+1},x)}{1+\mathcal{P}m(C\zeta_n,C\zeta_{n+1},x)}\Big\}\\ &= \max\Big\{\mathcal{P}m(\zeta_n,\zeta_{n+1},x),\mathcal{P}m(\zeta_n,\zeta_{n+1},x),\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x),\\ &\quad \frac{\mathcal{P}m(\zeta_n,\zeta_{n+1},x)\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)}{1+\mathcal{P}m(\zeta_n,\zeta_{n+1},x)},\frac{\mathcal{P}m(\zeta_n,\zeta_{n+1},x)\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)}{1+\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)}\Big\}\\ &= \max\big\{\mathcal{P}m(\zeta_n,\zeta_{n+1},x),\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)\big\}. \qquad (2.2)\end{aligned}$$

Let $\prod(\zeta_n,\zeta_{n+1},x)=\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)$. Then

$$\begin{aligned}\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x) &\le \psi\Big(\prod(\zeta_n,\zeta_{n+1},x)\Big)\\ &= \psi\big(\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)\big)\\ &\le \mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x), \qquad (2.3)\end{aligned}$$

which is impossible. Hence $\prod(\zeta_n,\zeta_{n+1},x)=\mathcal{P}m(\zeta_n,\zeta_{n+1},x)$ for all $n\in\mathbb{N}$, and

$$\begin{aligned}\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x) &\le \psi\Big(\prod(\zeta_n,\zeta_{n+1},x)\Big)\\ &= \psi(\mathcal{P}m(\zeta_n,\zeta_{n+1},x). \qquad (2.4)\end{aligned}$$

By property of $\psi$ we have

$$\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)\le\mathcal{P}m(\zeta_n,\zeta_{n+1},x) \qquad (2.5)$$

for every $n\in\mathbb{N}$. By (2.4) and (2.5) we have $\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)\le\psi^n\mathcal{P}m(\zeta_0,\zeta_1,x)$ for all $n\in\mathbb{N}$. By property of $\psi$ we have

$$\lim_{n\to\infty}\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)=0. \qquad (2.6)$$

Consider now (2.1) with $\zeta=\zeta_{n-1}$ and $\eta=\zeta_{n+1}$. We have

$$\begin{aligned}\mathcal{P}m(\zeta_n,\zeta_{n+2},x) &= \mathcal{P}m(C\zeta_{n-1},C\zeta_{n+1},x)\\ &\le \alpha(\zeta_{n-1},\zeta_{n+1},x)\mathcal{P}m(C\zeta_{n-1},C\zeta_{n+1},x)\\ &\le \psi\Big(\prod(\zeta_{n-1},\zeta_{n+1},x)\Big), \qquad (2.7)\end{aligned}$$

where

$$\begin{aligned}&\prod(\zeta_{n-1},\zeta_{n+1},x)\\ &\quad=\max\Big\{\mathcal{P}m(\zeta_{n-1},\zeta_{n+1},x),\mathcal{P}m(\zeta_{n-1},C\zeta_{n-1},x),\mathcal{P}m(\zeta_{n+1},C\zeta_{n+1},x),\end{aligned}$$

$$\left.\frac{\mathcal{P}m(\zeta_{n-1},C\zeta_{n-1},x)\mathcal{P}m(\zeta_{n+1},C\zeta_{n+1},x)}{1+\mathcal{P}m(\zeta_{n-1},\zeta_{n+1},x)},\frac{\mathcal{P},m(\zeta_{n-1},C\zeta_{n-1},x)\mathcal{P}m(\zeta_{n+1},C\zeta_{n+1},x)}{1+\mathcal{P}m(C\zeta_{n-1},T\zeta_{n+1},x)}\right\}$$

$$\begin{aligned}&=\max\biggl\{\mathcal{P}m(\zeta_{n-1},\zeta_{n+1},x),\mathcal{P}m(\zeta_{n-1},\zeta_n,x),\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x),\\&\frac{\mathcal{P}m(\zeta_{n-1},\zeta_n,x)\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)}{1+\mathcal{P}m(\zeta_{n-1},\zeta_{n+1},x)},\frac{\mathcal{P}m(\zeta_{n-1},\zeta_n,x)\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)}{1+\mathcal{P}m(\zeta_n,\zeta_{n+2},x)}\biggr\}.\end{aligned}\tag{2.8}$$

By (2.5), $\mathcal{P}m(\zeta_{n+1},\zeta_{n+2},x)<\mathcal{P}m(\zeta_{n-1},\zeta_n,x)$. Define $a_n=\mathcal{P}m(\zeta_n,\zeta_{n+2},x)$ and $b_n=\mathcal{P}m(\zeta_n,\zeta_{n+1},x)$. Then

$$\prod(\zeta_{n-1},\zeta_{n+1},x)=\max\left\{a_{n-1},b_{n-1},\frac{b_{n-1}b_{n+1}}{1+a_{n-1}},\frac{b_{n-1}b_{n+1}}{1+a_n}\right\}.$$

If $\prod(\zeta_{n-1},\zeta_{n+1},x)=b_{n-1}$ or $\frac{b_{n-1}b_{n+1}}{1+a_{n-1}}$ or $\frac{b_{n-1}b_{n+1}}{1+a_n}$, then in (2.8) taking $\limsup$ as $n\to+\infty$, by (2.7) and the upper semicontinuity of $\psi$ we have

$$\begin{aligned}0&\le\limsup_{n\to\infty}a_n\\&\le\limsup_{n\to\infty}\psi\Bigl(\prod(\zeta_{n-1},\zeta_{n+1},x)\Bigr)\\&=\psi\Bigl(\limsup_{n\to\infty}\prod(\zeta_{n-1},\zeta_{n+1},x)\Bigr)\\&=\psi(0)=0,\end{aligned}$$

and hence

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}P(\zeta_n,\zeta_{n+2},x)=0.$$

If $\prod(\zeta_{n-1},\zeta_{n+1},x)=a_{n-1}$, then by (2.8) we have

$$a_n\le\psi(a_{n-1})<a_{n-1}$$

by property of $\psi$. Also, $\{a_n\}$ being a positive decreasing sequence, it converges to some $t\ge0$. Let $t>0$. Then

$$t=\limsup_{n\to\infty}a_n=\limsup_{n\to\infty}\psi(a_{n-1})=\psi\Bigl(\limsup_{n\to\infty}a_{n-1}\Bigr)=\psi(t)<t,$$

a contradiction, and hence

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\mathcal{P}m(\zeta_n,\zeta_{n+2},x)=0.\tag{2.9}$$

For $n\ne m$, we will show that $\zeta_n\ne\zeta_m$. Conversely, let $\zeta_n=\zeta_m$ for some $m,n\in\mathbb{N}$, $n\ne m$. Since $\mathcal{P}m(\zeta_p,\zeta_{p+1},x)>0$ for each $p\in\mathbb{N}$, let $m>n+1$. Taking $\zeta=\zeta_n=\zeta_m$ and $\eta=\zeta_{n+1}=\zeta_{m+1}$ in (2.1) yields

$$\begin{aligned}\mathcal{P}m(\zeta_n,\zeta_{n+1},x)&=\mathcal{P}m(\zeta_n,C\zeta_n,x)=\mathcal{P}m(\zeta_m,C\zeta_m,x)\\&=\mathcal{P}m(C\zeta_{m-1},C\zeta_m,x)\end{aligned}$$

$$\begin{aligned}
&\le \alpha(\zeta_{m-1}, \zeta_m, x)\mathcal{P}m(C\zeta_{m-1}, C\zeta_m, x) \\
&\le \psi\Big(\prod(\zeta_{m-1}, \zeta_m, x)\Big),
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
&\prod(\zeta_{m-1}, \zeta_m, x) \\
&= \max\bigg\{\mathcal{P}m(\zeta_{m-1}, \zeta_m, x), \mathcal{P}m(\zeta_{m-1}, C\zeta_{m-1}, x), \mathcal{P}m(\zeta_m, C\zeta_m, x), \\
&\quad \frac{\mathcal{P}m(\zeta_{m-1}, C\zeta_{m-1}, x)\mathcal{P}m(\zeta_m, C\zeta_m, x)}{1 + \mathcal{P}m(\zeta_{m-1}, \zeta_m, x)}, \frac{\mathcal{P}m(\zeta_{m-1}, C\zeta_{m-1}, x)\mathcal{P}m(\zeta_m, C\zeta_m, x)}{1 + \mathcal{P}m(C\zeta_{m-1}, C\zeta_m, x)}\bigg\} \\
&= \max\bigg\{\mathcal{P}m(\zeta_{m-1}, \zeta_m, x), \mathcal{P}m(\zeta_{m-1}, \zeta_m, x), \mathcal{P}m(\zeta_m, \zeta_{m+1}, x), \\
&\quad \frac{\mathcal{P}m(\zeta_{m-1}, \zeta_m, x)\mathcal{P}m(\zeta_m, \zeta_{m+1}, x)}{1 + \mathcal{P}m(\zeta_{m-1}, \zeta_m, x)}, \frac{\mathcal{P}m(\zeta_{m-1}, \zeta_m, x)\mathcal{P}m(\zeta_m, \zeta_{m+1}, x)}{1 + \mathcal{P}m(\zeta_m, \zeta_{m+1}, x)}\bigg\} \\
&= \max\big\{\mathcal{P}m(\zeta_{m-1}, \zeta_m, x), \mathcal{P}m(\zeta_m, \zeta_{m+1}, x)\big\}.
\end{aligned} \tag{2.11}$$

If $\prod(\zeta_{m-1}, \zeta_m, x) = \mathcal{P}m(\zeta_{m-1}, \zeta_m, x)$, then (2.10) implies

$$\begin{aligned}
\mathcal{P}m(\zeta_n, \zeta_{n+1}, x) &\le \psi\big(\mathcal{P}m(\zeta_{m-1}, \zeta_m, x)\big) \\
&\le \psi^{m-n}\big(\mathcal{P}m(\zeta_n, \zeta_{n+1}, x)\big).
\end{aligned} \tag{2.12}$$

If, on the other hand, $\prod(\zeta_{m-1}, \zeta_m, x) = \mathcal{P}m(\zeta_m, \zeta_{m+1}, x)$, then from (2.10) we have

$$\begin{aligned}
\mathcal{P}m(\zeta_n, \zeta_{n+1}, x) &\le \psi\big(\mathcal{P}m(\zeta_m, \zeta_{m+1}, x)\big) \\
&\le \psi^{m-n+1}\big(\mathcal{P}m(\zeta_n, \zeta_{n+1}, x)\big).
\end{aligned} \tag{2.13}$$

By property of $\psi$, from (2.12) and (2.13) we have

$$\mathcal{P}m(\zeta_n, \zeta_{n+1}, x) < \mathcal{P}m(\zeta_n, \zeta_{n+1}, x),$$

which is true.

To prove that $\{\zeta_n\}$ is a Cauchy sequence, let $k \ge 3$, $k \in \mathbb{N}$, as the proof for $k = 1, 2$ is already done.

Case 1: Let $k = 2m + 1$ and $m \ge 1$. Then by (iii) of Definition 1.3

$$\begin{aligned}
\mathcal{P}m(\zeta_n, \zeta_{n+k}, x) &= \mathcal{P}m(\zeta_n, \zeta_{n+2m+1}, x) \\
&\le \mathcal{P}m(\zeta_n, \zeta_{n+1}, x) + \mathcal{P}m(\zeta_{n+1}, \zeta_{n+2}, x) + \cdots + \mathcal{P}m(\zeta_{n+2m}, \zeta_{n+2m+1}, x) \\
&\le \sum_{p=n}^{n+2m} \psi^p\big(\mathcal{P}m(\zeta_0, \zeta_1, x)\big) \\
&\le \sum_{p=n}^{+\infty} \psi^p\big(\mathcal{P}m(\zeta_0, \zeta_1, x)\big) \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{2.14}$$

Case 2: Let $k = 2m$ and $m \geq 2$. Then by (iii) of Definition 1.3

$$\begin{aligned}\mathcal{P}m(\zeta_n,\zeta_{n+k},x) &= \mathcal{P}m(\zeta_n,\zeta_{n+2m},x)\\ &\leq \mathcal{P}m(\zeta_n,\zeta_{n+2},x) + \mathcal{P}m(\zeta_{n+2},\zeta_{n+3},x) + \cdots + \mathcal{P}m(\zeta_{n+2m-1},\zeta_{n+2m},x)\\ &\leq \mathcal{P}m(\zeta_n,\zeta_{n+2},x) + \sum_{p=n+2}^{n+2m-1} \psi^p\big(\mathcal{P}m(\zeta_0,\zeta_1,x)\big)\\ &\leq \mathcal{P}m(\zeta_n,\zeta_{n+2},x) + \sum_{p=n}^{+\infty} \psi^p\big(\mathcal{P}m(\zeta_0,\zeta_1,x)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned}\tag{2.15}$$

Since $\lim_{n\to\infty} a_n = 0$ because of (2.9), in both cases above, we have $\lim_{n\to\infty} \mathcal{P}m(\zeta_n,\zeta_{n+k}, x) = 0$ for all $k \geq 3$. This shows that $\{\zeta_n\}$ is a Cauchy sequence in $(\Omega,d)$. By the completeness of $(\Omega,d)$ we have $\zeta^* \in \Omega$ satisfying

$$\lim_{n\to\infty} \mathcal{P}m\big(\zeta_n,\zeta^*,x\big) = 0. \tag{2.16}$$

Since $C$ is a continuous function, from (2.16) we get

$$\lim_{n\to\infty} \mathcal{P}m\big(C\zeta_n,C\zeta^*,x\big) = \lim_{n\to\infty} \mathcal{P}m\big(\zeta_{n+1},C\zeta^*,x\big) = 0.$$

By Proposition 1.8, $\zeta^* = C\zeta^*$, and hence $C$ has a fixed point $\zeta^*$.

Next, considering regular $\Omega$, there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ satisfying $\alpha(\zeta_{n_k-1}, \zeta^*,x) \geq 1$ for all $k \in \mathbb{N}$. From (2.1) with $\zeta = \zeta_{n_k}$ and $\eta = \zeta^*$ we have

$$\begin{aligned}\mathcal{P}m\big(\zeta_{n_k+1},C\zeta^*,x\big) &= \mathcal{P}m\big(C\zeta_{n_k},C\zeta^*,x\big)\\ &\leq \alpha\big(\zeta_{n_k},\zeta^*,x\big)\mathcal{P}m\big(C\zeta_{n_k},C\zeta^*,x\big)\\ &\leq \psi\Big(\prod\big(\zeta_{n_k},\zeta^*,x\big)\Big),\end{aligned}\tag{2.17}$$

where

$$\prod\big(\zeta_{n_k},\zeta^*,x\big) \tag{2.18}$$

$$= \max\bigg\{\mathcal{P}m\big(\zeta_{n_k},\zeta^*,x\big), \mathcal{P}m\big(\zeta_{n_k},C\zeta_{n_k},x\big), \mathcal{P}m\big(\zeta^*,C\zeta^*,x\big), \frac{\mathcal{P}m(\zeta_{n_k},C\zeta_{n_k},x)\mathcal{P}m(\zeta^*,C\zeta^*,x)}{1+\mathcal{P}m(\zeta_{n_k},\zeta^*,x)}, \frac{\mathcal{P}m(\zeta_{n_k},C\zeta_{n_k},x)\mathcal{P}m(\zeta^*,C\zeta^*,x)}{1+\mathcal{P}m(C\zeta_{n_k},C\zeta^*,x)}\bigg\}$$

$$= \max\bigg\{\mathcal{P}m\big(\zeta_{n_k},\zeta^*,x\big), \mathcal{P}m\big(\zeta_{n_k},\zeta_{n_k+1},x\big), \mathcal{P}m\big(\zeta^*,T\zeta^*,x\big) \frac{\mathcal{P}m(\zeta_{n_k},C\zeta_{n_k+1},x)\mathcal{P}m(\zeta^*,C\zeta^*,x)}{1+\mathcal{P}m(\zeta_{n_k},\zeta^*,x)}, \frac{\mathcal{P}m(\zeta_{n_k},\zeta_{n_k+1},x)\mathcal{P}m(\zeta^*,C\zeta^*,x)}{1+\mathcal{P}m(\zeta_{n_k+1},C\zeta^*,x)}\bigg\}. \tag{2.19}$$

Taking the limit as $k \to \infty$ in (2.19), we get $\prod(\zeta_{n_k},\zeta^*,x) = P(\zeta^*,C\zeta^*,x)$. Taking the limit as $k \to \infty$ in inequality (2.17), we get

$$\mathcal{P}m\big(\zeta^*,C\zeta^*,x\big) \leq \psi\big(\mathcal{P}m(\zeta^*,C\zeta^*,x)\big) \leq \mathcal{P}m\big(\zeta^*,C\zeta^*,x\big),$$

which implies $\zeta^* = C\zeta^*$, that is, $C$ has a fixed point $\zeta^*$.

Suppose $\zeta^*$ and $\eta^*$ are two fixed points of $C$ and $\zeta^* \neq \eta^*$. Then $\alpha(\zeta^*, \eta^*, x) \geq 1$. Taking $\zeta = \zeta^*$ and $\eta = \eta^*$ in (2.1), we get

$$\begin{aligned}\mathcal{P}m(\zeta^*, \eta^*, x) &= \mathcal{P}m(C\zeta^*, C\eta^*, x) \\ &\leq \alpha(\zeta^*, \eta^*, x)\mathcal{P}m(T\zeta^*, C\eta^*, x) \\ &\leq \psi\Big(\prod(\zeta^*, \eta^*, x)\Big),\end{aligned}$$

where

$$\begin{aligned}\prod(\zeta^*, \eta^*, x) &= \max\bigg\{\mathcal{P}m(\zeta^*, \eta^*, x), \mathcal{P}m(\zeta^*, C\zeta^*, x), \mathcal{P}m(\eta^*, C\eta^*, x), \\ &\qquad \frac{\mathcal{P}m(\zeta^*, C\zeta^*, x)\mathcal{P}m(\eta^*, C\eta^*, x)}{1 + \mathcal{P}m(\zeta^*, \zeta^*, x)}, \frac{\mathcal{P}m(\zeta^*, C\zeta^*, x)\mathcal{P}m(\eta*, C\eta^*, x)}{1 + \mathcal{P}m(C\zeta^*, C\eta^*, x)}\bigg\} \\ &= \mathcal{P}m(\zeta^*, \eta^*, x).\end{aligned} \tag{2.20}$$

Hence we get $\mathcal{P}m(\zeta^*, \eta^*, x) \leq \psi(\mathcal{P}m(\zeta^*, \eta^*, x)) < \mathcal{P}m(\zeta^*, \eta^*, x)$, which is possible only if $\mathcal{P}m(\zeta^*, \eta^*, x) = 0$, that is, $\zeta^* = \eta^*$. So, a fixed point of $C$ is unique. □

**Definition 2.3** Let $(\Omega, \mathcal{P}m)$ be a generalized parametric metric space, and let $C : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$. We say that $C$ is an $(\alpha, \psi)$-rational contractive mapping of type-*II* if for all $\zeta, \eta \in \Omega$ and $\psi \in \Psi$,

$$\alpha(\zeta, \eta, x)P(C\zeta, C\eta, x) \leq \psi\Big(\prod(\zeta, \eta, x)\Big), \tag{2.21}$$

where

$$\begin{aligned}\prod(\zeta, \eta, x) = \max\bigg\{&\mathcal{P}m(\zeta, \eta, x), \mathcal{P}m(\zeta, C\zeta, x), \mathcal{P}m(\eta, C\eta, x), \\ &\frac{\mathcal{P}m(\zeta, C\zeta, x)\mathcal{P}m(\eta, C\eta, x)}{1 + \mathcal{P}m(\zeta, \eta, x) + \mathcal{P}m(\zeta, C\eta, x) + \mathcal{P}m(\eta, C\zeta, x)}, \\ &\frac{\mathcal{P}m(\zeta, C\eta, x)\mathcal{P}m(\zeta, \eta, x)}{1 + \mathcal{P}m(\zeta, C\zeta, x) + \mathcal{P}m(\eta, C\zeta, x) + \mathcal{P}m(\eta, C\eta, x)}\bigg\}.\end{aligned}$$

**Theorem 2.4** *Let $(\Omega, \mathcal{P}m)$ be a generalized parametric metric space, and let $C : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$. Let $C$ be an $\alpha$-admissible mapping satisfying*

(i) *there exists $\zeta_0 \in \Omega$ satisfying $\alpha(\zeta_0, C\zeta_0, x) \geq 1$ and $\alpha(\zeta_0, C^2\zeta_0, x) \geq 1$;*
(ii) *$C$ is $(\alpha, \psi)$-rational contractive mapping of type-II;*
(iii) *$C$ is continuous, or $\Omega$ is $\alpha$-regular.*

*Then there is a fixed point $\zeta^* \in \Omega$ of $C$, and $\{C^n\zeta_0\}$ converges to $\zeta^*$. Further, if $\alpha(\zeta, \eta, x) \geq 1$ for all $\zeta, \eta \in F(C)$, then $C$ has a unique fixed point in $\Omega$.*

*Proof* Following the proof of Theorem 2.2, we can complete the proof. □

*Example* 2.5 Consider $\Omega = [0, +\infty)$ and

$$\mathcal{P}m(\zeta, \eta, x) = \begin{cases} x(\zeta + \eta)^2, & \zeta \neq \eta, \\ 0, & \zeta = \eta, \end{cases}$$

for all $\zeta,\eta \in \Omega$ and $x > 0$. Define $C : \Omega \to \Omega$ by

$$C\zeta = \begin{cases} \frac{1}{8}\zeta^2, & \zeta \in [0,1), \\ \frac{1}{8}\zeta, & \zeta \in [1,2), \\ \frac{1}{32}, & \zeta \in [2,\infty). \end{cases}$$

Also, define $\psi(t) = \frac{t}{2}$ and $\alpha(\zeta,\eta,x) = 1$ for $\zeta,\eta \in \Omega$ and $x > 0$. Clearly, $(\Omega,\mathcal{P}m)$ is a complete generalized parametric metric space.

Considering the following:

(i) Let $\zeta,\eta \in [0,1)$. Then

$$\begin{aligned} \alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) &= x\left(\frac{1}{8}\zeta^2 + \frac{1}{8}\eta^2\right)^2 = \frac{1}{64}x\left(\zeta^2+\eta^2\right)^2 \\ &\le \frac{1}{2}\left\{x(\zeta+\eta)^2\right\} = \psi\left(\mathcal{P}m(\zeta,\eta,x)\right) \\ &\le \psi\left(\prod(\zeta,\eta,x)\right). \end{aligned}$$

(ii) Let $\zeta,\eta \in [1,2)$ with $\zeta \le \eta$. Then

$$\begin{aligned} \alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) &= x\left(\frac{1}{8}\zeta + \frac{1}{8}\eta\right)^2 = \frac{1}{64}x(\zeta+\eta)^2 \\ &\le \frac{1}{2}x(\zeta+\eta)^2 = \psi\left(\mathcal{P}m(\zeta,\eta,x)\right) \\ &\le \psi\left(\prod(\zeta,\eta,x)\right). \end{aligned}$$

(iii) Let $\zeta,\eta \in [2,+\infty)$ with $\zeta \le \eta$. Then

$$\begin{aligned} \alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) &= x\left(\frac{1}{32} + \frac{1}{32}\right) = \frac{1}{16}x \le \frac{1}{8}x \\ &= \frac{1}{2}\left\{\frac{1}{4}(1+1)^2\right\} = \frac{1}{2}\mathcal{P}m(\zeta,\eta,x) \\ &\le \frac{1}{2}\left(\prod(\zeta,\eta,x)\right) = \psi\left(\prod(\zeta,\eta,x)\right). \end{aligned}$$

(iv) Let $\zeta \in [0,1)$ and $\eta \in [1,2)$ (clearly, $\zeta \le \eta$). Then

$$\begin{aligned} \alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) &= x\left(\frac{1}{8}\zeta^2 + \frac{1}{8}\eta\right)^2 \\ &\le x\left(\frac{1}{8}\zeta^2 + \frac{1}{8}\eta^2\right)^2 = \frac{1}{64}x\left(\zeta^2+\eta^2\right)^2 \\ &\le \frac{1}{2}\left\{x(\zeta+\eta)^2\right\} = \psi\left(\mathcal{P}m(\zeta,\eta,x)\right) \\ &\le \frac{1}{2}\left(\prod(\zeta,\eta,x)\right) = \psi\left(\prod(\zeta,\eta,x)\right). \end{aligned}$$

(v) Let $\zeta \in [0,1)$ and $\eta \in [2,+\infty)$ (clearly, $\zeta \leq \eta$). Then

$$\begin{aligned}\alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) &= x\left(\frac{1}{8}\zeta^2 + \frac{1}{32}\right)^2\\ &\leq x\left(\frac{1}{8}\zeta + \frac{1}{8}\eta\right)^2 = \frac{1}{64}x(\zeta+\eta)^2\\ &\leq \frac{1}{2}x(\zeta+\eta)^2 = \frac{1}{2}\big(\mathcal{P}m(\zeta,\eta,x)\big)\\ &\leq \frac{1}{2}\Big(\prod(\zeta,\eta,x)\Big) = \psi\Big(\prod(\zeta,\eta,x)\Big).\end{aligned}$$

(vi) Let $\zeta \in [0,1)$ and $\eta \in [2,+\infty)$ (clearly, $\zeta \leq \eta$). Then

$$\begin{aligned}\alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) &= x\left(\frac{1}{8}\zeta + \frac{1}{32}\right)^2\\ &\leq x\left(\frac{1}{8}\zeta + \frac{1}{8}\eta\right)^2 = \frac{1}{64}x(\zeta+\eta)^2\\ &\leq \frac{1}{2}x(\zeta+\eta)^2 = \frac{1}{2}\big(\mathcal{P}m(\zeta,\eta,x)\big)\\ &\leq \frac{1}{2}\Big(\prod(\zeta,\eta,x)\Big) = \psi\Big(\prod(\zeta,\eta,x)\Big).\end{aligned}$$

Therefore

$$\alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) \leq \psi\Big(\prod(\zeta,\eta,x)\Big)$$

for all $\zeta,\eta \in \Omega$ with $\zeta \leq \eta$ and all $x > 0$. Hence all the conditions of Theorem 2.2 hold, and $C$ has a unique fixed point.

## 3 Consequences

Here we derive various results in the literature as corollaries for generalized parametric metric spaces. In particular, we deduce the results of Aydi et al. [30] and Karapinar [31]. Now we give the following definitions.

**Definition 3.1** Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $C:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\times(0,+\infty)\to[0,+\infty)$. We call $C$ a generalized $(\alpha,\psi)$- contractive mapping of type $I$ if for all $\zeta,\eta\in\Omega$ and $\psi\in\Psi$,

$$\alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) \leq \psi\Big(\prod(\zeta,\eta,x)\Big), \quad x>0, \tag{3.1}$$

where

$$\prod(\zeta,\eta,x) = \max\big\{\mathcal{P}m(\zeta,\eta,x),\mathcal{P}m(\zeta,C\zeta,x),\mathcal{P}m(\eta,C\eta,x)\big\}. \tag{3.2}$$

**Definition 3.2** Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $C:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\times(0,+\infty)\to[0,+\infty)$ be mappings. We call $C$ a generalized $(\alpha,\psi)$- contractive mapping of type-$II$ if for all $\zeta,\eta\in\Omega$ and $\psi\in\Psi$,

$$\alpha(\zeta,\eta,x)\mathcal{P}m(C\zeta,C\eta,x) \leq \psi\big(N(\zeta,\eta,x)\big), \quad x>0, \tag{3.3}$$

where

$$N(\zeta,\eta,x) = \max\left\{\mathcal{P}m(\zeta,\eta,x), \frac{\mathcal{P}m(\zeta,C\zeta,x),\mathcal{P}m(\eta,C\eta,x)}{2}\right\}. \tag{3.4}$$

Now we state following theorem as a consequence of our Theorem 2.2, which extends the main results of Aydi et al. [30] (Theorems 15 and 17) and Karapinar [31] to the more general setting of generalized parametric metric spaces.

**Theorem 3.3** *Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $C:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\times(0,+\infty)\to[0,+\infty)$. Let $C$ be an $\alpha$-admissible mapping satisfying*

(i) *there exists $\zeta_0\in\Omega$ satisfying $\alpha(\zeta_0,C\zeta_0,x)\geq 1$ and $\alpha(\zeta_0,C^2\zeta_0,x)\geq 1$;*
(ii) *$C$ is a generalized $(\alpha,\psi)$-contractive mapping of type I;*
(iii) *$C$ is continuous, or $\Omega$ is $\alpha$-regular.*

*Then there exists $\mu$ in $\Omega$ satisfying $C\mu=\mu$.*

**Theorem 3.4** (see [30], Theorems 16 and 18) *Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $C:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\times(0,+\infty)\to[0,+\infty)$. Let $C$ be an $\alpha$-admissible mapping satisfying*

(i) *there exists $\zeta_0\in\Omega$ satisfying $\alpha(\zeta_0,C\zeta_0,x)\geq 1$ and $\alpha(\zeta_0,C^2\zeta_0,x)\geq 1$;*
(ii) *$C$ is a generalized $(\alpha,\psi)$-contractive mapping of type II;*
(iii) *$C$ is continuous or $\Omega$ is $\alpha$-regular.*

*Then there exists $\mu$ in $\Omega$ satisfying $C\mu=\mu$.*

Replace the continuity condition by "if $\{x_n\}$ is a sequence in $\Omega$ such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n$ and $x_n\to x\in\Omega$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x)\geq 1$, for all $k$". Then Theorem 3.3 remains true.

**Corollary 3.5** *Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $C:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\times(0,+\infty)\to[0,+\infty)$. Let $\psi\in\Psi$ be a function such that*

$$\mathcal{P}m(C\zeta,C\eta,x)\leq\psi\Big(\prod(\zeta,\eta,x)\Big),\quad x>0,$$

*for all $\zeta,\eta\in\Omega$. Then there exists a unique fixed point in $C$.*

*Proof* Take $\alpha(\zeta,\eta,x)=1$ in the proof of Theorem 2.2.
By taking $\psi(s)=\lambda s$, in Corollary 3.5, we have □

**Corollary 3.6** *Let $(\Omega,\mathcal{P}m)$ be a generalized parametric metric space, and let $C:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\times(0,+\infty)\to[0,+\infty)$. Let $\psi\in\Psi$ be a function such that*

$$\mathcal{P}m(C\zeta,C\eta,x)\leq\lambda\prod(\zeta,\eta,x)$$

*for all $\zeta,\eta\in\Omega$ and $x>0$. Then there exists a unique fixed point for $C$.*

**Definition 3.7** Define a partially ordered set $(\Omega,\preceq)$ and a mapping $C:\Omega\to\Omega$. We say that with respect to $\preceq$, $C$ is nondecreasing if $\zeta,\eta\in\Omega$ with $\zeta\preceq\eta$ implies $C\zeta\preceq C\eta$. A sequence $\zeta_n\in\Omega$ is called nondecreasing with respect to $\preceq$ if $\zeta_n\preceq\zeta_{n+1}$ for all $n$.

**Definition 3.8** Let $(\Omega, \mathcal{P}m)$ be a generalized parametric metric space, let $C : \Omega \to \Omega$, and let $(\Omega, \preceq)$ be a partially ordered set. We say that $(\Omega, \preceq, \mathcal{P}m)$ is regular if for every nondecreasing sequence $\zeta_n \in \Omega$ such that $\zeta_n$ converges to $\zeta \in \Omega$ as $n \to \infty$, there exists a subsequence $\zeta_{n_k}$ of $\zeta_n$ satisfying $\zeta_{n_k} \preceq \zeta$ for all $k$.

**Corollary 3.9** *Let $(\Omega, \mathcal{P}m)$ be a generalized parametric metric space, and let $C : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$. Let $(\Omega, \preceq)$ be a partially ordered set and suppose $(\Omega, \mathcal{P}m)$ is complete. Let $C$ be a nondecreasing mapping with respect to $\preceq$. Let $\psi \in \Psi$ be a function satisfying*

$$\mathcal{P}m(C\zeta, C\eta, x) \leq \psi\Big(\prod(\zeta, \eta, x)\Big), \quad x > 0,$$

*for all $\zeta, \eta \in \Omega$ with $\zeta \preceq \eta$. Also assume that the following conditions are satisfied.*

(i) *there exists $\zeta_0 \in \Omega$ satisfying $\zeta_0 \preceq C\zeta_0$ and $\zeta_0 \preceq C^2\zeta_0$;*
(ii) *$C$ is continuous, or $(\Omega, \preceq, \mathcal{P}m)$ is regular.*

*Then there exists a fixed point for $C$.*

*Proof* Let $\alpha : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$ be defined by $\alpha(\zeta, \eta, x) = 1$ for $x > 0$ if $\zeta \preceq \eta$ or $\zeta \succeq \eta$ and $\alpha(\zeta, \eta, x) = 0$ otherwise. As the conditions of Theorem 2.2 are satisfied, a fixed point of $C$ exists. □

## 4 Generalised fuzzy metric space

Here we establish relations of a generalized parametric metric space and a generalized fuzzy metric space.

**Definition 4.1** ([32]) Let $* : [0,1] \times [0,1] \to [0,1]$ be a binary operation that is commutative and associative. $*$ is called a continuous $t$-norm if

(i) $*$ is continuous;
(ii) for all $p \in [0,1]$, $p * 1 = p$;
(iii) If $p \leq r$, $q \leq s$, then $p * q \leq r * s$, where $p, q, r, s \in [0,1]$.

**Definition 4.2** ([2]) Let $\Omega$ be an arbitrary set, let $*$ be a continuous $t$-norm, and let $\prod$ be a fuzzy set on $\Omega^2 \times (0, +\infty)$. The triple $(\Omega, \prod, *)$ is called a fuzzy metric space if

(i) $\prod(\zeta, \eta, t) > 0$;
(ii) $\prod(\zeta, \eta, t) = 1$ for all $t > 0$ if and only if $\zeta = \eta$;
(iii) $\prod(\zeta, \eta, t) = \prod(\eta, \zeta, t)$;
(iv) $\prod(\zeta, \eta, t) * \prod(\eta, \xi, u) \leq \prod(\zeta, \xi, t + u)$;
(v) $\prod(\zeta, \eta, .) : (0, +\infty) \to [0,1]$ is continuous.

for all $\zeta, \eta, \xi \in \Omega$ and $t, u > 0$; $\prod(\zeta, \eta, t)$ expresses the rate of nearness of $\zeta$ and $\eta$ with respect to $t$.

**Definition 4.3** Let $\Omega$ be a nonempty set, let $*$ be a continuous $t$-norm, and let $\Delta$ be a fuzzy set on $\Omega \times \Omega \times (0, +\infty)$. Then the triple $(\Omega, \Delta, *)$ is called a generalized fuzzy metric space if it satisfies

(i) $\Delta(\zeta, \eta, t) > 0$;
(ii) $\Delta(\zeta, \eta, t) = 1$ if and only if $\zeta = \eta$;
(iii) $\Delta(\zeta, \eta, t) = \Delta(\eta, \zeta, t)$;

(iv) $\Delta(\zeta,\mu,u) * \Delta(\mu,\lambda,\nu) * \Delta(\lambda,\eta,t) \leq \Delta(\zeta,\zeta,u+\nu+t)$;
(v) $\Delta(\zeta,\eta,.) : (0,+\infty) \to (0,1]$ is left continuous
for all $\zeta,\eta \in \Omega$, distinct $\mu,\lambda \in \Omega - \{\zeta,\eta\}$, and $t,u,\nu > 0$.

**Definition 4.4** Let $(\Omega,\Delta,*)$ be a generalized fuzzy metric space. Then
(i) a sequence $\{\zeta_n\}$ converges to $\zeta \in \Omega$ if and only if $\lim_{n\to\infty}\Delta(\zeta_n,\zeta,t) = 1$ for all $t > 0$.
(ii) a sequence $\{\zeta_n\}$ in $\Omega$ is a Cauchy sequence if and only if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0$ such that $\Delta(\zeta_n,\zeta_m,t) > 1-\varepsilon$ for all $m,n \geq n_0$,
(iii) If every Cauchy sequence converges to some $\zeta \in \Omega$, then the generalized fuzzy metric space is said to be complete.

**Definition 4.5** Let $(\Omega,\Delta,*)$ be a generalized fuzzy metric space. The a generalized fuzzy metric $\Delta$ is said to be rectangular if

$$\frac{1}{\Delta(\zeta,\eta,t)} - 1 \leq \frac{1}{\Delta(\zeta,\mu,t)} - 1 + \frac{1}{\Delta(\mu,\lambda,t)} - 1 + \frac{1}{\Delta(\lambda,\eta,t)} - 1$$

for all $\zeta,\eta \in \Omega$ and distinct $\mu,\lambda \in \Omega - \{\zeta,\eta\}$ and $t > 0$.

*Example* 4.6 Let $(\Omega,d)$ be a generalized metric space, and let $\Delta : \Omega \times \Omega \times (0,+\infty) \to (0,+\infty)$ be such that

$$\Delta(\zeta,\eta,t) = \frac{t}{t + d(\zeta,\eta)}.$$

Let $p * q = \min\{p,q\}$. Then $(\Omega,\Delta,*)$ is a generalized fuzzy metric space, and $\Delta$ is a rectangular fuzzy metric.

*Remark* 4.7 Note that $\mathcal{P}m(\zeta,\eta,t) = \frac{1}{\Delta(\zeta,\eta,t)} - 1$ is a generalized parametric metric space, where $\Delta$ is a rectangular fuzzy metric.

**Definition 4.8** Let $(\Omega,\Delta,*)$ be a complete generalized fuzzy metric space, let $\Delta$ be a rectangular fuzzy metric on $\Omega$, and let $\alpha : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$ and $C : \Omega \to \Omega$. The mapping $C$ is said to be an $(\alpha,\psi)$-rational contractive mapping of type I if there exists a function $\psi \in \Psi$ satisfying

$$\alpha(\zeta,\eta,t)\Delta(C\zeta,C\eta,t) \leq \psi\Bigl(\prod(\zeta,\eta,t)\Bigr), \quad t > 0, \tag{4.1}$$

where

$$\prod(\zeta,\eta,t) = \max\left\{\frac{1}{\Delta(\zeta,\eta,t)} - 1, \frac{1}{\Delta(\zeta,C\zeta,t)} - 1, \frac{1}{\Delta(\eta,C\eta,t)} - 1, \frac{(\frac{1}{\Delta(\zeta,C\zeta,t)} - 1)(\frac{1}{\Delta(\eta,C\eta,t)} - 1)}{\frac{1}{\Delta(\zeta,\eta,t)}}, \frac{(\frac{1}{\Delta(\zeta,C\zeta,t)} - 1)(\frac{1}{\Delta(\eta,C\eta,t)} - 1)}{\frac{1}{\Delta(C\zeta,C\eta,t)}}\right\}$$

for all $\zeta,\eta \in \Omega$.

**Theorem 4.9** *Let $(\Omega,\Delta,*)$ be a complete generalized fuzzy metric space, let $\Delta$ be a rectangular fuzzy metric on $\Omega$. Suppose that mappings $\alpha : \Omega \times \Omega \times (0,+\infty) \to [0,+\infty)$ and $C : \Omega \to \Omega$ satisfy*

(i) *$C$ is $\alpha$-admissible*;
(ii) *$C$ is $(\alpha, \psi)$-rational contractive mapping of type I*;
(iii) *there exists $\zeta_0 \in X$ satisfying $\alpha(\zeta_0, C\zeta_0, t) \geq 1$ and $\alpha(\zeta_0, C^2\zeta_0, t) \geq 1$*;
(iv) *$C$ is continuous, or $\Omega$ is $\alpha$-regular.*

*Then $\{C^n\zeta_0\}$ converges to a fixed point $\zeta^* \in \Omega$ of $C$. Also, if for all $\zeta, \eta \in F(C)$, we have $\alpha(\zeta, \eta, t) \geq 1$, $t > 0$, then the fixed point of $C$ in $\Omega$ is unique.*

**Definition 4.10** Let $(\Omega, \Delta, *)$ be a complete generalized fuzzy metric space, let $\Delta$ be a triangular fuzzy metric on $\Omega$, and let $\alpha : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$ and $C : \Omega \to \Omega$. The mapping $C$ is said to be an $(\alpha, \psi)$-rational contractive mapping of type II if there exists a function $\psi \in \Psi$ such that

$$\alpha(\zeta, \eta, t)\Delta(C\zeta, C\eta, t) \leq \psi\Big(\prod(\zeta, \eta, t)\Big) \quad t > 0, \tag{4.2}$$

where

$$\begin{aligned}\prod(\zeta, \eta, t) = \max\biggl\{&\frac{1}{\Delta(\zeta, \eta, t)} - 1, \frac{1}{\Delta(\zeta, C\zeta, t)} - 1, \frac{1}{\Delta(\eta, C\eta, t)} - 1,\\ &\frac{(\frac{1}{\Delta(\zeta,C\zeta,t)} - 1)(\frac{1}{\Delta(\eta,C\eta,t)} - 1)}{\frac{1}{\Delta(\zeta,\eta,t)} + \frac{1}{\Delta(\zeta,C\eta,t)} + \frac{1}{\Delta(\eta,C\zeta,t)} - 2}, \frac{(\frac{1}{\Delta(\zeta,C\eta,t)} - 1)(\frac{1}{\Delta(\zeta,\eta,t)} - 1)}{\frac{1}{\Delta(\zeta,C\zeta,t)} + \frac{1}{\Delta(\eta,C\zeta,t)} + \frac{1}{\Delta(\eta,C\eta,t)} - 2}\biggr\}.\end{aligned}$$

**Theorem 4.11** *Let $(\Omega, \Delta, *)$ be a complete generalized fuzzy metric space, let $\Delta$ be a triangular fuzzy metric on $\Omega$. Suppose that mappings $\alpha : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$ and $C : \Omega \to \Omega$ satisfy*

(i) *$C$ is $\alpha$-admissible*;
(ii) *$C$ is an $(\alpha, \psi)$-rational contractive mapping of typeII*;
(iii) *there exists $\zeta_0 \in \Omega$ satisfying $\alpha(\zeta_0, C\zeta_0, t) \geq 1$ and $\alpha(\zeta_0, C^2\zeta_0, t) \geq 1$*;
(iv) *$C$ is continuous, or $\Omega$ is $\alpha$-regular.*

*Then $\{C^n\zeta_0\}$ converges to a fixed point $\zeta^* \in \Omega$ of $C$ Also, if for all $\zeta, \eta \in F(C)$, we have $\alpha(\zeta, \eta, t) \geq 1$, $t > 0$, then the fixed point of $C$ in $\Omega$ is unique.*

*Remark* 4.12 We can obtain results similar to Corollary 3.9 for fuzzy partially ordered generalized metric spaces.

**Acknowledgements**
The first author Thounaojam Stephen would like to thank Council of Scientific and Industrial Research (CSIR), New Delhi, for providing financial support in the form of research fellowship. The third author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

**Funding**
Not applicable.

**Availability of data and materials**
Not applicable

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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Received: 28 May 2021 Accepted: 8 July 2021 Published online: 23 July 2021

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