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Krasnoselski–Mann-type inertial method for solving split generalized mixed equilibrium and hierarchical fixed point problems

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Abstract
In this paper, we present Krasnoselski–Mann-type inertial method for solving split generalized mixed equilibrium and hierarchical fixed point problems for *k*-strictly pseudocontractive nonself-mappings. We establish that the weak convergence of the proposed accelerated iterative method with inertial terms involves a step size which does not require any prior knowledge of the operator norm under several suitable conditions in Hilbert spaces. Finally, the application to a Nash–Cournot oligopolistic market equilibrium model is discussed, and numerical examples are provided to demonstrate the effectiveness of our iterative method.

Keywords: Hierarchical fixed point problem; Split generalized mixed equilibrium problem; Maximal monotonicity; Nash–Cournot equilibrium model

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, and let C be a nonempty closed and convex subset of H. Let $f: C \to H$ be a nonlinear mapping, and let $\phi: C \to \mathbb{R}$ be a function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Then we consider the following generalized mixed equilibrium problem: Find $x^* \in C$ such that

$$F(x^*, y) + \langle fx^*, y - x^*\rangle + \phi(y) - \phi(x^*) \geq 0 \quad \text{for all } y \in C. \tag{1}$$

The set of solutions of (1) is denoted by GMEP(F, f, ϕ).

If $\phi = 0$, then the generalized mixed equilibrium problem (1) becomes the following mixed equilibrium problem: Find $x^* \in C$ such that

$$F(x^*, y) + \langle fx^*, y - x^*\rangle \geq 0 \quad \text{for all } y \in C. \tag{2}$$

The set of solutions of (2) is denoted by MEP(F, ϕ).

In particular, if $\phi = 0$ and $f = 0$, then the generalized mixed equilibrium problem (1) becomes the following equilibrium problem: Find $x^* \in C$ such that

$$F(x^*, y) \geq 0 \quad \text{for all } y \in C. \tag{3}$$

The set of solutions of (3) is denoted by $\mathrm{EP}(F)$.

On the other hand, if $F(x^*, y) = 0$, $\forall x, y \in C$, then (1) reduces to the following generalized vector variational inequality problem: Find $x^* \in C$ such that

$$\langle fx^*, y - x^* \rangle + \phi(y) - \phi(x^*) \geq 0 \quad \text{for all } y \in C. \tag{4}$$

Problem (4) was discussed in Sun and Chai [45]; it plays a critical role in algorithm design, can be used to measure how much the approximate solution fails to be in the solution set and to analyze the convergence rates of various methods. If we set $F(x^*, y) = 0$ and $\phi = 0$, $\forall x, y \in C$, then (1) reduces to the classical variational inequality problem (in short, VIP): Find $x^* \in C$ such that

$$\langle fx^*, y - x^* \rangle \geq 0 \quad \text{for all } y \in C, \tag{5}$$

which was first introduced by Giannessi [23]. Recently, several authors have studied and proposed many iterative algorithms for approximating the solutions of variational inequality problem and related optimization problems (see [25, 28, 29]).

The generalized mixed equilibrium problem is very general in the sense that it includes as a special case minimization problems, variational inequality problems, fixed point problems, Nash equilibrium problems in noncooperative games, and many others (see [2, 3, 7, 12, 16, 18, 21, 26, 48–50, 52]).

In 1994, Censor and Elfving [14] introduced the following split feasibility problem for modeling inverse problems which arise in phase retrievals and medical image reconstructions. Let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is formulated as finding a point

$$x \in C \quad \text{such that } Ax \in C.$$

The split feasibility problem has been applied extensively in many areas of science and engineering such as signal processing, image reconstruction, and intensity modulated radiation therapy. It has received attention of many authors, and various iterative methods have been proposed for finding its solutions (see [13, 15, 17]).

Next, we consider the split generalized mixed equilibrium problem (for short, S_pGMEP): Find $x^* \in C$ such that

$$F(x^*, x) + \langle fx^*, x^* - x \rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in C, \tag{6}$$

and such that

$$Ax^* \in Q \text{ solves} \quad G(Ax^*, y) + \langle g(Ax^*), Ax^* - y \rangle + \varphi(y) - \varphi(Ax^*) \geq 0, \quad \forall y \in Q, \tag{7}$$

where $F: C \times C \to \mathbb{R}$ and $G: Q \times Q \to \mathbb{R}$ are nonlinear bifunctions, $f: C \to H_1$ and $g: Q \to H_2$ are nonlinear mappings, $\phi: C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi: Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functions, and $A: H_1 \to H_2$ is a bounded linear operator. The set of solutions of the S_pGMEP is denoted by

$$\Omega = \{x^* \in \mathrm{GMEP}(F, f, \phi) : Ax^* \in \mathrm{GMEP}(G, g, \varphi)\}.$$

Jolaoso et al. [30] presented the following example to show that $\Omega \neq \emptyset$.

Example 1.1 ([30]) Let $H_1 = \mathbb{R}^2$ with the norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \in \mathbb{R}^2$, and let $H_2 = \mathbb{R}$. Let $C = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 - x_1 \geq 1\}$ and $Q \subseteq [1, \infty)$. Define $F(x, y) = y_2 - y_1 - x_2 + x_1$, where $x = (x_1, x_2)$ and $y = (y_1, y_2) \in C$. Then F is a bifunction from $C \times C$ to $\mathbb{R}$. Let $f(x) = \phi(x) = x_2 - x_1$, then $\mathrm{GMEP}(F, f, \phi) = \{q = (q_1, q_2) : q_2 - q_1 = 1\}$. Also define $G(u, v) = v - u$ for all $u, v \in Q$, so that G is a bifunction from $Q \times Q$ to $\mathbb{R}$, and let $g(u) = 2u$, $\varphi(u) = u$. For each $x = (x_1, x_2) \in H_1$, let $A(x) = x_2 - x_1$, so that A is a bounded linear operator from H_1 to H_2. Clearly, when $q \in \mathrm{GMEP}(F, f, \phi)$, we have $Aq = 1 \in \mathrm{GMEP}(G, g, \varphi)$. Thus $\Omega = \{q \in \mathrm{GMEP}(F, f, \phi) : Aq \in \mathrm{GMEP}(G, g, \varphi)\} \neq \emptyset$.

A nonself-mapping $T: C \to H_1$ is said to be k – strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H_1.$$

If $k = 0$, then T is a nonexpansive nonself-mapping.

It is known that if $\mathrm{Fix}(T) = \{x^* \in C : Tx^* = x^*\} \neq \emptyset$, then $\mathrm{Fix}(T)$ is closed and convex (see [22]).

The hierarchical fixed point problem (in short, HFPP) was introduced by Moudafi and Mainge [39] for a nonexpansive mapping T with respect to another mapping S, namely: Find $x^* \in \mathrm{Fix}(T)$ such that

$$\langle x^* - Sx^*, x^* - x\rangle \leq 0 \quad \text{for all } x \in \mathrm{Fix}(T), \tag{8}$$

where $S: C \to C$ is a nonexpansive mapping. By using the definition of the normal cone to $\mathrm{Fix}(T)$, i.e.,

$$N_{\mathrm{Fix}(T)} := \begin{cases} \{u \in H_1 : \langle y - x, u\rangle \leq 0, \forall y \in \mathrm{Fix}(T)\}, & \text{if } x \in \mathrm{Fix}(T), \\ \emptyset, & \text{otherwise}, \end{cases}$$

this amounts to saying that $x^* \in \mathrm{Fix}(T)$ satisfies a variational inequality depending on a given criterion S, namely: Find $x^* \in C$ such that

$$0 \in (I - S)x^* + N_{\mathrm{Fix}(T)}x^*, \tag{9}$$

where I is the identity on C. It is not hard to check that (8) is equivalent to the fixed point problem: Find $x^* \in C$ such that

$$x^* \in P_{\mathrm{Fix}(T)} \cdot Sx^*, \tag{10}$$

where $P_{\mathrm{Fix}(T)}$ stands for the metric projection on the closed convex set $\mathrm{Fix}(T)$. The solution set of HFPP (8) is denoted by $\Phi = \{x^* \in C : x^* = (P_{\mathrm{Fix}(T)} \cdot S)x^*\}$.

At the point, we wish to point out the link with monotone variational inequality on the fixed point set, minimization problems over equilibrium constraints, hierarchical minimization problems, etc.

By setting $S = I - rf$ where f is η-Lipschitzian and k-strongly monotone with $r \in (0, \frac{2k}{\eta^2})$, (8) reduces to: Find $x^* \in \mathrm{Fix}(T)$ such that

$$\langle x - x^*, fx^* \rangle \geq 0 \quad \text{for all } x \in \mathrm{Fix}(T),$$

which is a variational inequality in Yamada and Oqura [54]. Now, let M be a maximal monotone operator, by taking $T = J_\lambda^M = (I - \lambda M)^{-1}$ and $S = I - \gamma \nabla \psi$, where ψ is a convex function such that $\nabla \psi$ is η-Lipschitzian with $\gamma \in (0, \frac{2}{\eta})$ and using the fact that $\mathrm{Fix}(J_\lambda^M) = M^{-1}(0)$, then (8) reduces to the following mathematical program with generalized equation constraint:

$$\min_{0 \in M(x^*)} \psi(x^*), \tag{11}$$

which was considered by Luo et al. [36]. By taking $M = \partial \varphi$, where $\partial \varphi$ is the subdifferential of a lower semicontinuous convex function, problem (12) is reduced to the following hierarchical minimization problem considered by Cabot [11]:

$$\min_{x^* \in \arg\min \varphi} \psi(x^*). \tag{12}$$

We note that based upon relation (10), HFPP (8) has the iterative method $x_{n+1} = P_{\mathrm{Fix}(T)}(Sx_n)$. It will converge if a fixed point of the operator $P_{\mathrm{Fix}(T)} \cdot S$ exists, and if S is averaged, not just nonexpansive. But calculating $P_{\mathrm{Fix}(T)} \cdot S$ in this case is usually not easy. In 2007, Maudafi [39] introduced the following Krasnoselski–Mann iterative method for solving HFPP (8):

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\big(\sigma_n Sx_n + (1 - \sigma_n Tx_n)\big) \quad \text{for all } n \geq 0,$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two control sequences in $(0, 1)$. The main feature of its corresponding convergence theorems provides a unified frame for analyzing various concrete algorithms (see for instance [10, 55]). It is well known that problem (8) is often used in the area of optimization and related fields, such as signal processing and image reconstruction.

Define a mapping $S_{r_n}^F : H_1 \to C$ and $x \in H_1$ as follows:

$$S_{r_n}^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\},$$

where $F : C \times C \to \mathbb{R}$ is the bifunction and $r > 0$.

In 2017, Kazmi et al. [31] proposed a Krasnoselski–Mann-type iterative method to approximate a common solution set of a hierarchical fixed point problem for nonexpansive mappings S, T and a split mixed equilibrium problem which was defined as follows:

$$\begin{aligned} &x_0 \in C; \\ &u_n = (1-\alpha_n)x_n + \alpha_n\big(\beta_n Sx_n + (1-\beta_n)Tx_n\big); \\ &x_{n+1} = S_{r_n}^F(I - r_n f)\big(u_n + \lambda A^*\big(S_{r_n}^G(I - r_n g) - I\big)Au_n\big), \quad \forall n \geq 0, \end{aligned} \tag{13}$$

where the step size $\lambda \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of the bounded linear operator A. Under some suitable conditions on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$, they proved that the sequence $\{x_n\}$ converges weakly to a solution of hierarchical fixed point and split mixed equilibrium problems.

Recently, Kim and Majee [33] introduced a modified Krasnoselski–Mann iterative method for a common solution of split mixed equilibrium and hierarchical fixed point problems of k_i-strictly pseudocontractive nonself-mappings $\{T_i\}_{i=1}^N$ as follows: For starting point $x_0 \in C$, define $\{x_n\}$ by

$$\begin{aligned} &u_n = (1-\alpha_n)x_n + \alpha_n\big(\beta_n Sx_n + (1-\beta_n)T_n^N \cdots T_n^1 x_n\big); \\ &x_{n+1} = S_{r_n}^F(I - r_n f)\big(u_n + \delta_n A^*\big(S_{r_n}^G(I - r_n g) - I\big)Au_n\big), \quad n \geq 0, \end{aligned}$$

where

$$T_n^i = \big(1-\gamma_n^i\big)I + \gamma_n^i P_C\big(\tau_n^i I + \big(1-\tau_n^i\big)T^i\big), \quad 0 \leq k_i \leq \tau_n^i < 1, \gamma_n^i \in (0,1)$$

and the step size

$$\delta_n = \frac{\sigma_n \|(S_{r_n}^G(I - r_n g) - I)Ax_n\|}{\|A^*(S_{r_n}^G(I - r_n g) - I)Ax_n\|},$$

which does not require any prior knowledge of the operator norm. It is well known that the computation or an estimate of the spectral radius of a given operator is very difficult at times. They proved weak convergence to a solution of hierarchical fixed point and split equilibrium problems.

In general, the convergence rate of Krasnoselski–Mann iterative method and hybrid iterative method is slow. In particular, the term $\theta_n(x_n - x_{n-1})$ which is called the inertial extrapolation term was proposed as a remarkable tool for speeding up the convergence properties of iterative methods, and the inertial type algorithm has been studied and modified in various forms by many authors (see [1, 4, 6, 37, 42, 51]). For example, the inertial forward-backward splitting methods [35], the inertial Douglas–Rachford splitting methods [8] and the inertial proximal methods [40], the inertial extragradient methods [27], the inertial subgradient extragradient methods [43], the inertial shrinking projection methods [41].

Motivated and inspired by the work mentioned above, we propose a Krasnoselski–Mann-type inertial method for approximating a common solution of a hierarchical fixed point problem for a finite collection of k-strictly pseudocontractive nonself-mappings and a split generalized mixed equilibrium problem. Our iterative method combines the

Krasnoselski–Mann-type iterative method and the inertial term to obtain a new faster iterative method with a step size which does not require any prior knowledge of the operator norm and to prove a weak convergence results under some suitable conditions in real Hilbert spaces. Further, we apply our result to solve a common split mixed equilibrium and hierarchical fixed point problem. Finally, we apply generalized mixed equilibrium problems with Nash–Cournot oligopolistic market equilibrium problems and provide numerical experiments to compare the performances of our proposed iterative method with the method of Kim and Majee [33].

2 Preliminaries

In this section, we present some preliminary results that we will use in our results.

Definition 2.1 A mapping $f : H \to H$ is said to be:

(i) monotone, if

$$\langle fx - fy, x - y \rangle \geq 0, \quad \forall x, y \in H;$$

(ii) α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle fx - fy, x - y \rangle \geq \alpha \|fx - fy\|^2, \quad \forall x, y \in H;$$

(iii) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|fx - fy\| \leq \beta \|x - y\| \quad \forall x, y \in H.$$

The normal cone of a nonempty closed convex subset C of H at a point $x \in C$, denoted by $N_C(x)$, is defined as

$$N_C(x) = \{u \in H : \langle u, y - x \rangle \leq 0 \text{ for all } y \in C\}.$$

Let $M : H \to 2^H$ be a multivalued operator on H. Then the graph $G(M)$ of M is defined by

$$G(M) = \{(x, y) \in H \times H : y \in M(x)\},$$

and

(i) the operator M is called a monotone operator if

$$\langle u - v, x - y \rangle \geq 0, \quad \text{whenever } u \in M(x), v \in M(y);$$

(ii) the operator M is called a maximal monotone operator if M is monotone and the graph of M is not properly contained in the graph of other monotone mappings.

It is clear that a monotone mapping M is maximal if and only if for any $(x, u) \in H \times H$, if $\langle u - v, x - y \rangle \geq 0$ for all $(u, v) \in G(M)$, then $u \in M(x)$ (see [9]).

Lemma 2.2 ([9])

(i) *Let M be a maximal monotone mapping on H, then the $\{t_n^{-1}M\}$ graph converges to $N_{M^{-1}(0)}$ as $t_n \to 0$ provided that $M^{-1}(0) \neq \emptyset$;*

(ii) *Let $\{M_n\}$ be a sequence of maximal monotone mappings on H, the graph converges to a mapping M defined on H. If B is a Lipschitz maximal monotone mapping on H, then the $\{B + M_n\}$ graph converges to $B + M$ and $B + M$ is maximal monotone.*

Definition 2.3 A mapping $T : H_1 \to H_1$ is said to be an averaged mapping if there exists some number $\alpha \in (0,1)$ such that $T = (1-\alpha)I + \alpha S$, where $I : H_1 \to H_1$ is the identity mapping and $S : H_1 \to H_1$ is a nonexpansive mapping. An averaged mapping is also nonexpansive and $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.

Lemma 2.4 ([10]) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 \cdots T_N).$$

In particular, for $N = 2$, $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \mathrm{Fix}(T_1 T_2) = \mathrm{Fix}(T_2 T_1)$.

Lemma 2.5 ([56]) *Assume that $S : C \to H_1$ is a k-strictly pseudocontractive mapping. Define a mapping T by $Tx = \alpha x + (1-\alpha)Sx$ for all $x \in H_1$, where $\alpha \in [k,1)$. Then T is a nonexpansive mapping with $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.*

Lemma 2.6 ([56]) *Let $T : C \to H_1$ be a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $\mathrm{Fix}(P_C T) = \mathrm{Fix}(T)$.*

Lemma 2.7 ([20]) *Let $\{\psi_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ be the sequences in $[0,+\infty)$ such that $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$, and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 1$. Then the following hold:*

(i) *$\sum_{n\geq 1}[\psi_n - \psi_{n-1}]_+ < +\infty$, where $[t]_+ = \max\{t, 0\}$;*

(ii) *There exists $\psi^* \in [0,+\infty)$ such that $\lim_{n\to+\infty} \psi_n = \psi^*$.*

Lemma 2.8 ([5]) *Let C be a nonempty subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(i) *For any $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*

(ii) *Every sequential weak cluster point of $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

3 Convergence analysis

In this position, we establish the convergence of the Krasnoselski–Mann-type inertial method and assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following.

Assumption 3.1

L1. $F(x,x) = 0$ for all $x \in C$;

L2. F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

L3. For each $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;

L4. For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous.

The following properties are associated with a nonempty closed and convex subset C of H:

B1. For each $x \in H$ and $r > 0$, there exist a bounded subset $D \subseteq C$ and $y \in C$ such that, for any $z \in D$,

$$F(z,y) + \phi(y) + \frac{1}{r}\langle y - x, z - x\rangle < \phi(z);$$

B2. C is a bounded set.

It is easy to show that, under Assumption 3.1, the solution of GMEP(F, f, ϕ) is nonempty, closed, and convex (see, for instance, [44]). We present the following Krasnoselski–Mann-type inertial method for solving split generalized mixed equilibrium and hierarchical fixed point problems for k-strictly pseudocontractive nonself-mappings.

Algorithm 1 Krasnoselski–Mann-type inertial method

Initialization. Select $\theta \in [0,1)$ and a positive sequence $\{\epsilon_n\} \subset [0,\infty)$ satisfying $\sum_{n=0}^{\infty} \epsilon_n < \infty$. Choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}. \end{cases}$$

Choose initial iterates $x_0, x_1 \in C$, and set $n = 1$.

Step 1. Compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ u_n &= (1-\alpha_n)w_n + \alpha_n\bigl(\beta_n S w_n + (1-\beta_n)T_n^N \cdots T_n^1 w_n\bigr), \end{aligned}$$

where $T_n^i = (1-\gamma_n^i)I + \gamma_n^i P_C(\tau_n^i I + (1-\tau_n^i)T^i)$, $0 \le k_i \le \tau_n^i < 1$, $\gamma_n^i \in (0,1)$.

Step 2. Compute

$$x_{n+1} = U\bigl(u_n + \delta_n A^*(V-I)Au_n\bigr), \quad n \ge 1,$$

where $U = T_{r_n}^F(I - r_n f)$, $V = T_{r_n}^G(I - r_n g)$, and $\delta_n = \frac{\sigma_n \|(T_{r_n}^G(I-r_n g)-I)Au_n\|^2}{\|A^*(T_{r_n}^G(I-r_n g)-I)Au_n\|^2}$, $0 < a \le \sigma_n \le b < 1$.

Set $n = n + 1$, and return to **Step 1**.

Remark 1 From

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}, \end{cases},$$

we have

$$\sum_{n=1}^{\infty} \theta_n(x_n - x_{n-1}) < \infty.$$

Now, we prove a weak convergence theorem to approximate a common solution of split generalized mixed equilibrium and hierarchical fixed point problems for k-strictly pseudocontractive nonself-mappings.

Theorem 3.1 *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Assume that $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumption 3.1. Let $f : C \to H_1$ and $g : Q \to H_2$ be κ_1, κ_2-inverse strongly monotone mappings, $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $S : C \to C$ be a nonexpansive mapping and $\{T^i\}_{i=1}^N : C \to H_1$ be k_i-strictly pseudocontractive nonself-mappings. Assume that either* B1 *or* B2 *holds and $\Psi = \Omega \cap \Phi \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by Algorithm 1 and the following conditions are satisfied*:

(i) $\liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \beta_n < +\infty$;

(ii) $\lim_{n\to\infty} \frac{\|u_n - w_n\|}{\alpha_n \beta_n} = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ converges weakly to $x^* \in \Psi$.

Proof We now divide the remaining proof into several steps.

First, we will prove that $\{x_n\}$, $\{u_n\}$, and $\{w_n\}$ are bounded.

Since $f : C \to H_1$ is a κ_1-inverse strongly monotone mapping, then for any $x, y \in C$, we have

$$\begin{aligned} \bigl\|(I - r_n f)x - (I - r_n f)y\bigr\|^2 &= \bigl\|(x - y) - r_n(fx - fy)\bigr\|^2 \\ &= \|x - y\|^2 - r_n(2\kappa_1 - r_n)\|fx - fy\|^2 = \|x - y\|^2, \end{aligned}$$

which shows that $(I - r_n f)$ is nonexpansive. Similarly, $(I - r_n g)$ is nonexpansive. So $T_{r_n}^F(I - r_n f)$, $T_{r_n}^G(I - r_n g)$ are nonexpansive. Let $x^* \in \Psi$. Then $x^* \in \Phi$ and $x^* \in \Omega$ which imply that

$$\begin{aligned} \bigl\|w_n - x^*\bigr\| &= \bigl\|x_n + \theta_n(x_n - x_{n-1}) - x^*\bigr\| \\ &\le \bigl\|x_n - x^*\bigr\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{14}$$

From Lemma 2.4, Lemma 2.5, and Lemma 2.6, we get $x^* = T_n^2 T_n^1 x^*$. Hence, we have

$$\begin{aligned} \bigl\|u_n - x^*\bigr\| &= \bigl\|(1 - \alpha_n)w_n + \alpha_n\bigl(\beta_n S w_n + (1 - \beta_n)T_n^N \cdots T_n^1 w_n\bigr) - x^*\bigr\| \\ &\le (1 - \alpha_n)\bigl\|w_n - x^*\bigr\| + \alpha_n\bigl[\beta_n\bigl\|Sw_n - x^*\bigr\| + (1 - \beta_n)\bigl\|T_n^2 T_n^1 w_n - x^*\bigr\|\bigr] \\ &\le (1 - \alpha_n)\bigl\|w_n - x^*\bigr\| + \alpha_n\bigl[\beta_n\bigl\|w_n - x^*\bigr\| + (1 - \beta_n)\bigl\|T_n^2 T_n^1 w_n - x^*\bigr\| \\ &\quad + \alpha_n\beta_n\bigl\|Sx^* - x^*\bigr\|\bigr] \\ &= \bigl\|w_n - x^*\bigr\| + \alpha_n\beta_n\bigl\|Sx^* - x^*\bigr\|. \end{aligned} \tag{15}$$

Also, since $x^* \in \Psi$, we have $Ux^* = x^*$ and $VAx^* = Ax^*$. Let $y_n = u_n + \delta_n A^*(V - I)Au_n$. Then we get

$$\begin{aligned} \bigl\|y_n - x^*\bigr\|^2 &= \bigl\|u_n + \delta_n A^*(V - I)Au_n - x^*\bigr\|^2 \\ &= \bigl\|u_n - x^*\bigr\|^2 + 2\delta_n\bigl\langle u_n - x^*, A^*(V - I)Au_n\bigr\rangle + \lambda^2\|A\|^2\|(V - I)Au_n)\|^2. \end{aligned} \tag{16}$$

Observe that

$$\begin{aligned}
&\langle u_n - x^*, A^*(V-I)Au_n\rangle \\
&\quad = \langle Au_n - Ax^*, (V-I)Au_n\rangle \\
&\quad = \langle Au_n - Ax^* + (V-I)Au_n - (V-I)Au_n, (V-I)Au_n\rangle \\
&\quad = \left(\langle Au_n - Ax^*, (V-I)Au_n\rangle - \left\|(V-I)Au_n\right\|^2\right) \\
&\quad = \frac{1}{2}\left(\left\|Au_n - Ax^*\right\|^2 + \left\|(V-I)Au_n\right\|^2 - \left\|Au_n - Ax^*\right\|^2 - 2\left\|(V-I)Au_n\right\|^2\right) \\
&\quad = \frac{1}{2}\left(\left\|Au_n - Ax^*\right\|^2 - \left\|Au_n - Ax^*\right\|^2 - \left\|(V-I)Au_n\right\|^2\right) \\
&\quad = -\frac{1}{2}\left\|(V-I)Au_n\right\|^2.
\end{aligned} \tag{17}$$

From (16) and (17), we obtain

$$\left\|y_n - x^*\right\|^2 \leq \left\|u_n - x^*\right\|^2 - \delta_n\left(\left\|(V-I)Au_n\right\|^2 - \delta_n\left\|A^*(V-I)Au_n\right\|^2\right). \tag{18}$$

It follows that

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &= \left\|U\left(u_n + \delta_n A^*(V-I)Au_n\right) - x^*\right\|^2 \\
&= \left\|\left(u_n + \delta_n A^*(V-I)Au_n\right) - x^*\right\|^2 \\
&= \left\|y_n - x^*\right\|^2 \\
&\leq \left\|u_n - x^*\right\|^2 - \delta_n\left(\left\|(V-I)Au_n\right\|^2 - \delta_n\left\|A^*(V-I)Au_n\right\|^2\right).
\end{aligned} \tag{19}$$

From (14),(15), and (19), we get

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &\leq \left\|u_n - x^*\right\| \\
&\leq \left\|w_n - x^*\right\| + \alpha_n\beta_n\left\|Sx^* - x^*\right\| \\
&\leq \left\|x_n - x^*\right\| + \theta_n\|x_n - x_{n-1}\| + \alpha_n\beta_n\left\|Sx^* - x^*\right\|.
\end{aligned} \tag{20}$$

Since $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$, we have $\sum_{n=1}^{\infty}\alpha_n\beta_n < \infty$. By using Lemma 2.7, we conclude that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Hence $\{x_n\}$ is bounded, and so $\{u_n\}$ and $\{w_n\}$. Now, from (19), we get

$$\begin{aligned}
&\delta_n\left(\left\|(V-I)Au_n\right\|^2 - \delta_n\left\|A^*(V-I)Au_n\right\|^2\right) \\
&\quad \leq \left\|u_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&\quad \leq \left(\left\|w_n - x^*\right\| + \alpha_n\beta_n\left\|Sx^* - x^*\right\|\right)^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&\quad \leq \left(\left\|x_n - x^*\right\| + \theta_n\|x_n - x_{n-1}\| + \alpha_n\beta_n\left\|Sx^* - x^*\right\|\right)^2 \\
&\qquad - \left\|x_{n+1} - x^*\right\|^2 \\
&\quad \leq \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + \alpha_n\beta_n\left[2\left\|x_n - x^*\right\|\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n \beta_n \big\| Sx^* - x^* \big\| \big] + 2\theta_n \|x_n - x_{n-1}\| \big[\big\| x_n - x^* \big\| \\
&+ \alpha_n \beta_n \big\| Sx^* - x^* \big\| \big]. \qquad (21)
\end{aligned}
$$

Since $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$, we get

$$\lim_{n\to\infty} \delta_n \big(\big\| (V-I)Au_n \big\|^2 - \delta_n \big\| A^*(V-I)Au_n \big\|^2 \big) = 0,$$

which by the definition of δ_n implies that

$$\lim_{n\to\infty} \frac{\sigma_n(1-\sigma_n)\|(V-I)Au_n\|^4}{\|A^*(V-I)Au_n\|^2} = 0.$$

Since $0 < a \le \sigma_n \le b < 1$ and $\|(V-I)Au_n\|$ is bounded, we get

$$\lim_{n\to\infty} \big\| (V-I)Au_n \big\| = 0. \qquad (22)$$

Now

$$\lim_{n\to\infty} \big\| A^*(V-I)Au_n \big\| = \lim_{n\to\infty} \|A\| \big\| (V-I)Au_n \big\| = 0. \qquad (23)$$

So

$$\lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \big\| A^*(V-I)Au_n \big\| = 0. \qquad (24)$$

Now, we estimate

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \big\| \big(x_{n+1} - x^*\big) - \big(x_n - x^*\big) \big\|^2 \\
&\le \big\| x_{n+1} - x^* \big\|^2 - \big\| x_n - x^* \big\|^2 - 2\big\langle x_{n+1} - x_n, x_n - x^* \big\rangle \\
&\le \big\| x_{n+1} - x^* \big\|^2 - \big\| x_n - x^* \big\|^2 - 2\big\langle x_{n+1} - p, x_n - x^* \big\rangle \\
&\quad + 2\big\langle x_n - p, x_n - x^* \big\rangle, \qquad (25)
\end{aligned}
$$

where p is a weak cluster point of $\{x_n\}$. Since $\lim_{n\to\infty} \|x_n - x^*\|$ exists, then (25) implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (26)$$

From $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$, we obtain that

$$\|w_n - x_n\| = \big\| x_n + \theta_n(x_n - x_{n-1}) - x_n \big\| = \theta_n \|x_n - x_{n-1}\|$$

and

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \qquad (27)$$

Since $\liminf_{n\to\infty} r_n > 0$, there is a number $r > 0$ such that $r_n > r$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|T^F_{r_n}(I - rf)y_n - T^F_{r_n}(I - rf)x^*\|^2 \\
&= \|(I - rf)y_n - (I - rf)x^*\|^2 \\
&= \|y_n - x^*\|^2 - 2r\langle y_n - x^*, fy_n - fx^*\rangle + r^2\|fy_n - fx^*\|^2 \\
&\le \|y_n - x^*\|^2 - 2r\kappa_1\|fy_n - fx^*\|^2 + r^2\|fy_n - fx^*\|^2 \\
&= \|y_n - x^*\|^2 - r(2\kappa_1 - r)\|fy_n - fx^*\|^2 \\
&\le \|y_n - x^*\|^2 - r(2\kappa_1 - r)\|fy_n - fx^*\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
r(2\kappa_1 - r)\|fy_n - fx^*\|^2 &\le \|y_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \|u_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \left(\|w_n - x^*\| + \alpha_n\beta_n\|Sx^* - x^*\|\right)^2 - \|x_{n+1} - x^*\|^2 \\
&\le \left(\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\| + \alpha_n\beta_n\|Sx^* - x^*\|\right)^2 \\
&\quad - \|x_{n+1} - x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\beta_n\big[2\|x_n - x^*\| \\
&\quad + \alpha_n\beta_n\|Sx^* - x^*\|\big] + 2\theta_n\|x_n - x_{n-1}\|\big[\|x_n - x^*\| \\
&\quad + \alpha_n\beta_n\|Sx^* - x^*\|\big].
\end{aligned}$$

Since $r(2\kappa_1 - r) > 0$, $\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0$, $\sum_{n=1}^{\infty} \beta_n < +\infty$, and $\lim_{n\to\infty} \|x_n - x^*\|$ exists, we have

$$\lim_{n\to\infty} \|fy_n - fx^*\| = 0. \tag{28}$$

Since $T^F_{r_n}$ is firmly nonexpansive, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|T^F_{r_n}(I - r_nf)y_n - T^F_{r_n}(I - r_nf)x^*\|^2 \\
&\le \langle (I - rf)y_n - (I - rf)x^*, x_{n+1} - x^*\rangle \\
&= \frac{1}{2}\big[\|(I - rf)y_n - (I - rf)x^*\|^2 + \|x_{n+1} - x^*\|^2 \\
&\quad - \|y_n - x_{n+1} - r(fy_n - fx^*)\|^2\big] \\
&\le \frac{1}{2}\big[\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \\
&\quad - \|y_n - x_{n+1}\|^2 + 2r\langle y_n - x_{n+1}, fy_n - fx^*\rangle \\
&\quad - r^2\|fy_n - fx^*\|^2\big] \\
&\le \frac{1}{2}\big[\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \\
&\quad - \|y_n - x_{n+1}\|^2 + 2r\|y_n - x_{n+1}\|\|fy_n - fx^*\|\big],
\end{aligned}$$

which in turn yields

$$\|x_{n+1}-x^*\|^2 \le \|y_n-x^*\|^2 - \|y_n-x_{n+1}\|^2 + 2r\|y_n-x_{n+1}\|\|fy_n-fx^*\|,$$

and this together with (14), (15), (16), and (18) implies that

$$\begin{aligned}
\|y_n-x_{n+1}\|^2 &\le \|y_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + 2r\|y_n-x_{n+1}\|\|fy_n-fx^*\| \\
&\le \|u_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + 2r\|y_n-x_{n+1}\|\|fy_n-fx^*\| \\
&\le \left(\|w_n-x^*\| + \alpha_n\beta_n\|Sx^*-x^*\|\right)^2 - \|x_{n+1}-x^*\|^2 \\
&\quad + 2r\|y_n-x_{n+1}\|\|fy_n-fx^*\| \\
&\le \left(\|x_n-x^*\| + \theta_n\|x_n-x_{n-1}\| + \alpha_n\beta_n\|Sx^*-x^*\|\right)^2 - \|x_{n+1}-x^*\|^2 \\
&\quad + 2r\|y_n-x_{n+1}\|\|fy_n-fx^*\| \\
&\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \alpha_n\beta_n\big[2\|x_n-x^*\| \\
&\quad + \alpha_n\beta_n\|Sx^*-x^*\|\big] + 2\theta_n\|x_n-x_{n-1}\|\big[\|x_n-x^*\| \\
&\quad + \alpha_n\beta_n\|Sx^*-x^*\|\big] + 2r\|y_n-x_{n+1}\|\|fy_n-fx^*\|.
\end{aligned}$$

From (28), we get

$$\lim_{n\to\infty}\|y_n-x_{n+1}\| = 0. \tag{29}$$

It follows that

$$\|x_n-y_n\| \le \|x_n-x_{n+1}\| + \|x_{n+1}-y_n\|.$$

Thus

$$\lim_{n\to\infty}\|y_n-x_n\| = 0. \tag{30}$$

Since

$$\|x_n-u_n\| \le \|x_n-y_n\| + \|y_n-u_n\|,$$

from (24) and (30), we have

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{31}$$

Furthermore,

$$\|u_n-w_n\| \le \|u_n-x_n\| + \|x_n-w_n\|,$$

from (24) and (27), we have

$$\lim_{n\to\infty}\|u_n-w_n\| = 0. \tag{32}$$

Next, we will show that $x^* \in \Psi$. Since $T_n^2 T_n^1$ is an averaged mapping, it is nonexpansive. Using the boundedness of $\{w_n\}$ and the nonexpansivity of S, there exists $K > 0$ such that $\|Sw_n - T_n^2 T_n^1 w_n\| \leq K$ for all $n \geq 1$. Consider

$$\begin{aligned}\|u_n - T_n^2 T_n^1 w_n\| &= \|(1-\alpha_n)w_n + \alpha_n(\beta_n Sw_n + (1-\beta_n)T_n^2 T_n^1 w_n) - T_n^2 T_n^1 w_n\| \\ &\leq (1-\alpha_n)\|w_n - T_n^2 T_n^1 w_n\| + \alpha_n\beta_n\|Sw_n - T_n^2 T_n^1 w_n\| \\ &\leq (1-\alpha_n)\|u_n - w_n\| + (1-\alpha_n)\|u_n - T_n^2 T_n^1 w_n\| \\ &\quad + \alpha_n\beta_n\|Sw_n - T_n^2 T_n^1 w_n\|,\end{aligned}$$

which implies that

$$\begin{aligned}\alpha_n\|u_n - T_n^2 T_n^1 w_n\| &\leq (1-\alpha_n)\|u_n - w_n\| + \alpha_n\beta_n\|Sw_n - T_n^2 T_n^1 w_n\|. \\ &\leq (1-\alpha_n)\|u_n - w_n\| + \alpha_n\beta_n K.\end{aligned}$$

So

$$\|u_n - T_n^2 T_n^1 w_n\| \leq \frac{\|u_n - w_n\|}{\alpha_n} + \beta_n K. \tag{33}$$

It follows from conditions (I)–(II) that

$$\lim_{n\to\infty}\|u_n - T_n^2 T_n^1 w_n\| = 0. \tag{34}$$

Since

$$\|w_n - T_n^2 T_n^1 w_n\| \leq \|w_n - u_n\| + \|u_n - T_n^2 T_n^1 w_n\|,$$

from (32) and (34), we get

$$\lim_{n\to\infty}\|w_n - T_n^2 T_n^1 w_n\| = 0. \tag{35}$$

Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_k}\}$ that weakly converges to x^*. Using the boundedness of $\{\gamma_n^i\}$ for $i = 1, 2$, we can assume that $\gamma_{n_k}^i \to \gamma_\infty^i$ as $k \to \infty$, where $0 < \gamma_\infty^i < 1$ for $i = 1, 2$. Let

$$T_\infty^i = (1-\gamma_\infty^i)I + \gamma_\infty^i P_C(\tau^i I + (1-\tau^i)T^i), \quad \forall i = 1, 2.$$

Now, by Lemma 2.5 and Lemma 2.6, $\operatorname{Fix}(P_C(\tau^i I + (1-\tau^i)T^i)) = \operatorname{Fix}(T^i)$. Again, since $P_C(\tau^i I + (1-\tau^i)T^i)$ is a nonexpansive mapping, T_∞^i is averaged and $\operatorname{Fix}(T_\infty^i) = \operatorname{Fix}(T^i)$ for $i = 1, 2$. Since

$$\operatorname{Fix}(T_\infty^1) \cap \operatorname{Fix}(T_\infty^2) = \operatorname{Fix}(T^1) \cap \operatorname{Fix}(T^2) = \operatorname{Fix}(\Psi) \neq \emptyset,$$

by Lemma 2.4, we get

$$\operatorname{Fix}(T_\infty^1 T_\infty^2) = \operatorname{Fix}(T_\infty^1) \cap \operatorname{Fix}(T_\infty^2) = \operatorname{Fix}(\Psi).$$

Since

$$\left\| T^i_{n_k}(t) - T^i_\infty(t) \right\| \le \left| \gamma^i_{n_k} - \gamma^i_\infty \right| \left(\|t\| + \left\| P_C\left(\tau^i t + \left(1 - \tau^i\right) T^i(t) \right) \right\| \right),$$

we get

$$\lim_{k\to\infty} \sup_{t\in B} \left\| T^i_{n_k}(t) - T^i_\infty(t) \right\| = 0, \tag{36}$$

where B is an arbitrary bounded subset of H_1. Also, we have

$$\begin{aligned}
\left\| w_{n_k} - T^2_\infty T^1_\infty w_{n_k} \right\| &\le \left\| w_{n_k} - T^2_{n_k} T^1_{n_k} w_{n_k} \right\| + \left\| T^2_{n_k} T^1_{n_k} w_{n_k} - T^2_\infty T^1_{n_k} w_{n_k} \right\| \\
&\quad + \left\| T^2_\infty T^1_{n_k} w_{n_k} - T^2_\infty T^1_\infty w_{n_k} \right\| \\
&\le \left\| w_{n_k} - T^2_{n_k} T^1_{n_k} w_{n_k} \right\| + \left\| T^2_{n_k} T^1_{n_k} w_{n_k} - T^2_\infty T^1_{n_k} w_{n_k} \right\| \\
&\quad + \left\| T^1_{n_k} w_{n_k} - T^1_\infty w_{n_k} \right\| \\
&\le \left\| w_{n_k} - T^2_{n_k} T^1_{n_k} w_{n_k} \right\| + \sup_{t\in B_1} \left\| T^2_{n_k} t - T^2_\infty t \right\| \\
&\quad + \sup_{t\in B_2} \left\| T^1_{n_k} t - T^1_\infty t \right\|,
\end{aligned} \tag{37}$$

where B_1 is a bounded subset of $\{T^1_{n_k} x_{n_k}\}$ and B_2 is a bounded subset of $\{x_{n_k}\}$. It follows from (35), (36), and (37) that

$$\lim_{k\to\infty} \left\| w_{n_k} - T^2_\infty T^1_\infty w_{n_k} \right\| = 0.$$

Hence, from Lemma 2.4, we get $x^* \in \operatorname{Fix}(T^2_\infty T^1_\infty) = \operatorname{Fix}(T^1) \cap \operatorname{Fix}(T^2)$. Again from Algorithm 1, we have

$$u_n - w_n = \alpha_n \left(\beta_n (S w_n - w_n) - (1 - \beta_n) \left(T^2_n T^1_n w_n - w_n \right) \right),$$

and hence

$$\frac{1}{\alpha_n \beta_n} (w_n - u_n) = \left((I - S) + \left(\frac{1 - \beta_n}{\beta_n} \right) \left(I - T^2_n T^1_n \right) \right) w_n. \tag{38}$$

By Lemma 2.2(i) guarantees that the operator sequence $\{(\frac{1-\beta_n}{\beta_n})(I - T^2_n T^1_n)\}$ graph converges to $N_{\operatorname{Fix}(T^1)\cap\operatorname{Fix}(T^2)}$, and hence it follows from Lemma 2.2(ii) that the operator sequence $\{(I - S) + (\frac{1-\beta_n}{\beta_n})(I - T^2_n T^1_n)\}$ graph converges to $(I - S) + N_{\operatorname{Fix}(T^1)\cap\operatorname{Fix}(T^2)}$. Now, by replacing n with n_k and by passing to the limit in (38) as $k \to \infty$ and by taking into account the fact that $\lim_{n\to\infty} \frac{1}{\alpha_n\beta_n} \|w_n - u_n\| = 0$ and that the graph of $(I - S) + N_{\operatorname{Fix}(T^1)\cap\operatorname{Fix}(T^2)}$ is weakly-strongly closed, we obtain that $0 \in (I - S)x^* + N_{\operatorname{Fix}(T^1)\cap\operatorname{Fix}(T^2)} x^*$, so $x^* \in \Phi$.

Finally, we show that $x^* \in \Omega$. Since $x_{n+1} = U(y_n) = T^F_{r_n}(I - r_n f)(y_n)$, we have

$$F(x_{n+1}, y) + \langle f y_n, y - x_{n+1} \rangle + \phi(y) - \phi(x_{n+1}) + \frac{1}{r_n} \langle y - x_{n+1}, x_{n+1} - y_n \rangle \ge 0, \quad \forall y \in C.$$

Since F is monotone, the above inequality implies

$$\langle fy_n, y - x_{n+1}\rangle + \phi(y) - \phi(x_{n+1}) + \frac{1}{r_n}\langle y - x_{n+1}, x_{n+1} - y_n\rangle \geq F(y, x_{n+1}), \quad \forall y \in C,$$

and hence replacing n with n_k in the above inequality, we have

$$\begin{aligned} &\langle fy_{n_k}, y - x_{n_k+1}\rangle + \left\langle y - x_{n_k+1}, \frac{x_{n_k+1} - y_{n_k}}{r_{n_k}}\right\rangle \\ &\quad \geq F(y, x_{n_k+1}) - \phi(y) + \phi(x_{n_k} + 1), \quad \forall y \in C. \end{aligned} \tag{39}$$

Further, for any $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)x^*$. Since $x^* \in C$ and $y \in C$, then $y_t \in C$. From the monotonicity of F, the above inequality implies

$$\begin{aligned} \langle y_t - x_{n_k+1}, fy_t\rangle &\geq \langle y_t - x_{n_k+1}, fy_t - fx_{n_k+1}\rangle + \langle y_t - x_{n_k+1}, fx_{n_k+1} - fy_{n_k}\rangle \\ &\quad - \left\langle y_t - x_{n_k+1}, \frac{x_{n_k+1} - y_{n_k}}{r_{n_k}}\right\rangle + F(y_t, x_{n_k+1}) - \phi(y_t) + \phi(x_{n_k+1}). \end{aligned} \tag{40}$$

Since the sequences $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ have the same asymptotic behavior and $x_{n_k} \to x^*$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to x^*$. Since $\lim_{k\to\infty}\|x_{n_k+1} - y_{n_k}\| = 0$ and f is Lipschitz continuous, we have $\lim_{k\to\infty}\|fx_{n_k+1} - fy_{n_k}\| = 0$. From the condition of $\liminf_{n\to\infty} r_n > 0$, there exists a number $r > 0$ such that $\liminf_{n\to\infty} r_n = r$, which implies that

$$\begin{aligned} \lim_{k\to\infty}\frac{\|x_{n_k+1} - y_{n_k}\|}{r_{n_k}} &\leq \frac{\lim_{k\to\infty}\|x_{n_k+1} - y_{n_k}\|}{\lim_{k\to\infty} r_{n_k}} \\ &= \frac{1}{r}\lim_{k\to\infty}\|x_{n_k+1} - y_{n_k}\| = 0. \end{aligned}$$

From the monotonicity of f, we have

$$\langle y_t - x_{n_k+1}, fy_t - fx_{n_k+1}\rangle \geq 0.$$

Therefore, by L4 and the weak lower semicontinuity of ϕ, taking the limit of (40) as $k \to \infty$, we get

$$\langle y_t - x^*, fy_t\rangle \geq F(y_t, x^*) + \phi(x^*) - \phi(y_t). \tag{41}$$

The convexity of F and (41) imply that

$$\begin{aligned} 0 &= F(y_t, y_t) + \phi(y_t) - \phi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, x^*) + tF(y) + (1-t)\phi(x^*) - \phi(y_t) \\ &= t\big(F(y_t, y) + \phi(y) - \phi(y_t)\big) + (1-t)\big(F(y_t, x^*) + \phi(x^*) - \phi(y_t)\big) \\ &\leq t\big(F(y_t, y) + \phi(y) - \phi(y_t)\big) + (1-t)\langle y_t - x^*, fy_t\rangle \\ &\leq t\big(F(y_t, y) + \phi(y) - \phi(y_t)\big) + (1-t)t\langle y - x^*, fy_t\rangle, \end{aligned}$$

so

$$F(y_t,y)+\phi(y)-\phi(y_t)+(1-t)\langle y-x^*,fy_t\rangle\geq 0.$$

Letting $t\to 0$, then for each $y\in C$, we have

$$F\bigl(x^*,y\bigr)+\phi(y)-\phi\bigl(x^*\bigr)+\langle y-x^*,fx^*\rangle\geq 0.$$

This implies that $x^*\in \mathrm{GMEP}(F,f,\phi)$. Since A is a bounded linear operator, then $Au_{n_k}\rightharpoonup Ax^*$. Now, setting $s_{n_k}=Au_{n_k}-VAu_{n_k}$, it follows from (22) that $\lim_{k\to\infty}s_{n_k}=0$ and $Au_{n_k}-s_{n_k}=VAu_{n_k}$. Therefore,

$$\begin{aligned}&G(Au_{n_k}-s_{n_k},z)+\langle gAu_{n_k},z-(Au_{n_k}-s_{n_k})\rangle+\varphi(z)-\varphi(Au_{n_k}-s_{n_k})\\&\quad+\frac{1}{r_n}\langle z-(Au_{n_k}-s_{n_k}),Au_{n_k}-s_{n_k}-Au_{n_k}\rangle\geq 0,\quad \forall z\in Q.\end{aligned}$$

Since G is upper semicontinuous in the first argument and taking lim sup of the above inequality as $k\to\infty$, we get

$$G\bigl(Ax^*,z\bigr)+\langle z-Ax^*,gAx^*\rangle+\varphi(z)-\varphi\bigl(Ax^*\bigr)\geq 0\quad \forall z\in Q.$$

This implies that $Ax^*\in \mathrm{GMEP}(G,g,\phi)$. Therefore $x^*\in\Omega$, and we can conclude that $x^*\in\Psi$. It follows from the existence of $\lim_{n\to\infty}\|x_n-x^*\|$ and the Opial condition that $\{x_n\}$ has only one weak cluster point and hence $\{x_n\}$ converges weakly to $x^*\in\Psi$. □

The following consequence is a weak convergence theorem for computing a common solution of a split mixed equilibrium problem (for short $\mathrm{S}_p\mathrm{MEP}$) and a hierarchical fixed point problem in Hilbert spaces.

If we set $\phi=0$ and $\varphi=0$ in (6)–(7), then $\mathrm{S}_p\mathrm{GMEP}$ is reduced to the split mixed equilibrium problem (in short $\mathrm{S}_p\mathrm{MEP}$): Find $x^*\in C$ such that

$$F\bigl(x^*,u\bigr)+\langle fx^*,u-x^*\rangle\geq 0\quad \forall u\in C, \tag{42}$$

and such that

$$Ax^*\in Q \text{ solves}\quad G\bigl(Ax^*,v\bigr)+\langle g\bigl(Ax^*\bigr),v-Ax^*\rangle\geq 0,\quad \forall v\in Q. \tag{43}$$

The solution set of the $\mathrm{S}_p\mathrm{MEP}$ is denoted by

$$\Omega_1=\bigl\{x^*\in \mathrm{Sol}\bigl(\mathrm{MEP}\,(42)\bigr):Ax^*\in \mathrm{Sol}\bigl(\mathrm{MEP}\,(43)\bigr)\bigr\}.$$

Corollary 3.2 *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A:H_1\to H_2$ be a bounded linear operator with its adjoint operator A^*. Assume that $F:C\times C\to\mathbb{R}$ and $G:Q\times Q\to\mathbb{R}$ are bifunctions satisfying Assumption 3.1. Let $f:C\to H_1$ and $g:Q\to H_2$ be κ_1,κ_2-inverse strongly monotone mappings, and let $S:C\to C$ be a nonexpansive mapping and $\{T^i\}_{i=1}^N:C\to H_1$ be k_i-strictly* *pseudocontractive nonself-mappings. Assume that* $\Psi_1 = \Omega_1 \cap \Phi \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}); \\ u_n = (1-\alpha_n)w_n + \alpha_n(\beta_n S w_n + (1-\beta_n)T_n^N \cdots T_n^1 w_n); \\ x_{n+1} = U(u_n + \delta_n A^*(V-I)Au_n), \quad n \geq 1, \end{cases} \tag{44}$$

where $U = S_{r_n}^F(I - r_n f)$, $V = S_{r_n}^G(I - r_n g)$, $T_n^i = (1-\gamma_n^i)I + \gamma_n^i P_C(\tau_n^i I + (1-\tau_n^i)T^i)$, $0 \leq k_i \leq \tau_n^i < 1$, $\gamma_n^i \in (0,1)$, $\delta_n = \frac{\sigma_n \|(T_{r_n}^G(I-r_n g)-I)Ax_n\|^2}{\|A^*(T_{r_n}^G(I-r_n g)-I)Ax_n\|^2}$, $0 < a \leq \sigma_n \leq b < 1$, *and* θ_n *satisfies Algorithm 1, and the following conditions are satisfied*:

(i) $\liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \beta_n < +\infty$;

(ii) $\lim_{n\to\infty} \frac{\|u_n - w_n\|}{\alpha_n \beta_n} = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *converges weakly to* $x^* \in \Psi_1$.

4 Nash–Cournot oligopolistic market equilibrium problem

In this section, we apply generalized mixed equilibrium problems with Nash–Cournot oligopolistic market equilibrium problems which have been introduced by Cournot and studied by some authors [19, 20, 34]. An oligopolistic market model considers n firms (producers) that produce a common homogeneous commodity. Each firm has a profit function which is the difference between the price and the cost. Each firm attempts to maximize its profit by choosing the corresponding production level on its strategy set.

Consider that there are n-firms which produce a common homogenous commodity and the price p_i of firm i depends on the total quantity $\sigma = \sum_{i=1}^n x_i$ of the commodity. Let $h_i(x_i)$ denote the cost of the firm i when its production level is x_i. Suppose that the profit of firm i is given by

$$f_i(x_1, \ldots, x_n) = x_i p_i\left(\sum_{i=1}^n x_i\right) - h_i(x_i), \quad i = 1, \ldots, n, \tag{45}$$

where h_i is the cost function of firm i that is assumed to be dependent only on its production level.

Let $U_i \subset \mathbb{R}$ for $i = 1, \ldots, n$ denote the strategy set of the firm i. Each firm seeks to maximize its own profit by choosing the corresponding production level under the hypothesis that the production of other firms is parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in U = U_1 \times \cdots \times U_n$ is said to be a Nash-equilibrium if

$$f_i(x_1^*, \ldots, x_{i-1}^*, y_i, \ldots, x_n^*) = f_i(x_1^*, \ldots, x_n^*), \quad \forall y_i \in U_i, \forall i = 1, \ldots, n. \tag{46}$$

When h_i is affine, this market problem can be formulated as a special Nash equilibrium problem in n-person noncooperative game theory.

In classical Cournot models [20, 34], the price and the cost functions for each firm are assumed to be affine of the forms:

$$\begin{aligned} p_i(\sigma) &= p(\sigma) = \alpha_0 - \beta\sigma, \quad \alpha_0 \geq 0, \beta > 0, \text{with } \sigma = \sum_{i=1}^{n} x_i, \\ h_i(x_i) &= \mu_i x_i + \xi_i, \quad \mu_i \geq 0, \xi_i \geq 0 \text{ for } i = 1, \dots, n. \end{aligned} \tag{47}$$

In this case, we take

$$B = \begin{bmatrix} 2\beta & 0 & 0 & \dots & 0 \\ 0 & 2\beta & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2\beta \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 & \beta & \beta & \dots & \beta \\ \beta & 0 & \beta & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots \\ \beta & \beta & \beta & \dots & 0 \end{bmatrix},$$

$$\alpha = (\alpha_0, \dots, \alpha_0)^T, \qquad \mu = (\mu_1, \dots, \mu_n)^T.$$

For (45), (46), (47), it has been shown in [20, 34], that the problem can be formulated equivalently as the convex quadratic problem

$$\min_{x \in U} \left\{ \frac{1}{2} x^T Q x + (\mu - \alpha)^T x \right\}, \tag{48}$$

where

$$Q = \begin{bmatrix} 2\beta & \beta & \beta & \dots & \beta \\ \beta & 2\beta & \beta & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots \\ \beta & \beta & \beta & \dots & 2\beta \end{bmatrix}.$$

Since $\beta > 0$, we have Q is a symmetric and positive definite matrix. Hence problem (48) has a unique optimal solution which is also the unique equilibrium point of the classical oligopolistic market equilibrium model.

Let

$$\alpha^T = (\alpha_1, \alpha_2, \dots,), \qquad F(x,y) = \bigl(-Bx + (B + \tilde{B})y\bigr)^T (y - x),$$

$$f(x) = \tilde{B}x + \alpha, \qquad \phi(x) = h(x).$$

Then the problem of finding a Nash equilibrium point defined by (46) with $F(x,y), f(x)$ given by (45) becomes the following generalized mixed equilibrium:

$$\begin{cases} \text{find a point } x^* \in U \text{ such that} \\ \Phi(x^*, y) = F(x^*, y) + \langle fx^*, x^* - y\rangle + \phi(y) - \phi(x^*) \geq 0 \quad \text{for all } y \in C, \end{cases} \tag{49}$$

where F is a bifunction, f is affine, and ϕ is a convex function (see [24]). In the literature problem (49) is often called the generalized mixed equilibrium which is more general than the mixed variational inequality (see [20]) because of the appearance of the bifunction F.

5 Numerical example

Numerical results are presented in this section to show the efficiency of the proposed method. The MATLAB codes was run in MATLAB version 9.5 (R2018b) on MacBook Pro 13-inch, 2019 with 2.4 GHz Quad-Core Intel Core i5 processor. RAM 8.00 GB. and the stopping criteria $\frac{\|x_{n+1}-x_n\|}{\max\{1,\|x_n\|\}} \leq 10^{-5}$.

Example 5.1 We apply Theorem 3.1 with Nash–Cournot oligopolistic market equilibrium problem in Sect. 4 when $H_1 = \mathbb{R}^N$ and $H_2 = \mathbb{R}^M$, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}^N$ and the induced usual norm $|\cdot|$, the linear operator $A : \mathbb{R}^N \to \mathbb{R}^M$ is given by an $M \times N$ matrix. The bifunction F is given as follows:

$$F_i(x,y) = \bigl(-Bx + (B+\tilde{B})y\bigr)^T (y-x),$$

where B and $\tilde{B}$ are symmetric positive semidefinite matrices such that $2B + \tilde{B}$ is a symmetric negative semidefinite matrix. Since $F_i(x,y) + F_i(y,x) = (y-x)^T(2B+\tilde{B})(y-x)$, then F_i is monotone. Similarly, define the bifunction of G_j by

$$G_j(u,v) = (Du + \tilde{D}v)^T (v-u),$$

where $\tilde{D}$ is a symmetric positive semidefinite matrix such that $\tilde{D} - D$ is a symmetric negative semidefinite matrix (see [53]).

The feasible sets are

$$C = \bigl\{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N : -5 \leq x_i \leq 5, \forall i = 1, \dots, N\bigr\},$$

and

$$Q = \bigl\{(u_1, u_2, \dots, u_M) \in \mathbb{R}^M : -2 \leq u_j \leq 2, \forall j = 1, \dots, M\bigr\}.$$

The mappings $f_i : C \to H_1$ and $g_j : Q \to H_2$ are defined by $f_i(x) = \tilde{B}x + b$, $g_j(u) = \tilde{D}u + d$, $\phi(x) = bx$ for all $x \in C$ and $\varphi(y) = dy$ for all $y \in Q$. Let $S : C \to C$ be a mapping defined to be $Sx = x$, and

$$T^n x = \begin{cases} \frac{1}{n}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We can see that T^n is $\frac{1}{3}$-strictly pseudocontractive with $\bigcap_{n=1}^{\infty} F(T^n) = \{0\}$. Further, we observe that T^n is not nonexpansive (see [32]). Set

$$\beta_n = \frac{1}{n^3}, \qquad \alpha_n = \frac{1}{2}, \qquad \theta = 0.6 \quad \text{for all } n \in \mathbb{N}.$$

We use the starting point $x_0 = (2, 2, \dots, 2)^T \in \mathbb{R}^N$ and $x_1 = (-2, -2, \dots, -2)^T \in \mathbb{R}^N$. The main subproblems were solved with the MATLAB Optimization Toolbox by using the QUADPROG function for the positive semidefinite quadratic function. The entries of matrices $B, \tilde{B}, D, \tilde{D}$ and vectors b, d are generated randomly in the interval $[-1, 5]$ respectively. Define matrix $A = \frac{1}{2}G$, where $G_{i,j} = 0$ when $i \neq j$. The numerical and graphical results of our algorithm are shown in Figures 1–4 and Table 1.

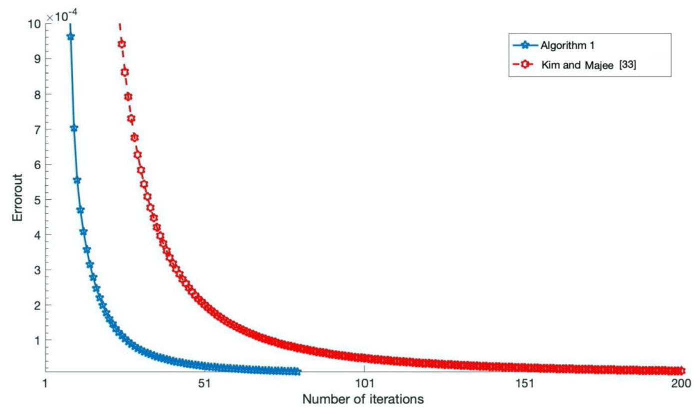

Figure 1 The graph of $N = 5$, $M = 5$

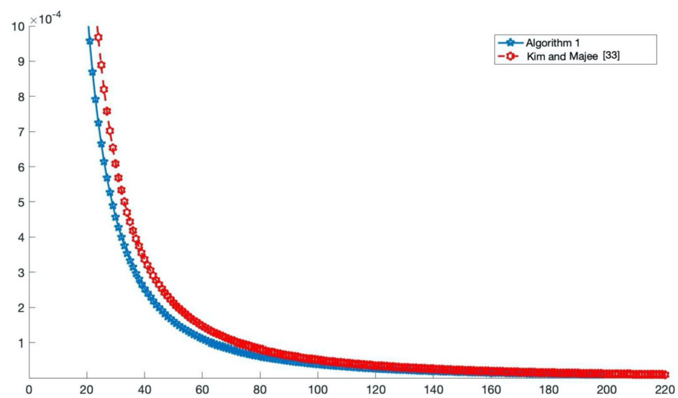

Figure 2 The graph of $N = 5$, $M = 10$

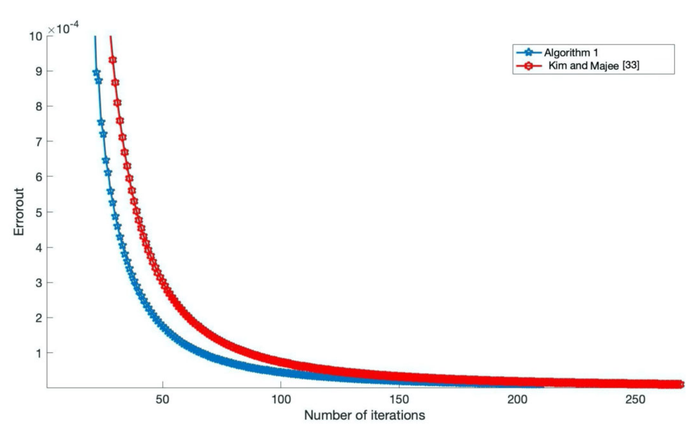

Figure 3 The graph of $N = 10$, $M = 20$

Example 5.2 We consider an example in infinite dimensional Hilbert spaces. Assume $H_1 = H_2 = L^2([0,1])$ with the inner product $\langle x,y\rangle = \int_0^1 x(t)y(t)\,dt$ and the induced norm $\|x\| =$

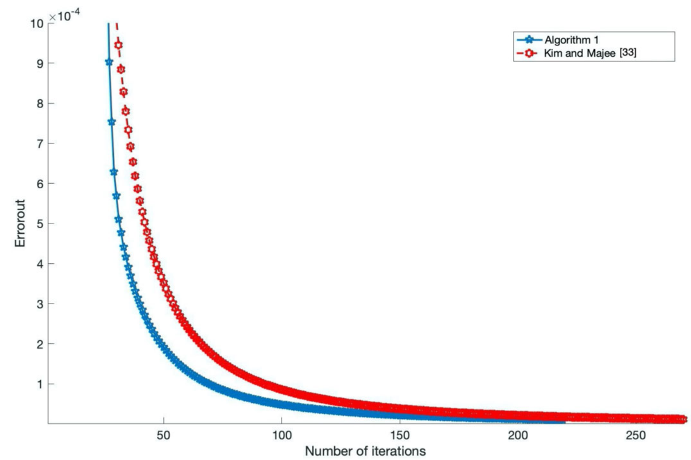

Figure 4 The graph of $N = 50$, $M = 100$

Table 1 The results computed on Algorithm 1 and the method in [33]

N	M	Average iteration		Average times	
		Algorithm 1	Method in [33]	Algorithm 1	Method in [33]
5	5	77	215	1.4100	3.7300
5	10	197	227	4.0800	4.2300
10	20	209	269	4.1000	4.3900
50	100	218	290	8.1200	10.6600

$\sqrt{\int_0^1 |x(t)|^2\,dt}$ for all $x, y \in L^2([0,1])$. Let

$$C = \left\{x \in L^2\big([0,1]\big) \,\Big|\, \int_0^1 x(t)\,dt \leq 1\right\} \quad \text{and} \quad Q = \left\{u \in L^2\big([0,1]\big) \,\Big|\, \int_0^1 tu(t)\,dt \leq 2\right\}.$$

Therefore C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Define an operator $A : L^2([0,1]) \to L^2([0,1])$ by $(Ax)(t) = \frac{1}{2}x(t)$. Thus A is a bounded linear operator. We define $F : C \times C \to \mathbb{R}$ by

$$F(x,y) = \big\langle P(x), y - x\big\rangle, \quad \text{where } P\big(x(t)\big) = \frac{x(t)}{2}.$$

For the purpose of our numerical computation, we use the following formula for the projection onto C (see [5]):

$$P_C(x) = \begin{cases} 1 - a + x, & \text{if } a > 1, \\ x & \text{if } a \leq 1, \end{cases}$$

where $a = \int_0^1 x(t)\,dt$. Similarly, define the bifunction $G : Q \times Q \to \mathbb{R}$ by

$$G(u,v) = \big\langle H(u), v - u\big\rangle, \quad \text{where } H\big(u(t)\big) = \frac{u(t)}{3}.$$

It is observed that F and G are monotone satisfying conditions L1–L4. Let $S : C \to C$ be a mapping defined by $(Sx)(t) = x(t)$ and $(T^n x)(t) = \frac{3}{n+3}x(t)$. We can see that T^n is 0-

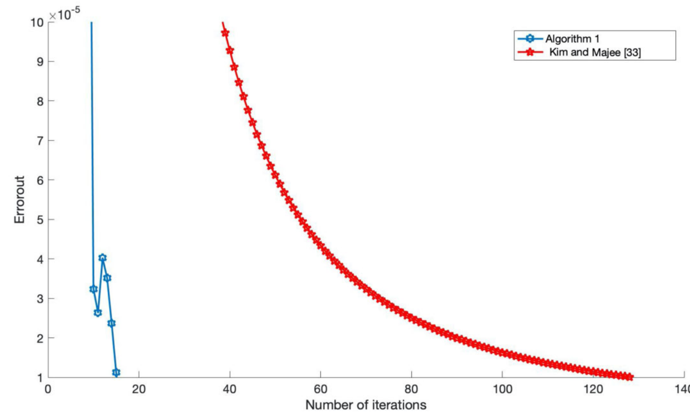

Figure 5 Numerical behaviour when $x_0 = t, x_1 = 2t$

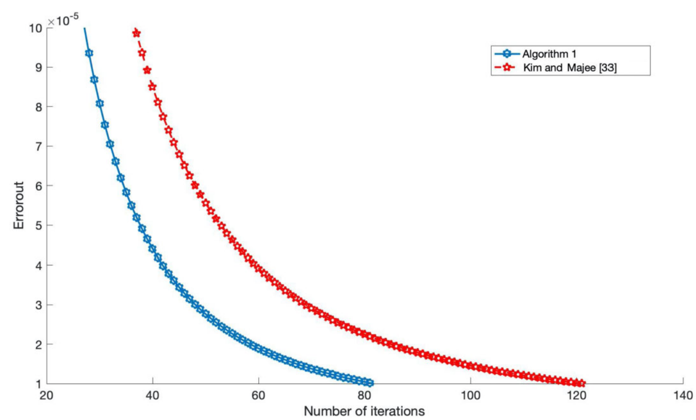

Figure 6 Numerical behaviour when $x_0 = t, x_1 = 2t^2$

strictly pseudocontractive for all $n \geq 1$. Set $\beta_n = \frac{1}{n^3}$, $\alpha_n = \frac{1}{2}$, $\sigma_n = 0.8$, $\theta = 0.9$ for all $n \in \mathbb{N}$. The mappings $f : C \to H_1$ and $g : Q \to H_2$ are defined by $(fx)(t) = \frac{1}{2}x(t)$, $(gu)(s) = \frac{1}{n}u(s)$, $(\phi(x))(t) = 0$ for all $x(t) \in C$ and $(\varphi(u))(t) = 0$ for all $u(t) \in Q$. Numerical results are reported in Figs. 5 and 6.

6 Conclusion

This paper discussed the modified Krasnoselski–Mann-type iterative method based on the idea of inertial technique. Weak convergence results have been obtained under some suitable conditions. Numerical conclusions have been drawn to explain the numerical efficiency of our algorithm in comparison to another method. Note that our algorithm and results presented in this paper can summarize and improve some known results in the area. Our future work will focus on obtaining the results to robust equilibrium problems by our algorithm in [38, 46, 47].

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