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Wirtinger integral inequalities for pseudo-integrals and pseudo-additive measure

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Abstract

The main purpose of this paper is to show Wirtinger type inequalities for the pseudo-integral. We are concerned with pseudo-integrals based on the following three canonical cases: in the first case, the real semiring with pseudo-operation is generated by a strictly monotone continuous function *g*; in the second case, the pseudo-operations include a pseudo-multiplication and a power arithmetic addition; in the last case, ⊕-measures are interval-valued. Examples are given to illustrate these equalities.

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1 Introduction

It is well known that if $u \in C^1([0,T],\mathbb{R})$, $u(0) = u(T)$ and $\int_0^T u(t)\,dt = 0$, then the following inequality holds:

$$\int_0^T \left|u(t)\right|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T \left|u'(t)\right|^2 dt, \tag{1}$$

with equality if and only if $u(t) = A\sin(\frac{2\pi t}{T}) + B\cos(\frac{2\pi t}{T})$, where A and B are constants. If $T = 2\pi$, then Inequality (1) is known in the literature as Wirtinger's inequality; see [1, 2]. The Wirtinger's inequality and its generalizations have wide applications in p-Laplacian systems [3, 4], time-delay systems [5–7], Lurie systems [8], stability criteria [9, 10], discrete-time systems [11, 12] and so on.

Pseudo-analysis is chosen as the research background of this paper, because it presents a contemporary mathematical theory which has been successfully applied in many practical fields. The last decade has shown an increasing research activity on pseudo-analysis [13–17]. As a method to promote the classical mathematical analysis, pseudo-analysis extends the concept of traditional operation to pseudo-operation including pseudo-addition and pseudo-multiplication. Later on, the researchers present pseudo-integrals [16] based on the important theory of pseudo-analysis, namely pseudo-operations and interval-valued measure. The pseudo-integral is now emerging as one of the hottest mathematical subjects, many scholars have studied its application and promotion on generalizations of integral inequalities [18–22].

Since the Wirtinger inequality is one of the most important inequalities, this paper studies three Wirtinger type integral inequalities in pseudo-analysis environment. In the first case, we consider a Wirtinger type integral inequality for an applied pseudo-integration equipped with a monotonic continuous mapping g. In the second case, we study a Wirtinger type integral inequality for the pseudo-integration adopting a semiring $([0.1], \sup, \odot)$ to design the theory. In the last case, we show a Wirtinger type integral inequality with respect to interval-valued $\oplus$-measures. Moreover, several examples are provided for validation.

The paper is organized as follows: Sect. 2 contains some of preliminaries. Section 3 provides generalizations of two Wirtinger type inequalities to pseudo-integrals on a g-semiring. Section 4 proves a Wirtinger integral inequality for the pseudo-integral of a real-valued function with respect to the pseudo-additive measure. The conclusion is shown in Sect. 5.

2 Preliminaries

In this section, we review some basic notions about pseudo-operations and pseudo-integrals, the relevant literature includes [15, 23].

An interval $[c,d]$ is a closed subset of $[-\infty,+\infty]$. The complete order in $[c,d]$ is expressed as $\preceq$ which can be the usual order of the real line or can be another order. Also, the total order $\preceq$ is closely connected to the choice of pseudo-addition. Namely, for the semirings of the first and the third class, i.e., for the semirings with idempotent pseudo-addition, total order is induced by the following

$$x \preceq y \quad \text{if and only if} \quad x \oplus y = x.$$

For the semiring of the second class given by a generator g, total order is given by

$$x \preceq y \quad \text{if and only if} \quad g(x) \leq g(y).$$

Let $[c,d]^+ = \{x | x \in [c,d], \mathbf{0} \preceq x\}$. The structure $([c,d], \oplus, \odot)$, endowed with a pseudo-addition $\oplus$ and a pseudo-multiplication $\odot$, is a semiring (see [16, 24]). The equation $\mathbf{0} \odot i = \mathbf{0}$ holds.

We consider the semiring $([c,d], \oplus, \odot)$ in two situations. In the first situation, let $g : [c,d] \to [0,\infty]$ be a monotone and continuous mapping and let

$$x \oplus y = g^{-1}(g(x) + g(y)), \qquad x \odot y = g^{-1}(g(x) \cdot g(y)) \quad \text{and} \quad x_{\odot}^{(n)} = g^{-1}(g^n(x)).$$

The pseudo-integral for a function $p : [a,b] \to [c,d]$ is expressed as

$$\int_{[c,d]}^{\oplus} p(t)\, dt = g^{-1}\left(\int_c^d g\bigl(p(t)\bigr)\, dt \right);$$

see for details [17]. The second situation is when

$$x \oplus y = \max(x, y) \quad \text{and} \quad x \odot y = g^{-1}\big(g(x)g(y)\big),$$

the pseudo-integral for a function $p : \mathbb{R} \to [c, d]$ is given as

$$\int_{\mathbb{R}}^{\oplus} p \odot dm = \sup\big(p(x) \odot \psi(x)\big), \tag{2}$$

where the function ψ means sup-measure m; see [14].

Theorem 1 ([14]) *Let m be a* sup*-measure on $([0,\infty], \mathcal{D}([0,\infty]))$, where $\mathcal{D}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(C) = \operatorname{ess\,sup}_{\mu}(\psi(x)|x \in A)$, and $\mu : [0,\infty] \to [0,\infty]$ is a continuous density. Then, for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0,\infty), \mathcal{D})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0,\infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.*

Theorem 2 ([14]) *Let $([0,\infty], \sup, \odot)$ be a semiring, when $\odot$ is generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0,\infty)$. Let m be the same as in Theorem 1. Then there exists a family m_λ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0,\infty)$ such that, for every continuous function $p : [0,\infty] \to [0,\infty]$,*

$$\int^{\sup} p \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} p \odot dm_\lambda = \lim_{n\to\infty} \big(g^\lambda\big)^{-1}\left(\int g^\lambda\big(p(t)\big)\, dt\right). \tag{3}$$

Pseudo-operations on nonempty subsets C and D of $[c, d]$ adopts the methods similar to the pseudo-addition and the pseudo-multiplication [14]. If C and D stand for two arbitrary nonempty subsets of $[c, d]$ and $\beta \in [c, d]^+$, then

$$C \oplus D = \{x \oplus y | x \in C \text{ and } y \in D\},$$
$$C \odot D = \{x \odot y | x \in C \text{ and } y \in D\},$$
$$\beta \odot C = \{\beta \odot x | x \in C\}.$$

The present paper focuses on the interval operation. Let I be the class of all closed subintervals of $[c, d]^+$, i.e.,

$$I = \big\{[w, e] | w \le e \text{ and } [w, e] \subseteq [c, d]^+\big\}.$$

It can be shown (see [13]) that for $C, D \in I$, where $C = [d_1, h_1]$ and $D = [d_2, h_2]$,

$$C \oplus D = [d_1 \oplus d_2, h_1 \oplus h_2], \qquad C \odot D = [d_1 \odot d_2, h_1 \odot h_2].$$

For $\beta \in [c, d]^+$ and $D = [d_2, h_2] \in I$, we have $\beta \odot D = [\beta \odot d_2, \beta \odot h_2]$.

Theorem 3 ([13]) *Let $([c, d], \oplus, \odot)$ be a semiring that belongs to one kind basic categories, where $\oplus = \sup$ or $\oplus$ is equivalent to an increasing generator g. Let $p : X \to [c, d]^+$ be a* *measurable function. An interval-valued set-function $\bar{\mu}^{p}_{\mathcal{M}}$ based on the pseudo-integral of p with respect to interval-valued ⊕-measure given by*

$$\bar{\mu}^{p}_{\mathcal{M}}(C) = \int_C^{\oplus} p \odot d\bar{\mu}_{\mathcal{M}} = \left[\int_C^{\oplus} p \odot d\mu_d, \int_C^{\oplus} p \odot d\mu_u\right], \tag{4}$$

where $C \subseteq X$, the properties can be summarized as follows:

(i) $\bar{\mu}^{p}_{\mathcal{M}}(\emptyset) = [0, 0]$,
(ii) $\bar{\mu}^{p}_{\mathcal{M}}$ *is monotone with respect to* $\preceq_S$,
(iii) $\bar{\mu}^{p}_{\mathcal{M}}$ *is additive*,
(iv) $\bar{\mu}^{p}_{\mathcal{M}}$ *is* σ-⊕-*additive*.

In Eq. (4), *the set-function* $\bar{\mu}^{p}_{\mathcal{M}}$ *is an interval-valued ⊕-measure*.

3 Wirtinger integral inequality for pseudo-integrals

In this section, we discuss Wirtinger integral inequality for pseudo-integrals based on the semiring $([0,T], \oplus, \odot)$ and semiring $([0,T], \sup, \odot)$ separately.

Let $U(t) = u'(t)$, then $u(t) = \int_0^t U(s)\,ds$, and Inequality (1) implies that

$$\int_0^T U(t)^2\,dt \geq \frac{4\pi^2}{T^2}\int_0^T \left(\int_0^t U(s)\,ds\right)^2 dt. \tag{5}$$

Theorem 4 *Suppose that $U : [0,T] \to [0,T]$ $(T \geq 0)$ is a measurable function. If a generator $g : [0,T] \to [0,+\infty]$ of the pseudo-addition and the pseudo-multiplication is an increasing function and if $\int_0^t g(U(s))\,ds \in C^1([0,T],\mathbb{R})$, $\int_0^T g(U(t))\,dt = 0$, $\int_0^T (\int_0^t g(U(s))\,ds)\,dt = 0$. Then for any δ-⊕-measure*

$$\int_{[0,T]}^{\oplus} \big(U(t)\big)_{\odot}^{(2)}\,dt \geq \left(\frac{4\pi^2}{T^2}\right)\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U(s)\,ds\right)_{\odot}^{(2)} dt. \tag{6}$$

Proof By (5), we obtain

$$\int_0^T \big(g\big(U(t)\big)\big)^2\,dt \geq \frac{4\pi^2}{T^2}\left(\int_0^T \left(\int_0^t \big(g\big(U(s)\big)\big)\,ds\right)^2 dt\right).$$

Since g^{-1} is an increasing function, one has

$$g^{-1}\left(\int_0^T \big(g\big(U(t)\big)\big)^2\,dt\right) \geq \left(\frac{4\pi^2}{T^2}\right)\left(g^{-1}\left(\int_0^T \left(\int_0^t \big(g\big(U(s)\big)\big)\,ds\right)^2 dt\right)\right),$$

which implies that

$$\begin{aligned} &g^{-1}\left(\int_0^T \big(g\big(g^{-1}\big(g\big(U(t)\big)\big)^2\big)\big)\right)dt) \\ &\quad\geq \left(\frac{4\pi^2}{T^2}\right)g^{-1}\left(\int_0^T \left(g\left(g^{-1}\left(\int_0^t \big(g\big(U(s)\big)\big)\,ds\right)^2\right)\right)dt\right). \end{aligned}$$

Hence, we have

$$\int_{[0,T]}^{\oplus} \big(U(t)\big)_{\odot}^{(2)}\,dt$$

$$\begin{aligned}
&\geq \left(\frac{4\pi^2}{T^2}\right) g^{-1}\left(\int_0^T \left(g\left(g^{-1}\left(\int_0^t \left(g\big(U(s)\big)\right) ds\right)^2\right)\right) dt\right) \\
&= \left(\frac{4\pi^2}{T^2}\right) \int_{[0,T]}^{\oplus} \left(g^{-1}\left(\int_0^t \left(g\big(U(s)\big)\right) ds\right)^2\right) dt \\
&= \left(\frac{4\pi^2}{T^2}\right) \int_{[0,T]}^{\oplus} \left(g^{-1}\left(g\left(g^{-1}\left(\int_0^t \left(g\big(U(s)\big)\right) ds\right)\right)\right)^2\right) dt \\
&= \left(\frac{4\pi^2}{T^2}\right) \int_{[0,T]}^{\oplus} \left(g^{-1}\left(\int_0^t \left(g\big(U(s)\big)\right) ds\right)\right)_{\odot}^{(2)} dt \\
&= \left(\frac{4\pi^2}{T^2}\right) \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U(s)\, ds\right)_{\odot}^{(2)} dt.
\end{aligned}$$

The proof is therefore complete. □

Example 5 Consider that $g(x) = \ln x$, $U(t) = \mathrm{e}^{\cos t}$, and $T = 2\pi$. The corresponding pseudo-operations are $x \oplus y = xy$, $x \odot y = \mathrm{e}^{\ln x \cdot \ln y}$. By direct calculation, one has

$$\begin{aligned}
&\int_{[0,t]}^{\oplus} U(s)\, ds = \mathrm{e}^{\int_0^t \ln U(t)\, dt} = \mathrm{e}^{\sin t}, \\
&\int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U(s)\, ds\right)_{\odot}^{(2)} dt = g^{-1}\left(\int_0^{2\pi} (\sin t)^2\right) = \mathrm{e}^{\pi}, \\
&\int_{[0,T]}^{\oplus} \big(U(t)\big)_{\odot}^{(2)} dt = g^{-1}\left(\int_0^{2\pi} (\cos t)^2\right) = \mathrm{e}^{\pi}.
\end{aligned}$$

So Inequality (6) holds.

Example 6 Consider that $g(x) = \tan x$, $U(t) = \arctan(\cos t)$, and $T = 2\pi$. The corresponding pseudo-operations are $x \oplus y = \arctan(\tan x + \tan y)$, $x \odot y = \arctan(\tan x \cdot \tan y)$. Since

$$\int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U(s)\, ds\right)_{\odot}^{(2)} dt = \arctan \pi = \int_{[0,T]}^{\oplus} \big(U(t)\big)_{\odot}^{(2)} dt,$$

Inequality (6) holds.

Now, we give an extension of Wirtinger integral inequality with semiring $([0,T], \sup, \odot)$.

Theorem 7 *Let $\odot$ be represented by an increasing generator g and let m be a complete* sup-*measure. Then for any functions $U : [0,T] \to [0,T]$, $T \geq 0$,*

$$\int_{[0,T]}^{\sup} U_{\odot}^{(2)} \odot dm \geq \left(\frac{4\pi^2}{T^2}\right) \int_{[0,T]}^{\sup} \left(\left(\int_{[0,t]}^{\sup} U \odot dm\right)_{\odot}^{(2)}\right) \odot dm.$$

Proof According to Theorem 2, one has

$$\int_{[0,T]}^{\sup} U_{\odot}^{(2)} \odot dm = \lim_{\lambda \to \infty}\left(\int_{[0,T]}^{\oplus_\lambda} U_{\odot}^{(2)} \odot dm_\lambda\right) = \lim_{\lambda \to \infty} \left(g^\lambda\right)^{-1}\left(\int_0^T \left(g^\lambda\left(\big(U(t)\big)^2\right)\right) dt\right).$$

Since g is increasing, $g^{-1}, g^{\lambda}, (g^{\lambda})^{-1}$ are also increasing. Thus, by Theorem 4, we have

$$\begin{aligned}
&\lim_{\lambda\to\infty}(g^{\lambda})^{-1}\left(\int_0^T \left(g^{\lambda}\left(\left(U(t)\right)^2\right)\right)dt\right)\\
&\geq \lim_{\lambda\to\infty}\left(\frac{4\pi^2}{T^2}\right)(g^{\lambda})^{-1}\int_0^T\left(\left(\int_0^t \left(g^{\lambda}\left(\left(U(s)\right)ds\right)^2\right)\right)dt\\
&= \lim_{\lambda\to\infty}\left(\frac{4\pi^2}{T^2}\right)(g^{\lambda})^{-1}\int_0^T\left(g^{\lambda}\left((g^{\lambda})^{-1}\left(\left(\int_0^t \left(g^{\lambda}\left(\left(U(s)\right)ds\right)_{\odot}^{(2)}\right)\right)\right)\right)dt\\
&= \lim_{\lambda\to\infty}\left(\frac{4\pi^2}{T^2}\right)\int_{[0,T]}^{\oplus_\lambda}\left((g^{\lambda})^{-1}\left(\left(\int_0^t \left(g^{\lambda}\left(\left(U(s)\right)ds\right)_{\odot}^{(2)}\right)\right)\right)\right)\odot dm_\lambda\\
&= \lim_{\lambda\to\infty}\left(\frac{4\pi^2}{T^2}\right)\int_{[0,T]}^{\oplus_\lambda}\left((g^{\lambda})^{-1}\left(g^{\lambda}\left((g^{\lambda})^{-1}\left(\int_0^t \left(g^{\lambda}\left(\left(U(s)\right)ds\right)\right)\right)^2\right)\right)\right)\odot dm_\lambda\\
&= \lim_{\lambda\to\infty}\left(\frac{4\pi^2}{T^2}\right)\int_{[0,T]}^{\oplus_\lambda}\left(\left(\int_{[0,t]}^{\oplus_\lambda}U\odot dm_\lambda\right)_{\odot}^{(2)}\right)\odot dm_\lambda\\
&= \left(\frac{4\pi^2}{T^2}\right)\int_{[0,T]}^{\sup}\left(\left(\int_{[0,t]}^{\sup}U\odot dm\right)_{\odot}^{(2)}\right)\odot dm.
\end{aligned}$$

This ends the proof. □

4 Inequalities of Wirtinger type for pseudo-integrals with respect to interval-valued ⊕-measures

This section contains the further results of this paper, i.e., Wirtinger type inequalities based on the interval-valued ⊕-measure [19].

Theorem 8 *Let $([a,b],\oplus,\odot)$ be a g-semiring. If $u\in C^1([0,T],\mathbb{R})$, $\int_0^T u(t)\,dt = 0$, $U = u'(t)$, and $U:[a,b]\to[a,b]$ is a measurable function, then*

$$\int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\odot d\bar{\mu}_{\mathcal{M}} \succeq_S \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_{\mathcal{M}}\right)_{\odot}^{(2)}\odot d\bar{\mu}_{\mathcal{M}}. \tag{7}$$

Proof By Theorem 3, we have

$$\int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\odot d\bar{\mu}_{\mathcal{M}} = \left[\int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\odot d\bar{\mu}_d, \int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\odot d\bar{\mu}_h\right].$$

On the other hand,

$$\begin{aligned}
&\frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_{\mathcal{M}}\right)_{\odot}^{(2)}\odot d\bar{\mu}_{\mathcal{M}}\\
&= \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left[\int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_d, \int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_h\right]\\
&\quad\odot\left[\int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_d, \int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_h\right]\odot d\bar{\mu}_{\mathcal{M}}\\
&= \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left[\left(\int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_d\right)\odot\left(\int_{[0,t]}^{\oplus}U\odot d\bar{\mu}_d\right),\right.
\end{aligned}$$

$$\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h\right) \odot \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h\right)\Bigg] \odot d\bar{\mu}_{\mathcal{M}}$$
$$= \frac{4\pi^2}{T^2}\left[\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d\right)_{\odot}^{(2)} \odot d\bar{\mu}_d, \int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h\right)_{\odot}^{(2)} \odot d\bar{\mu}_h\right].$$

Let

$$d_1 = \int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\, d\bar{\mu}_d, \qquad d_2 = \int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d\right)_{\odot}^{(2)} \odot d\bar{\mu}_d,$$
$$h_1 = \int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\, d\bar{\mu}_h, \qquad h_2 = \int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h\right)_{\odot}^{(2)} \odot d\bar{\mu}_h.$$

Since the interval $[\int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d, \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_h]$ is pseudo-convex, an arbitrary element

$$x \in \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}}$$

can be written in the form

$$x = \alpha \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d \oplus \beta \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)}\, d\bar{\mu}_h,$$

where $\alpha, \beta \in [a,b]^+$, $\alpha \oplus \beta = 1$.

Based on Wirtinger inequality for pseudo-integrals (6), one has

$$\int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d \succeq \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d\right)_{\odot}^{(2)} \odot d\bar{\mu}_d.$$

Since $\alpha \in [a,b]^+$ and $\odot$ is a positively non-decreasing function, we have

$$\alpha \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d \succeq \alpha \odot \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d\right)_{\odot}^{(2)} \odot d\bar{\mu}_d.$$

Similarly, for $\beta \in [a,b]^+$ and $\odot$-measure μ_h, $\odot$ is a positively non-decreasing function, and then, by Inequality (6), we have

$$\beta \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_h \succeq \beta \odot \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h\right)_{\odot}^{(2)} \odot d\bar{\mu}_h.$$

For

$$y = \alpha \odot \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d\right)_{\odot}^{(2)} \odot d\bar{\mu}_d$$
$$\oplus \beta \odot \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h\right)_{\odot}^{(2)} \odot d\bar{\mu}_h,$$

we have $x \succeq y$ and

$$y \in \frac{4\pi^2}{T^2}\int_{[0,T]}^{\oplus}\left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}}\right)_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}}.$$

Therefore, we have completed the first part of the proof. Furthermore, due to the property of the pseudo-convexity of the subinterval of $[a,b]^+$,

$$y \in \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}} \right)_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}}$$

can be written in the form

$$\begin{aligned} y = \alpha \odot \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d \right)_{\odot}^{(2)} \odot d\bar{\mu}_d \\ \oplus \beta \odot \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h \right)_{\odot}^{(2)} \odot d\bar{\mu}_h, \end{aligned}$$

for some $\alpha, \beta \in [a,b]^+$, $\alpha \oplus \beta = 1$. It follows from (6) that

$$\begin{aligned} \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d \succeq \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d \right)_{\odot}^{(2)} \odot d\bar{\mu}_d, \\ \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_h \succeq \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h \right)_{\odot}^{(2)} \odot d\bar{\mu}_h. \end{aligned}$$

Since $\oplus$ is a non-decreasing function, and $\odot$ is a positively non-decreasing function, one has

$$\begin{aligned} &\alpha \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d \oplus \beta \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \, d\bar{\mu}_h \\ &\quad \succeq \alpha \odot \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d \right)_{\odot}^{(2)} \odot d\bar{\mu}_d \\ &\qquad \oplus \beta \odot \frac{4\pi^2}{T^2} \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h \right)_{\odot}^{(2)} \odot d\bar{\mu}_h. \end{aligned}$$

This yields

$$x = \alpha \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d \oplus \beta \odot \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \, d\bar{\mu}_h \in \int_{[0,T]}^{\oplus} U_{\odot}^{(2)} \, d\bar{\mu}_{\mathcal{M}}$$

satisfying $x \succeq y$. Hence, the following form holds:

$$[d_1, h_1] \succeq_S [d_2, h_2].$$

This completes the proof. □

Remark 9 The proof for a decreasing generator g is similar. In this case, since the total order is opposite to the usual order on the real line, Inequality (7) is reduced to

$$[d_1, h_1] \preceq_S [d_2, h_2].$$

Example 10 Consider the g-semiring with the generating function $g(x) = x^2$, the interval-valued $\oplus$-measure $\bar{\mu}((a,b]) = [\sqrt{\frac{2}{3}(b-a)}, \sqrt{b-a}]$, $(a,b] \in B[a,b]$, the function $u(x) = \frac{1}{2}$, if $0 \le x \le \frac{1}{2}$, $u(x) = 1$, if $\frac{1}{2} \le x \le 1$ and $T = 1$. Then, the Wirtinger inequality (7) is of the form

$$\int_{[0,1]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}} \succeq_S 4\pi^2 \int_{[0,1]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}} \right)_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}}. \tag{8}$$

The pseudo-integral of the function U with respect to the interval-valued $\oplus$-measure $\bar{\mu}_{\mathcal{M}}$ is

$$\int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}} = \left[\int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_d, \int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_h \right].$$

Since $g(x) = x^2$, $x \odot y = x \cdot y$, $x \oplus y = \sqrt[\alpha]{x^\alpha + y^\alpha}$, one has

$$\begin{aligned}
\int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_d &= \frac{1}{2} \odot \bar{\mu}_d\left(\left[0, \frac{1}{2}\right]\right) \oplus 1 \odot \bar{\mu}_d\left(\left(\frac{1}{2}, 1\right]\right) \\
&= \frac{1}{2} \odot \sqrt{\frac{2}{3} \cdot \frac{1}{2}} \oplus 1 \odot \sqrt{\frac{2}{3} \cdot \frac{1}{2}} = \sqrt{\frac{5}{12}}, \\
\int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_h &= \frac{1}{2} \odot \bar{\mu}_h\left(\left[0, \frac{1}{2}\right]\right) \oplus 1 \odot \bar{\mu}_h\left(\left(\frac{1}{2}, 1\right]\right) \\
&= \frac{1}{2} \odot \sqrt{\frac{1}{2}} \oplus 1 \odot \sqrt{\frac{1}{2}} = \sqrt{\frac{5}{8}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}} = \left[\sqrt{\frac{5}{12}}, \sqrt{\frac{5}{8}} \right], \\
&\left(\int_{[0,1]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}} \right)_{\odot}^{(2)} = \left[\frac{5}{12}, \frac{5}{8} \right], \\
&\int_{[0,1]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d \right)_{\odot}^{(2)} \odot d\bar{\mu}_d = \frac{5}{12} \odot \bar{\mu}_d[0,1] = \frac{5}{12}\sqrt{\frac{2}{3}}, \\
&\int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h \right)_{\odot}^{(2)} \odot d\bar{\mu}_h = \frac{5}{8} \odot \bar{\mu}_h[0,1] = \frac{5}{8}.
\end{aligned}$$

For the right hand side of (8), we have

$$\begin{aligned}
&4\pi^2 \int_{[0,1]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_{\mathcal{M}} \right)_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}} \\
&= \frac{4\pi^2}{T^2} \left[\int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_d \right)_{\odot}^{(2)} \odot d\bar{\mu}_d, \int_{[0,T]}^{\oplus} \left(\int_{[0,t]}^{\oplus} U \odot d\bar{\mu}_h \right)_{\odot}^{(2)} \odot d\bar{\mu}_h \right] \\
&= \left[\frac{3}{5}\sqrt{\frac{2}{3}}\pi, \frac{5}{2}\pi \right].
\end{aligned}$$

Since $u(x) = \frac{1}{4}$, $0 \le x \le \frac{1}{2}$, $u(x) = 1$, $\frac{1}{2} \le x \le 1$, the following equations hold:

$$\begin{aligned}\int_{[0,1]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d &= \frac{1}{4} \odot \bar{\mu}_d\left(\left[0, \frac{1}{2}\right]\right) \oplus 1 \odot \bar{\mu}_d\left(\left(\frac{1}{2}, 1\right]\right) \\ &= \frac{1}{4} \odot \sqrt{\frac{2}{3} \cdot \frac{1}{2}} \oplus 1 \odot \sqrt{\frac{2}{3} \cdot \frac{1}{2}} = \frac{1}{4}\sqrt{\frac{17}{3}} \\ \int_{[0,1]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_h &= \frac{1}{4} \odot \bar{\mu}_h\left(\left[0, \frac{1}{2}\right]\right) \oplus 1 \odot \bar{\mu}_h\left(\left(\frac{1}{2}, 1\right]\right) \\ &= \frac{1}{4} \odot \sqrt{\frac{1}{2}} \oplus 1 \odot \sqrt{\frac{1}{2}} = \frac{1}{4}\sqrt{\frac{17}{2}}.\end{aligned}$$

For the left hand of (8), we have

$$\int_{[0,1]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_{\mathcal{M}} = \left[\int_{[0,1]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_d, \int_{[0,1]}^{\oplus} U_{\odot}^{(2)} \odot d\bar{\mu}_h\right] = \left[\frac{1}{4}\sqrt{\frac{17}{3}}, \frac{1}{4}\sqrt{\frac{17}{2}}\right].$$

The final form of (8) is

$$\left[\frac{3}{5}\sqrt{\frac{2}{3}}\pi, \frac{5}{2}\pi\right] \succeq_S \left[\frac{1}{4}\sqrt{\frac{17}{3}}, \frac{1}{4}\sqrt{\frac{17}{2}}\right].$$

5 Conclusion

In this paper, we present three Wirtinger type integral inequalities for pseudo-integrals with g-semirings and interval-valued $\oplus$-measure, respectively. It is well known that pseudo-integrals cover Lebesgue integrals, Sugeno fuzzy integrals, so the presented process can extended to other integrals. Since the Wirtinger integral inequalities are important basic tools widely used in engineering technology, and that the pseudo-integral is applicable in many fields by helping to explain some practical problems, our inequalities could become essential tools in several practical areas, such as stability analysis of delayed stochastic neural networks and the Takagi–Sugeno fuzzy time-delay system. Thus, in future work, we intend to refine our proposed inequalities and to modify the Wirtinger type double integral inequality for pseudo-integrals, and then apply them to fuzzy time-varying delay system.

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