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Correction to: Generalized Ponce's inequality

Julio Muñoz[1*]

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*Correspondence:
Julio.Munoz@uclm.es
[1]Departamento de Matemáticas, Universidad de Castilla-La Mancha, Toledo, Spain

Following publication of the original article [2], the author reported errors. The revised text is indicated hereafter.

1 Introduction

We assume that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary; $(k_\delta)_{\delta>0}$ is a set of radial positive functions such that supp $k_\delta \subset B(0,\delta)$, $\frac{1}{C_N}\int_{B(0,\delta)} k_\delta(|s|)\,ds = 1$, where $C_N = \frac{1}{\operatorname{meas}(S^{N-1})}\int_{S^{N-1}} |\sigma\cdot\mathbf{e}|^p\,d\mathcal{H}^{N-1}(\sigma)$, $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure on the unit sphere S^{N-1}, $\mathbf{e}$ is any unit vector in $\mathbb{R}^N$, $p>1$, and $B(x,\delta)$ is the the ball with center x and radius δ.

In [2], under the assumptions above, the following compactness is recalled (see [2] and references therein):

Theorem 1 *Assume Ω is an open bounded domain with Lipschitz boundary. Let $(u_\delta)_\delta$ be a sequence uniformly bounded in $L^p(\Omega)$, and let C be a positive constant such that*

$$\int_\Omega\int_\Omega \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x')-u_\delta(x)\right|^p dx'\,dx \le C \tag{1.1}$$

for any δ. Then, from $(u_\delta)_\delta$ we can extract a subsequence, still denoted by $(u_\delta)_\delta$, and we can find $u\in W^{1,p}(\Omega)$ such that $u_\delta \to u$ strongly in $L^p(\Omega)$ as $\delta\to 0$ and

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x')-u_\delta(x)\right|^p dx'\,dx \ge \int_\Omega \left|\nabla u(x)\right|^p dx. \tag{1.2}$$

Even though several authors are involved in the proof, we refer to estimate (1.2) as Ponce's inequality.

The goal of [2] is to prove the following extension of (1.2):

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x')-u_\delta(x)\right|^p dx'\,dx \ge \int_\Omega h(x)\left|\nabla u(x)\right|^p dx, \tag{1.3}$$

where Ω is an open bounded set with Lipschitz boundary, $H(x',x)=\frac{h(x')+h(x)}{2}$, and h is a nonnegative function from $L^\infty(\Omega)$.

Alternatively, the goal is to check the inequality (1.2) for measurable sets, that is,

$$\lim_{\delta\to 0}\int_E\int_E \frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x')-u_\delta(x)\right|^p dx'\,dx \\ \geq \int_E \left|\nabla u(x)\right|^p dx, \quad \text{for any measurable } E\subset\Omega. \tag{1.4}$$

It must be remarked that both inequalities are true but some basis for the proofs is false. Concretely, Proposition 1 from [2, p. 3] is wrong and, consequently, those parts where it is used have to be modified. Let us go through the steps and distinguish which parts are faulty.

2 First proof

Proposition 2 from [2, p. 4] is true and its proof is correct. The analysis application derived that proposition establishes

$$\liminf_{\delta\to 0}\int_O\int_O H(x',x)\frac{k_\delta(|x'-x|)}{|x'-x|^p}\left|u_\delta(x')-u_\delta(x)\right|^p dx'\,dx \geq \int_O h(x)\left|\nabla u(x)\right|^p dx \tag{2.1}$$

for any symmetric nonnegative continuous function $F\in L^\infty(O\times O)$ and any smooth open set O such that $|\partial O| = 0$. However, the proof extending (1.3) to the case where H is a measurable function of $L^\infty(\Omega)$ is invalid because it relies on Proposition 1.

The extension to the case of measurable functions is possible because Proposition 2 from [2, p. 4] is also true for the case $p=\infty$ and $q=1$. Let us check it. By looking back at the original work where the idea of the proof comes from, we can check that this result is valid for all $f\in L^p$ and $\xi\in L^q(\Omega)$, with $\frac{1}{p}+\frac{1}{q}=1$, even for the case $p=\infty$ and $q=1$ (see [3, p. 126]). Namely, in [3, p. 130], given $f\in L^p$, we can select a family of disjoint sets $\{a_{kj}+\epsilon_{kj}\overline{\Omega}\}_j$ covering Ω such that

$$\int_\Omega f(x)\psi(x)\,dx \leq \sum_i f(a_{ki})\int_{a_{ki}+\epsilon_{ki}\Omega}\psi(x)\,dx - \frac{1}{k}|\Omega|^{1/p}\|\psi\|_{L^q(\Omega)}$$

for any $\psi\in L^q$.

Now, for simplicity, we assume $f\in L^\infty$ and $\xi\in L^1$ are nonnegative functions. Since $\xi^{1/q}\in L^q$ for any q, and $f\in L^p$ for any p, the above inequality for $\psi=\xi^{1/q}$ reads as

$$\int_\Omega f(x)\xi^{1/q}(x)\,dx \leq \sum_i f(a_{ki})\int_{a_{ki}+\epsilon_{ki}\Omega}\xi^{1/q}(x)\,dx - \frac{1}{k}|\Omega|^{1/p}\|\xi\|^{1/q}_{L^1(\Omega)}.$$

If we pass to the limit as $p\uparrow\infty$, then $q=\frac{p}{p-1}\downarrow 1$ and $\xi^{1/q}(x)\to\xi(x)$, and, consequently, by monotone and dominated convergence for series and integrals, we infer

$$\int_\Omega f(x)\xi(x)\,dx \leq \sum_i f(a_{ki})\int_{a_{ki}+\epsilon_{ki}\Omega}\xi(x)\,dx - \frac{1}{k}\|\xi\|_{L^1(\Omega)}.$$

Using this inequality and following the previous procedure, then we can conclude that (2.1) remains valid for any symmetric and nonnegative function $F\in L^\infty(O\times O)$ and any smooth domain $O\subset\Omega$ such that $|\partial O| = 0$.

Finally, in order to circumvent the assumption $|\partial\Omega| = 0$, we simplify as follows: for the given domain Ω, we consider $\widetilde{\Omega}$, a regular domain containing Ω whose boundary is a null set, and we extend H by zero in $(\widetilde{\Omega} \times \widetilde{\Omega}) \setminus (\Omega \times \Omega)$. We denote this extended function of H by H_0, which is measurable, symmetric, and nonnegative. In the same way, we also appropriately extend u_δ to $\widetilde{\Omega}$, so that (1.1) still holds. To do that, we first note that Ω is smooth and, therefore, we can extend u to $\widetilde{u} \in W^{1,p}(\widetilde{\Omega})$. Then, we define $\widetilde{u}_\delta(x) = u(x)$ if $x \in \widetilde{\Omega} \setminus \Omega$ and $\widetilde{u}_\delta(x) = u_\delta(x)$ if $x \in \Omega$. It is immediate to check that $(\widetilde{u}_\delta)_\delta$ is uniformly bounded in L^p and

$$\int_{\widetilde{\Omega}} \int_{\widetilde{\Omega}} H_0(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|\widetilde{u}_\delta(x') - \widetilde{u}_\delta(x)\right|^p dx'\, dx \le C.$$

Then, by Theorem 1, we obtain

$$\liminf_{\delta\to 0} \int_{\widetilde{\Omega}} \int_{\widetilde{\Omega}} H_0(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^p} \left|\widetilde{u}_\delta(x') - \widetilde{u}_\delta(x)\right|^p dx'\, dx \ge \int_{\widetilde{\Omega}} H_0(x,x) \left|\nabla \widetilde{u}(x)\right|^p dx.$$

Now we realize that the above inequality coincides with (2.1) for any open and bounded set Ω.

The analysis performed proving a corollary in Sect. 2.3 in [2, p. 7] is correct and therefore serves to establish that (1.4) is valid for all measurable sets $G \subset \Omega$.

3 A second proof

This part of the paper deserves a stark modification because the proof given in [2] is based entirely on Proposition 1.

We first prove (1.4) and then (1.3). We assume Ω is open and $|\partial\Omega| = 0$. By hypothesis, $(\xi_\delta)_\delta$ is a sequence uniformly bounded in $L^1(\Omega \times \Omega)$ and, under these circumstances, we can use Chacon's biting lemma (see [1]) to ensure the existence of a subsequence of $\delta's$, not relabeled, a decreasing sequence of measurable sets $\mathcal{E}_n \subset \Omega \times \Omega$, such that $|\mathcal{E}_n| \downarrow 0$, and a function $\xi \in L^1(\Omega \times \Omega)$ such that $\xi_\delta \rightharpoonup \xi$ weakly in $L^1(\Omega \times \Omega \setminus \mathcal{E}_n)$ for all n. Since we are dealing with a sequence of symmetric functions, we can ensure $(\Omega \times \Omega) \setminus \mathcal{E}_n = (\Omega \setminus E_n) \times (\Omega \setminus E_n)$ where the sequence of sets $E_n \subset \Omega$ is decreasing and $|E_n| \downarrow 0$ if $n \to \infty$.

Let O_n be any open set such that $E_n \subset O_n \subset \Omega$, $|\partial O_n| = 0$, $|\overline{O}_n| \downarrow 0$ if $n \to \infty$, and $\overline{O}_n \subset \Omega$ except for a null subset of $\overline{O}_n$. To achieve these properties, we solely need to take $\overline{O}_n$ as the infimum of the unions of open balls containing E_n.

We apply Chacon's biting lemma to guarantee

$$\lim_{\delta\to 0} \iint_{A\times A} \xi_\delta(x',x)\, dx'\, dx = \iint_{A\times A} \xi(x',x)\, dx'\, dx \tag{3.1}$$

for any measurable $A \times A \subset (\Omega \setminus \overline{O}_n) \times (\Omega \setminus \overline{O}_n)$. Also, inequality (1.4) for open sets provides

$$\lim_{\delta\to 0} \iint_{A\times A} \xi_\delta(x',x)\, dx'\, dx \ge \int_A \left|\nabla u(x)\right|^p dx, \tag{3.2}$$

for any measurable set $A \subset \Omega \setminus \overline{O}_n$ (here we are considering the subsequence of $\delta's$ for which (1.4) holds).

Now, we first consider $A = B(x_0, r) \subset \Omega \setminus \overline{O}_n$ for any $x_0 \in \Omega \setminus \overline{O}_n$. Then, on the one hand, by (3.2)we have

$$\lim_{\delta\to 0} \iint_{B(x_0,r)\times B(x_0,r)} \xi_\delta\left(x',x\right) dx'\, dx = \iint_{B(x_0,r)\times B(x_0,r)} \xi\left(x',x\right) dx'\, dx. \tag{3.3}$$

On the other hand, since $B(x_0, r) \times B(x_0, r)$ is a smooth domain, (1.4) can be applied and hence

$$\lim_{\delta\to 0} \iint_{B(x_0,r)\times B(x_0,r)} \xi_\delta\left(x',x\right) dx'\, dx \geq \int_{B(x_0,r)} \left|\nabla u(x)\right|^p dx. \tag{3.4}$$

By using (3.3) and (3.4), we arrive at this crucial inequality for any $B(x_0, r) \subset \Omega \setminus \overline{O}_n$:

$$\iint_{B(x_0,r)\times B(x_0,r)} \xi\left(x',x\right) dx'\, dx \geq \int_{B(x_0,r)} \left|\nabla u(x)\right|^p dx\, dx'. \tag{3.5}$$

Thus, (3.2) holds for any measurable set $A \subset \Omega \setminus \overline{O}_n$.

Finally, we analyze $\lim_{\delta\to 0} \iint_{G\times G} \xi_\delta(x', x)\, dx'\, dx$, where $G \subset \Omega$ is any measurable set. We note that

$$\iint_{G\times G} \xi_\delta\left(x',x\right) dx'\, dx \geq \iint_{(G\setminus\overline{O}_n)\times(G\setminus\overline{O}_n)} \xi_\delta\left(x',x\right) dx'\, dx$$

which, thanks to Chacon's biting lemma, provides the estimate

$$\lim_{\delta\to 0} \iint_{G\times G} \xi_\delta\left(x',x\right) dx'\, dx \geq \iint_{(G\setminus\overline{O}_n)\times(G\setminus\overline{O}_n)} \xi\left(x',x\right) dx'\, dx.$$

Since $G \setminus \overline{O}_n$ is a measurable set included in $\Omega \setminus \overline{O}_n$, (3.2) for measurable sets provides the estimate

$$\iint_{(G\setminus\overline{O}_n)\times(G\setminus\overline{O}_n)} \xi\left(x',x\right) dx'\, dx \geq \int_{G\setminus\overline{O}_n} \left|\nabla u(x)\right|^p dx,$$

which straightforwardly implies

$$\lim_{\delta\to 0} \iint_{G\times G} \xi_\delta\left(x',x\right) dx'\, dx \geq \int_{G\setminus\overline{O}_n} \left|\nabla u(x)\right|^p dx.$$

By letting $n \to \infty$, we finish the proof of (1.4).

To avoid the hypothesis $|\partial\Omega| = 0$, we proceed as in the previous section.

The analysis performed when proving a corollary in Sect. 3.1 from [2, p. 8] is correct and therefore serves to assert that (1.3) is valid for all measurable functions h.

All the changes requested are implemented in this correction.

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References

1. Brooks, J.K., Chacon, R.V.: Continuity and compactness of measures. Adv. Math. **37**, 16–26 (1980)
2. Muñoz, J.: Generalized Ponce's inequality. J. Inequal. Appl. **2021**, 11 (2021). https://doi.org/10.1186/s13660-020-02543-1
3. Pedregal, P.: Parametrized Measures and Variational Principles. Birkhäuser, Basel (1997)