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Hermite–Hadamard-type inequalities for interval-valued preinvex functions via Riemann–Liouville fractional integrals

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Abstract

In this paper, we introduce (h_1, h_2)-preinvex interval-valued function and establish the Hermite–Hadamard inequality for preinvex interval-valued functions by using interval-valued Riemann–Liouville fractional integrals. We obtain Hermite–Hadamard-type inequalities for the product of two interval-valued functions. Further, some examples are given to confirm our theoretical results.

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1 Introduction

Hanson [8] introduced one of the most important generalizations of convex functions as the class of invex functions. In 1988, Weir and Mond [30] established sufficient optimality conditions and duality in nonlinear programming by using the concept of preinvex functions. Wang *et al.* [28] investigated fractional integral identities for a differentiable mapping involving Riemann–Liouville fractional integrals and Hadamard fractional integrals and gave some inequalities via standard convex, r-convex, s-convex, m-convex, (s, m)-convex, (β, m)-convex functions, etc. Further, Işcan [10] obtained some Hermite–Hadamard type inequalities using fractional integrals for preinvex functions. For more generalizations of the Hermite–Hadamard inequality, see [2, 6, 16, 23].

Moore [18] computed arbitrarily sharp upper and lower bounds on exact solutions of many problems in applied mathematics by using interval arithmetic, interval-valued functions, and integrals of interval-valued functions. Moore [18] showed that if a real-valued function $\xi(x)$ satisfies an ordinary Lipschitz condition in X, $|\xi(x) - \xi(y)| \leqq L|x - y|$ for $x, y \in X$, then the united extension of ξ is a Lipschitz interval extension in X.

Hilger [9] proposed a theory on time scales that can unify the study of the discrete and continuous dynamical system. The prolific increase of the applications of the dynamic equations and integral inequalities on time scales in the various fields, such as electrical engineering, quantum mechanics, heat transfer, neural network, combinatorics, and population dynamics [1], made visible to us the requirement of this theory. Agarwal *et*

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al. [1] discussed dynamic inequalities on time scales such as Young's inequality, Hölder's inequality, Minkoswki's inequality, Jensen's inequality, Steffensen's inequality, Hermite–Hadamard inequality, Čebyšv's inequality, and Opial type inequality. In 2010, Srivastava *et al.* [24] established some general weighted Opial type inequalities on time scales. Further, Srivastava *et al.* [25] presented some extensions and generalizations of Maroni's inequality to hold true on time. Wei *et al.* [29] established local fractional integral analogue of Anderson's inequality on fractal space under some suitable conditions and also showed that the local fractional integral inequality on fractal space is a new generalization of the classical Anderson's inequality. Further, Tunç *et al.* [27] established an identity for local fractional integrals and derived several generalizations of the celebrated Steffensen's inequality associated with local fractional integrals. For more details, we can refer to [1, 26] and the references therein.

Bhurjee and Panda [3] defined the interval-valued function in the parametric form and developed a methodology to study the existence of the solution of a general interval optimization problem. Lupulescu [14] introduced the differentiability and integrability for the interval-valued functions on time scales by using the concept of the generalized Hukuhara difference. In 2015, Cano *et al.* [7] proposed a new Ostrowski type inequalities for *gH*-differentiable interval-valued functions and obtained generalization of the class of real functions which is not necessarily differentiable. Cano *et al.* [7] obtained error bounds to quadrature rules for *gH*-differentiable interval-valued functions. Further, Roy and Panda [22] introduced the concept of μ-monotonic property of interval-valued function in the higher dimension and derived some results by using generalized Hukuhara differentiability. For more details of interval-valued functions, we refer to [4, 5, 11, 13, 15, 22] and the references therein.

Recently, An *et al.* [2] introduced (h_1, h_2)-convex interval-valued function and obtained some interval Hermite–Hadamard type inequalities. Further, Budak *et al.* [6] established the Hermite–Hadamard inequality for the convex interval-valued function and for the product of two convex interval-valued functions.

Motivated by the above works and ideas, we introduce the concept of (h_1, h_2)-preinvex interval-valued function and establish the Hermite–Hadamard inequality for preinvex interval-valued functions and for the product of two preinvex interval-valued functions via interval-valued Riemann–Liouville fractional integrals. Also, we give some examples in the support of our theory.

2 Preliminaries
In this section, we mention some definitions and related results required for this manuscript.

2.1 Interval arithmetic
The rules for interval addition, subtraction, product, and quotient [18] are

$$[\underline{X}, \overline{X}] + [\underline{Y}, \overline{Y}] = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}],$$

$$[\underline{X}, \overline{X}] - [\underline{Y}, \overline{Y}] = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}],$$

$$X.Y = \{xy : x \in X, y \in Y\}.$$

It is easy to see that X.Y is again an interval, whose end points can be computed from

$$\underline{X}.\underline{Y} = \min\{\underline{XY}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{XY}\}$$

and

$$\overline{X}.\overline{Y} = \max\{\underline{XY}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{XY}\}.$$

The reciprocal of an interval is as follows:

$$1/X = \{1/x : x \in X\}. \tag{1}$$

If X is an interval not containing the number 0, then

$$1/X = [1/\overline{X}, 1/\underline{X}].$$

$$X/Y = X.(1/Y) = \{x/y : x \in X, y \in Y\},$$

where $1/y$ is defined by (1).

Scalar multiplication of the interval X is defined by

$$\lambda X = \lambda[\underline{X}, \overline{X}] = \begin{cases} [\lambda \underline{X}, \lambda \overline{X}], & \text{if } \lambda > 0, \\ \{0\}, & \text{if } \lambda = 0, \\ [\lambda \overline{X}, \lambda \underline{X}], & \text{if } \lambda < 0, \end{cases}$$

where $\lambda \in \mathbb{R}$.

Let $\mathbb{R}_I$, $\mathbb{R}_I^+$, and $\mathbb{R}_I^-$ be the sets of all closed intervals of $\mathbb{R}$, sets of all positive closed intervals of $\mathbb{R}$, and sets of all negative closed intervals of $\mathbb{R}$, respectively. Now, we discuss some algebraic properties of interval arithmetic [18].

(1) (Associativity of addition) $(X + Y) + Z = X + (Y + Z)$, $\forall X, Y, Z \in \mathbb{R}_I$.

(2) (Additive element) $X + 0 = 0 + X = X$, $\forall X \in \mathbb{R}_I$.

(3) (Commutativity of addition) $X + Y = Y + X$, $\forall X, Y \in \mathbb{R}_I$.

(4) (Cancellation law) $X + Z = Y + Z \implies X = Y$, $\forall X, Y, Z \in \mathbb{R}_I$.

(5) (Associativity of multiplication) $(X.Y).Z = X.(Y.Z)$, $\forall X, Y, Z \in \mathbb{R}_I$.

(6) (Commutativity of multiplication) $X.Y = Y.X$, $\forall X, Y \in \mathbb{R}_I$.

(6) (Unit element) $X.1 = 1.X = X$, $\forall X \in \mathbb{R}_I$.

(7) (Associate law) $\lambda(\mu X) = (\lambda \mu) X$, $\forall X \in \mathbb{R}_I$, and $\forall \lambda, \mu \in \mathbb{R}$.

(8) (First distributive law) $\lambda(X + Y) = \lambda X + \lambda Y$, $\forall X, Y \in \mathbb{R}_I$, and $\forall \lambda \in \mathbb{R}$.

(9) (Second distributive law) $(\lambda + \mu)X = \lambda X + \mu X$, $\forall X \in \mathbb{R}_I$, and $\forall \lambda, \mu \in \mathbb{R}$.

However, the distributive law does not always hold.

Example 1

$$X = [-2, -1], \qquad Y = [-1, 0] \quad \text{and} \quad Z = [1, 3].$$

$$X.(Y + Z) = [-2, -1].\big([-1, 0] + [1, 3]\big) = [-6, 0],$$

whereas

$$X.Y + X.Z = [-2,-1].[-1,0] + [-2,-1].[1,3] = [-6,1].$$

2.2 Integral of interval-valued functions:

A function ξ is said to be an interval-valued function of δ on $[c,d]$ if it assigns a nonempty interval to each $\delta \in [c,d]$

$$\xi(\delta) = \left[\underline{\xi}(\delta), \overline{\xi}(\delta)\right],$$

where $\underline{\xi}$ and $\overline{\xi}$ are real-valued functions. A partition of $[c,d]$ is any finite ordered subset P having the form

$$P : c = t_0 < t_1 < \cdots < t_n = d.$$

The mesh of a partition P is defined by

$$\mathrm{mesh}(P) = \max\{t_i - t_{i-1} : i = 1, 2, \ldots, n\}.$$

The set of all partitions of $[c,d]$ is denoted by $\mathfrak{P}([c,d])$. Let $\mathfrak{P}(\rho,[c,d])$ be the set of all $P \in \mathfrak{P}([c,d])$ such that $\mathrm{mesh}(P) < \rho$. Choose an arbitrary point x_i in the interval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$, and we define the sum

$$\mathcal{S}(\xi, P, \rho) = \sum_{i=1}^{n} \xi(x_i)[t_i - t_{i-1}],$$

where $\xi : [c,d] \to \mathbb{R}_I$. $\mathcal{S}(\xi, P, \rho)$ denotes the Riemann sum of ξ corresponding to $P \in \mathfrak{P}(\rho, [c,d])$.

Definition 1 ([21]) A function $\xi : [c,d] \to \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[c,d]$ if there exists $K \in \mathbb{R}_I$ such that, for each $\epsilon > 0$, there exists $\rho > 0$ such that

$$d\big(\mathcal{S}(\xi, P, \rho), K\big) < \epsilon$$

for every Riemann sum $\mathcal{S}$ of ξ corresponding to each $P \in \mathfrak{P}(\rho, [c,d])$ and independent of the choice of $x_i \in [t_{i-1}, t_i]$ for $1 \le i \le n$. K is called the *IR*-integral of ξ on $[c,d]$ and is denoted by

$$K = (IR) \int_c^d \xi(\delta)\, d\delta.$$

The collection of all (IR)-integrable functions on $[c,d]$ is denoted by $\mathcal{IR}_{([c,d])}$.

Theorem 1 ([19]) *Let* $\xi : [c,d] \to \mathbb{R}_I$ *be an interval-valued function such that* $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$. $\xi \in IR_{([c,d])}$ *if and only if* $\underline{\xi}(\delta), \overline{\xi}(\delta) \in R_{([c,d])}$ *and*

$$(IR) \int_c^d \xi(\delta)\, d\delta = \left[(R) \int_c^d \underline{\xi}(\delta)\, d\delta, (R) \int_c^d \overline{\xi}(\delta)\, d\delta \right],$$

where $R_{([c,d])}$ *denotes the R-integrable function.*

Definition 2 ([12]) Let $\xi \in L_1[c,d]$. The Riemann–Liouville fractional integrals $J_{c^+}^\alpha \xi$ and $J_{d^-}^\alpha \xi$ of order $\alpha > 0$ with $c \geq 0$ are defined by

$$J_{c^+}^\alpha \xi(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x - \delta)^{(\alpha-1)} \xi(\delta)\, d\delta, \quad x > c,$$

and

$$J_{d^-}^\alpha \xi(x) = \frac{1}{\Gamma(\alpha)} \int_x^d (\delta - x)^{(\alpha-1)} \xi(\delta)\, d\delta, \quad x < d,$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $J_{c^+}^0 \xi(x) = J_{d^-}^0 \xi(x) = \xi(x)$.

Definition 3 ([15]) Let $\xi : [c,d] \to \mathbb{R}_I$ be an interval-valued function such that $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$ and $\xi \in IR_{([c,d])}$. The interval-valued left-sided Riemann–Liouville fractional integral of function ξ is defined by

$$J_{c^+}^\alpha \xi(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_c^x (x - \delta)^{(\alpha-1)} \xi(\delta)\, d\delta, \quad x > c, \alpha > 0,$$

where $\Gamma(\alpha)$ is the gamma function.

Definition 4 ([6]) Let $\xi : [c,d] \to \mathbb{R}_I$ be an interval-valued function such that $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$ and $\xi \in IR_{([c,d])}$. The interval-valued right-sided Riemann–Liouville fractional integral of function ξ is defined by

$$J_{d^-}^\alpha \xi(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_x^d (\delta - x)^{(\alpha-1)} \xi(\delta)\, d\delta, \quad x < d, \alpha > 0,$$

where $\Gamma(\alpha)$ is the gamma function.

Corollary 1 ([6]) *If $\xi : [c,d] \to \mathbb{R}_I$ is an interval-valued function such that $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$ with $\underline{\xi}(\delta), \overline{\xi}(\delta) \in R_{([c,d])}$, then we have*

$$J_{c^+}^\alpha \xi(x) = \left[J_{c^+}^\alpha \underline{\xi}(x), J_{c^+}^\alpha \overline{\xi}(x) \right]$$

and

$$J_{d^-}^\alpha \xi(x) = \left[J_{d^-}^\alpha \underline{\xi}(x) J_{d^-}^\alpha \overline{\xi}(x) \right].$$

Definition 5 ([30]) The set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if

$$y + \delta \eta(x,y) \in A, \quad \forall x, y \in A, \delta \in [0,1].$$

It is well known that every convex set is invex with respect to $\eta(x,y) = x - y$ but not conversely.

Definition 6 ([30]) The function ξ on the invex set A is said to be preinvex with respect to η if

$$\xi\big(y + \delta\eta(x,y)\big) \leq (1 - \delta)\xi(y) + \delta\xi(x), \quad \forall x,y \in A, \delta \in [0,1].$$

It is well known that every convex function is preinvex with respect to $\eta(x,y) = x - y$ but not conversely.

Condition C ([17]) Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. We say that the function η satisfies Condition C if, for any $x,y \in A$ and any $\delta \in [0,1]$,

$$\eta\big(y, y + \delta\eta(x,y)\big) = -\delta\eta(x,y),$$
$$\eta\big(x, y + \delta\eta(x,y)\big) = (1 - \delta)\eta(x,y).$$

Note that $\forall x,y \in A$ and $\delta \in [0,1]$, then from Condition C we have

$$\eta\big(y + \delta_2\eta(x,y), y + \delta_1\eta(x,y)\big) = (\delta_2 - \delta_1)\eta(x,y).$$

Theorem 2 ([20]) *Let $\xi : A = [c, c + \eta(d,c)] \to (0,\infty)$ be a preinvex function on the interval of real numbers* $\mathrm{int}(A)$ *and* $c,d \in \mathrm{int}(A)$ *with* $c < c + \eta(d,c)$. *Then the following inequality holds*:

$$\xi\left(\frac{2c + \eta(d,c)}{2}\right) \leq \frac{1}{\eta(d,c)} \int_c^{c+\eta(d,c)} \xi(x)\,dx \leq \frac{\xi(c) + \xi(d)}{2}. \tag{2}$$

3 Main results

In this section, first, we give the definition of interval-valued h-preinvex function and discuss some special cases of interval-valued h-preinvex functions.

Definition 7 Let $h : [a,b] \to \mathbb{R}$ be a nonnegative function, $(0,1) \subseteq [a,b]$ and $h \not\equiv 0$. Let $A \subseteq \mathbb{R}$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}$, $\xi(x) = [\underline{\xi}(x), \overline{\xi}(x)]$ be an interval-valued function defined on A. We say that ξ is h-preinvex at x with respect to η if

$$\xi\big(y + \delta\eta(x,y)\big) \supseteq h(\delta)\xi(x) + h(1 - \delta)\xi(y), \quad \forall \delta \in [0,1] \text{ and } \forall x \in A.$$

Now, we discuss some special cases of interval-valued h-preinvex functions.
   (1) If $h(\delta) = 1$, then we have the definition of interval-valued P-preinvex functions.
   (2) If $h(\delta) = \delta$, then we have the definition of interval-valued preinvex functions.
   (3) If $h(\delta) = \delta^{-1}$, then we have the definition of interval-valued Q-preinvex functions.
   (4) If $h(\delta) = \delta^s$ with $s \in (0,1)$, then we have the definition of interval-valued s-preinvex functions.

Theorem 3 *Let $h : [a,b] \to \mathbb{R}$ be a nonnegative function, $(0,1) \subseteq [c,d]$ and $h \not\equiv 0$. Let A be an invex subset of $\mathbb{R}$ with respect to $\eta : A \times A \to \mathbb{R}$ and ξ be an interval-valued function defined on A. Then ξ is h-preinvex at x if and only if $\underline{\xi}$ and $\overline{\xi}$ are h-preinvex at x with respect to η i.e.*

$$\underline{\xi}\big(y + \delta\eta(x,y)\big) \leq h(\delta)\underline{\xi}(x) + h(1 - \delta)\underline{\xi}(y)$$

and

$$\overline{\xi}\big(y + \delta\eta(x,y)\big) \le h(\delta)\overline{\xi}(x) + h(1 - \delta)\overline{\xi}(y), \quad \forall \delta \in [0,1] \text{ and } \forall x \in A.$$

Now, we establish the Hermite–Hadamard inequalities for the preinvex interval-valued functions.

Theorem 4 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $c, d \in A$ with $c < c + \eta(d,c)$. If $\xi : [c, c + \eta(d,c)] \to \mathbb{R}_I^+$ is a preinvex interval-valued function such that $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$. $\xi \in L[c, c + \eta(d,c)]$ and η satisfies Condition* C *and $\alpha > 0$, then we have*

$$\xi\left(c + \frac{\eta(d,c)}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(d,c)}\left[J^\alpha_{c^+}\xi\big(c + \eta(d,c)\big) + J^\alpha_{(c+\eta(d,c))^-}\xi(c)\right]$$
$$\supseteq \frac{\xi(c) + \xi(c + \eta(d,c))}{2} \supseteq \frac{\xi(c) + \xi(d)}{2}. \tag{3}$$

Proof Since ξ is a preinvex interval-valued function, we have

$$\xi\left(x + \frac{1}{2}\eta(y,x)\right) \supseteq \frac{\xi(x) + \xi(y)}{2}, \quad \forall x, y \in \big[c, c + \eta(d,c)\big]. \tag{4}$$

Using $x = c + (1 - \delta)\eta(d,c)$, $y = c + \delta\eta(d,c)$ and Condition C in (4), we get

$$\xi\left(c + (1 - \delta)\eta(d,c) + \frac{1}{2}\eta\big(c + \delta\eta(d,c), c + (1 - \delta)\eta(d,c)\big)\right)$$
$$\supseteq \frac{\xi(c + (1 - \delta)\eta(d,c)) + \xi(c + \delta\eta(d,c))}{2}.$$

This implies

$$\xi\left(c + \frac{1}{2}\eta(d,c)\right) \supseteq \frac{\xi(c + (1 - \delta)\eta(d,c)) + \xi(c + \delta\eta(d,c))}{2}. \tag{5}$$

Multiplying by $\delta^{\alpha-1}$, $\alpha > 0$ on both sides in (5), we have

$$\delta^{\alpha-1}\xi\left(c + \frac{1}{2}\eta(d,c)\right) \supseteq \frac{\delta^{\alpha-1}}{2}\left[\xi\big(c + (1 - \delta)\eta(d,c)\big) + \xi\big(c + \delta\eta(d,c)\big)\right]. \tag{6}$$

Integrating the above inequality on $[0,1]$, we get

$$(IR)\int_0^1 \delta^{\alpha-1}\xi\left(c + \frac{1}{2}\eta(d,c)\right)d\delta \supseteq \frac{1}{2}\left[(IR)\int_0^1 \delta^{\alpha-1}\xi\big(c + (1 - \delta)\eta(d,c)\big)\,d\delta\right.$$
$$\left. + (IR)\int_0^1 \delta^{\alpha-1}\xi\big(c + \delta\eta(d,c)\big)\,d\delta\right]. \tag{7}$$

Applying Theorem 1 in the above relation, we get

$$(IR)\int_0^1 \delta^{\alpha-1}\xi\left(c + \frac{1}{2}\eta(d,c)\right)d\delta$$
$$= \left[(R)\int_0^1 \delta^{\alpha-1}\underline{\xi}\left(c + \frac{1}{2}\eta(d,c)\right)d\delta, (R)\int_0^1 \delta^{\alpha-1}\overline{\xi}\left(c + \frac{1}{2}\eta(d,c)\right)d\delta\right]$$

$$= \left[\underline{\xi}\left(c + \frac{1}{2}\eta(d,c)\right)(R)\int_0^1 \delta^{\alpha-1}\,d\delta, \overline{\xi}\left(c + \frac{1}{2}\eta(d,c)\right)(R)\int_0^1 \delta^{\alpha-1}\,d\delta \right]$$

$$= \left[\frac{1}{\alpha}\underline{\xi}\left(c + \frac{1}{2}\eta(d,c)\right), \frac{1}{\alpha}\overline{\xi}\left(c + \frac{1}{2}\eta(d,c)\right) \right]$$

$$= \frac{1}{\alpha}\xi\left(c + \frac{1}{2}\eta(d,c)\right), \tag{8}$$

$$(IR)\int_0^1 \delta^{\alpha-1}\xi\left(c + \delta\eta(d,c)\right)d\delta$$

$$= \left[(R)\int_0^1 \delta^{\alpha-1}\underline{\xi}\left(c + \delta\eta(d,c)\right)d\delta, (R)\int_0^1 \delta^{\alpha-1}\overline{\xi}\left(c + \delta\eta(d,c)\right)d\delta \right].$$

This implies

$$(IR)\int_0^1 \delta^{\alpha-1}\xi\left(c + \delta\eta(d,c)\right)d\delta$$

$$= \left[\frac{1}{\eta^\alpha(d,c)}(R)\int_c^{c+\eta(d,c)} (v-c)^{\alpha-1}\underline{\xi}(v)dv, \frac{1}{\eta^\alpha(d,c)}(R)\int_c^{c+\eta(d,c)} (v-c)^{\alpha-1}\overline{\xi}(v)dv \right]$$

$$= \frac{\Gamma(\alpha)}{\eta^\alpha(d,c)}\left[J^\alpha_{(c+\eta(d,c))^-}\underline{\xi}(c), J^\alpha_{(c+\eta(d,c))^-}\overline{\xi}(c) \right]$$

$$= \frac{\Gamma(\alpha)}{\eta^\alpha(d,c)}J^\alpha_{(c+\eta(d,c))^-}\xi(c). \tag{9}$$

Similarly,

$$(IR)\int_0^1 \delta^{\alpha-1}\xi\left(c + (1-\delta)\eta(d,c)\right)d\delta = \frac{\Gamma(\alpha)}{\eta^\alpha(d,c)}J^\alpha_{c^+}\xi\left(c + \eta(d,c)\right). \tag{10}$$

Using (8), (9), and (10) in (7), we have

$$\xi\left(c + \frac{1}{2}\eta(d,c)\right) \supseteq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(d,c)}\left[J^\alpha_{c^+}\xi\left(c + \eta(d,c)\right) + J^\alpha_{(c+\eta(d,c))^-}\xi(c) \right]. \tag{11}$$

Now, we prove the second pair of inequalities.

Since ξ is an interval-valued preinvex function on $[c, c + \eta(d,c)]$. Therefore,

$$\xi\left(c + \delta\eta(d,c)\right) = \xi\left(c + \eta(d,c) + (1-\delta)\eta\left(c, c + \eta(d,c)\right)\right)$$

$$\supseteq \delta\xi\left(c + \eta(d,c)\right) + (1-\delta)\xi(c) \tag{12}$$

and

$$\xi\left(c + (1-\delta)\eta(d,c)\right) = \xi\left(c + \eta(d,c) + \delta\eta\left(c, c + \eta(d,c)\right)\right)$$

$$\supseteq (1-\delta)\xi\left(c + \eta(d,c)\right) + \delta\xi(c). \tag{13}$$

Adding (12) and (13), we have

$$\xi\left(c + \delta\eta(d,c)\right) + \xi\left(c + (1-\delta)\eta(d,c)\right) \supseteq \xi(c) + \xi\left(c + \eta(d,c)\right). \tag{14}$$

Multiplying by $\delta^{\alpha-1}$ and integrating on $[0,1]$, we have

$$(IR)\int_0^1 \delta^{\alpha-1}\xi\big(c+\delta\eta(d,c)\big)\,d\delta + (IR)\int_0^1 \delta^{\alpha-1}\xi\big(c+(1-\delta)\eta(d,c)\big)\,d\delta$$

$$\supseteq (IR)\int_0^1 \delta^{\alpha-1}\big[\xi(c)+\xi\big(c+\eta(d,c)\big)\big]. \tag{15}$$

Applying Theorem 1 in the above relation, we get

$$(IR)\int_0^1 \delta^{\alpha-1}\big[\xi(c)+\xi\big(c+\eta(d,c)\big)\big]$$

$$= \left[(R)\int_0^1 \delta^{\alpha-1}\big[\underline{\xi}(c)+\underline{\xi}\big(c+\eta(d,c)\big)\big]\,d\delta, (R)\int_0^1 \delta^{\alpha-1}\big[\overline{\xi}(c)+\overline{\xi}\big(c+\eta(d,c)\big)\big]\,d\delta\right]$$

$$= \left[\big[\underline{\xi}(c)+\underline{\xi}\big(c+\eta(d,c)\big)\big](R)\int_0^1 \delta^{\alpha-1}\,d\delta, \big[\overline{\xi}(c)+\overline{\xi}\big(c+\eta(d,c)\big)\big](R)\int_0^1 \delta^{\alpha-1}\,d\delta\right]$$

$$= \left[\frac{1}{\alpha}\big[\underline{\xi}(c)+\underline{\xi}\big(c+\eta(d,c)\big)\big], \frac{1}{\alpha}\big[\overline{\xi}(c)+\overline{\xi}\big(c+\eta(d,c)\big)\big]\right]$$

$$= \frac{1}{\alpha}\big[\xi(c)+\xi\big(c+\eta(d,c)\big)\big]. \tag{16}$$

Using (9), (10), and (16) in (15), we have

$$\frac{\Gamma(\alpha+1)}{2\eta^\alpha(d,c)}\big[J_{c^+}^\alpha\xi\big(c+\eta(d,c)\big) + J_{(c+\eta(d,c))^-}^\alpha\xi(c)\big]$$

$$\supseteq \frac{\xi(c)+\xi(c+\eta(d,c))}{2} \supseteq \frac{\xi(c)+\xi(d)}{2}. \tag{17}$$

From (11) and (17), we get

$$\xi\left(c+\frac{\eta(d,c)}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(d,c)}\big[J_{c^+}^\alpha\xi\big(c+\eta(d,c)\big) + J_{(c+\eta(d,c))^-}^\alpha\xi(c)\big]$$

$$\supseteq \frac{\xi(c)+\xi(c+\eta(d,c))}{2} \supseteq \frac{\xi(c)+\xi(d)}{2}.$$

This completes the proof. □

Example 2 Let $\xi(x) = [x, 2x]$, $\eta(d,c) = d - 2c$, $\alpha = 1$, $c = 0$, and $d = 2$, then all assumptions of the above theorem are satisfied.

Remark 1 When $\eta(d,c) = d - c$, then the above theorem reduces to Theorem 3.4 of [6].

We prove Hermite–Hadamard type inequalities for the product of two preinvex interval-valued functions.

Theorem 5 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $c,d \in A$ with $c < c + \eta(d,c)$. If $\xi,\phi : [c, c + \eta(d,c)] \to \mathbb{R}_I^+$ is a preinvex interval-valued function such that $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$ and $\phi(\delta) = [\underline{\phi}(\delta), \overline{\phi}(\delta)]$. $\xi, \phi \in L[c, c + \eta(d,c)]$ and η satisfies*

Condition C *and* $\alpha > 0$, *then we have*

$$\frac{\Gamma(\alpha + 1)}{2\eta^\alpha(d,c)}\big[J^\alpha_{c^+}\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big) + J^\alpha_{(c+\eta(d,c))^-}\xi(c)\phi(c)\big]$$

$$\supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)}\right)M\big(c, c + \eta(d,c)\big) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)}N\big(c, c + \eta(d,c)\big),$$

where $M(c, c + \eta(d,c)) = \xi(c)\phi(c) + \xi(c + \eta(d,c))\phi(c + \eta(d,c))$ *and* $N(c, c + \eta(d,c)) = \xi(c)\phi(c + \eta(d,c)) + \xi(c + \eta(d,c))\phi(c)$.

Proof Since ξ and ϕ are two preinvex interval-valued functions for $\delta \in [0, 1]$, we have

$$\xi\big(c + \delta\eta(d,c)\big) = \xi\big(c + \eta(d,c) + (1 - \delta)\eta\big(c, c + \eta(d,c)\big)\big)$$

$$\supseteq \delta\xi\big(c + \eta(d,c)\big) + (1 - \delta)\xi(c) \tag{18}$$

and

$$\phi\big(c + \delta\eta(d,c)\big) = \phi\big(c + \eta(d,c) + (1 - \delta)\eta\big(c, c + \eta(d,c)\big)\big)$$

$$\supseteq \delta\phi\big(c + \eta(d,c)\big) + (1 - \delta)\phi(c). \tag{19}$$

Since $\xi(x), \phi(x) \in \mathbb{R}^+_I$, $\forall x \in [c, d]$, then from (18) and (19) we have

$$\xi\big(c + \delta\eta(d,c)\big)\phi\big(c + \delta\eta(d,c)\big)$$

$$\supseteq \delta^2\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big) + (1 - \delta)^2\xi(c)\phi(c)$$

$$+ \delta(1 - \delta)\big[\xi\big(c + \eta(d,c)\big)\phi(c) + \xi(c)\phi\big(c + \eta(d,c)\big)\big]. \tag{20}$$

Similarly,

$$\xi\big(c + (1 - \delta)\eta(d,c)\big)\phi\big(c + (1 - \delta)\eta(d,c)\big)$$

$$\supseteq \delta^2\xi(c)\phi(c) + (1 - \delta)^2\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big)$$

$$+ \delta(1 - \delta)\big[\xi\big(c + \eta(d,c)\big)\phi(c) + \xi(c)\phi\big(c + \eta(d,c)\big)\big]. \tag{21}$$

Adding (20) and (21), we have

$$\xi\big(c + \delta\eta(d,c)\big)\phi\big(c + \delta\eta(d,c)\big) + \xi\big(c + (1 - \delta)\eta(d,c)\big)\phi\big(c + (1 - \delta)\eta(d,c)\big)$$

$$\supseteq \delta^2\big[\xi(c)\phi(c) + \xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big)\big]$$

$$+ (1 - \delta)^2\big[\xi(c)\phi(c) + \xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big)\big]$$

$$+ 2\delta(1 - \delta)\big[\xi\big(c + \eta(d,c)\big)\phi(c) + \xi(c)\phi\big(c + \eta(d,c)\big)\big]$$

$$= \big[\delta^2 + (1 - \delta)^2\big]\big[\xi(c)\phi(c) + \xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big)\big]$$

$$+ 2\delta(1 - \delta)\big[\xi\big(c + \eta(d,c)\big)\phi(c) + \xi(c)\phi\big(c + \eta(d,c)\big)\big]$$

$$= \big[2\delta^2 - 2\delta + 1\big]M\big(c, c + \eta(d,c)\big) + 2\delta(1 - \delta)N\big(c, c + \eta(d,c)\big). \tag{22}$$

Multiplying by $\delta^{\alpha-1}$ on both sides and integrating on $[0,1]$, we have

$$
(IR) \int_0^1 \delta^{\alpha-1} \xi\big(c + \delta\eta(d,c)\big)\phi\big(c + \delta\eta(d,c)\big)\,d\delta
$$

$$
+ (IR) \int_0^1 \delta^{\alpha-1} \xi\big(c + (1-\delta)\eta(d,c)\big)\phi\big(c + (1-\delta)\eta(d,c)\big)\,d\delta
$$

$$
\supseteq (IR) \int_0^1 \big[2\delta^{\alpha+1} - 2\delta^\alpha + \delta^{\alpha-1}\big] M\big(c, c + \eta(d,c)\big)\,d\delta
$$

$$
+ (IR) \int_0^1 2\big[\delta^\alpha - \delta^{\alpha+1}\big] N\big(c, c + \eta(d,c)\big)\,d\delta. \tag{23}
$$

Since

$$
(IR) \int_0^1 \delta^{\alpha-1} \xi\big(c + \delta\eta(d,c)\big)\phi\big(c + \delta\eta(d,c)\big)\,d\delta = \frac{\Gamma(\alpha)}{\eta^\alpha(d,c)} J^\alpha_{(c+\eta(d,c))^-} \xi(c)\phi(c), \tag{24}
$$

$$
(IR) \int_0^1 \delta^{\alpha-1} \xi\big(c + (1-\delta)\eta(d,c)\big)\phi\big(c + (1-\delta)\eta(d,c)\big)\,d\delta
$$

$$
= \frac{\Gamma(\alpha)}{\eta^\alpha(d,c)} J^\alpha_{c^+} \xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big), \tag{25}
$$

$$
(IR) \int_0^1 \big[2\delta^{\alpha+1} - 2\delta^\alpha + \delta^{\alpha-1}\big] M\big(c, c + \eta(d,c)\big)\,d\delta
$$

$$
= \frac{2}{\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) M\big(c, c + \eta(d,c)\big), \tag{26}
$$

and

$$
(IR) \int_0^1 2\big[\delta^\alpha - \delta^{\alpha+1}\big] N\big(c, c + \eta(d,c)\big)\,d\delta = \frac{2}{(\alpha+1)(\alpha+2)} N\big(c, c + \eta(d,c)\big). \tag{27}
$$

Using (24), (25), (26), and (27) in (23), we have

$$
\frac{\Gamma(\alpha)}{\eta^\alpha(d,c)}\big[J^\alpha_{(c+\eta(d,c))^-}\xi(c)\phi(c) + J^\alpha_{c^+}\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big)\big]
$$

$$
\supseteq \frac{2}{\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) M\big(c, c + \eta(d,c)\big) + \frac{2}{(\alpha+1)(\alpha+2)} N\big(c, c + \eta(d,c)\big). \tag{28}
$$

This implies

$$
\frac{\Gamma(\alpha+1)}{2\eta^\alpha(d,c)}\big[J^\alpha_{c^+}\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big) + J^\alpha_{(c+\eta(d,c))^-}\xi(c)\phi(c)\big]
$$

$$
\supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) M\big(c, c + \eta(d,c)\big) + \frac{\alpha}{(\alpha+1)(\alpha+2)} N\big(c, c + \eta(d,c)\big). \qquad \square
$$

Remark 2 When $\eta(d,c) = d - c$, then the above theorem reduces to Theorem 3.5 of [6].

Theorem 6 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $c, d \in A$ with $c < c + \eta(d,c)$. If $\xi, \phi : [c, c + \eta(d,c)] \rightarrow \mathbb{R}_I^+$ is a preinvex interval-valued function such that $\xi(\delta) = [\underline{\xi}(\delta), \overline{\xi}(\delta)]$ and $\phi(\delta) = [\underline{\phi}(\delta), \overline{\phi}(\delta)]$. $\xi, \phi \in L[c, c + \eta(d,c)]$ and η satisfies Condition C and $\alpha > 0$, then we have*

$$
2\xi\left(c + \frac{1}{2}\eta(d,c)\right)\phi\left(c + \frac{1}{2}\eta(d,c)\right)
$$
$$
\supseteq \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(d,c)}\left[J_{c^+}^\alpha \xi\left(c + \eta(d,c)\right)\phi\left(c + \eta(d,c)\right) + J_{(c+\eta(d,c))^-}^\alpha \xi(c)\phi(c)\right]
$$
$$
+ \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)}\right)N\left(c, c + \eta(d,c)\right) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)}M\left(c, c + \eta(d,c)\right),
$$

where $M(c, c + \eta(d,c))$ and $N(c, c + \eta(d,c))$ are defined as previously.

Proof Since ξ is a preinvex interval-valued function, we have

$$
\xi\left(x + \frac{1}{2}\eta(y,x)\right) \supseteq \frac{\xi(x) + \xi(y)}{2}, \quad \forall x, y \in \left[c, c + \eta(d,c)\right].
$$

Using $x = c + (1 - \delta)\eta(d,c)$, $y = c + \delta\eta(d,c)$ and Condition C, we get

$$
\xi\left(c + (1 - \delta)\eta(d,c) + \frac{1}{2}\eta\left(c + \delta\eta(d,c), c + (1 - \delta)\eta(d,c)\right)\right)
$$
$$
\supseteq \frac{\xi(c + (1 - \delta)\eta(d,c)) + \xi(c + \delta\eta(d,c))}{2}.
$$

This implies

$$
\xi\left(c + \frac{1}{2}\eta(d,c)\right) \supseteq \frac{\xi(c + (1 - \delta)\eta(d,c)) + \xi(c + \delta\eta(d,c))}{2}. \tag{29}
$$

Similarly,

$$
\phi\left(c + \frac{1}{2}\eta(d,c)\right) \supseteq \frac{\phi(c + (1 - \delta)\eta(d,c)) + \phi(c + \delta\eta(d,c))}{2}. \tag{30}
$$

From (29) and (30), we get

$$
\xi\left(c + \frac{1}{2}\eta(d,c)\right)\phi\left(c + \frac{1}{2}\eta(d,c)\right)
$$
$$
\supseteq \frac{1}{4}\left[\xi\left(c + (1 - \delta)\eta(d,c)\right) + \xi\left(c + \delta\eta(d,c)\right)\right]
$$
$$
\times \left[\phi\left(c + (1 - \delta)\eta(d,c)\right) + \phi\left(c + \delta\eta(d,c)\right)\right]
$$
$$
= \frac{1}{4}\left[\xi\left(c + (1 - \delta)\eta(d,c)\right)\phi\left(c + (1 - \delta)\eta(d,c)\right) + \xi\left(c + \delta\eta(d,c)\right)\phi\left(c + \delta\eta(d,c)\right)\right.
$$
$$
\left. + \xi\left(c + (1 - \delta)\eta(d,c)\right)\phi\left(c + \delta\eta(d,c)\right) + \xi\left(c + \delta\eta(d,c)\right)\phi\left(c + (1 - \delta)\eta(d,c)\right)\right]. \tag{31}
$$

Since ξ and $\phi \in \mathbb{R}_{\mathbb{I}}^+$, $\forall x \in [c, c + \eta(d,c)]$ are two preinvex interval-valued functions for $\delta \in [0,1]$, we have

$$
\begin{aligned}
\xi\big(c + (1-\delta)\eta(d,c)\big)&\phi\big(c + \delta\eta(d,c)\big) \\
&\supseteq \delta^2 \xi(c)\phi\big(c + \eta(d,c)\big) + (1-\delta)^2 \xi\big(c + \eta(d,c)\big)\phi(c) \\
&\quad + \delta(1-\delta)\big[\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big) + \xi(c)\phi(c)\big].
\end{aligned}
\tag{32}
$$

Similarly,

$$
\begin{aligned}
\xi\big(c + \delta\eta(d,c)\big)&\phi\big(c + (1-\delta)\eta(d,c)\big) \\
&\supseteq \delta^2 \xi\big(c + \eta(d,c)\big)\phi(c) + (1-\delta)^2 \xi(c)\phi\big(c + \eta(d,c)\big) \\
&\quad + \delta(1-\delta)\big[\xi\big(c + \eta(d,c)\big)\phi\big(c + \eta(d,c)\big) + \xi(c)\phi(c)\big].
\end{aligned}
\tag{33}
$$

Adding (32) and (33), we obtain

$$
\begin{aligned}
\xi\big(c + (1-\delta)\eta(d,c)\big)&\phi\big(c + \delta\eta(d,c)\big) + \xi\big(c + \delta\eta(d,c)\big)\phi\big(c + (1-\delta)\eta(d,c)\big) \\
&\supseteq \big[2\delta^2 - 2\delta + 1\big]N\big(c, c + \eta(d,c)\big) + 2\delta(1-\delta)M\big(c, c + \eta(d,c)\big).
\end{aligned}
\tag{34}
$$

From (31) and (34), we have

$$
\begin{aligned}
\xi\bigg(c + \frac{1}{2}\eta(d,c)\bigg)&\phi\bigg(c + \frac{1}{2}\eta(d,c)\bigg) \\
&\supseteq \frac{1}{4}\big[(2\delta^2 - 2\delta + 1)N\big(c, c + \eta(d,c)\big) + 2\delta(1-\delta)M\big(c, c + \eta(d,c)\big)\big] \\
&\quad + \frac{1}{4}\big[\xi\big(c + (1-\delta)\eta(d,c)\big)\phi\big(c + (1-\delta)\eta(d,c)\big) + \xi\big(c + \delta\eta(d,c)\big)\phi\big(c + \delta\eta(d,c)\big)\big].
\end{aligned}
$$

Multiplying by $\delta^{\alpha-1}$ on both sides, then integrating on $[0,1]$, we obtain

$$
\begin{aligned}
(IR)\int_0^1 \xi\bigg(c + &\frac{1}{2}\eta(d,c)\bigg)\phi\bigg(c + \frac{1}{2}\eta(d,c)\bigg)\delta^{\alpha-1}\,d\delta \\
&\supseteq \frac{1}{4}(IR)\int_0^1 \delta^{\alpha-1}\big(2\delta^2 - 2\delta + 1\big)N\big(c, c + \eta(d,c)\big)\,d\delta \\
&\quad + \frac{1}{2}(IR)\int_0^1 \delta^{\alpha}(1-\delta)M\big(c, c + \eta(d,c)\big)\,d\delta \\
&\quad + \frac{1}{4}(IR)\int_0^1 \xi\big(c + (1-\delta)\eta(d,c)\big)\phi\big(c + (1-\delta)\eta(d,c)\big)\delta^{\alpha-1}\,d\delta \\
&\quad + \frac{1}{4}(IR)\int_0^1 \xi\big(c + \delta\eta(d,c)\big)\phi\big(c + \delta\eta(d,c)\big)\delta^{\alpha-1}\,d\delta.
\end{aligned}
$$

This implies

$$2\xi\left(c + \frac{1}{2}\eta(d,c)\right)\phi\left(c + \frac{1}{2}\eta(d,c)\right)$$
$$\supseteq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(d,c)}\left[J^\alpha_{c^+}\xi\left(c + \eta(d,c)\right)\phi\left(c + \eta(d,c)\right) + J^\alpha_{(c+\eta(d,c))^-}\xi(c)\phi(c)\right]$$
$$+ \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N\left(c, c + \eta(d,c)\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}M\left(c, c + \eta(d,c)\right).$$

This completes the proof. □

Remark 3 When $\eta(d,c) = d - c$, then the above theorem reduces to Theorem 3.6 of [6].

4 Conclusion

In this paper, we introduced the concept of interval-valued h-preinvex functions. We established the Hermite–Hadamard inequalities for the preinvex interval-valued functions and Hermite–Hadamard type inequalities for the product of two preinvex interval-valued functions.

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