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Post-quantum Hermite–Hadamard type inequalities for interval-valued convex functions

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Abstract

In this research, we introduce the notions of (p, q)-derivative and integral for interval-valued functions and discuss their fundamental properties. After that, we prove some new inequalities of Hermite–Hadamard type for interval-valued convex functions employing the newly defined integral and derivative. Moreover, we find the estimates for the newly proved inequalities of Hermite–Hadamard type. It is also shown that the results proved in this study are the generalization of some already proved research in the field of Hermite–Hadamard inequalities.

Keywords: Hermite–Hadamard inequality; (p, q)-integral; Quantum calculus; Interval-valued calculus; Interval-valued convex functions

1 Introduction

Many studies have recently been carried out in the field of q-analysis, starting with Euler due to a high demand for mathematics that models quantum computing q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other sciences, quantum theory, mechanics, and the theory of relativity [1–5]. Apparently, Euler was the founder of this branch of mathematics by using the parameter q in Newton's work of infinite series. Later, Jackson was the first to develop q-calculus known as without limits calculus in a systematic way [2]. In 1908–1909, Jackson defined the general q-integral and q-difference operator [4]. In 1969, Agarwal described the q-fractional derivative for the first time [6]. In 1966–1967 Al-Salam introduced q-analogues of the Riemann–Liouville fractional integral operator and q-fractional integral operator [7]. In 2004, Rajkovic gave a definition of the Riemann-type q-integral which was a generalization of Jackson q-integral. In 2013, Tariboon introduced ${}_aD_q$-difference operator [8].

Many integral inequalities well known in classical analysis, such as Hölder's inequality, Simpson's inequality, Newton's inequality, Hermite–Hadamard inequality, Ostrowski inequality, Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss–Cebysev, and other integral inequalities, have been proved and applied for q-calculus using classical convexity. Many mathematicians have done studies in q-calculus analysis, the interested reader can check [9–25].

Inspired by these ongoing studies, we give the idea about the post-quantum derivative and integral in the setting of interval-valued calculus. We also prove some new inequalities of Hermite–Hadamard type and find their estimates.

2 Interval calculus

We give notation and preliminary information about the interval analysis in this section. Let the space of all closed intervals of $\mathbb{R}$ denoted by I_c and K be a bounded element of I_c, we have the representation

$$K = [\underline{k}, \overline{k}] = \{t \in \mathbb{R} : \underline{k} \leq t \leq \overline{k}\},$$

where $\underline{k}, \overline{k} \in \mathbb{R}$ and $\underline{k} \leq \overline{k}$. The length of the interval $K = [\underline{k}, \overline{k}]$ can be stated as $L(K) = \overline{k} - \underline{k}$. The numbers $\underline{k}$ and $\overline{k}$ are called the left and the right endpoints of interval K, respectively. When $\overline{k} = \underline{k}$, the interval K is said to be degenerate, and we use the form $K = k = [k, k]$. Also, we can say that K is positive if $\underline{k} > 0$, or we can say that K is negative if $\overline{k} < 0$. The sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ are denoted by I_c^+ and I_c^-, respectively. The Pompeiu–Hausdorff distance between the intervals K and M is defined by

$$d_H(K, M) = d_H\bigl([\underline{k}, \overline{k}], [\underline{m}, \overline{m}]\bigr) = \max\bigl\{|\underline{k} - \underline{m}|, |\overline{k} - \overline{m}|\bigr\}. \tag{2.1}$$

(I_c, d) is known to be a complete metric space (see [26]).

The absolute value of K, denoted by $|K|$, is the maximum of the absolute values of its endpoints:

$$|K| = \max\bigl\{|\underline{k}|, |\overline{k}|\bigr\}.$$

Now, we mention the definitions of fundamental interval arithmetic operations for the intervals K and M as follows:

$$\begin{aligned}
&K + M = [\underline{k} + \underline{m}, \overline{k} + \overline{m}],\\
&K - M = [\underline{k} - \overline{m}, \overline{k} - \underline{m}],\\
&K \cdot M = [\min U, \max U], \quad \text{where } U = \{\underline{k}\underline{m}, \underline{k}\overline{m}, \overline{k}\underline{m}, \overline{k}\overline{m}\},\\
&K / M = [\min V, \max V], \quad \text{where } V = \{\underline{k}/\underline{m}, \underline{k}/\overline{m}, \overline{k}/\underline{m}, \overline{k}/\overline{m}\} \text{ and } 0 \notin M.
\end{aligned}$$

Scalar multiplication of the interval K is defined by

$$\mu K = \mu[\underline{k}, \overline{k}] = \begin{cases} [\mu\underline{k}, \mu\overline{k}], & \mu > 0;\\ \{0\}, & \mu = 0;\\ [\mu\overline{k}, \mu\underline{k}], & \mu < 0, \end{cases}$$

where $\mu \in \mathbb{R}$.

The opposite of the interval K is

$$-K := (-1)K = [-\overline{k}, -\underline{k}],$$

where $\mu = -1$.

The subtraction is given by

$$K - M = K + (-M) = [\underline{k} - \overline{m}, \overline{k} - \underline{m}].$$

In general, $-K$ is not additive inverse for K, i.e., $K - K \neq 0$.

The definitions of operations cause a great many algebraic features which allows I_c to be a quasilinear space (see [27]). These properties can be listed as follows (see [26–30]):

(1) (Associativity of addition) $(K + M) + N = K + (M + N)$ for all $K, M, N \in I_c$,
(2) (Additivity element) $K + 0 = 0 + K = K$ for all $K \in I_c$,
(3) (Commutativity of addition) $K + M = M + K$ for all $K, M \in I_c$,
(4) (Cancellation law) $K + N = M + N \Longrightarrow K = M$ for all $K, M, N \in I_c$,
(5) (Associativity of multiplication) $(K \cdot M) \cdot N = K \cdot (M \cdot N)$ for all $K, M, N \in I_c$,
(6) (Commutativity of multiplication) $K \cdot M = M \cdot K$ for all $K, M \in I_c$,
(7) (Unity element) $K \cdot 1 = 1 \cdot K$ for all $K \in I_c$,
(8) (Associativity law) $\lambda(\mu K) = (\lambda\mu)K$ for all $K \in I_c$ and all $\lambda, \mu \in \mathbb{R}$,
(9) (First distributivity law) $\lambda(K + M) = \lambda K + \lambda M$ for all $K, M \in I_c$ and all $\lambda \in \mathbb{R}$,
(10) (Second distributivity law) $(\lambda + \mu)K = \lambda K + \mu K$ for all $K \in I_c$ and all $\lambda, \mu \in \mathbb{R}$.

In addition to all these features, the distributive law is not always true for intervals. As an example, $K = [1, 2]$, $M = [2, 3]$, and $N = [-2, -1]$.

$$K \cdot (M + N) = [0, 4],$$

whereas

$$K \cdot M + K \cdot N = [-2, 5].$$

Definition 1 ([31]) For the intervals K and M, we state that the $g\mathcal{H}$-difference of K and M is the interval T such that

$$K \ominus_g M = T \Leftrightarrow \begin{cases} K = M + T, \\ \text{or} \\ T = K + (-M). \end{cases}$$

It looks beyond dispute that

$$K \ominus_g M = \begin{cases} [\underline{k} - \underline{m}, \overline{k} - \overline{m}], & \text{if } L(K) \geq L(M), \\ [\overline{k} - \overline{m}, \underline{k} - \underline{m}], & \text{if } L(K) < L(M). \end{cases}$$

Particularly, if $M = m \in \mathbb{R}$ is a constant, we have

$$K \ominus_g M = [\underline{k} - m, \overline{k} - m].$$

Moreover, another set feature is the inclusion $\subseteq$ that is defined by

$$K \subseteq M \quad \Longleftrightarrow \quad \underline{k} \leq \underline{m} \quad \text{and} \quad \overline{k} \leq \overline{m}.$$

Throughout this paper, $0 < q < 1$ and a function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ is called L-increasing (or L-decreasing) if $L(f) : [a,b] \to [0,\infty)$ is increasing (or decreasing) on $[a,b]$. Also, $F = [\underline{F}, \overline{F}]$ is said to be L-monotone on $[a,b]$ if $L(f)$ is monotone on $[a,b]$. For condensation, interval-valued quantum calculus and interval-valued post-quantum calculus are denoted by Iq-calculus and $I(p,q)$-calculus, respectively.

3 Preliminaries of *Iq*-calculus and inequalities

In this section, we recollect some formerly regarded concepts about the q-calculus and Iq-calculus. Moreover, here and further we use the following notation (see [5]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0,1).$$

In [4], Jackson gave the q-Jackson integral from 0 to b for $0 < q < 1$ as follows:

$$\int_0^b f(x)\, d_q x = (1-q)b \sum_{n=0}^{\infty} q^n f\bigl(bq^n\bigr) \tag{3.1}$$

provided the sum converges absolutely.

Definition 2 ([32]) For a function $f : [a,b] \to \mathbb{R}$, the q_a-derivative of f at $x \in [a,b]$ is characterized by the expression

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{3.2}$$

Moreover, we have ${}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x)$. The function f is said to be q- differentiable on $[a,b]$ if ${}_aD_q f(x)$ exists for all $x \in [a,b]$. If $a = 0$ in (3.2), then ${}_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is familiar q-derivative of f at $x \in [0,b]$ defined by the expression (see [5])

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$

Definition 3 ([33]) For a function $f : [a,b] \to \mathbb{R}$, the q^b-derivative of f at $x \in [a,b]$ is characterized by the expression

$${}^bD_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \quad x \neq b.$$

Definition 4 ([32]) Let $f : [a,b] \to \mathbb{R}$ be a function. Then the q_a-definite integral on $[a,b]$ is defined as

$$\begin{aligned} \int_a^b f(x)\, {}_ad_q x &= (1-q)(b-a) \sum_{n=0}^{\infty} q^n f\bigl(q^n b + \bigl(1-q^n\bigr)a\bigr) \\ &= (b-a) \int_0^1 f\bigl((1-t)a + tb\bigr)\, d_q t. \end{aligned}$$

Definition 5 ([33]) Let $f:[a,b]\to\mathbb{R}$ be a function. Then the q^b-definite integral on $[a,b]$ is defined as

$$\int_a^b f(x)\,{}^b d_q x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f\bigl(q^n a + \bigl(1-q^n\bigr)b\bigr)$$
$$= (b-a)\int_0^1 f\bigl(ta+(1-t)b\bigr)\,d_q t.$$

On the other hand, recently, Lou et al. introduced the notions of $I(q)$-calculus. They gave the following definitions of $I(q)$-derivative and integral, and proved some inequalities of $I(q)$-Hermite–Hadamard type for interval-valued convex functions.

Definition 6 ([31]) For an interval-valued function $F=[\underline{F},\overline{F}]:[a,b]\to I_c$, the Iq_a-derivative of F at $x\in[a,b]$ is defined by

$$ {}_aD_qF(x)=\frac{F(x)\ominus_g F(qx+(1-q)a)}{(1-q)(x-a)},\quad x\neq a. \tag{3.3}$$

Since $F=[\underline{F},\overline{F}]:[a,b]\to I_c$ is a continuous function, we can state

$${}_aD_qF(a)=\lim_{x\to a}{}_aD_qF(x).$$

The function F is said to be Iq-differentiable on $[a,b]$ if ${}_aD_qF(x)$ exist for all $x\in[a,b]$. If we set $a=0$ in (3.3), then ${}_0D_qF(a)=D_qF(a)$, where $D_qF(a)$ is called Iq-Jackson derivative of F at $x\in[a,b]$ defined by the expression

$$D_qF(x)=\frac{F(x)\ominus_g F(qx)}{(1-q)x}.$$

Definition 7 ([31]) For an interval-valued function $F=[\underline{F},\overline{F}]:[a,b]\to I_c$, the Iq_a-definite integral is defined by

$$\int_a^x F(s)\,{}_ad_q^I s=(1-q)(x-a)\sum_{n=0}^{\infty}q^nF\bigl(q^nx+\bigl(1-q^n\bigr)a\bigr) \tag{3.4}$$

for all $x\in[a,b]$.

Remark 1 If we set $a=0$ in (3.4), then we have Iq-Jackson integral defined by the following equation:

$$\int_0^x F(s)\,{}_0d_q^I s=(1-q)x\sum_{n=0}^{\infty}q^nF\bigl(q^nx\bigr)$$

for all $x\in[0,\infty)$.

Theorem 1 ([31]) *Let $F=[\underline{F},\overline{F}]:[a,b]\to I_c$ be Iq_a-differentiable and convex on $[a,b]$. Then the Iq_a-Hermite–Hadamard inequality is expressed as follows*:

$$F\left(\frac{qa+b}{[2]_q}\right)\supseteq\frac{1}{b-a}\int_a^b F(x)\,{}_ad_q^I x\supseteq\frac{qF(a)+F(b)}{[2]_q}.$$

In [34], Alp et al. gave the definition of Iq^b-integral and proved inequalities of Hermite–Hadamard type for interval-valued convex functions by using Iq^b-integral.

Definition 8 For an interval-valued function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$, the Iq^b-definite integral is defined by

$$\int_x^b F(s)\,{}^b d_q^I s = (1-q)(b-x)\sum_{n=0}^{\infty} q^n F\bigl(q^n x + \bigl(1-q^n\bigr)b\bigr) \tag{3.5}$$

for all $x \in [a,b]$.

Theorem 2 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ be interval-valued convex on $[a,b]$. Then the Iq^b-Hermite–Hadamard inequality is expressed as follows*:

$$F\left(\frac{a+qb}{[2]_q}\right) \supseteq \frac{1}{b-a}\int_a^b F(x)\,{}^b d_q^I x \supseteq \frac{F(a)+qF(b)}{[2]_q}.$$

4 $I(p,q)$-calculus

In this section, the notions and results about the (p,q)-calculus are reviewed, and we are interested in introducing the concepts of $I(p,q)$-calculus.

The $[n]_{p,q}$ is said to be (p,q)-integers and expressed as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}$$

with $0 < q < p \le 1$. The $[n]_{p,q}!$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]!$ are called (p,q)-factorial and (p,q)-binomial, respectively, and expressed as

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \ge 1, \qquad [0]_{p,q}! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}! = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

Definition 9 ([35]) For a function $f : [a,b] \to \mathbb{R}$, the (p,q)-derivative of f at $x \in [a,b]$ is given by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \ne 0 \tag{4.1}$$

with $0 < q < p \le 1$.

On the other hand, Tunç and Göv gave the following new definitions of (p,q)-derivative and integrals.

Definition 10 ([36]) For a function $f : [a,b] \to \mathbb{R}$, the $(p,q)_a$-derivative of f at $x \in [a,b]$ is given by

$${}_aD_{p,q}f(x) = \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)}, \quad x \ne a, \tag{4.2}$$

with $0 < q < p \le 1$.

For $x = a$, we state ${}_aD_{p,q}f(a) = \lim_{x\to a} {}_aD_{p,q}f(x)$ if it exists and is finite.

Definition 11 ([36]) For a function $f : [a,b] \to \mathbb{R}$, the definite $(p,q)_a$-integral of f on $[a,b]$ is stated as

$$\int_a^x f(t)\, {}_ad_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)a\right) \tag{4.3}$$

with $0 < q < p \leq 1$.

On the other hand, Ali et al. gave the following new definition of (p,q)-derivative and integral, and proved some related inequalities.

Definition 12 ([37]) For a continuous function $f : [a,b] \to \mathbb{R}$, the definite $(p,q)^b$-integral of f on $[a,b]$ is stated as

$$\int_x^b f(t)\, {}^bd_{p,q}t = (p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)b\right) \tag{4.4}$$

with $0 < q < p \leq 1$.

Definition 13 ([37]) For a continuous function $f : [a,b] \to \mathbb{R}$, the $(p,q)^b$-derivative of f at $x \in [a,b]$ is given as follows:

$${}^bD_{p,q}f(x) = \frac{f(qx+(1-q)b) - f(px+(1-p)b)}{(p-q)(b-x)}, \quad x \neq b. \tag{4.5}$$

For $x = b$, we state ${}^bD_{p,q}f(b) = \lim_{x\to b} {}^bD_{p,q}f(x)$ if it exists and is finite.

Theorem 3 ([37]) *Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$. Then the following inequalities hold for $(p,q)^b$-integrals:*

$$f\left(\frac{pa+qb}{[2]_{p,q}}\right) \leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\, {}^bd_{p,q}x \leq \frac{pf(a)+qf(b)}{[2]_{p,q}}, \tag{4.6}$$

where $0 < q < p \leq 1$.

Theorem 4 ([37]) *Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$. Then the following inequalities hold for $(p,q)^b$-integrals:*

$$\begin{aligned} f\left(\frac{qa+pb}{[2]_{p,q}}\right) + \frac{(p-q)(b-a)}{[2]_{p,q}}f'\left(\frac{qa+pb}{[2]_{p,q}}\right) &\leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\, {}^bd_{p,q}x \\ &\leq \frac{pf(a)+qf(b)}{[2]_{p,q}}, \end{aligned} \tag{4.7}$$

where $0 < q < p \leq 1$.

Theorem 5 ([37]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and ${}^bD_{p,q}f$ be continuous and integrable on $[a,b]$. If $|{}^bD_{p,q}f|$ is a convex function over $[a,b]$, then we have the following (p,q)-midpoint inequality*:

$$\begin{aligned}&\left|\int_{ap+(1-p)b}^{b} f(x)\,{}^bd_{p,q}x - f\left(\frac{pa+qb}{[2]_{p,q}}\right)\right| \\ &\quad\le (b-a)\bigl[\bigl(\bigl|{}^bD_{p,q}f(a)\bigr|A_1(p,q)+\bigl|{}^bD_{p,q}f(b)\bigr|A_2(p,q)\bigr) \\ &\qquad+\bigl(\bigl|{}^bD_{p,q}f(a)\bigr|A_3(p,q)+\bigl|{}^bD_{p,q}f(b)\bigr|A_4(p,q)\bigr)\bigr],\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}A_1(p,q)&=\frac{qp^3}{([2]_{p,q})^3[3]_{p,q}},\\ A_2(p,q)&=\frac{q(p^3(p^2+q^2-p)+p^2[3]_{p,q})}{([2]_{p,q})^4[3]_{p,q}},\\ A_3(p,q)&=\frac{q(q+2p)}{[2]_{p,q}}-\frac{q^2(q^2+3p^2+3pq)}{([2]_{p,q})^3[3]_{p,q}},\\ A_4(p,q)&=\frac{q}{[2]_{p,q}}-\frac{q^2(q+2p)}{([2]_{p,q})^4}-A_3(p,q),\end{aligned}$$

and $0<q<p\le 1$.

Theorem 6 ([37]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and ${}^bD_{p,q}f$ be integrable on $[a,b]$. If $|{}^bD_{p,q}f|$ is a convex function over $[a,b]$, then we have the following new (p,q)-trapezoidal inequality*:

$$\begin{aligned}&\left|\frac{pf(a)+qf(b)}{[2]_{p,q}}-\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\,{}^bd_{p,q}x\right| \\ &\quad\le \frac{q(b-a)}{[2]_{p,q}}\bigl[\bigl|{}^bD_{p,q}f(a)\bigr|A_5(p,q)+\bigl|{}^bD_{p,q}f(b)\bigr|A_6(p,q)\bigr],\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}A_5(p,q)&=\frac{2([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}^3[3]_{p,q}}+\frac{[2]_{p,q}^2-[3]_{p,q}}{[2]_{p,q}[3]_{p,q}},\\ A_6(p,q)&=\frac{2([2]_{p,q}-1)}{[2]_{p,q}^2}-A_5(p,q).\end{aligned}$$

Now, we are able to introduce the concepts of $I(p,q)^b$- derivative and integrals.

4.1 $I(p,q)^b$-derivative

Definition 14 For a continuous interval-valued function $F=[\underline{F},\overline{F}]:[a,b]\to I_c$, the $I(p,q)^b$-derivative of F at $x\in[a,b]$ is given as follows:

$${}^bD_{p,q}F(x)=\frac{F(qx+(1-q)b)\ominus_g F(px+(1-p)b)}{(p-q)(b-x)},\quad x\neq b, \tag{4.10}$$

with $0<q<p\le 1$. For $x=b$, we state ${}^bD_{p,q}F(b)=\lim_{x\to b}{}^bD_{p,q}F(x)$ if it exists and is finite.

Remark 2 If we choose $p = 1$ in (4.10), then we have Iq^b-derivative defined as follows:

$$^{b}D_{q}F(x) = \frac{F(qx + (1-q)b) \ominus_{g} F(x)}{(1-q)(b-x)}, \quad x \neq b.$$

Theorem 7 *An interval-valued function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ is said to be $I(p,q)^b$-differentiable at $x \in [a,b]$ if and only if $\underline{F}$ and $\overline{F}$ are $(p,q)^b$-differentiable at $x \in [a,b]$. Moreover,*

$$^{b}D_{p,q}F(x) = \left[\min\left\{{}^{b}D_{p,q}\underline{F}(x), {}^{b}D_{p,q}\overline{F}(x)\right\}, \max\left\{{}^{b}D_{p,q}\underline{F}(x), {}^{b}D_{p,q}\overline{F}(x)\right\}\right]. \tag{4.11}$$

Proof Let F be an $I(p,q)^b$-differentiable function at $x \in [a,b]$, there exist $\underline{G}$ and $\overline{G}$ such that ${}^{b}D_{p,q}F(x) = [\underline{G}, \overline{G}]$. From Definition 14, we have that

$$\underline{G}(x) = \min\left\{\frac{\underline{F}(qx + (1-q)b) - \underline{F}(px + (1-p)a)}{(p-q)(b-x)}, \frac{\overline{F}(qx + (1-q)b) - \overline{F}(px + (1-p)a)}{(p-q)(b-x)}\right\}$$

and

$$\overline{G}(x) = \max\left\{\frac{\underline{F}(qx + (1-q)b) - \underline{F}(px + (1-p)a)}{(p-q)(b-x)}, \frac{\overline{F}(qx + (1-q)b) - \overline{F}(px + (1-p)a)}{(p-q)(b-x)}\right\}$$

exist. So, it is clear that the $\underline{F}$ and $\overline{F}$ are $(p,q)^b$-differentiable at $x \in [a,b]$.

To prove conversely, we suppose that $\underline{F}$ and $\overline{F}$ are $(p,q)^b$-differentiable at $x \in [a,b]$. Then we have two possibilities ${}^{b}D_{p,q}\underline{F}(x) \leq {}^{b}D_{p,q}\overline{F}(x)$ or ${}^{b}D_{p,q}\underline{F}(x) \geq {}^{b}D_{p,q}\overline{F}(x)$ for all $x \in [a,b]$.

If ${}^{b}D_{p,q}\underline{F}(x) \leq {}^{b}D_{p,q}\overline{F}(x)$, then we have following relation:

$$\begin{aligned}
&\left[{}^{b}D_{p,q}\underline{F}(x), {}^{b}D_{p,q}\overline{F}(x)\right] \\
&\quad = \left[\frac{\underline{F}(qx + (1-q)b) - \underline{F}(px + (1-p)a)}{(p-q)(b-x)}, \frac{\overline{F}(qx + (1-q)b) - \overline{F}(px + (1-p)a)}{(p-q)(b-x)}\right] \\
&\quad = \frac{F(qx + (1-q)b) \ominus_{g} F(px + (1-p)a)}{(p-q)(b-x)} \\
&\quad = {}^{b}D_{p,q}F(x).
\end{aligned}$$

Thus, $F(x)$ is $I(p,q)^b$-differentiable at $x \in [a,b]$. Now, if ${}^{b}D_{p,q}\underline{F}(x) \geq {}^{b}D_{p,q}\overline{F}(x)$, then

$$^{b}D_{p,q}F(x) = \left[{}^{b}D_{p,q}\overline{F}(x), {}^{b}D_{p,q}\underline{F}(x)\right],$$

and by applying the similar concepts, we can prove $F(x)$ is $I(p,q)^b$-differentiable at $x \in [a,b]$. □

Theorem 8 *Let $F = [\underline{F}, \overline{F}] \to I_c$ be an $I(p,q)^b$-differentiable function on $[a,b]$. Then the following equalities hold for all $x \in [a,b]$:*

1. ${}^{b}D_{p,q}F(x) = [{}^{b}D_{p,q}\underline{F}(x), {}^{b}D_{p,q}\overline{F}(x)]$, *if F is L-increasing;*
2. ${}^{b}D_{p,q}F(x) = [{}^{b}D_{p,q}\overline{F}(x), {}^{b}D_{p,q}\underline{F}(x)]$, *if F is L-decreasing.*

Proof To prove the first equality, we suppose that F is $I(p,q)^b$-differentiable and L-decreasing on $[a,b]$. So, we have

$$F\big(qx+(1-q)b\big) > F\big(px+(1-p)a\big)$$

for any $x \in [a,b]$. Since $L(f)$ is increasing, then we have

$$\begin{aligned}
&\big[\overline{F}\big(qx+(1-q)b\big) - \underline{F}\big(qx+(1-q)b\big)\big] \\
&\quad -\big[\overline{F}\big(px+(1-p)b\big) - \underline{F}\big(px+(1-p)b\big)\big] > 0, \\
&\overline{F}\big(qx+(1-q)b\big) - \overline{F}\big(px+(1-p)b\big) \\
&\quad > \underline{F}\big(qx+(1-q)b\big) - \underline{F}\big(px+(1-p)b\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&{}^bD_{p,q}F(x) \\
&\quad = \frac{[\underline{F}(qx+(1-q)b), \overline{F}(qx+(1-q)b)] \ominus_g [\underline{F}(px+(1-p)b), \overline{F}(px+(1-p)b)]}{(p-q)(b-x)} \\
&\quad = \left[\frac{\underline{F}(qx+(1-q)b) - \underline{F}(px+(1-p)b)}{(p-q)(b-x)}, \frac{\overline{F}(qx+(1-q)b) - \overline{F}(px+(1-p)b)}{(p-q)(b-x)}\right] \\
&\quad = \big[{}^bD_{p,q}\underline{F}(x), {}^bD_{p,q}\overline{F}(x)\big].
\end{aligned}$$

With the similar steps, the second equality can be done. □

Theorem 9 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ be a $I(p,q)^b$-differentiable function on $[a,b]$. Then, for all $C = [\underline{C}, \overline{C}] \in I_c$ and $\alpha \in \mathbb{R}$, the functions $F + C$ and αF are $I(p,q)^b$-differentiable functions on $[a,b]$. Moreover,*

$${}^bD_{p,q}(F + C) = {}^bD_{p,q}F(x)$$

and

$${}^bD_{p,q}\alpha F(x) = \alpha\, {}^bD_{p,q}F(x).$$

Proof The proof can be easily done using Definition 14, hence we leave the proof for the readers. □

Theorem 10 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ be an $I(p,q)^b$-differentiable function on $[a,b]$. Then, for all $C = [\underline{C}, \overline{C}] \in I_c$, if $L(F) - L(C)$ has a constant sign on $[a,b]$, then $F \ominus_g C$ is an $I(p,q)^b$-differentiable function and ${}^bD_{p,q}(F \ominus_g C) = {}^bD_{p,q}F(x)$.*

Proof The proof can be easily done using Definition 14, hence we leave the proof for the readers. □

4.2 $I(p,q)^b$-integral

Definition 15 For a continuous interval-valued function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$, the definite $I(p,q)^b$-integral of F on $[a,b]$ is stated as

$$\int_x^b F(t)\,{}^b d^I_{p,q} t = (p-q)(b-x)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} F\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) b\right) \tag{4.12}$$

with $0 < q < p \leq 1$.

Remark 3 If we set $p = 1$ in (4.12), then we have the definition of Iq^b-integral that we reviewed in the last section.

The following theorem provides us a relation between $I(p,q)^b$-integral and $(p,q)^b$-integral.

Theorem 11 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ be a continuous function on $[a,b]$, the function F is $I(p,q)^b$-integrable on $[a,b]$ if and only if $\underline{F}$ and $\overline{F}$ are $(p,q)^b$-integrable functions on $[a,b]$. Furthermore,*

$$\int_a^b F(x)\,{}^b d^I_{p,q} x = \left[\int_a^b \underline{F}(x)\,{}^b d_{p,q} x, \int_a^b \overline{F}(x)\,{}^b d_{p,q} x\right]. \tag{4.13}$$

Example 1 Let $F = [\underline{F}, \overline{F}] : [0,1] \to I_c$, defined by $F = [x^2, x]$. For $0 < q < p \leq 1$, we obtain that

$$\int_a^b F(x)\,{}^b d^I_{p,q} x = \left[\frac{1}{[3]_{p,q}}, \frac{1}{[2]_{p,q}}\right].$$

Theorem 12 *Let $F, G : [a,b] \to I_c$ be two continuous and $I(p,q)^b$-integrable functions on $[a,b]$ such that $F = [\underline{F}, \overline{F}]$ and $G = [\underline{G}, \overline{G}]$. Then, for $\alpha \in \mathbb{R}$, the following properties hold:*

1. $\int_a^b [F(x) + G(x)]\,{}^b d^I_{p,q} x = \int_a^b F(x)\,{}^b d^I_{p,q} x + \int_a^b G(x)\,{}^b d^I_{p,q} x$;
2. $\int_a^b \alpha F(x)\,{}^b d^I_{p,q} x = \alpha \int_a^b F(x)\,{}^b d^I_{p,q} x$;
3. $\int_a^b F(x)\,{}^b d^I_{p,q} x \ominus_g \int_a^b G(x)\,{}^b d^I_{p,q} x \subseteq \int_a^b F(x) \ominus_g G(x)\,{}^b d^I_{p,q} x$.

Moreover, if $L(F) - L(G)$ has a constant sign, then we have

$$\int_a^b F(x)\,{}^b d^I_{p,q} x \ominus_g \int_a^b G(x)\,{}^b d^I_{p,q} x = \int_a^b F(x) \ominus_g G(x)\,{}^b d^I_{p,q} x.$$

Theorem 13 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$ be a continuous function on $[a,b]$, if F is an $I(p,q)^b$-differentiable function on $[a,b]$, then ${}^b D_{p,q} F$ is $I(p,q)^b$-integrable. Furthermore, if F is L-increasing on $[a,b]$, then the following equality holds for $c \in [a,b]$:*

$$F(c) \ominus_g F(x) = \int_x^c {}^b D_{p,q} F(s)\,{}^b d^I_{p,q} s.$$

Proof The proof of Theorem 13 can be easily done by using Theorems 15 and 7. □

5 Hermite–Hadamard inequalities for $I(p,q)^b$-integral

In this section, we review the concept of interval-valued convex functions and prove inequalities of Hermite–Hadamard type for an interval-valued convex function by using the newly defined $I(p,q)$-integral.

Definition 16 ([31]) A function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c^+$ is said to be interval-valued convex if, for all $x, y \in [a,b]$ and $t \in (0,1)$, we have

$$tF(x) + (1-t)F(y) \subseteq F\big(tx + (1-t)y\big).$$

Theorem 14 *A function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c^+$ is said to be interval-valued convex if and only if $\underline{F}$ is a convex function $[a,b]$ and $\overline{F}$ is a concave function on $[a,b]$.*

Theorem 15 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c^+$ be a differentiable interval-valued convex function, then the following inequalities hold for the $I(p,q)^b$-integral*:

$$F\left(\frac{pa+qb}{[2]_{p,q}}\right) \supseteq \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} F(x)\,{}^b d_{p,q}^I x \supseteq \frac{pF(a)+qF(b)}{[2]_{p,q}}. \tag{5.1}$$

Proof Since $F = [\underline{F}, \overline{F}] : [a,b] \to I_c^+$ is an interval-valued convex function, therefore $\underline{F}$ is a convex function and $\overline{F}$ is a concave function. So, from $\underline{F}$ and inequality (4.6), we have

$$\underline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right) \le \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} \underline{F}(x)\,{}^b d_{p,q} x \le \frac{p\underline{F}(a)+q\underline{F}(b)}{[2]_{p,q}}, \tag{5.2}$$

and from the concavity of $\overline{F}$ and (4.6), we have

$$\frac{p\overline{F}(a)+q\overline{F}(b)}{[2]_{p,q}} \le \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} \overline{F}(x)\,{}^b d_{p,q} x \le \overline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right). \tag{5.3}$$

From (5.2) and (5.3), we obtain

$$\begin{aligned}\underline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right) &\le \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} \underline{F}(x)\,{}^b d_{p,q} x \\ &\le \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} \overline{F}(x)\,{}^b d_{p,q} x \le \overline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right),\end{aligned}$$

and hence, we have

$$F\left(\frac{pa+qb}{[2]_{p,q}}\right) \supseteq \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} F(x)\,{}^b d_{p,q}^I x. \tag{5.4}$$

Also, from (5.2) and (5.3), we obtain

$$\begin{aligned}\frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} \underline{F}(x)\,{}^b d_{p,q} x &\le \frac{p\underline{F}(a)+q\underline{F}(b)}{[2]_{p,q}} \\ &\le \frac{p\overline{F}(a)+q\overline{F}(b)}{[2]_{p,q}} \le \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} \overline{F}(x)\,{}^b d_{p,q} x,\end{aligned}$$

and hence, we have

$$\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}F(x)\,{}^{b}d_{p,q}^{I}x \supseteq \frac{p\underline{F}(a)+q\underline{F}(b)}{[2]_{p,q}}. \tag{5.5}$$

By combining (5.4) and (5.5), we obtain the required inequality, which accomplishes the proof. □

Theorem 16 *Let $F=[\underline{F},\overline{F}]:[a,b]\to I_c^+$ be a differentiable interval-valued convex function on $[a,b]$, then the following inequalities hold for the $I(p,q)^b$-integral*:

$$\begin{aligned}F\left(\frac{qa+pb}{[2]_{p,q}}\right)+\frac{(p-q)(b-a)}{[2]_{p,q}}F'\left(\frac{qa+pb}{[2]_{p,q}}\right) &\supseteq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}F(x)\,{}^{b}d_{p,q}^{I}x\\ &\supseteq \frac{pF(a)+qF(b)}{[2]_{p,q}}.\end{aligned} \tag{5.6}$$

Proof Since $F=[\underline{F},\overline{F}]:[a,b]\to I_c^+$ is an interval-valued convex function, therefore $\underline{F}$ is a convex function and $\overline{F}$ is a concave function. Because of the convexity of $\underline{F}$, from inequalities (4.7), we obtain that

$$\begin{aligned}\underline{F}\left(\frac{qa+pb}{[2]_{p,q}}\right)+\frac{(p-q)(b-a)}{[2]_{p,q}}\underline{F}'\left(\frac{qa+pb}{[2]_{p,q}}\right) &\le \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\underline{F}(x)\,{}^{b}d_{p,q}x\\ &\le \frac{p\underline{F}(a)+q\underline{F}(b)}{[2]_{p,q}}.\end{aligned} \tag{5.7}$$

Now, using the fact that $\overline{F}$ is a concave function, and from inequality (4.7), we obtain that

$$\begin{aligned}\overline{F}\left(\frac{qa+pb}{[2]_{p,q}}\right)+\frac{(p-q)(b-a)}{[2]_{p,q}}\overline{F}'\left(\frac{qa+pb}{[2]_{p,q}}\right) &\ge \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\overline{F}(x)\,{}^{b}d_{p,q}x\\ &\ge \frac{p\overline{F}(a)+q\overline{F}(b)}{[2]_{p,q}}.\end{aligned} \tag{5.8}$$

The rest of the proof can be done by applying the same lines of the previous theorem and considering inequalities (5.7) and (5.8). Thus, the proof is completed. □

Theorem 17 *Let $F=[\underline{F},\overline{F}]:[a,b]\to I_c^+$ be a differentiable interval-valued convex function on $[a,b]$, then the following inequalities hold for the $I(p,q)^b$-integral*:

$$\max\{A_1,A_2\}\supseteq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}F(x)\,{}^{b}d_{p,q}^{I}x \supseteq \frac{pF(a)+qF(b)}{[2]_{p,q}}, \tag{5.9}$$

where

$$A_1=F\left(\frac{pa+qb}{[2]_{p,q}}\right)$$

and

$$F\left(\frac{qa+pb}{[2]_{p,q}}\right)+\frac{(p-q)(b-a)}{[2]_{p,q}}F'\left(\frac{qa+pb}{[2]_{p,q}}\right).$$

Proof From inequalities (5.1) and (5.2), we have the required inequalities (5.9). Thus, the proof is finished. □

6 Midpoint and trapezoidal type inequalities for $I(p,q)^b$-integral

In this section, some new inequalities of midpoint and trapezoidal type for interval-valued functions are obtained.

Theorem 18 *Let $F=[\underline{F},\overline{F}]:[a,b]\to I_c^+$ be a $I(p,q)^b$-differentiable function. If $|{}^bD_{p,q}\underline{F}|$ and $|{}^bD_{p,q}\overline{F}|$ are convex functions on $[a,b]$, then the following $I(p,q)$ midpoint inequality holds for interval-valued functions*:

$$\begin{aligned} &d_H\left(\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}F(x)\,{}^bd_{p,q}^Ix, F\left(\frac{pa+qb}{[2]_{p,q}}\right)\right) \\ &\quad\le (b-a)\bigl[\bigl(\bigl|{}^bD_{p,q}F(a)\bigr|A_1(p,q)+\bigl|{}^bD_{p,q}F(b)\bigr|A_2(p,q)\bigr) \\ &\qquad+\bigl(\bigl|{}^bD_{p,q}F(a)\bigr|A_3(p,q)+\bigl|{}^bD_{p,q}F(b)\bigr|A_4(p,q)\bigr)\bigr], \end{aligned} \tag{6.1}$$

where $A_1(p,q)$–$A_4(p,q)$ are defined in Theorem 5 and d_H is a Pompeiu–Hausdorff distance between the intervals.

Proof Using the definition of d_H distance between intervals, one can easily obtain that

$$\begin{aligned} &d_H\left(\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}F(x)\,{}^bd_{p,q}^Ix, F\left(\frac{pa+qb}{[2]_{p,q}}\right)\right) \\ &\quad= d_H\left(\left[\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\underline{F}(x)\,{}^bd_{p,q}x, \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\overline{F}(x)\,{}^bd_{p,q}x\right],\right. \\ &\qquad\left.\left[\underline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right),\overline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right]\right) \\ &\quad= \max\left\{\left|\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\underline{F}(x)\,{}^bd_{p,q}x-\underline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right|,\right. \\ &\qquad\left.\left|\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\overline{F}(x)\,{}^bd_{p,q}x-\overline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right|\right\}. \end{aligned}$$

Now, using the fact that $|{}^bD_{p,q}\underline{F}|$ is a convex function, and from inequality (4.8), we have

$$\begin{aligned} &\left|\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\underline{F}(x)\,{}^bd_{p,q}x-\underline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right| \\ &\quad\le (b-a)\bigl[\bigl(\bigl|{}^bD_{p,q}\underline{F}(a)\bigr|A_1(p,q)+\bigl|{}^bD_{p,q}\underline{F}(b)\bigr|A_2(p,q)\bigr) \\ &\qquad+\bigl(\bigl|{}^bD_{p,q}\underline{F}(a)\bigr|A_3(p,q)+\bigl|{}^bD_{p,q}\underline{F}(b)\bigr|A_4(p,q)\bigr)\bigr]. \end{aligned} \tag{6.2}$$

Similarly, considering that $|{}^bD_{p,q}\overline{F}|$ is convex on $[a,b]$ and using inequality (4.8), we have

$$\begin{aligned} &\left|\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\overline{F}(x)\,{}^bd_{p,q}x-\overline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right| \\ &\quad\le (b-a)\bigl[\bigl(\bigl|{}^bD_{p,q}\overline{F}(a)\bigr|A_1(p,q)+\bigl|{}^bD_{p,q}\overline{F}(b)\bigr|A_2(p,q)\bigr) \\ &\qquad+\bigl(\bigl|{}^bD_{p,q}\overline{F}(a)\bigr|A_3(p,q)+\bigl|{}^bD_{p,q}\overline{F}(b)\bigr|A_4(p,q)\bigr)\bigr]. \end{aligned} \tag{6.3}$$

So, from inequalities (6.2) and (6.3), we have

$$
\begin{aligned}
& d_H\left(\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} F(x)\,{}^{b}d_{p,q}^{I}x, F\left(\frac{pa+qb}{[2]_{p,q}}\right)\right) \\
&= \max\left\{\left|\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} \underline{F}(x)\,{}^{b}d_{p,q}x - \underline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right|,\right. \\
&\quad \left.\left|\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} \overline{F}(x)\,{}^{b}d_{p,q}x - \overline{F}\left(\frac{pa+qb}{[2]_{p,q}}\right)\right|\right\} \\
&\leq \max\left\{(b-a)\left[\left(\left|{}^{b}D_{p,q}\underline{F}(a)\right|A_1(p,q) + \left|{}^{b}D_{p,q}\underline{F}(b)\right|A_2(p,q)\right)\right.\right. \\
&\quad \left.+\left(\left|{}^{b}D_{p,q}\underline{F}(a)\right|A_3(p,q) + \left|{}^{b}D_{p,q}\underline{F}(b)\right|A_4(p,q)\right)\right], \\
&\quad (b-a)\left[\left(\left|{}^{b}D_{p,q}\overline{F}(a)\right|A_1(p,q) + \left|{}^{b}D_{p,q}\overline{F}(b)\right|A_2(p,q)\right)\right. \\
&\quad \left.\left.+\left(\left|{}^{b}D_{p,q}\overline{F}(a)\right|A_3(p,q) + \left|{}^{b}D_{p,q}\overline{F}(b)\right|A_4(p,q)\right)\right]\right\} \\
&= (b-a)\left[\left(\left|{}^{b}D_{p,q}F(a)\right|A_1(p,q) + \left|{}^{b}D_{p,q}F(b)\right|A_2(p,q)\right)\right. \\
&\quad \left.+\left(\left|{}^{b}D_{p,q}F(a)\right|A_3(p,q) + \left|{}^{b}D_{p,q}F(b)\right|A_4(p,q)\right)\right]
\end{aligned}
$$

since

$$
\begin{aligned}
\left|{}^{b}D_{p,q}F(a)\right| &= \max\left\{\left|{}^{b}D_{p,q}\underline{F}(a)\right|, \left|{}^{b}D_{p,q}\overline{F}(a)\right|\right\}, \\
\left|{}^{b}D_{p,q}F(b)\right| &= \max\left\{\left|{}^{b}D_{p,q}\underline{F}(b)\right|, \left|{}^{b}D_{p,q}\overline{F}(b)\right|\right\}.
\end{aligned}
$$

Therefore, the proof is completed. □

Corollary 1 *If we set $p = 1$ in Theorem 18, then we have the following new q-midpoint inequality for interval-valued functions*:

$$
\begin{aligned}
& d_H\left(\frac{1}{(b-a)}\int_{a}^{b} F(x)\,{}^{b}d_{q}^{I}x, F\left(\frac{a+qb}{[2]_{q}}\right)\right) \\
&\leq (b-a)\left[\left(\left|{}^{b}D_{q}F(a)\right|A_1(1,q) + \left|{}^{b}D_{q}F(b)\right|A_2(1,q)\right)\right. \\
&\quad \left.+\left(\left|{}^{b}D_{q}F(a)\right|A_3(1,q) + \left|{}^{b}D_{q}F(b)\right|A_4(1,q)\right)\right],
\end{aligned}
$$

where $|{}^{b}D_{q}\underline{F}|$ and $|{}^{b}D_{q}\overline{F}|$ both are convex functions.

Corollary 2 *If we set $p = 1$ and $q \to 1^{-}$ in Theorem 18, then we have the following midpoint inequality for interval-valued functions*:

$$
\begin{aligned}
& d_H\left(\frac{1}{b-a}\int_{a}^{b} F(x)\,d^{I}x, F\left(\frac{a+b}{2}\right)\right) \\
&\leq (b-a)\left[\left(\left|F'(a)\right|A_1(1,1) + \left|F'(b)\right|A_2(1,1)\right)\right. \\
&\quad \left.+\left(\left|F'(a)\right|A_3(1,1) + \left|F'(b)\right|A_4(1,1)\right)\right],
\end{aligned}
$$

where $|\underline{F}'(a)|$ and $|\overline{F}'(a)|$ both are convex functions.

Theorem 19 *Let $F = [\underline{F}, \overline{F}] : [a,b] \to I_c^+$ be an $I(p,q)^b$-differentiable function. If $|{}^bD_{p,q}\underline{F}|$ and $|{}^bD_{p,q}\overline{F}|$ are convex functions on $[a,b]$, then the following $I(p,q)$ trapezoidal inequality holds for interval-valued functions*:

$$d_H\left(\frac{pF(a)+qF(b)}{[2]_{p,q}}, \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} F(x)\,{}^bd_{p,q}^I x\right) \le \frac{q(b-a)}{[2]_{p,q}}\left[\left|{}^bD_{p,q}F(a)\right|A_5(p,q) + \left|{}^bD_{p,q}F(b)\right|A_6(p,q)\right], \tag{6.4}$$

where A_5 and A_6 are defined in Theorem 6 and d_H is the Pompeiu–Hausdorff distance between the intervals.

Proof From the definition of d_H distance between the intervals and inequality (4.9), and using the strategies followed in the last theorem, one can easily obtain inequality (6.4). □

Corollary 3 *If we set $p = 1$ in Theorem 19, then we have the following new q-trapezoidal inequality for interval-valued functions*:

$$d_H\left(\frac{F(a)+qF(b)}{[2]_q}, \frac{1}{b-a}\int_a^b F(x)\,{}^bd_q^I x\right) \le \frac{q(b-a)}{[2]_q}\left[\left|{}^bD_qF(a)\right|A_5(1,q) + \left|{}^bD_qF(b)\right|A_6(1,q)\right],$$

where $|{}^bD_q\underline{F}|$ and $|{}^bD_q\overline{F}|$ both are convex functions.

Corollary 4 *If we set $p = 1$ and $q \to 1^-$ in Theorem 19, then we have the following new trapezoidal inequality for interval-valued functions*:

$$d_H\left(\frac{F(a)+F(b)}{2}, \frac{1}{b-a}\int_a^b F(x)\,d^I x\right) \le \frac{(b-a)}{2}\left[\left|F'(a)\right|A_5(1,1) + \left|F'(b)\right|A_6(1,1)\right],$$

where $|\underline{F}'(a)|$ and $|\overline{F}'(a)|$ both are convex functions.

7 Conclusion

In this study, we have introduced the notions of (p,q)-derivative and integral for interval-valued functions and discussed their basic properties. We have proved some new Hermite–Hadamard type inequalities for interval-valued convex functions by using newly given concepts of (p,q)-derivative and integral. Moreover, we have proved midpoint and trapezoidal estimates for newly established (p,q)-Hermite–Hadamard inequalities. It is an interesting and new problem that the upcoming researchers can establish Simpson type inequalities, Newton type inequalities, and Ostrowski type inequalities for interval-valued functions by employing the techniques of this research in their future work.

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Authors' contributions
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