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On paranormed Ideal convergent sequence spaces defined by Jordan totient function

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Abstract

The study of sequence spaces and summability theory has been an important aspect in defining new notions of convergence for the sequences that do not converge in the usual sense. Paving the way into the applications of law of large numbers and theory of functions, it has proved to be an essential tool. In this paper we generalise the classical Maddox sequence spaces $c_0(p)$, $c(p)$, $\ell(p)$ and $\ell_\infty(p)$ and define new ideal paranormed sequence spaces $c_0^I(\Upsilon^r,p)$, $c^I(\Upsilon^r,p)$, $\ell_\infty^I(\Upsilon^r,p)$ and $\ell_\infty(\Upsilon^r,p)$ defined with the aid of Jordan's totient function and a bounded sequence of positive real numbers. We develop isomorphism between certain maps and also find their α-, β- and γ-duals. We examine algebraic and topological properties of these corresponding spaces. Further we study some standard inclusion relations and prove the decomposition theorem.

Keywords: Jordan totient operator; Paranorm; α, β, γ duals; *I*-convergence; Solidity; Convergence free; Decomposition theorem

1 Introduction and background

In 1964, S. Simons [1] generalised the classical bounded sequence spaces defined by Maddox [2] and Nakano [3] using a sequence of positive real numbers (p_k) bounded above by 1. He showed that these spaces were linear under coordinate wise addition and scalar multiplication. Later on, in 1970, Lascarides and Maddox [4] extrapolated new spaces with the restriction on the sequence of positive real numbers (p_k) relaxed from being bounded.

Ever since the concept of defining "convergence" for sequences and series that do not converge in the Cauchy sense was debated, researchers have been proactively involved in coming up with their own notions of convergence, see [5–9, 23, 24]. Summability methods have proved to be an efficient tool in it by defining a linear transformation from a sequence space into another. The most commonly used linear transformation on a sequence space is given by an infinite matrix. Let $M = (m_{st})$ be an infinite matrix of real numbers for $s, t \in \mathbb{N}$ and X and Y be two sequence spaces. If $Mx = (M_s(x)) \in X$ $\forall x = (x_n) \in X$, then M is a matrix mapping from X into Y, denoted by

$$M : X \to Y, \quad \text{where } M_s(x) = \sum m_{st}x_t \ \forall m \in \mathbb{N}.$$

The approach to constructing new sequence spaces defined as the domain of matrix operators has been studied by numerous authors, see [4, 10–12, 25, 28] with Euler totient matrix operator [13, 26, 27] being one of them. Recently a generalisation of Euler totient function ϕ, namely Jordan totient function J_r, was introduced in [14]. Denoted by J_r, it is an arithmetic function with the domain and codomain as $\mathbb{N}$. It is the number of r tuples $(h_1, h_2, \ldots, h_r)$ such that $1 \leq h_i \leq n$ and $\gcd(h_1, h_2, \ldots, h_r, n) = 1$. Formally, it is defined as $J_r(n) = n^r \prod_{p|n}(1 - \frac{1}{p^r})$, where $n = p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ for $\alpha \geq 1$, is the unique prime decomposition of n. More literature on Jordan totient function can be found in [15–18, 22].

Consequently the Jordan totient matrix operator denoted by $\Upsilon^r = (\nu_{nk}^r)$ was defined in [14] as follows:

$$\nu_{nk}^r = \begin{cases} \frac{J_r(k)}{n^r}, & \text{if } k|n, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

and its inverse $(\Upsilon^r)^{-1}$ is given by

$$\left(\Upsilon^r\right)^{-1} = \begin{cases} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r & \text{if } k|n, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

where μ is the Möbius function defined as follows:

$$\mu(n) = \begin{cases} 0, & \text{if } p^2|n \text{ for some prime } p, \\ 1, & \text{if } n = 1, \\ (-1)^i, & \text{if } n = \prod_{k=1}^{i} p_k \text{ where } p_k\text{s are distinct.} \end{cases}$$

Recently Khan et al. defined ideal convergence on sequence spaces as a domain of Jordan totient function, namely $c_0^I(\Upsilon^r)$, $c^I(\Upsilon^r)$ and $\ell_\infty^I(\Upsilon^r)$. Moving forward, we define ideal convergence on a paranormed space with the aid of Jordan totient matrix operator and study its properties after mentioning the following definitions and lemmas that will be put to use in the paper later.

2 Preliminaries

Definition 2.1 ([19]) If $b = (b_k)$ is a sequence in ω, then b is said to be I-Cauchy if, for every $\epsilon > 0, \exists$ a number $N = N(\epsilon) \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |b_k - b_N| \geq \epsilon\} \in I$.

Definition 2.2 ([20]) If I is an ideal and $b = (b_k)$ is a sequence in ω, then b is said to be I-convergent to a number $b_0 \in \mathbb{R}$ if, for every $\epsilon > 0$, we have $\{k \in \mathbb{N} : |b_k - b_0| \geq \epsilon\} \in I$, and we represent it by $I\text{-}\lim b_k = b_0$. If $b_0 = 0$, then $(b_k) \in \omega$ is called I-null.

Definition 2.3 ([20]) A sequence $b = (b_k) \in \omega$ is I-bounded if there exists $L > 0$ such that $\{k \in \mathbb{N} : |b_k| > L\} \in I$.

Definition 2.4 ([19]) Let $b = (b_k)$ and $c = (c_k)$ be two sequences, then we can say that $b_k = c_k$ for almost all k relative to I ($a.a.k.r.I$) if the set $\{k \in \mathbb{N} : b_k \neq c_k\} \in I$.

Definition 2.5 ([19]) A sequence space S is said to be solid (or normal) if $(\beta_k b_k) \in S$ whenever $(b_k) \in S$ and (β_k) is a sequence of scalar in ω such that $|\beta_k| < 1$ and $k \in \mathbb{N}$.

Definition 2.6 ([19]) Let $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ and S be a sequence space. A K-step space of S is a sequence space

$$\lambda_K^S = \{(b_{k_n}) \in \omega : (b_k) \in S\}.$$

A canonical pre-image of a sequence $(b_{k_i}) \in \lambda_K^S$ is a sequence $(c_k) \in \omega$ defined as follows:

$$c_k = \begin{cases} b_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of λ_K^S is a set of canonical pre-images of all elements in λ_K^S, i.e. c is in canonical pre-image of λ_K^S iff c is a canonical pre-image of some element $b \in \lambda_K^S$.

Definition 2.7 ([19]) If a sequence space S contains the canonical pre-images of its step space, then S is known as a monotone sequence space.

Definition 2.8 ([29]) A paranorm space, denoted by (X, G), is defined as a function $G : X \to \mathbb{R}$ such that

(a) $G(u) \geq 0 \ \forall u \in X$,
(b) $G(u) = 0$ if $u = \theta$,
(c) $G(-u) = G(u) \ \forall u \in X$,
(d) $G(u + v) \leq G(u) + G(v) \ \forall u, v \in X$,
(e) If λ_k is a sequence of scalars such that $\lambda_k \to \lambda$ and (u_k) is a sequence such that $G(u_k - u) \to 0$ as $k \to \infty$, then $G(\lambda_k u_k - \lambda u) \to 0$ as $k \to \infty$.

Lemma 2.1 ([19]) *Every solid space is monotone.*

Lemma 2.2 ([19]) *Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

The following lemmas are required to find the duals of the space $\ell(\Upsilon^r, p)$. Let $\mathscr{A}$ denote the family of all finite subsets of $\mathbb{N}$. Throughout this paper we denote a bounded sequence of positive real numbers by (p_n) such that $\sup_{n\in\mathbb{N}} p_n = P$ and $H = \max\{1, P\}$.

Lemma 2.3 ([21])

(i) $T = (t_{nk}) \in (\ell_p, \ell_1)$ *if and only if*

$$\sup_{N\in\mathscr{A}} \sum_k \left| \sum_{n\in\mathbb{N}} t_{nk} L^{-1} \right|^{p_n} < \infty, \tag{2.1}$$

where $L > 1$ is an integer and $1 < p_n \leq P < \infty$.

(ii) *For $0 < p_n \leq 1$, $T = (t_{nk}) \in (\ell_p, \ell_1)$ if and only if*

$$\sup_{N\in\mathscr{A}} \sup_k \left| \sum_{n\in\mathbb{N}} t_{nk} \right|^{p_n} < \infty. \tag{2.2}$$

(iii) *For $1 < p_n < \infty$, $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ if and only if*

$$\sup_n \sum_k \left| t_{nk} L^{-1} \right|^{p_n} < \infty. \tag{2.3}$$

(iv) *For* $0 < p_n \leq 1$, $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ *if and only if*

$$\sup_{n,k\in\mathbb{N}} |t_{nk}|^{p_n} < \infty. \tag{2.4}$$

(v) *For* $0 < p_n \leq \infty$, $T = (t_{nk}) \in (\ell_p, c)$ *if and only if*

$$\lim_n t_{nk} = l_n. \tag{2.5}$$

We denote the sequence spaces of all bounded, convergent and null sequences as ℓ_∞, $\mathscr{C}$ and $\mathscr{C}_0$ and the respective series as $\mathscr{BS}$, $\mathscr{CS}$ and $\mathscr{CS}_0$ respectively. The p-absolutely convergent series will be denoted by ℓ_p.

Given a sequence space X, its duals are defined as follows:

$$\begin{aligned}
X^\alpha &= \{a = (a_k) \in \omega : au = (a_k u_k) \in \ell_1 \text{ for all } u = (u_k) \in X\},\\
X^\beta &= \{a = (a_k) \in \omega : au = (a_k u_k) \in \mathscr{CS} \text{ for all } u = (u_k) \in X\},\\
X^\gamma &= \{a = (a_k) \in \omega : au = (a_k u_k) \in \mathscr{BS} \text{ for all } u = (u_k) \in X\}.
\end{aligned}$$

New Banach sequence spaces $l_p(\Phi)$ and $l_\infty(\Phi)$ derived by using a matrix operator, defined as

$$\begin{aligned}
l_p(\Phi) &= \left\{u = (u_n) \in \omega : \sum_n \left|\frac{1}{n}\sum_{k|n} \phi(k) u_k\right|^p < \infty\right\},\\
l_\infty(\Phi) &= \left\{u = (u_n) \in \omega : \sup \left|\frac{1}{n}\sum_{k|n} \phi(k) u_k\right|^p < \infty\right\},
\end{aligned}$$

were introduced by Ilkhan and Kara in the space ℓ_p for $1 \leq p < \infty$ [13]. They later generalised these spaces on a paranormed space defined by using a bounded sequence of strictly positive numbers (p_k) and defined $\ell(\Phi, p)$. Further a new generalised form of Euler operator, namely Jordan totient operator, was introduced [14], following which they introduced the matrix domain of the matrix Υ^r in the space of ℓ_p for $1 \leq p < \infty$.

Since paranormed spaces and ideal convergence are more general than normed spaces and usual convergence, respectively, therefore in this paper we first generalise the norm space $\ell_p(\Upsilon^r)$ to the paranorm space $\ell(\Upsilon^r, p)$, define its duals, and then we construct the ideal convergent sequence spaces, show that they are complete linear paranormed space defined by its paranorm and study diverse riveting properties of these resulting spaces.

3 Main results

3.1 The paranormed sequence space $\ell(\Upsilon^r, p)$

In the current section we propose the sequence space $\ell(\Upsilon^r, p)$ defined by using Jordan totient matrix operator Υ^r as

$$\ell(\Upsilon^r, p) = \left\{u = (u_k) \in \omega : \sum_n \left|\frac{1}{n^r}\sum_{k|n} J_r(k) u_k\right|^{p_n} < \infty\right\}.$$

The above defined space reduces to $\ell_p(\Upsilon^r)$ when $p_n = p$.

Theorem 3.1 *The sequence space $l(\Upsilon^r,p)$ is a complete paranorm space with the paranorm defined by*

$$G_{\Upsilon^r}(u)=\left(\sum_n\left|\frac{1}{n^r}\sum_{k|n}J_r(k)u_k\right|^{p_n}\right)^{\frac{1}{H}}$$

for all $u=(u_k)\in\ell(\Upsilon^r,p)$.

Proof It is obvious that $G_{\Upsilon^r}(\theta)=0$, where θ is the zero sequence and $G_{\Upsilon^r}(-u)=G_{\Upsilon^r}(u)$. Let $u=(u_k)$ and $v=(v_k)$ be two sequences in $l(\Upsilon^r,p)$, then

$$\begin{aligned}G_{\Upsilon^r}(u_k+v_k)&=\left(\sum_n\left|\frac{1}{n^r}\sum_{k|n}J_r(k)(u_k+v_k)\right|^{p_n}\right)^{\frac{1}{H}}\\&\le\left(\sum_n\left|\frac{1}{n^r}\sum_{k|n}J_r(k)u_k\right|^{p_n}\right)^{\frac{1}{H}}+\left(\sum_n\left|\frac{1}{n^r}\sum_{k|n}J_r(k)v_k\right|^{p_n}\right)^{\frac{1}{H}}\\&=G_{\Upsilon^r}(u_k)+G_{\Upsilon^r}(v_k).\end{aligned}$$

Therefore G is subadditive. Now, to show the continuity of multiplication by scalars, consider a sequence $(u^n)\in\ell(\Upsilon^r,p)$ such that $G_{\Upsilon^r}(u^n-u)\to 0$, and let (a_n) be a sequence of scalars such that $a_n\to a$. Since G is subadditive, therefore $G_{\Upsilon^r}(u^n)\le G_{\Upsilon^r}(u)+G_{\Upsilon^r}(u^n-u)$. Thus, $G_{\Upsilon^r}(u^n)$ is bounded, following which we conclude that

$$\begin{aligned}G_{\Upsilon^r}\left(a_nu^n-au\right)&=\left(\sum_n\left|\frac{1}{n^r}\sum_{k|n}J_r(k)\left(a_nu_k^n-au_k\right)\right|^{p_n}\right)^{\frac{1}{H}}\\&\le|a_n-a|G_{\Upsilon^r}\left(u^n\right)+|a|G_{\Upsilon^r}\left(u^n-u\right)\end{aligned}$$

which tends to 0 as $n\to\infty$. Therefore, G_{Υ^r} is a paranorm on $l(\Upsilon^r,p)$.

To show the completeness of the space, it suffices to show convergence of a Cauchy sequence. Let $(u^l)=(u_1^l,u_2^l,u_3^l,\dots)$ be any Cauchy sequence in $\ell(\Upsilon^r,p)$ for all $\in\mathbb{N}$. Then given $\epsilon>0$, there exists an integer $\mathbb{N}_0(\epsilon)>0$ such that

$$G_{\Upsilon^r}\left(u^l-u^m\right)<\epsilon \tag{3.1}$$

for all $l,m\ge N_0(\epsilon)$. Using the definition of G_{Υ^r}, we get that for each fixed $i\in\mathbb{N}$

$$\left|\Upsilon_i^r\left(u^l\right)-\Upsilon_i^r\left(u^m\right)\right|\le\left[\sum_i\left|\Upsilon_i^r\left(u^l\right)-\Upsilon_i^r\left(u^m\right)\right|^{p_n}\right]^{\frac{1}{H}}<\epsilon.$$

This shows that $\{\Upsilon_i^r(u^1),\Upsilon_i^r(u^2),\Upsilon_i^r(u^3),\dots\}$ is a real Cauchy sequence and is therefore complete, which implies $\Upsilon_i^r(u^l)\to\Upsilon_i^r(u)$ as $l\to\infty$ for every fixed $i\in\mathbb{N}$. For different values of i, we consider the sequence $\{\Upsilon_1^r(u),\Upsilon_2^r(u),\Upsilon_3^r(u),\dots\}$. Thus from equation (3.1) we can write for each fixed $t\in\mathbb{N}$ and $l,m\ge N_0(\epsilon)$

$$\sum_{i=1}^t\left|\Upsilon_i^r\left(u^l\right)-\Upsilon_i^r\left(u^m\right)\right|^{p_n}\le G_{\Upsilon^r}\left(u^l-u^m\right)^H<\epsilon^H. \tag{3.2}$$

Let $l \geq N_0(\epsilon)$. Taking $m, t \to \infty$, we have $G_{\Upsilon^r}(u^l - u) \leq \epsilon$. Fixing $\epsilon = 1$ and taking $l \geq N_0(1)$, we can see that for any fixed $t \in \mathbb{N}$

$$\begin{aligned}\left(\sum_{i=1}^{t} \left|\Upsilon_i^r(u)\right|^{p_n}\right)^{\frac{1}{H}} &\leq G_{\Upsilon^r}(u) = G_{\Upsilon^r}\left(u^l - u + u^l\right) \\ &\leq G_{\Upsilon^r}\left(u^l - u\right) + G_{\Upsilon^r}\left(u^l\right) \leq 1 + G_{\Upsilon^r}\left(u^l\right).\end{aligned}$$

Thus $u \in \ell(\Upsilon^r, p)$ and therefore the space is complete. □

Theorem 3.2 *The sequence space $l(\Upsilon^r, p)$ is linearly isomorphic to the space $l(p)$.*

Proof To establish linear isomorphism, we define a map $A : \ell(\Upsilon^r, p) \to \ell(p)$ such that $A(u) = \Upsilon^r u$ for all $u \in l(\Upsilon^r, p)$. It is evident that the map is linear and is injective too since $\ker(l(\Upsilon^r, p)) = \{v \in l(\Upsilon^r, p) : A(v) = 0\} = \{0\}$.

To show that the mapping is surjective, consider $v = (v_n) \in l(p)$ and define a sequence $u = (u_n)$ by

$$u_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r v_k \quad \text{for all } n \in \mathbb{N},$$

$$\begin{aligned}G_{\Upsilon^r}(u) &= \left(\sum_n \left|\frac{1}{n^r} \sum_{k|n} J_r(k) u_k\right|^{p_n}\right)^{\frac{1}{H}} \\ &= \left(\sum_n \left|\frac{1}{n^r} \sum_{k|n} J_r(k) \sum_{j|k} \frac{\mu(\frac{k}{j})}{J_r(k)} j^r v_k\right|^{p_n}\right)^{\frac{1}{H}} \\ &= \left(\sum_n \left|\frac{1}{n^r} \sum_{k|n} \sum_{j|k} \mu\left(\frac{k}{j}\right) j^r v_j\right|^{p_n}\right)^{\frac{1}{H}} \\ &= \left(\sum_n \left|\frac{1}{n^r} \sum_{k|n} \left(\sum_{j|k} \mu(j)\right) \left(\frac{n}{k}\right)^r v_{\frac{n}{k}}\right|^{p_n}\right)^{\frac{1}{H}} \\ &= \left(\sum_n |v_n|^{p_n}\right)^{\frac{1}{H}} < \infty.\end{aligned}$$

Thus $u \in l(\Upsilon^r, p)$ and the map preserves the norm and therefore the spaces are isomorphic.

It is well established that $\ell(p) \subseteq \ell(t)$ for $1 \leq p_n \leq t_n$, thenceforth we conclude $\ell(\Upsilon^r, p) \subseteq \ell(\Upsilon^r, t)$. □

Theorem 3.3 *The α dual of the space $l(\Upsilon^r, p)$ for $0 < p_n \leq 1$ and $1 < p_n \leq P < \infty$ can be defined as*

$$\left(l\left(\Upsilon^r, p\right)\right)_1^{\alpha} = \left\{v = (v_n) \in \omega : \sup_{N \in \mathscr{A}} \sup_k \left|\sum_{k|n, k \in N} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r v_n\right|^{p_k} < \infty\right\},$$

$$\left(l\left(\Upsilon^r, p\right)\right)_2^{\alpha} = \bigcup_{L > 1} \left\{v = (v_n) \in \omega : \sup_{N \in \mathscr{A}} \sum_k \left|\sum_{k|n, k \in N} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r v_n L^{-1}\right|^{p_k} < \infty\right\},$$

respectively.

Proof We prove our result for the second case and the first can be followed similarly.

Consider a sequence $a = (a_n) \in \omega$ and for a fixed sequence $u = (u_n) \in l(\Upsilon^r, p)$ for $1 < p_n < \infty$, we have

$$\begin{aligned} a_n u_n &= a_n \left(\sum_{k|n} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r v_n \right), \quad \text{where } v = (v_n) \in \ell(p) \\ &= \sum_{k|n} \left(\frac{\mu(\frac{n}{k})}{J_r(n)} k^r a_n \right) v_n = T_n v, \end{aligned}$$

where the matrix $T = (t_{nk})$ is defined by

$$t_{nk} = \begin{cases} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r a_n, & \text{if } k|n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore by (2.1), we conclude that $au = (a_n u_n) \in \ell_1$ whenever $u_n \in \ell(\Upsilon^r, p)$ if and only if $Tv \in \ell_1$ whenever $v \in \ell(p)$. Thus $a = (a_n) \in \{\ell(\Upsilon^r, p)\}^\alpha$ iff $T \in (\ell(p), \ell_1)$. □

Theorem 3.4 *Define the following sets*:

$$\begin{aligned} \mathscr{X}^\beta &= \left\{ a = (a_k) \in \omega \lim_{n\to\infty} \sum_{j=k,k|j}^{n} \frac{\mu(\frac{j}{k})}{J_r(j)} k^r a_j, \textit{ exists } \forall k \in \mathbb{N} \right\}, \\ \mathscr{Y}^\beta &= \bigcup_{L>1} \left\{ a = (a_k) \in \omega \sup_n \sum_{k=1}^{n} \left| \sum_{j=k,k|j}^{n} \frac{\mu(\frac{j}{k})}{J_r(j)} k^r a_j \right|^{p_k} < \infty \right\}, \\ \mathscr{Z}^\beta &= \left\{ a = (a_k) \in \omega \sup_{n,k} \left| \sum_{j=k,k|j}^{n} \frac{\mu(\frac{j}{k})}{J_r(j)} k^r a_j \right|^{p_k} < \infty \right\}. \end{aligned}$$

Then $\{\ell(\Upsilon^r, p)\}^\beta = \mathscr{X}^\beta \cup \mathscr{Y}^\beta \cup \mathscr{Z}^\beta$.

Proof Let $a = (a_k) \in \omega$ and $c = (c_k)$ be the Υ^r transform of $b = (b_k)$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k \left(\sum_{j|k} \frac{\mu(\frac{k}{j})}{J_r(k)} j^r c_k \right) = \sum_{k=1}^{n} \left(\sum_{j=k,k|j}^{n} a_j \frac{\mu(\frac{j}{k})}{J_r(j)} k^r \right) c_k = T_n v,$$

where $T = (t_{nk})$ is an infinite matrix given by

$$t_{nk} = \begin{cases} \sum_{j=k,k|j}^{n} \frac{\mu(\frac{j}{k})}{J_r(j)} k^r a_j, & \text{if } 1 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Thus, for any $b = (b_k) \in \ell(\Upsilon^r, p)$, $ab = (a_k b_k) \in \mathscr{CS}$ if and only if $Tc \in \mathscr{C}$ for $c = (c_k) \in \ell_p$. Thus $a = (a_k) \in \{\ell(\Upsilon^r, p)\}^\beta$ iff $T \in (\ell(p), c)$. Thus from (2.3), (2.4) and (2.5) we can conclude that $\{\ell(\Upsilon^r, p)\}^\beta = \mathscr{X}^\beta \cup \mathscr{Y}^\beta \cup \mathscr{Z}^\beta$. □

Theorem 3.5 *γ dual of the space $\ell(\Upsilon^r,p)$ denoted by $\{\ell(\Upsilon^r,p)\}^\gamma$ is given by*

$$\{\ell(\Upsilon^r,p)\}^\gamma = \begin{cases} \mathscr{Y}^\beta & \text{for } 1 < p_n < \infty, \\ \mathscr{Z}^\beta & \text{if } 0 < p_n \le 1. \end{cases}$$

Proof The proof similarly follows from the theorem with the aid of equations (2.3) and (2.4) from Lemma (2.3). □

3.2 Paranormed Ideal sequence spaces defined by Jordan totient matrix operator

In this section we define ideal convergence on a paranorm space with the aid of Jordan totient matrix and further study properties of these spaces.

Let $p = (p_k)$ be a bounded sequence of positive real numbers, then for given $\epsilon > 0$

$$c_0^I(\Upsilon^r,p) = \{(u_k) \in \omega : \{k \in \mathbb{N} : |\Upsilon_n^r(u_k)|^{p_k} \ge \epsilon\} \in I\}, \tag{3.3}$$

$$c^I(\Upsilon^r,p) = \{(u_k) \in \omega : \{k \in \mathbb{N} : |\Upsilon_n^r(u_k) - \ell|^{p_k} \ge \epsilon \text{ for some } \ell \in \mathbb{R}\} \in I\}, \tag{3.4}$$

$$\ell_\infty^I(\Upsilon^r,p) = \{(u_k) \in \omega : \exists M > 0 \text{ s.t } \{k \in \mathbb{N} : |\Upsilon_n^r(u_k)|^{p_k} > M\} \in I\}, \tag{3.5}$$

$$\ell_\infty(\Upsilon^r,p) = \Big\{(u_k) \in \ell_\infty : \sup_k(|\Upsilon_n^r(u_k)|)^{p_k} < \infty\Big\}. \tag{3.6}$$

We write

$$m_0^I(\Upsilon^r,p) := c_0^I(\Upsilon^r,p) \cap \ell_\infty(\Upsilon^r,p) \tag{3.7}$$

and

$$m^I(\Upsilon^r,p) := c^I(\Upsilon^r,p) \cap \ell_\infty(\Upsilon^r,p). \tag{3.8}$$

Theorem 3.6 *The sequence spaces $c_0^I(\Upsilon^r,p)$, $c^I(\Upsilon^r,p)$, $\ell_\infty^I(\Upsilon^r,p)$, $m_0^I(\Upsilon^r,p)$ and $m^I(\Upsilon^r,p)$ are linear spaces over $\mathbb{R}$.*

Proof Consider two sequences $u = (u_k)$, $v = (v_k)$ in $c^I(\Upsilon^r,p)$, and let λ, μ be scalars. Thus, for any given $\epsilon > 0$, there exist $l_1, l_2 \in \mathbb{R}$ such that

$$\Big\{k \in \mathbb{N} : |\Upsilon_n^r(u_k) - l_1|^{p_k} \ge \frac{\epsilon}{2N_1}\Big\} \in I,$$

$$\Big\{n \in \mathbb{N} : |\Upsilon_n^r(v_k) - l_2|^{p_k} \ge \frac{\epsilon}{2N_2}\Big\} \in I,$$

where

$$N_1 = H.\max\Big\{1, \sup_k |\lambda|^{p_k}\Big\},$$

$$N_2 = H.\max\Big\{1, \sup_k |\mu|^{p_k}\Big\},$$

and

$$H = \max\{1, 2^{D-1}\} \quad \text{and} \quad D = \sup_k p_k \ge 0.$$

Let

$$J_1 = \left\{ n \in \mathbb{N} : \left| \Upsilon_n^r(u_k) - l_1 \right|^{p_k} < \frac{\epsilon}{2N_1} \right\} \in \mathcal{F}(I),$$
$$J_2 = \left\{ n \in \mathbb{N} : \left| \Upsilon_n^r(v_k) - l_2 \right|^{p_k} < \frac{\epsilon}{2N_2} \right\} \in \mathcal{F}(I),$$

be such that $J_1^c, J_2^c \in I$. Then

$$\begin{aligned} J_3 &= \left\{ k \in \mathbb{N} : \left| \Upsilon_n^r(\lambda u_k + \mu v_k) - (\lambda l_1 + \mu l_2) \right|^{p_k} < \epsilon \right\} \\ &\supseteq \left[\left\{ k \in \mathbb{N} : |\lambda|^{p_k} \left| \Upsilon_n^r(u_k) - l_1 \right|^{p_k} < \frac{\epsilon}{2N_1} |\lambda|^{p_k} H \right\} \right. \\ &\quad \left. \cap \left\{ k \in \mathbb{N} : |\mu|^{p_k} \left| \Upsilon_n^r(v_k) - l_2 \right|^{p_k} < \frac{\epsilon}{2N_2} |\mu|^{p_k} H \right\} \right]. \end{aligned} \tag{3.9}$$

Clearly $J_3 \in \mathcal{F}(I)$ and hence $J_3^c = J_1^c \cup J_2^c \in I$, following which we can conclude that $(\lambda u_k + \mu v_k) \in c^I(\Upsilon^r, p)$. Hence $c^I(\Upsilon^r, p)$ is a linear space. □

Theorem 3.7 *The inclusions $c_0^I(\Upsilon^r, p) \subset c^I(\Upsilon^r, p) \subset \ell_\infty^I(\Upsilon^r, p)$ are strict.*

Proof It is evident that $c_0^I(\Upsilon^r, p) \subset c^I(\Upsilon^r, p)$. Consider $u = (u_k) \in \omega$ such that $\Upsilon_n^r(u) = 1 - \frac{1}{k}$ and $p_k = 1$ if k is even and $p_k = 2$ when k is odd. Thus $\Upsilon_n^r(u) \in c^I$ but $\Upsilon_n^r(u) \notin c_0^I$. This implies $u \in c^I(\Upsilon^r, p) \backslash c_0^I(\Upsilon^r, p)$. Next, let $u = (u_k) \in c^I(\Upsilon^r, p)$. Then there exists $a \in \mathbb{R}$ such that $I\text{-}\lim \Upsilon_n^r(u) = a$, that is,

$$\left\{ n \in \mathbb{N} : \left| \Upsilon_n^r(u) - a \right|^{p_k} \geq \epsilon \right\} \in I.$$

We can write

$$\left| \Upsilon_n^r(u) \right|^{p_k} = \left| \Upsilon_n^r(x) - a + a \right|^{p_k} \leq \left| \Upsilon_n^r(x) - a \right|^{p_k} + |a|^{p_k}.$$

Thus $(u_k) \in \ell_\infty^I(\Upsilon^r)$. We show the strictness of $c^I(\Upsilon^r, p) \subset \ell_\infty^I(\Upsilon^r, p)$ by fashioning the following example. □

Example 3.1 Let $(p_k) = 1$ and $u = (u_k) \in \omega$ be a sequence such that

$$\Upsilon_n^r(u) = \begin{cases} \sqrt[3]{n} & \text{if } n \text{ is a cube;} \\ 1 & \text{if } n \text{ is odd non-cube;} \\ 0 & \text{if } n \text{ is even non-cube.} \end{cases}$$

Then $\Upsilon_n^r(u) \in \ell_\infty^I$ but $\Upsilon_n^r(u) \notin c^I$ thus $u \in \ell_\infty^I(\Upsilon^r, p) \backslash c^I(\Upsilon^r, p)$.

Thus, we get that $c_0^I(\Upsilon^r, p) \subset c^I(\Upsilon^r, p) \subset \ell_\infty^I(\Upsilon^r, p)$ is strict.

Theorem 3.8 *The sets $m^I(\Upsilon^r, p)$ and $m_0^I(\Upsilon^r, p)$ are closed subspaces of $\ell_\infty(\Upsilon^r, p)$.*

Proof Consider a Cauchy sequence $(u_k^{(l)})$ in $m^I(\Upsilon^r,p) \subset \ell_\infty(\Upsilon^r,p)$ such that $u^{(l)} \to u$. To show that $u \in m^I(\Upsilon^r,p)$. Since $(u_k^{(l)}) \in m^I(\Upsilon^r,p)$, there exists (b_l) such that $\{k \in \mathbb{N} : |u_k^{(l)} - b_l|^{p_k} \geq \epsilon\} \in I$. To prove our result, we show the following:

(i) (b_l) converges to b,

(ii) If $S = \{k \in \mathbb{N} : |\Upsilon_n^r(u_k) - b|^{p_k} \leq \epsilon\}$, then $S^c \in I$.

(i) Since $(u_k^{(l)})$ is a Cauchy sequence in $m^I(\Upsilon^r,p) \implies$ given $\epsilon > 0$ we can get $p_o \in \mathbb{N}$ such that

$$\sup_k \left|\Upsilon_n^r\left(u_k^{(l)}\right) - \Upsilon_n^r\left(u_k^{(m)}\right)\right|^{\frac{p_k}{M}} < \frac{\epsilon}{3} \quad \text{for all } l, m \geq p_o. \tag{3.10}$$

For $\epsilon > 0$, consider the following sets:

$$U_{lm} = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\left(u_k^{(l)}\right) - \Upsilon_n^r\left(u_k^{(m)}\right)\right|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\}; \tag{3.11}$$

$$U_m = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\left(u_k^{(m)}\right) - b_m\right|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\}; \tag{3.12}$$

$$U_l = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\left(u_k^{(l)}\right) - b_l\right|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\}. \tag{3.13}$$

Then $U_{lm}^c, U_m^c, U_l^c \in I$. Let $U^c = U_{lm}^c \cup U_m^c \cup U_l^c$, where $U = \{k \in \mathbb{N} : |b_m - b_l|^{p_k} < \epsilon\}$. Then $U^c \in I$.

Choose $k_0 \in U^c$, then for each $l, m \geq k_0$, we have

$$\begin{aligned}\left\{k \in \mathbb{N} : |b_m - b_l|^{p_k} < \epsilon\right\} \supseteq \Bigg[&\left\{k \in \mathbb{N} : \left|b_m - \Upsilon_n^r\left(u_k^{(m)}\right)\right|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\} \\ &\cap \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\left(u_k^{(m)}\right) - \Upsilon_n^r\left(u_k^{(l)}\right)\right|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\} \\ &\cap \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\left(u_k^{(l)}\right) - b_l\right|^{p_k} < \left(\frac{\epsilon}{3}\right)^M\right\}\Bigg].\end{aligned}$$

Thus (b_l) is a Cauchy sequence of scalars in the field F, therefore there exists $b \in F$ such that $b_l \to b$ as $l \to \infty$.

(ii) Let $0 < \delta < 1$ be a given number. We show that if

$$A = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r(u_k) - b\right|^{p_k} \leq \delta\right\},$$

then $A^c \in I$. Since $u^{(l)} \to u$, there exists $q_0 \in \mathbb{N}$ such that

$$X = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\left(u_k^{(q_0)}\right) - \Upsilon_n^r(u_k)\right|^{p_k} < \left(\frac{\delta}{3H}\right)^M\right\} \quad \implies \quad X^c \in I.$$

$$Y = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r(b_{q_0}) - b\right|^{p_k} < \left(\frac{\delta}{3H}\right)^M\right\} \quad \text{such that } Y^c \in I.$$

Since $\{k \in \mathbb{N} : |\Upsilon_n^r(u_k^{(q_0)}) - \Upsilon_n^r(b_{q_0})|^{p_k} \geq \delta\} \in I$. Thus we get a subset $Z = \{k \in \mathbb{N} : |\Upsilon_n^r(u_k^{(q_0)}) - \Upsilon_n^r(b_{q_0})|^{p_k} < (\frac{\delta}{3H})^M\}$ of $\mathbb{N}$ such that $Z^c \in I$. Let $W^c = X^c \cup Y^c \cup Z^c$, where $W = \{k \in \mathbb{N} : |\Upsilon_n^r(u_k) - b|^{p_k} \leq \delta\}$.

Thus, for each $k \in W^c$, we have

$$\begin{aligned}\left\{k \in \mathbb{N} : \left|\Upsilon_n^r(u_k) - b\right|^{p_k} < \delta\right\} \supseteq \Bigg[&\left\{k \in \mathbb{N} : \left|\Upsilon_n^r(u_k) - \Upsilon_n^r\big(u_k^{(q_0)}\big)\right|^{p_k} < \left(\frac{\delta}{3H}\right)^M\right\} \\ &\cap \left\{k \in \mathbb{N} : \left|\Upsilon_n^r\big(u_k^{(q_0)}\big) - \Upsilon_n^r(b_{q_0})\right|^{p_k} < \left(\frac{\delta}{3H}\right)^M\right\} \\ &\cap \left\{k \in \mathbb{N} : \left|\Upsilon_n^r(b_{q_0}) - b\right|^{p_k} < \left(\frac{\delta}{3H}\right)^M\right\}\Bigg].\end{aligned}$$

The result thus follows. □

Remark 3.1 The sets $m^I(\Upsilon^r, p)$ and $m_0^I(\Upsilon^r, p)$ are nowhere dense subsets of $\ell_\infty(\Upsilon^r, p)$.

Theorem 3.9 *For $(p_k) \in \ell_\infty$, the spaces $m_0^I(\Upsilon^r, p)$ and $m^I(\Upsilon^r, p)$ are paranormed spaces with paranorm defined by*

$$G(u_k) = \sup_k \left|\Upsilon_n^r(u_k)\right|^{\frac{p_k}{M}}, \quad \textit{where } M = \max\left\{1, \sup_k p_k\right\}.$$

Proof Consider two sequences $a = (a_k)$ and $b = (b_k) \in m^I(\Upsilon^r, p)$.

(i) It is evident that $G(u) = 0$ if and only if $u = 0$.

(ii) Also $G(-u) = G(u)$.

(iii) Using Minkowski's relation on a paranorm function G and since $\frac{p_k}{M} \leq 1$, we have

$$\begin{aligned}G(a_k + b_k) &= \sup_k \left|\Upsilon_n^r(a_k + b_k)\right|^{\frac{p_k}{M}} \\ &= \sup_k \left|\Upsilon_n^r(a_k) + \Upsilon_n^r(b_k)\right|^{\frac{p_k}{M}} \\ &\leq \sup_k \left|\Upsilon_n^r(a_k)\right|^{\frac{p_k}{M}} + \sup_k \left|\Upsilon_n^r(b_k)\right|^{\frac{p_k}{M}}.\end{aligned}$$

Therefore $G(a_k + b_k) \leq G(a_k) + G(b_k)$.

(iv) Let (μ_k) be a sequence of scalars such that $\mu_k \to \mu$ as $k \to \infty$. Let $a_k \to l$ with respect to the paranorm, that is,

$$G(a_k - l) \to 0 \quad (k \to \infty);$$

using sub-additivity of the paranorm, we have

$$G(a_k) = G(a_k - l + l) \leq G(a_k - l) + G(l).$$

Thus the sequence $G(a_k)$ is bounded. Consider

$$\begin{aligned}G\big[(\mu_k a_k - \mu l)\big] &= G[\mu_k a_k - \mu a_k + \mu a_k - \mu l] \\ &= G\big[(\mu_k - \mu)a_k + \mu(a_k - l)\big]\end{aligned}$$

$$\begin{aligned}&\leq G\bigl[(\mu_k - \mu)a_k\bigr] + G\bigl[\mu(a_k - l)\bigr]\\&\leq |\mu_k - \mu|^{\frac{p_k}{M}} G(a_k) + |\mu|^{\frac{p_k}{M}} G(a_k - l) \to 0.\end{aligned}$$

Hence the space $c^I(\Upsilon^r, p)$ is a paranormed space with respect to the norm defined as above. □

Theorem 3.10 *The spaces $c_0^I(\Upsilon^r, p)$ and $m_0^I(\Upsilon^r, p)$ are solid and monotone.*

Proof Let $u = (u_k) \in c_0^I(\Upsilon^r, p)$ and $|\lambda_k| \leq 1$ be a sequence of scalars.

Since $|\lambda_k|^{p_k} \leq \max\{1, |\lambda_k|^D\} \leq 1$ for all $k \in \mathbb{N}$, we have

$$\bigl|\lambda_k \Upsilon_n^r(u_k)\bigr|^{p_k} \leq \bigl|\Upsilon_n^r(u_k)\bigr|^{p_k} \quad \forall k \in \mathbb{N}.$$

Thus

$$\bigl\{k \in \mathbb{N} : \bigl|\Upsilon_n^r(u_k)\bigr|^{p_k} \geq \epsilon\bigr\} \supseteq \bigl\{k \in \mathbb{N} : \bigl|\lambda_k \Upsilon_n^r(u_k)\bigr|^{p_k} \geq \epsilon\bigr\}.$$

Thus $(\lambda_k u_k) \in c_0^I(\Upsilon^r, p)$ and is therefore both solid and thus monotone. □

Remark 3.2 The spaces $c^I(\Upsilon^r, p)$, $m^I(\Upsilon^r, p)$ are neither solid nor monotone if I is neither maximal nor finite.

Example 3.2 Let $I = I_\delta$ and let (p_k) be the sequence of real numbers defined as

$$p_k = \begin{cases} 1, & \text{if } k = 2t, \\ 3, & \text{if } k = 2t + 1. \end{cases} \tag{3.14}$$

Now define a sequence $(v_k) \in E_k$ (the K step space) by

$$\Upsilon_n^r(v_k) = \begin{cases} \Upsilon_n^r(u_k), & k = 2t + 1, \\ 0, & k = 2t. \end{cases}$$

Consider the sequence (u_k) such that $\Upsilon_n^r(u_k) = 1 + \frac{1}{k}$. Thus $(u_k) \in c^I(\Upsilon^r, p)$ but its K-step space's pre-image is not in $c^I(\Upsilon^r, p)$.

Thus $c^I(\Upsilon^r, p)$ and $m^I(\Upsilon^r, p)$ are not monotone and therefore not solid.

Theorem 3.11 *Let $H = \sup_k p_k < \infty$ and I be an admissible ideal. The following statements are equivalent*:

(i) $(u_k) \in c^I(\Upsilon^r, p)$;

(ii) $\exists\, (v_k) \in c(\Upsilon^r, p)$ *so that* $u_k = v_k$ *for a.a.k.r.I*;

(iii) $\exists\, (v_k) \in c(\Upsilon^r, p)$ *and* $w_k \in c_0^I(\Upsilon^r, p)$ *such that* $u_k = v_k + w_k\ \forall\, k \in \mathbb{N}$ *and* $\{k \in \mathbb{N} : |\Upsilon_n^r(v_k) - l|^{p_k} \geq \epsilon\} \in I$;

(iv) *There exists a subset* $A = \{a_1 < a_2 < a_3 < a_4 \cdots\}$ *of* $\mathbb{N}$ *such that* $A \in \mathcal{F}(I)$ *and* $\lim_{j\to\infty} |\Upsilon_n^r(u_{k_j}) - l|^{p_{k_n}} = 0$.

Proof (i) $\implies$ (ii) Let $u = (u_k) \in c^I(\Upsilon^r, p)$. Thus $\exists\, l \in \mathbb{C}$ so that

$$\left\{k \in \mathbb{N} : \left|\Upsilon_n^r(u_k) - l\right|^{p_k} \geq \epsilon\right\} \in I.$$

Consider an increasing sequence $(b_m) \in \mathbb{N}$ such that

$$\left\{k \leq b_m : \left|\Upsilon_n^r(u_k) - l\right|^{p_k} \geq \frac{1}{m}\right\} \in I.$$

Define another sequence v_k as

$$u_k = v_k \quad \text{for all } k \leq b_1,$$

for $b_m < k \leq b_m + 1$, $t \in \mathbb{N}$,

$$v_k = \begin{cases} u_k, & \text{if } |\Upsilon_n^r(u_k) - l|^{p_k} < \frac{1}{m}, \\ l, & \text{otherwise.} \end{cases}$$

Then $(v_k) \in c(\Upsilon^r, p)$, and using inclusion

$$\{k \leq b_m : u_k \neq v_k\} \subseteq \left\{k \leq b_m : \left|\Upsilon_n^r(u_k) - l\right|^{p_k} \geq \epsilon\right\} \in I,$$

we conclude that $u_k = v_k$ for $a.a.k.r.I$.

(ii) $\implies$ (iii) For any sequence $u = (u_k) \in c^I(\Upsilon^r, p)$, there exists $v_k \in c(\Upsilon^r, p)$ such that $u_k = v_k$ for $a.a.k.r.I$. Let $A = \{k \in \mathbb{N} : u_k \neq v_k\}$, then $A \in I$.

Illustrate a sequence (w_k) as follows:

$$w_k = \begin{cases} u_k - v_k, & \text{if } k \in A, \\ 0, & \text{if } k \in A^c. \end{cases}$$

Then $(w_k) \in c_0^I(\Upsilon^r, p)$ and $(v_k) \in c(\Upsilon^r, p)$.

(iii) $\implies$ (iv) Let (ii) hold and for given $\epsilon > 0$, consider the sets

$$B = \left\{k \in \mathbb{N} : \left|\Upsilon_n^r(w_k)\right|^{p_k} \geq \epsilon\right\} \in I$$

and

$$K = B^c = \{k_1 < k_2 < k_3 < k_4 \cdots\} \in \mathcal{F}(I).$$

Then we have

$$\lim_{j \to \infty} \left|\Upsilon_n^r(u_{k_j}) - l\right|^{p_{k_j}} = 0.$$

(iv) $\implies$ (i) Let $A = \{k_1 < k_2 < k_3 < k_4 \cdots\} \in \mathcal{F}(I)$ and $\lim_{j\to\infty} |\Upsilon_n^r(u_{k_j}) - l|^{p_{k_j}} = 0$.

Thus, for any $\epsilon > 0$ and Lemma (2.2), we have

$$\left\{k \in \mathbb{N} : \left|\Upsilon_n^r(u_k) - l\right|^{p_k} \geq \epsilon\right\} \subseteq A^c \cup \left\{k \in \mathbb{N} : \left|\Upsilon_n^r(u_k) - l\right|^{p_k} \geq \epsilon\right\}.$$

Thus $(u_k) \in c^I(\Upsilon^r, p)$. $\square$

Theorem 3.12 *The sequence spaces $m^I(\Upsilon^r, p)$ and $m_0^I(\Upsilon^r, p)$ are not separable.*

Proof Consider an infinite subset $H = \{h_1 < h_2 < h_3 \cdots\}$ of natural numbers $\mathbb{N}$ so that $H \in I$. Let for $\lambda, \mu \in \mathbb{R}$

$$p_k = \begin{cases} \lambda, & \text{if } k \in H, \\ \mu, & \text{if } k \in H^c. \end{cases}$$

Let P be the set containing all sequences (u_k) such that

$$u_k = \begin{cases} 0 \text{ or } 1, & \text{if } k \in H, \\ 0, & \text{if } k \in H^c. \end{cases}$$

Evidently P is uncountable since H is infinite. Regard the set of open balls as

$$B_1 = \left\{ B\left(\nu, \frac{1}{\mu}\right), \nu \in P \right\}.$$

Let C be an open cover of $m^I(\Upsilon^r, p)$ containing B_1. Since B_1 is uncountable, therefore C cannot be a countable sub-cover of $m^I(\Upsilon^r, p)$. Hence the result follows. □

Theorem 3.13 $m_0^I(\Upsilon^r, p) \supseteq m_0^I(\Upsilon^r, q)$ *if* $\lim_{k\in A} \inf \frac{p_k}{q_k} > 0$, (p_k) *and* (q_k) *being sequences of positive real numbers and* $A \subseteq \mathbb{N}$ *such that* $A \in \mathcal{F}(I)$.

Proof Let $\lim_{k\in A} \inf \frac{p_k}{q_k} > 0$ and $(u_k) \in m_0^I(\Upsilon^r, q)$. Then $\exists\ \alpha > 0$ so that $p_k > \alpha q_k$ for amply large $k \in A$. Since $(u_k) \in m_0^I(\Upsilon^r, q)$, for given $\epsilon > 0$,

$$C = \left\{ k \in \mathbb{N} : \left| \Upsilon_n^r(u_k) \right|^{q_k} \geq \epsilon \right\} \in I.$$

Since $A \in \mathcal{F}(I)$ therefore $A^c \in I$. Constructing a set as a union of A^c and C, we have $S = (A^c \cup C) \in I$. Thus, for sufficiently large $k \in S$,

$$\left\{ k \in \mathbb{N} : \left| \Upsilon_n^r(u_k) \right|^{p_k} \geq \epsilon \right\} \subseteq \left\{ k \in \mathbb{N} : \left| \Upsilon_n^r(u_k) \right|^{\alpha q_k} \geq \epsilon \right\} \in I.$$

Thus $(u_k) \in m_0^I(\Upsilon^r, p)$. □

Corollary 3.1 *The spaces $m_0^I(\Upsilon^r, p)$ and $m_0^I(\Upsilon^r, q)$ are identical if* $\lim_{k\in A} \inf \frac{p_k}{q_k} > 0$ *and* $\lim_{k\in A} \inf \frac{q_k}{p_k} > 0$ *hold simultaneously where* $A \subseteq \mathbb{N}$ *such that* $A^c \in I$.

Theorem 3.14 *The spaces $c_0^I(\Upsilon^r, p)$, $c^I(\Upsilon^r, p)$, $m_0^I(\Upsilon^r, p)$ and $m^I(\Upsilon^r, p)$ are not symmetric if I is neither maximal nor $I = I_f$.*

Proof We provide the following counter example in support of our statement. □

Example 3.3 Let $I = I_f$, $B = \{k \in \mathbb{N} : k = m \text{ for } m \in \mathbb{N}\}$ and $C = \{k \in \mathbb{N} : k = n^2 \text{ for } n \in \mathbb{N}\}$.

$$A = B \cup C = \left\{ k \in \mathbb{N} : k = m \text{ or } n^2 \text{ for } m, n \in \mathbb{N} \right\}.$$

Let for $t \in \mathbb{N}$

$$p_k = \begin{cases} 1, & \text{if } k = 2t, \\ 2, & \text{if } k = 2t - 1. \end{cases}$$

Consider a sequence $(u_k) \in \omega$ such that

$$\Upsilon_n^r(u) = \begin{cases} \frac{1}{\sqrt{k}} & \text{if } k = n^2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $(u_k) \in c_0^I(\Upsilon^r, p)$. Now consider another sequence $(v_k) \in \omega$ formed after rearranging the terms of $\Upsilon_n^r(u)$ so that $(v_k) \notin c_0^I(\Upsilon^r, p)$. This concludes our result.

4 Conclusion

The paper is aimed at developing a novel generalised paranormed sequence space $\ell(\Upsilon^r, p)$ of the classical Maddox sequence space $\ell(p)$ in terms of Jordan totient operator and developing its completeness and isomorphism. Advancing forward we define the α, β and γ duals of the proposed space. The paper further explores the ideal convergent paranormed sequence spaces $c_0^I(\Upsilon^r, p)$, $c^I(\Upsilon^r, p)$, $\ell_\infty^I(\Upsilon^r, p)$ aided by Jordan totient operator. We analysed some of the spaces' topological and algebraic properties, presented compelling examples, standard inclusion relations, decomposition theorem, monotonicity and symmetricity. Researchers can employ this methodology to enhance the study on existing operators and relate it with other metric and topological spaces.

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