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New characterizations of weights on dynamic inequalities involving a Hardy operator

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Abstract

In this paper, we establish some new characterizations of weighted functions of dynamic inequalities containing a Hardy operator on time scales. These inequalities contain the characterization of Ariňo and Muckenhoupt when $\mathbb{T} = \mathbb{R}$, whereas they contain the characterizations of Bennett–Erdmann and Gao when $\mathbb{T} = \mathbb{N}$.

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1 Introduction

In [10], Muckenhoupt characterized the weights such that the inequality

$$\left(\int_0^\infty \Re^k(x)\left(\int_0^x \hbar(\varsigma)\,d\varsigma\right)^k dx\right)^{1/k} \leq C\left(\int_0^\infty \omega^k(x)\hbar^k(x)\,dx\right)^{1/k}$$

holds for all measurable $\hbar \geq 0$ and the constant C is independent of $\hbar$ (here $1 < k < \infty$). The characterization reduces to the condition that the nonnegative functions $\Re$ and ω satisfy

$$\sup_{x>0}\left(\int_x^\infty \Re^k(\varsigma)\,d\varsigma\right)^{1/k}\left(\int_0^x \omega^{-k^*}(\varsigma)\,d\varsigma\right)^{1/k^*} = K < \infty, \quad k^* = \frac{k}{k-1},$$

and $K \leq C \leq Kk^{1/k}(k^*)^{1/k^*}$.

In [7], Bradley gave new characterizations of weights such that the general inequality

$$\left(\int_0^\infty \Re^q(x)\left(\int_0^x \hbar(\varsigma)\,d\varsigma\right)^q dx\right)^{1/q} \leq C\left(\int_0^\infty \omega^k(x)\hbar^k(x)\,dx\right)^{1/k}$$

holds for all measurable $\hbar \geq 0$ and the constant C is independent of $\hbar$ (here $1 \leq k \leq q \leq \infty$). The characterization reduces to the condition that the nonnegative functions $\Re$ and

ω satisfy

$$\sup_{x>0}\left(\int_x^\infty \Re^q(\varsigma)\,d\varsigma\right)^{1/q}\left(\int_0^x \omega^{-k^*}(\varsigma)\,d\varsigma\right)^{1/k^*} = K < \infty,$$

and $K \le C \le Kk^{1/q}(k^*)^{1/k^*}$ for $1 < k < q < \infty$ and $K = C$ if $k = 1$ and $q = \infty$.

In [3], Ariňo and Muckenhoupt characterized the weight function such that the inequality

$$\int_0^\infty \varphi(\varsigma)\left(\frac{1}{\varsigma}\int_0^\varsigma \hbar(x)\,dx\right)^k d\varsigma \le C\int_0^\infty \varphi(\varsigma)\hbar^k(\varsigma)\,d\varsigma \tag{1}$$

holds for all nonnegative nonincreasing measurable function $\hbar$ on $(0,\infty)$ with a constant $C > 0$ independent on $\hbar$ (here $1 \le k < \infty$). The characterization reduces to the condition that the function φ satisfies

$$\int_\varsigma^\infty \frac{\varphi(x)}{x^k}\,dx \le \frac{B}{\varsigma^k}\int_0^\varsigma \varphi(x)\,dx \quad \text{for all } \varsigma \in (0,\infty) \text{ and } B > 0.$$

In [12], Sinnamon characterized the weights such that the inequality

$$\left(\int_{\varsigma_0}^b \Re(x)\left(\int_{\varsigma_0}^x \hbar(\varsigma)\,d\varsigma\right)^q dx\right)^{1/q} \le C\left(\int_{\varsigma_0}^b \omega(x)\hbar^k(x)\,dx\right)^{1/k}$$

holds for all measurable $\hbar \ge 0$ and the constant C is independent of $\hbar$ (here $0 < q < 1 < k$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{k}$). The characterization reduces to the condition that the nonnegative functions $\Re$ and ω satisfy

$$\int_{\varsigma_0}^b\left(\int_x^b \Re(\varsigma)\,d\varsigma\right)^{r/k}\left(\int_{\varsigma_0}^x \omega^{1-k'}(\varsigma)\,d\varsigma\right)^{r/k'}\Re(x)\,dx = K < \infty, \quad k' = \frac{k}{k-1}.$$

For the discrete case, however, Bennett and Erdmann [4] characterized the weights such that the inequality

$$\sum_{n=1}^\infty \varphi_n\left(\frac{1}{n}\sum_{k=1}^n z_k\right)^k \le C\sum_{n=1}^\infty \varphi_n z_n^k, \quad 1 \le k < \infty, \tag{2}$$

holds for all nonnegative nonincreasing sequence z_n. The characterization reduces to the condition that the nonnegative sequence φ_n satisfies

$$\sum_{k=n}^\infty \frac{\varphi_k}{k^k} \le \frac{B}{n^k}\sum_{k=1}^n \varphi_k \quad \text{for all } n \in \mathbb{N} \text{ and } B > 0.$$

In [8], Gao extended the results of Bennett and Erdmann and characterized the weights such that the inequality

$$\sum_{n=1}^\infty \frac{\varphi_n}{A_n^k}\left(\sum_{k=1}^n a_k z_k\right)^k \le C\sum_{n=1}^\infty \varphi_n z_n^k, \quad k \ge 1, \tag{3}$$

holds for all nonnegative and nonincreasing sequences z_n and a_n with $a_1 > 0$, where the constant C is independent of a_n and z_n. The characterization reduces to the condition that the nonnegative sequences a_n and φ_n satisfy

$$\sum_{k=n}^{\infty} \frac{\varphi_k}{A_k^k} \leq \frac{B}{A_n^k} \sum_{k=1}^{n} \varphi_k \quad \text{for all } n \in \mathbb{N} \text{ and } B > 0,$$

where $A_n = \sum_{k=1}^{n} a_k$.

In this paper, we are concerned with proving some dynamic inequalities on time scales; see [1, 2]. The general idea is to prove our results where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In [11], the authors characterized the weights such that the dynamic inequality

$$\left(\int_{\varsigma_0}^{b} \Re(x)\left(\int_{\varsigma_0}^{\sigma(x)} \hbar(\varsigma)\Delta\varsigma\right)^{q} \Delta x\right)^{1/q} \leq C\left(\int_{\varsigma_0}^{b} \upsilon(x)\hbar^{k}(x)\Delta x\right)^{1/k} \tag{4}$$

on a time scale $\mathbb{T}$ holds for all nonnegative rd-continuous function $\hbar$ on $[\varsigma_0, b]_{\mathbb{T}}$ with ς_0, $b \in \mathbb{T}$, $1 < k \leq q < \infty$. The characterization reduces to the condition that the nonnegative functions $\Re$ and υ satisfy

$$\sup_{\varsigma_0 < x < b} \left(\int_{x}^{b} \Re(\varsigma)\Delta\varsigma\right)^{1/q} \left(\int_{\varsigma_0}^{\sigma(x)} \upsilon^{1-k'}(\varsigma)\Delta\varsigma\right)^{1/k'} = K < \infty, \quad k' = \frac{k}{k-1}.$$

Moreover, the estimate for the constant C in (4) is given by

$$K \leq C \leq \left(1 + \frac{q}{k'}\right)^{1/q} \left(1 + \frac{k'}{q}\right)^{1/k'} K.$$

As a particular case of (4) if $k = q$, $\Re(\varsigma) = \varphi(\varsigma)(\sigma(\varsigma) - \varsigma_0)^{-k}$ and $\upsilon = \varphi$, then we get the inequality

$$\int_{\varsigma_0}^{b} \varphi(x)\left(\frac{1}{\sigma(\varsigma) - \varsigma_0} \int_{\varsigma_0}^{\sigma(x)} \hbar(\varsigma)\Delta\varsigma\right)^{k} \Delta x \leq C \int_{\varsigma_0}^{b} \varphi(x)\hbar^{k}(x)\Delta x, \tag{5}$$

and the characterization reduces to the condition that the nonnegative function φ satisfies

$$\sup_{\varsigma_0 < x < b} \left(\int_{x}^{b} \frac{\varphi(\varsigma)}{(\sigma(\varsigma) - \varsigma_0)^k} \Delta\varsigma\right)^{1/k} \left(\int_{\varsigma_0}^{\sigma(x)} \varphi^{1-k'}(\varsigma)\Delta\varsigma\right)^{1/k'} = K < \infty.$$

Our aim in this paper is to establish some new characterizations of the weights for the dynamic inequalities of the form (5) and for the general inequalities of the form

$$\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^c} \left(\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(x)\hbar(x)\Delta x\right)^{k} \Delta\varsigma \leq D \int_{\varsigma_0}^{\infty} \varphi(\varsigma)\hbar^{k}(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c} \Delta\varsigma, \tag{6}$$

where $\Upsilon(\varsigma) = \int_{\varsigma_0}^{\varsigma} \psi(x)\Delta x$, $1 < k < \infty$, and $c > 0$.

The paper is organized as follows. In Sect. 2, we present some definitions and basic concepts of time scales and prove essential lemmas needed in Sect. 3 where the main results are proved. Our findings significantly recover particular cases. Indeed, the proposed theorems contain the characterizations of inequalities (2) and (3) proved by Bennett and Erdmann and Gao when $\mathbb{T} = \mathbb{N}$, whereas they give the characterizations of inequality (1) proved by Ariño and Muckenhoupt when $\mathbb{T} = \mathbb{R}$.

2 Preliminaries and basic lemmas

For completeness, we recall the following concepts related to the notion of time scales. We refer the reader to the two books by Bohner and Peterson [5, 6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by: $\sigma(\varsigma) := \inf\{s \in \mathbb{T} : s > \varsigma\}$ and $\rho(\varsigma) := \sup\{s \in \mathbb{T} : s < \varsigma\}$, respectively. A point $\varsigma \in \mathbb{T}$ is said to be left-dense if $\rho(\varsigma) = \varsigma$, right-dense if $\sigma(\varsigma) = \varsigma$, left-scattered if $\rho(\varsigma) < \varsigma$, and right-scattered if $\sigma(\varsigma) > \varsigma$. A function $z : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided z is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(\varsigma) := \sigma(\varsigma) - \varsigma \geq 0$, and for any function $\hbar : \mathbb{T} \to \mathbb{R}$ the notation $\hbar^\sigma(\varsigma)$ denotes $\hbar(\sigma(\varsigma))$. The three most popular examples of calculus on time scales are when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^\varsigma : \varsigma \in \mathbb{N}_0\}$, where $q > 1$. The derivative of the product $\hbar z$ and the quotient $\hbar/z$ (where $zz^\sigma \neq 0$) of two differentiable functions $\hbar$ and z are given by

$$(\hbar z)^\Delta = \hbar^\Delta z + \hbar^\sigma z^\Delta = \hbar z^\Delta + \hbar^\Delta z^\sigma, \qquad \left(\frac{\hbar}{z}\right)^\Delta = \frac{\hbar^\Delta z - \hbar z^\Delta}{zz^\sigma}. \tag{7}$$

In this paper, we refer to the (delta) integral which is defined as follows: If $Z^\Delta(\varsigma) = z(\varsigma)$, then $\int_{\varsigma_0}^{\varsigma} z(s)\Delta s := Z(\varsigma) - Z(\varsigma_0)$. It can be shown (see [5]) that if $z \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, then the Cauchy integral $Z(\varsigma) := \int_{\varsigma_0}^{\varsigma} z(s)\Delta s$ exists, $\varsigma_0 \in \mathbb{T}$, and satisfies $Z^\Delta(\varsigma) = z(\varsigma)$, $\varsigma \in \mathbb{T}$. An improper integral is defined by $\int_{\varsigma_0}^{\infty} z(\varsigma)\Delta\varsigma = \lim_{b\to\infty} \int_{\varsigma_0}^{b} z(\varsigma)\Delta\varsigma$, and the integration by parts formula on time scales is given by

$$\int_{\varsigma_0}^{b} \Re(\varsigma)\omega^\Delta(\varsigma)\Delta\varsigma = \big[\Re(\varsigma)\omega(\varsigma)\big]_{\varsigma_0}^{b} - \int_{\varsigma_0}^{b} \Re^\Delta(\varsigma)\omega^\sigma(\varsigma)\Delta\varsigma. \tag{8}$$

The time scales chain rule (see [5, Theorem 1.87]) is given by

$$(z \circ \delta)^\Delta(\varsigma) = z'\big(\delta(d)\big)\delta^\Delta(\varsigma), \quad \text{where } d \in \big[\varsigma, \sigma(\varsigma)\big], \tag{9}$$

where it is assumed that $z : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $\delta : \mathbb{T} \to \mathbb{R}$ is delta differentiable. A simple consequence of Keller's chain rule [5, Theorem 1.90] is given by

$$\big(x^\gamma(\varsigma)\big)^\Delta = \gamma \int_0^1 \big[hx^\sigma(\varsigma) + (1-h)x(\varsigma)\big]^{\gamma-1}\,dh\,x^\Delta(\varsigma). \tag{10}$$

The Hölder inequality, see [5, Theorem 6.13], on time scales is given by

$$\int_{\varsigma_0}^{b} \left|\hbar(\varsigma)z(\varsigma)\right|\Delta\varsigma \le \left[\int_{\varsigma_0}^{b} \left|\hbar(\varsigma)\right|^{\gamma}\Delta\varsigma\right]^{\frac{1}{\gamma}}\left[\int_{\varsigma_0}^{b} \left|z(\varsigma)\right|^{\nu}\Delta\varsigma\right]^{\frac{1}{\nu}}, \tag{11}$$

where ς_0, $b \in \mathbb{T}$, $\hbar$, $z \in \mathrm{C}_{\mathrm{rd}}(\mathbb{I},\mathbb{R})$, $\gamma > 1$, and $\frac{1}{\gamma} + \frac{1}{\nu} = 1$. The special case $\gamma = \nu = 2$ in (11) yields the time scales Cauchy–Schwarz inequality.

Throughout the paper, we assume (without mentioning) that the functions are nonnegative rd-continuous functions on $[\varsigma_0,\infty)_{\mathbb{T}}$ and the integrals considered are assumed to exist (finite i.e. convergent). We define $[\varsigma_0,b]_{\mathbb{T}}$ by $[\varsigma_0,b]_{\mathbb{T}} := [\varsigma_0,b] \cap \mathbb{T}$ and call it the time scale interval. The following lemma is adopted from [11].

Lemma 2.1 *Assume that $\mathbb{T}$ is a time scale with ς_0, $\varsigma \in \mathbb{T}$ and $z \in C_{\mathrm{rd}}([\varsigma_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$. If $k \ge 1$, then*

$$\left(\int_{\varsigma_0}^{\sigma(\varsigma)} z(x)\Delta x\right)^{k} \le k\int_{\varsigma_0}^{\sigma(\varsigma)} z(x)\left(\int_{\varsigma_0}^{\sigma(x)} z(\tau)\Delta\tau\right)^{k-1}\Delta x.$$

The following lemmas are needed in Sect. 3.

Lemma 2.2 *Assume that φ, ψ are nonnegative rd-continuous functions defined on $[\varsigma_0,\infty)_{\mathbb{T}}$. Then*

$$\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\left(\int_{\varsigma}^{\infty} \psi(x)\Delta x\right)\Delta\varsigma = \int_{\varsigma_0}^{\infty} \psi(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(x)\Delta x\right)\Delta\varsigma.$$

Proof Let $\Upsilon(\varsigma) = \int_{\varsigma}^{\infty} \psi(x)\Delta x$. Applying formula (8) on the term $\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\Upsilon(\varsigma)\Delta\varsigma$ with $\Re(\varsigma) = \Upsilon(\varsigma)$ and $\omega^{\Delta}(\varsigma) = \varphi(\varsigma)$, we see that

$$\begin{aligned}&\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\left(\int_{\varsigma}^{\infty} \psi(x)\Delta x\right)\Delta\varsigma \\ &\quad= \int_{\varsigma_0}^{\infty} \varphi(\varsigma)\Upsilon(\varsigma)\Delta\varsigma = \Upsilon(\varsigma)\omega(\varsigma)\big|_{\varsigma_0}^{\infty} - \int_{\varsigma_0}^{\infty} \Upsilon^{\Delta}(\varsigma)\omega^{\sigma}(\varsigma)\Delta\varsigma,\end{aligned}$$

where $\omega(\varsigma) = \int_{\varsigma_0}^{\varsigma} \varphi(x)\Delta x$. Using $\omega(\varsigma_0) = \Upsilon(\infty) = 0$ (recall all integrals are assumed to be convergent), we obtain that

$$\begin{aligned}\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\left(\int_{\varsigma}^{\infty} \psi(x)\Delta x\right)\Delta\varsigma &= \int_{\varsigma_0}^{\infty} \left[-\Upsilon^{\Delta}(\varsigma)\right]\omega^{\sigma}(\varsigma)\Delta\varsigma = \int_{\varsigma_0}^{\infty} \psi(\varsigma)\omega^{\sigma}(\varsigma)\Delta\varsigma \\ &= \int_{\varsigma_0}^{\infty} \psi(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(x)\Delta x\right)\Delta\varsigma.\end{aligned}$$

The proof is complete. □

Lemma 2.3 *Let $\phi, z, \upsilon \in C_{\mathrm{rd}}([\varsigma_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$. If*

$$\begin{aligned}&\int_{\varsigma_0}^{\sigma(\varsigma)} \upsilon(s)\Delta s \le \int_{\varsigma_0}^{\sigma(\varsigma)} \phi(s)\Delta s \quad \textit{for } \varsigma \in (\varsigma_0,\infty)_{\mathbb{T}}, \quad \textit{and} \\ &\int_{\varsigma_0}^{\infty} \upsilon(s)\Delta s \le \int_{\varsigma_0}^{\infty} \phi(s)\Delta s,\end{aligned} \tag{12}$$

then

$$\int_{\varsigma_0}^{\infty} \upsilon(\varsigma)z(\varsigma)\Delta\varsigma \le \int_{\varsigma_0}^{\infty} \phi(\varsigma)z(\varsigma)\Delta\varsigma,$$

where z is a nonincreasing function.

Proof Integrating the term $\int_{\varsigma_0}^{\infty} \upsilon(\varsigma)z(\varsigma)\Delta\varsigma$ and using (8) with $\Re(\varsigma) = z(\varsigma)$ and $\omega^{\Delta}(\varsigma) = \upsilon(\varsigma)$, we have

$$\int_{\varsigma_0}^{\infty} \upsilon(\varsigma)z(\varsigma)\Delta\varsigma = z(\varsigma)\omega(\varsigma)|_{\varsigma_0}^{\infty} - \int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\omega^{\sigma}(\varsigma)\Delta\varsigma,$$

where $\omega(\varsigma) = \int_{\varsigma_0}^{\varsigma} \upsilon(s)\Delta s$. Using the fact that $\omega(\varsigma_0) = 0$, we get that

$$\begin{aligned}\int_{\varsigma_0}^{\infty} \upsilon(\varsigma)z(\varsigma)\Delta\varsigma &= \lim_{\varsigma\to\infty}\left(z(\varsigma)\int_{\varsigma_0}^{\varsigma} \upsilon(s)\Delta s\right) - \int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\omega^{\sigma}(\varsigma)\Delta\varsigma \\ &= \lim_{\varsigma\to\infty}\left(z(\varsigma)\int_{\varsigma_0}^{\varsigma} \upsilon(s)\Delta s\right) - \int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \upsilon(s)\Delta s\right)\Delta\varsigma.\end{aligned}$$

Using (12) (note that $z^{\Delta}(\varsigma) \le 0$), we obtain

$$\int_{\varsigma_0}^{\infty} \upsilon(\varsigma)z(\varsigma)\Delta\varsigma \le \lim_{\varsigma\to\infty}\left(z(\varsigma)\int_{\varsigma_0}^{\varsigma} \phi(s)\Delta s\right) - \int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \phi(s)\Delta s\right)\Delta\varsigma. \quad (13)$$

Applying the integration by parts formula (8) on the term

$$\int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \phi(s)\Delta s\right)\Delta\varsigma,$$

with $\Re^{\Delta}(\varsigma) = z^{\Delta}(\varsigma)$ and $\omega^{\sigma}(\varsigma) = \int_{\varsigma_0}^{\sigma(\varsigma)} \phi(s)\Delta s$, we have that

$$\int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \phi(s)\Delta s\right)\Delta\varsigma = z(\varsigma)\omega(\varsigma)|_{\varsigma_0}^{\infty} - \int_{\varsigma_0}^{\infty} z(\varsigma)\omega^{\Delta}(\varsigma)\Delta\varsigma,$$

where $\omega(\varsigma) = \int_{\varsigma_0}^{\varsigma} \phi(s)\Delta s$. Using the fact that $\omega(\varsigma_0) = 0$, we see that

$$\int_{\varsigma_0}^{\infty} z^{\Delta}(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \phi(s)\Delta s\right)\Delta\varsigma = \lim_{\varsigma\to\infty}\left(z(\varsigma)\int_{\varsigma_0}^{\varsigma} \phi(s)\Delta s\right) - \int_{\varsigma_0}^{\infty} z(\varsigma)\phi(\varsigma)\Delta\varsigma. \quad (14)$$

Substituting (14) into (13), we observe that

$$\int_{\varsigma_0}^{\infty} \upsilon(\varsigma)z(\varsigma)\Delta\varsigma \le \int_{\varsigma_0}^{\infty} \phi(\varsigma)z(\varsigma)\Delta\varsigma.$$

The proof is complete. □

Remark 2.1 Note in the above lemma that

$$\lim_{\varsigma\to\infty}\left(z(\varsigma)\int_{\varsigma_0}^{\varsigma} \upsilon(s)\Delta s\right) \quad \text{and} \quad \lim_{\varsigma\to\infty}\left(z(\varsigma)\int_{\varsigma_0}^{\varsigma} \phi(s)\Delta s\right)$$

exist since $\int_{\varsigma_0}^{\infty} \upsilon(s)\Delta s$ and $\int_{\varsigma_0}^{\infty} \phi(s)\Delta s$ are convergent and $z \in C_{\mathrm{rd}}([\varsigma_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ is nondecreasing.

Lemma 2.4 *Assume that $\psi, \hbar \in C_{\mathrm{rd}}([\varsigma_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $\hbar$ is nonincreasing and $k \geq 1$. Let*

$$\hbar(\varsigma) = \frac{1}{\Upsilon(\varsigma)} \int_{\varsigma_0}^{\varsigma} \psi(x)\hbar(x)\Delta x \quad \textit{and} \quad \Upsilon(\varsigma) = \int_{\varsigma_0}^{\varsigma} \psi(x)\Delta x.$$

Then $H(\varsigma) = \hbar(\varsigma)(\hbar^{\sigma}(\varsigma))^{k-1}$ is nonincreasing on $[\varsigma_0,\infty)_{\mathbb{T}}$.

Proof Using the quotient rule (7), we get for $\varsigma \in (\varsigma_0,\infty)_{\mathbb{T}}$ that

$$\hbar^{\Delta}(\varsigma) = \frac{\psi(\varsigma)\hbar(\varsigma)\Upsilon(\varsigma) - \psi(\varsigma)\int_{\varsigma_0}^{\varsigma} \psi(x)\hbar(x)\Delta x}{\Upsilon(\varsigma)\Upsilon^{\sigma}(\varsigma)}. \tag{15}$$

Since $\hbar$ is nonnegative and nonincreasing, then

$$\int_{\varsigma_0}^{\varsigma} \psi(x)\hbar(x)\Delta x \geq \hbar(\varsigma)\int_{\varsigma_0}^{\varsigma} \psi(x)\Delta x,$$

and then

$$\psi(\varsigma)\hbar(\varsigma)\Upsilon(\varsigma) - \psi(\varsigma)\int_{\varsigma_0}^{\varsigma} \psi(x)\hbar(x)\Delta x \leq 0. \tag{16}$$

Substituting (16) into (15), we have that $\hbar^{\Delta}(\varsigma) \leq 0$. Since $\sigma^2(\varsigma) \geq \sigma(\varsigma)$, then $\hbar^{\sigma^2}(\varsigma) \leq \hbar^{\sigma}(\varsigma)$ and then

$$\bigl[\hbar^{\sigma^2}(\varsigma)\bigr]^{k-1} \leq \bigl[\hbar^{\sigma}(\varsigma)\bigr]^{k-1}. \tag{17}$$

Using the definition of H and (17), we see that

$$\begin{aligned} H^{\sigma}(\varsigma) &= \hbar^{\sigma}(\varsigma)\bigl[\hbar^{\sigma^2}(\varsigma)\bigr]^{k-1} \leq \hbar(\varsigma)\bigl[\hbar^{\sigma^2}(\varsigma)\bigr]^{k-1} \\ &\leq \hbar(\varsigma)\bigl[\hbar^{\sigma}(\varsigma)\bigr]^{k-1} = H(\varsigma), \end{aligned}$$

which shows that $H(\varsigma)$ is nonincreasing. The proof is complete. □

3 Main results

This section is devoted to state and prove our main results.

Theorem 3.1 *Assume $1 \leq k < \infty$. Furthermore, assume that $\hbar$ is nonincreasing and $\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma < \infty$. Suppose that there is a constant $D > 0$ with*

$$\int_{\varsigma}^{\infty} \frac{\varphi(s)}{(\sigma(s)-\varsigma_0)^k}\Delta s \leq \frac{D}{(\sigma(\varsigma)-\varsigma_0)^k}\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(s)\Delta s \quad \textit{for all } \varsigma \in (\varsigma_0,\infty)_{\mathbb{T}}. \tag{18}$$

Then

$$\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\biggl(\frac{1}{\sigma(\varsigma)-\varsigma_0}\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x\biggr)^k \Delta\varsigma \leq k^k(D+1)^k \int_{\varsigma_0}^{\infty} \varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma. \tag{19}$$

Proof Suppose that (18) holds. Apply Lemma 2.1 with $z = \hbar$, and we have that

$$\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x\right)^k \le k\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)} \hbar(\tau)\Delta\tau\right)^{k-1}\Delta x. \tag{20}$$

Substituting (20) into the left-hand side of (19), we get that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x\right)^k \Delta\varsigma \\
&\quad \le k\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left[\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)} \hbar(\tau)\Delta\tau\right)^{k-1}\Delta x\right]\Delta\varsigma.
\end{aligned} \tag{21}$$

Applying Lemma 2.2, with

$$\varphi(x) = \hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)} \hbar(\tau)\Delta\tau\right)^{k-1} \quad \text{and} \quad \psi(\varsigma) = \varphi(\varsigma)/\left(\sigma(\varsigma)-\varsigma_0\right)^k,$$

on the right hand side of (21), we see that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x\right)^k \Delta\varsigma \\
&\quad \le k\int_{\varsigma_0}^{\infty} \hbar(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x\right)^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta\varsigma \\
&\quad = k\int_{\varsigma_0}^{\infty} \left(\sigma(\varsigma)-\varsigma_0\right)^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x}{\sigma(\varsigma)-\varsigma_0}\right)^{k-1}\Delta\varsigma.
\end{aligned} \tag{22}$$

Using the additive property of integrals [5, Theorem 1.77(iv)] on time scales, we have for $\varsigma \in (\varsigma_0,\infty)_{\mathbb{T}}$ that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\left(\int_{s}^{\infty} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\quad = \int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\left(\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\qquad + \int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\left(\int_{\sigma(\varsigma)}^{\infty} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\quad \le \int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\left(\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\qquad + \int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\quad = \int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\left(\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\qquad + \left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\int_{\varsigma_0}^{\sigma(\varsigma)} \left(\sigma(s)-\varsigma_0\right)^{k-1}\Delta s.
\end{aligned} \tag{23}$$

Substituting (18) into the right-hand side of (23), we see that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\left(\int_s^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\quad\le\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\left(\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\qquad+\frac{D}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(x)\Delta x\right)\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\Delta s.
\end{aligned}\tag{24}$$

Applying integration by parts formula (8) on the term

$$\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\left(\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s,$$

with

$$\Re(s)=\Upsilon(s)=\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x,\quad\text{and}\quad \upsilon^{\Delta}(s)=\big(\sigma(s)-\varsigma_0\big)^{k-1},$$

we get that

$$\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\left(\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s=\Upsilon(s)\upsilon(s)\big|_{\varsigma_0}^{\sigma(\varsigma)}-\int_{\varsigma_0}^{\sigma(\varsigma)}\Upsilon^{\Delta}(s)\upsilon^{\sigma}(s)\Delta s,$$

where $\upsilon(s)=\int_{\varsigma_0}^{s}(\sigma(x)-\varsigma_0)^{k-1}\Delta x$. Using $\Upsilon^{\sigma}(\varsigma)=0$ and $\upsilon(\varsigma_0)=0$, we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\left(\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\quad=\int_{\varsigma_0}^{\sigma(\varsigma)}\big[-\Upsilon^{\Delta}(s)\big]\upsilon^{\sigma}(s)\Delta s \\
&\quad=\int_{\varsigma_0}^{\sigma(\varsigma)}\frac{\varphi(s)}{(\sigma(s)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(s)}\big(\sigma(x)-\varsigma_0\big)^{k-1}\Delta x\right)\Delta s.
\end{aligned}\tag{25}$$

Substituting (25) into the right-hand side of (24), we get that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\left(\int_s^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s \\
&\quad\le\int_{\varsigma_0}^{\sigma(\varsigma)}\frac{\varphi(s)}{(\sigma(s)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(s)}\big(\sigma(x)-\varsigma_0\big)^{k-1}\Delta x\right)\Delta s \\
&\qquad+\frac{D}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(x)\Delta x\right)\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(s)-\varsigma_0\big)^{k-1}\Delta s.
\end{aligned}\tag{26}$$

Since

$$\int_{\varsigma_0}^{\sigma(\varsigma)}\big(\sigma(x)-\varsigma_0\big)^{k-1}\Delta x\le\big(\sigma(\varsigma)-\varsigma_0\big)^k,\tag{27}$$

inequality (26) becomes

$$\begin{aligned}\int_{\varsigma_0}^{\sigma(\varsigma)}\bigl(\sigma(s)-\varsigma_0\bigr)^{k-1}\left(\int_s^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s &\leq \int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(s)\Delta s\\ &\quad + D\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(x)\Delta x\\ &= (D+1)\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(s)\Delta s. \end{aligned}\tag{28}$$

Applying Lemma 2.2 on the term

$$\int_{\varsigma_0}^{\infty}\bigl(\sigma(s)-\varsigma_0\bigr)^{k-1}\left(\int_s^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s,$$

we see that

$$\begin{aligned}&\int_{\varsigma_0}^{\infty}\bigl(\sigma(s)-\varsigma_0\bigr)^{k-1}\left(\int_s^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s\\ &\quad=\int_{\varsigma_0}^{\infty}\frac{\varphi(s)}{(\sigma(s)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(s)}\bigl(\sigma(x)-\varsigma_0\bigr)^{k-1}\Delta x\right)\Delta s.\end{aligned}\tag{29}$$

From (27) and (29), we have

$$\begin{aligned}&\int_{\varsigma_0}^{\infty}\bigl(\sigma(s)-\varsigma_0\bigr)^{k-1}\left(\int_s^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\Delta s\\ &\quad\leq\int_{\varsigma_0}^{\infty}\varphi(x)\Delta x\leq (D+1)\int_{\varsigma_0}^{\infty}\varphi(x)\Delta x.\end{aligned}\tag{30}$$

Putting $\psi(x)=1$ in Lemma 2.4, we see that the function

$$\hbar(\varsigma)\left(\frac{1}{\sigma(\varsigma)-\varsigma_0}\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^{k-1}$$

is nonincreasing. Now, by applying Lemma 2.3, with

$$\upsilon(\varsigma)=\bigl(\sigma(\varsigma)-\varsigma_0\bigr)^{k-1}\left(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\quad\text{and}\quad\phi(\varsigma)=(D+1)\varphi(\varsigma),$$

and

$$z(\varsigma)=\hbar(\varsigma)\left(\frac{1}{\sigma(\varsigma)-\varsigma_0}\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^{k-1},$$

we have from (28) and (30) that

$$\begin{aligned}&\int_{\varsigma_0}^{\infty}\bigl(\sigma(\varsigma)-\varsigma_0\bigr)^{k-1}\left(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{(\sigma(x)-\varsigma_0)^k}\Delta x\right)\hbar(\varsigma)\left(\frac{1}{\sigma(\varsigma)-\varsigma_0}\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^{k-1}\Delta\varsigma\\ &\quad\leq (D+1)\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar(\varsigma)\left[\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x}{\sigma(\varsigma)-\varsigma_0}\right]^{k-1}\Delta\varsigma.\end{aligned}\tag{31}$$

Substituting (31) into the right-hand side of (22), we see that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma \\
&\quad\le k(D+1)\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar(\varsigma)\left[\frac{1}{(\sigma(\varsigma)-\varsigma_0)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)\right]^{k-1}\Delta\varsigma \\
&\quad= k(D+1)\int_{\varsigma_0}^{\infty}\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)\right]^{k-1}\Delta\varsigma. \qquad (32)
\end{aligned}$$

Applying Hölder's inequality (11) on the term

$$\int_{\varsigma_0}^{\infty}\left[\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\right]\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{\sigma(\varsigma)-\varsigma_0}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)\right]^{k-1}\Delta\varsigma,$$

with indices k and $k/(k-1)$, we get that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\left[\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\right]\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{\sigma(\varsigma)-\varsigma_0}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)\right]^{k-1}\Delta\varsigma \\
&\quad\le\left(\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma\right)^{\frac{1}{k}}\left[\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma\right]^{\frac{k-1}{k}}. \qquad (33)
\end{aligned}$$

Finally substituting (33) into (32), we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma \\
&\quad\le k(D+1)\left(\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma\right)^{\frac{1}{k}}\left[\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma\right]^{\frac{k-1}{k}}.
\end{aligned}$$

This implies that

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma\le k^k(D+1)^k\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma,$$

which is (19). The proof is complete. □

Remark 3.1 In Theorem 3.1 we could replace $\hbar$ is rd-continuous with $\hbar$ is integrable.

Remark 3.2 Suppose that $\hbar$ is integrable and also assume that

$$\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\left(\frac{1}{\sigma(\varsigma)-\varsigma_0}\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma\le C\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma \qquad (34)$$

holds for some constant $C > 0$. Then (34) holds when

$$\hbar(x)=\begin{cases}1, & \text{if } x\in[\varsigma_0,\sigma(s)]_{\mathbb{T}},\\ 0, & \text{if } x\notin[\varsigma_0,\sigma(s)]_{\mathbb{T}},\end{cases}$$

for any fixed $s \in (\varsigma_0, \infty)_{\mathbb{T}}$. For this $\hbar$, the left-hand side of (34) becomes

$$\begin{aligned}\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \hbar(x)\Delta x\right)^k \Delta\varsigma &= \int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\left(\int_{\varsigma_0}^{\sigma(s)} \Delta x\right)^k \Delta\varsigma \\ &= \left(\sigma(s)-\varsigma_0\right)^k \int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\Delta\varsigma \\ &\geq \left(\sigma(s)-\varsigma_0\right)^k \int_{s}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\Delta\varsigma, \end{aligned} \tag{35}$$

and the right-hand side of (34) now becomes

$$C\int_{\varsigma_0}^{\infty} \varphi(\varsigma)\hbar^k(\varsigma)\Delta\varsigma = C\int_{\varsigma_0}^{\sigma(s)} \varphi(\varsigma)\Delta\varsigma. \tag{36}$$

From (35) and (36), we obtain

$$\left(\sigma(s)-\varsigma_0\right)^k \int_{s}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\Delta\varsigma \leq C\int_{\varsigma_0}^{\sigma(s)} \varphi(\varsigma)\Delta\varsigma,$$

and then

$$\int_{s}^{\infty} \frac{\varphi(\varsigma)}{(\sigma(\varsigma)-\varsigma_0)^k}\Delta\varsigma \leq \frac{C}{(\sigma(s)-\varsigma_0)^k}\int_{\varsigma_0}^{\sigma(s)} \varphi(s)\Delta s.$$

For particular cases of Theorem 3.1, we have the following results.

Remark 3.3 In the case when $\mathbb{T} = \mathbb{R}$, inequality (19) in Theorem 3.1 reduces to the continuous inequality (1) of Ariňo and Muckenhoupt [3].

Remark 3.4 In the case when $\mathbb{T} = \mathbb{N}$, inequality (19) in Theorem 3.1 reduces to inequality (2) of Bennett and Gross–Erdmann [4].

Remark 3.5 If $\varphi(\varsigma) = 1$, $k > 1$, and $\frac{\sigma(\varsigma)-\varsigma_0}{\varsigma-\varsigma_0} \leq K$ for $\varsigma \in (\varsigma_0, \infty)_{\mathbb{T}}$, we see that inequality (18) holds with

$$D = \frac{K^{k-1}}{k-1}. \tag{37}$$

Applying the chain rule formula (10), we see that

$$-\left((s-\varsigma_0)^{-k+1}\right)^{\Delta} \geq (k-1)\left(\sigma(s)-\varsigma_0\right)^{-k},$$

and therefore

$$\begin{aligned}\int_{\varsigma}^{\infty} \frac{1}{(\sigma(s)-\varsigma_0)^k}\Delta s &\leq \frac{1}{k-1}\int_{\varsigma}^{\infty}\left(\frac{-1}{(s-\varsigma_0)^{k-1}}\right)^{\Delta}\Delta s \\ &\leq \frac{K^{k-1}}{k-1}\frac{1}{(\sigma(\varsigma)-\varsigma_0)^{k-1}}.\end{aligned}$$

Thus we get the inequality

$$\int_{\varsigma_0}^{\infty}\left(\frac{1}{\sigma(\varsigma)-\varsigma_0}\int_{\varsigma_0}^{\sigma(\varsigma)}\hbar(x)\Delta x\right)^k\Delta\varsigma \le k^k\left(\frac{K^{k-1}}{k-1}+1\right)^k\int_{\varsigma_0}^{\infty}\hbar^k(\varsigma)\Delta\varsigma. \tag{38}$$

Remark 3.6 In the case when $\mathbb{T}=\mathbb{R}$ and $\varsigma_0=0$, then $K=1$ in the previous remark and from (38) we have

$$\int_0^{\infty}\left(\frac{1}{\varsigma}\int_0^{\varsigma}\hbar(x)\,dx\right)^k d\varsigma \le \left(\frac{k^2}{k-1}\right)^k\int_0^{\infty}\hbar^k(\varsigma)\,d\varsigma,$$

which is a Hardy-type inequality with constant $(k^2/(k-1))^k$ (see [9]).

Theorem 3.2 *Assume that $k\ge 1$ and $c>0$. Furthermore, assume that $\hbar$ is nonincreasing and*

$$\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^k(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma<\infty.$$

Suppose that there is a constant $D>0$ with

$$\int_{\varsigma}^{\infty}\frac{\varphi(x)}{[\Upsilon^{\sigma}(x)]^c}\Delta x \le \frac{D}{[\Upsilon^{\sigma}(\varsigma)]^k}\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(x)\left[\Upsilon^{\sigma}(x)\right]^{k-c}\Delta x \quad \textit{for all } \varsigma\in(\varsigma_0,\infty)_{\mathbb{T}}; \tag{39}$$

here $\Upsilon(\varsigma)=\int_{\varsigma_0}^{\varsigma}\psi(x)\Delta x$. Then

$$\begin{aligned}&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^c}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^k\Delta\varsigma\\&\quad\le k^k(D+1)^k\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^k(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma.\end{aligned} \tag{40}$$

Proof Suppose that (39) holds. Apply Lemma 2.1 with $z=\psi\hbar$, and we get that

$$\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^k \le k\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\left[\int_{\varsigma_0}^{\sigma(x)}\psi(\tau)\hbar(\tau)\Delta\tau\right]^{k-1}\Delta x. \tag{41}$$

Substituting (41) into the left-hand side of (40), we get that

$$\begin{aligned}&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^c}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^k\Delta\varsigma\\&\quad\le k\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^c}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\Delta x\right)\Delta\varsigma.\end{aligned} \tag{42}$$

Applying Lemma 2.2 on the term

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^c}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\Delta x\right)\Delta\varsigma,$$

we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{[\Upsilon^\sigma(\varsigma)]^c}\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(x)\hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)} \psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\Delta x\right)\Delta\varsigma \\
&\quad= \int_{\varsigma_0}^{\infty} \psi(\varsigma)\hbar(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta\varsigma. \qquad (43)
\end{aligned}$$

Substituting (43) into (42), we obtain

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{[\Upsilon^\sigma(\varsigma)]^c}\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma \\
&\quad\le k\int_{\varsigma_0}^{\infty} \psi(\varsigma)\hbar(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta\varsigma \\
&\quad= k\int_{\varsigma_0}^{\infty} \psi(\varsigma)\left[\Upsilon^\sigma(\varsigma)\right]^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^\sigma(\varsigma)}\right)^{k-1}\Delta\varsigma. \qquad (44)
\end{aligned}$$

Using the additive property of integrals [5, Theorem 1.77(iv)] on time scales, we see for any $\varsigma \in (\varsigma_0, \infty)_{\mathbb{T}}$ that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left[\Upsilon^\sigma(s)\right]^{k-1}\left(\int_{s}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\
&\quad= \int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left[\Upsilon^\sigma(s)\right]^{k-1}\left(\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x + \int_{\sigma(\varsigma)}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\
&\quad\le \int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left[\Upsilon^\sigma(s)\right]^{k-1}\left(\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\
&\qquad+ \int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left[\Upsilon^\sigma(s)\right]^{k-1}\left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\
&\quad= \int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left[\Upsilon^\sigma(s)\right]^{k-1}\left(\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\
&\qquad+ \left(\int_{\varsigma}^{\infty} \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left[\Upsilon^\sigma(s)\right]^{k-1}\Delta s. \qquad (45)
\end{aligned}$$

Integrating the term

$$\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\left(\Upsilon^\sigma(s)\right)^{k-1}\left[\int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{(\Upsilon^\sigma(x))^c}\Delta x\right]\Delta s,$$

using the parts formula (8) with

$$\Re(s) = \Phi(s) = \int_{s}^{\sigma(\varsigma)} \frac{\varphi(x)}{(\Upsilon^\sigma(x))^c}\Delta x \quad \text{and} \quad \omega^\Delta(s) = \psi(s)\left(\Upsilon^\sigma(s)\right)^{k-1},$$

we have that

$$\begin{aligned}&\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl[\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\Upsilon^\sigma(x))^c}\Delta x\biggr]\Delta s\\&\quad=\Phi(s)\omega(s)|_{\varsigma_0}^{\sigma(\varsigma)}-\int_{\varsigma_0}^{\sigma(\varsigma)}\Phi^\Delta(s)\omega^\sigma(s)\Delta s,\end{aligned}$$

where $\omega(s)=\int_{\varsigma_0}^{s}\psi(x)(\Upsilon^\sigma(x))^{k-1}\Delta x$. Using $\Phi^\sigma(\varsigma)=\omega(\varsigma_0)=0$, we get that

$$\begin{aligned}&\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl[\int_s^{\sigma(\varsigma)}\frac{\varphi(x)}{(\Upsilon^\sigma(x))^c}\Delta x\biggr]\Delta s\\&\quad=\int_{\varsigma_0}^{\sigma(\varsigma)}\bigl[-\Phi^\Delta(s)\bigr]\omega^\sigma(s)\Delta s\\&\quad=\int_{\varsigma_0}^{\sigma(\varsigma)}\frac{\varphi(s)}{(\Upsilon^\sigma(s))^c}\biggl(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\biggr)\Delta s.\end{aligned}\tag{46}$$

Substituting (46) into (45), we have that

$$\begin{aligned}&\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\biggl(\int_s^{\infty}\frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\biggr)\Delta s\\&\quad\le\int_{\varsigma_0}^{\sigma(\varsigma)}\frac{\varphi(s)}{(\Upsilon^\sigma(s))^c}\biggl(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\biggr)\Delta s\\&\qquad+\biggl(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\biggr)\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\Delta s.\end{aligned}\tag{47}$$

Note

$$\begin{aligned}&\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\\&\quad=\int_{\varsigma_0}^{\varsigma}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x+\int_{\varsigma}^{\sigma(\varsigma)}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\\&\quad=\int_{\varsigma_0}^{\varsigma}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x+\mu(\varsigma)\psi(\varsigma)\bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\\&\quad=\int_{\varsigma_0}^{\varsigma}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x+\bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\int_{\varsigma}^{\sigma(\varsigma)}\psi(x)\Delta x.\end{aligned}\tag{48}$$

Since $\Upsilon^\Delta(\varsigma)=\psi(\varsigma)\ge 0$ and σ is an increasing function, we have (for $x\le\varsigma$) that $\sigma(x)\le\sigma(\varsigma)$, and then $\Upsilon^\sigma(x)\le\Upsilon^\sigma(\varsigma)$, and then $(\Upsilon^\sigma(x))^{k-1}\le(\Upsilon^\sigma(\varsigma))^{k-1}$. Therefore

$$\int_{\varsigma_0}^{\varsigma}\psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\le\bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\int_{\varsigma_0}^{\varsigma}\psi(x)\Delta x.\tag{49}$$

Substituting (49) into the right-hand side of (48), we have that

$$\begin{aligned} &\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x \\ &\quad\le \bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\int_{\varsigma_0}^{\varsigma}\psi(x)\Delta x + \bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\int_{\varsigma}^{\sigma(\varsigma)}\psi(x)\Delta x \\ &\quad= \bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\Delta x = \bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k}. \end{aligned} \tag{50}$$

Substituting (50) into (47), we obtain

$$\begin{aligned} &\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\left(\int_s^\infty \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\ &\quad\le \int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-c}\Delta s + \left(\int_{\varsigma}^\infty \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k}. \end{aligned} \tag{51}$$

Substituting (39) into (51), we get that

$$\begin{aligned} &\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\left(\int_s^\infty \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\ &\quad\le \int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-c}\Delta s + D\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(x)\bigl[\Upsilon^\sigma(x)\bigr]^{k-c}\Delta x \\ &\quad= (D+1)\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-c}\Delta s. \end{aligned} \tag{52}$$

Applying Lemma 2.2 on the term

$$\int_{\varsigma_0}^{\infty} \psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\left(\int_s^\infty \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s,$$

we have

$$\begin{aligned} &\int_{\varsigma_0}^{\infty} \psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\left(\int_s^\infty \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\ &\quad= \int_{\varsigma_0}^{\infty} \frac{\varphi(s)}{[\Upsilon^\sigma(s)]^c}\left(\int_{\varsigma_0}^{\sigma(s)} \psi(x)\bigl[\Upsilon^\sigma(x)\bigr]^{k-1}\Delta x\right)\Delta s. \end{aligned} \tag{53}$$

From (50) and (53), we see that

$$\begin{aligned} &\int_{\varsigma_0}^{\infty} \psi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-1}\left(\int_s^\infty \frac{\varphi(x)}{[\Upsilon^\sigma(x)]^c}\Delta x\right)\Delta s \\ &\quad\le \int_{\varsigma_0}^{\infty} \varphi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-c}\Delta s \\ &\quad\le (D+1)\int_{\varsigma_0}^{\infty} \varphi(s)\bigl[\Upsilon^\sigma(s)\bigr]^{k-c}\Delta s. \end{aligned} \tag{54}$$

Since from Lemma 2.4, the function

$$\hbar(\varsigma)\left(\frac{1}{\Upsilon^{\sigma}(\varsigma)}\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}$$

is nonincreasing, then by applying Lemma 2.3 on the term

$$\int_{\varsigma_0}^{\infty}\left[\psi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-1}\left(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{[\Upsilon^{\sigma}(x)]^{c}}\Delta x\right)\right]\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1}\Delta\varsigma,$$

with

$$\upsilon(\varsigma)=\psi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-1}\left(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{[\Upsilon^{\sigma}(x)]^{c}}\Delta x\right),\qquad \phi(\varsigma)=(D+1)\varphi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c},$$

and

$$z(\varsigma)=\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1},$$

we get from (52) and (54) that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\left[\psi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-1}\left(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{[\Upsilon^{\sigma}(x)]^{c}}\Delta x\right)\right]\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1}\Delta\varsigma\\
&\quad\le (D+1)\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1}\Delta\varsigma\\
&\quad= (D+1)\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{1-c}\hbar(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\Delta\varsigma.
\end{aligned}\tag{55}$$

Substituting (55) into the right-hand side of (44), we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma\\
&\quad\le k(D+1)\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{1-c}\hbar(\varsigma)\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\Delta\varsigma\\
&\quad= k(D+1)\int_{\varsigma_0}^{\infty}\varphi^{\frac{k-1}{k}}(\varsigma)\frac{(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau)^{k-1}}{[\Upsilon^{\sigma}(\varsigma)]^{c(\frac{k-1}{k})}}\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{\frac{k-c}{k}}\Delta\varsigma.
\end{aligned}\tag{56}$$

Applying Hölder's inequality (11) on the term

$$\int_{\varsigma_0}^{\infty}\left[\varphi^{\frac{k-1}{k}}(\varsigma)\frac{(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau)^{k-1}}{[\Upsilon^{\sigma}(\varsigma)]^{c(\frac{k-1}{k})}}\right]\left[\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{\frac{k-c}{k}}\right]\Delta\varsigma,$$

with indices k and $k/(k-1)$, we see that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\left[\varphi^{\frac{k-1}{k}}(\varsigma)\frac{(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau)^{k-1}}{[\Upsilon^{\sigma}(\varsigma)]^{c(\frac{k-1}{k})}}\right]\left[\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{\frac{k-c}{k}}\right]\Delta\varsigma \\
&\quad\le\left(\int_{\varsigma_0}^{\infty}\left[\varphi^{\frac{k-1}{k}}(\varsigma)\frac{(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau)^{k-1}}{[\Upsilon^{\sigma}(\varsigma)]^{c(\frac{k-1}{k})}}\right]^{\frac{k}{k-1}}\Delta\varsigma\right)^{\frac{k-1}{k}} \\
&\qquad\times\left(\int_{\varsigma_0}^{\infty}\left[\varphi^{\frac{1}{k}}(\varsigma)\hbar(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{\frac{k-c}{k}}\right]^{k}\Delta\varsigma\right)^{\frac{1}{k}} \\
&\quad=\left(\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\frac{(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau)^{k}}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\Delta\varsigma\right)^{\frac{k-1}{k}}\left(\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^{k}(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma\right)^{\frac{1}{k}}. \qquad (57)
\end{aligned}$$

Finally, substituting (57) into the right-hand side of (56), we obtain

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma \\
&\quad\le k(D+1)\left(\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\frac{(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau)^{k}}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\Delta\varsigma\right)^{\frac{k-1}{k}} \\
&\qquad\times\left(\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^{k}(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma\right)^{\frac{1}{k}},
\end{aligned}$$

and then

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma\le k^{k}(D+1)^{k}\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^{k}(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma,$$

which is (40). The proof is complete. □

Remark 3.7 In Theorem 3.2 we could replace $\hbar$ is rd-continuous with $\hbar$ is integrable.

Remark 3.8 Suppose that $\hbar$ is integrable, and also assume that

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma\le D\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^{k}(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma \qquad (58)$$

holds for some constant $D>0$. Then (58) holds when

$$\hbar(x)=\begin{cases}1, & x\in[\varsigma_0,\sigma(s)]_{\mathbb{T}},\\ 0, & x\notin[\varsigma_0,\sigma(s)]_{\mathbb{T}},\end{cases}$$

for any fixed $s\in(\varsigma_0,\infty)_{\mathbb{T}}$. For this $\hbar$ in (58), we obtain

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\Delta x\right)^{k}\Delta\varsigma\le D\int_{\varsigma_0}^{\sigma(s)}\varphi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma. \qquad (59)$$

Note

$$\begin{aligned}\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\Delta x\right)^{k}\Delta\varsigma &\geq \int_{s}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\left(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\Delta x\right)^{k}\Delta\varsigma\\ &= \left(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\Delta x\right)^{k}\int_{s}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\Delta\varsigma\\ &= \left[\Upsilon^{\sigma}(s)\right]^{k}\int_{s}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\Delta\varsigma. \end{aligned}\tag{60}$$

From (59) and (60), we have that

$$\int_{s}^{\infty}\frac{\varphi(\varsigma)}{[\Upsilon^{\sigma}(\varsigma)]^{c}}\Delta\varsigma \leq \frac{D}{[\Upsilon^{\sigma}(s)]^{k}}\int_{\varsigma_0}^{\sigma(s)}\varphi(\varsigma)\left[\Upsilon^{\sigma}(\varsigma)\right]^{k-c}\Delta\varsigma.$$

As a particular case of Theorem 3.2 when $k = c$, we have the following result.

Theorem 3.3 *Assume that $1 \leq k < \infty$. Furthermore, assume that $\hbar$ is nonincreasing and $\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^{k}(\varsigma)\Delta\varsigma < \infty$. Suppose that there is a constant $D > 0$ with*

$$\int_{\varsigma}^{\infty}\frac{\varphi(x)}{(\Upsilon^{\sigma}(x))^{k}}\Delta x \leq \frac{D}{(\Upsilon^{\sigma}(\varsigma))^{k}}\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(x)\Delta x \quad \text{for all } \varsigma \in (\varsigma_0,\infty)_{\mathbb{T}};\tag{61}$$

here $\Upsilon(\varsigma) = \int_{\varsigma_0}^{\varsigma}\psi(x)\Delta x$. Then

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{(\Upsilon^{\sigma}(\varsigma))^{k}}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma \leq k^{k}(D+1)^{k}\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\hbar^{k}(\varsigma)\Delta\varsigma.\tag{62}$$

Theorem 3.4 *Assume that $1 \leq k < \infty$ and $c > 0$. Furthermore, assume that $\hbar$ is nonincreasing and*

$$\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\frac{(\Upsilon^{\sigma}(\varsigma))^{k}}{\Theta^{c}(\varsigma)}\hbar^{k}(\varsigma)\Delta\varsigma < \infty.$$

Suppose that there is a constant $D > 0$ with

$$\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^{c}(x)}\Delta x \leq \frac{D}{(\Upsilon^{\sigma}(\varsigma))^{k}}\int_{\varsigma_0}^{\sigma(\varsigma)}\varphi(x)\frac{(\Upsilon^{\sigma}(x))^{k}}{\Theta^{c}(x)}\Delta x \quad \text{for all } \varsigma \in (\varsigma_0,\infty)_{\mathbb{T}};\tag{63}$$

here $\Theta(\varsigma) = \int_{\varsigma}^{\infty}\psi(x)\Delta x$ and $\Upsilon(\varsigma) = \int_{\varsigma_0}^{\varsigma}\psi(x)\Delta x$. Then

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma \leq k^{k}(D+1)^{k}\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\frac{(\Upsilon^{\sigma}(\varsigma))^{k}}{\Theta^{c}(\varsigma)}\hbar^{k}(\varsigma)\Delta\varsigma.\tag{64}$$

Proof Suppose that (63) holds. From (41), the left-hand side of (64) becomes

$$\begin{aligned}&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma\\ &\quad\leq k\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\left(\int_{\varsigma_0}^{\sigma(x)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\Delta x\right)\Delta\varsigma.\end{aligned}\tag{65}$$

Applying Lemma 2.2 on the term

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^c(\varsigma)}\biggl(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\biggl(\int_{\varsigma_0}^{\sigma(x)}\psi(\tau)\hbar(\tau)\Delta\tau\biggr)^{k-1}\Delta x\biggr)\Delta\varsigma,$$

we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^c(\varsigma)}\biggl[\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\biggl(\int_{\varsigma_0}^{\sigma(x)}\psi(\tau)\hbar(\tau)\Delta\tau\biggr)^{k-1}\Delta x\biggr]\Delta\varsigma\\
&\quad=\int_{\varsigma_0}^{\infty}\psi(\varsigma)\hbar(\varsigma)\biggl(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\biggr)^{k-1}\biggl(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta\varsigma\\
&\quad=\int_{\varsigma_0}^{\infty}\psi(\varsigma)\bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\biggl(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\hbar(\varsigma)\biggl(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^\sigma(\varsigma)}\biggr)^{k-1}\Delta\varsigma. \qquad (66)
\end{aligned}$$

Substituting (66) into (65), we see that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^c(\varsigma)}\biggl(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\biggr)^{k}\Delta\varsigma\\
&\quad\le k\int_{\varsigma_0}^{\infty}\psi(\varsigma)\bigl(\Upsilon^\sigma(\varsigma)\bigr)^{k-1}\biggl(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\hbar(\varsigma)\biggl(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^\sigma(\varsigma)}\biggr)^{k-1}\Delta\varsigma. \qquad (67)
\end{aligned}$$

Using the additive property of integrals [5, Theorem 1.77(iv)] on time scales, we obtain for any $\varsigma\in(\varsigma_0,\infty)_{\mathbb{T}}$ that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl(\int_{s}^{\infty}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta s\\
&\quad=\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl(\int_{s}^{\sigma(\varsigma)}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta s\\
&\qquad+\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl(\int_{\sigma(\varsigma)}^{\infty}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta s\\
&\quad\le\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl(\int_{s}^{\sigma(\varsigma)}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta s\\
&\qquad+\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta s. \qquad (68)
\end{aligned}$$

Integrating the term

$$\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\biggl(\int_{s}^{\sigma(\varsigma)}\frac{\varphi(x)}{\Theta^c(x)}\Delta x\biggr)\Delta s,$$

using the parts formula (8) with

$$\Re(s)=\Phi(s)=\int_{s}^{\sigma(\varsigma)}\frac{\varphi(x)}{\Theta^c(x)}\Delta x,\quad\text{and}\quad\omega^\Delta(s)=\psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1},$$

we have that

$$\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\left(\int_s^{\sigma(\varsigma)} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\Delta s = \Phi(s)\omega(s)|_{\varsigma_0}^{\sigma(\varsigma)} - \int_{\varsigma_0}^{\sigma(\varsigma)} \Phi^\Delta(s)\omega^\sigma(s)\Delta s,$$

where $\omega(s) = \int_{\varsigma_0}^s \psi(x)(\Upsilon^\sigma(x))^{k-1}\Delta x$. Using $\Phi^\sigma(\varsigma) = \omega(\varsigma_0) = 0$, we get that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\left(\int_s^{\sigma(\varsigma)} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\Delta s \\
&\quad = \int_{\varsigma_0}^{\sigma(\varsigma)} \bigl[-\Phi^\Delta(s)\bigr]\omega^\sigma(s)\Delta s \\
&\quad = \int_{\varsigma_0}^{\sigma(\varsigma)} \frac{\varphi(s)}{\Theta^c(s)}\left(\int_{\varsigma_0}^{\sigma(s)} \psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\right)\Delta s.
\end{aligned} \tag{69}$$

Substituting (69) into (68), we obtain

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\left(\int_s^{\infty} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\Delta s \\
&\quad \le \int_{\varsigma_0}^{\sigma(\varsigma)} \frac{\varphi(s)}{\Theta^c(s)}\left(\int_{\varsigma_0}^{\sigma(s)} \psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\right)\Delta s \\
&\qquad + \left(\int_\varsigma^{\infty} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\Delta s.
\end{aligned} \tag{70}$$

Substituting (63) into (70), we get that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\left(\int_s^{\infty} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\Delta s \\
&\quad \le \int_{\varsigma_0}^{\sigma(\varsigma)} \frac{\varphi(s)}{\Theta^c(s)}\left(\int_{\varsigma_0}^{\sigma(s)} \psi(x)\bigl(\Upsilon^\sigma(x)\bigr)^{k-1}\Delta x\right)\Delta s \\
&\qquad + \frac{D}{(\Upsilon^\sigma(\varsigma))^k}\left(\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(x)\frac{(\Upsilon^\sigma(x))^k}{\Theta^c(x)}\Delta x\right)\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\Delta s.
\end{aligned} \tag{71}$$

Substituting (50) into (71), we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\sigma(\varsigma)} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\left(\int_s^{\infty} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\Delta s \\
&\quad \le \int_{\varsigma_0}^{\sigma(\varsigma)} \frac{\varphi(s)}{\Theta^c(s)}\bigl(\Upsilon^\sigma(s)\bigr)^k\Delta s + D\int_{\varsigma_0}^{\sigma(\varsigma)} \varphi(x)\frac{(\Upsilon^\sigma(x))^k}{\Theta^c(x)}\Delta x \\
&\quad = (D+1)\int_{\varsigma_0}^{\sigma(\varsigma)} \frac{\varphi(s)}{\Theta^c(s)}\bigl[\Upsilon^\sigma(s)\bigr]^k\Delta s.
\end{aligned} \tag{72}$$

Applying Lemma 2.2 on the term

$$\int_{\varsigma_0}^{\infty} \psi(s)\bigl(\Upsilon^\sigma(s)\bigr)^{k-1}\left(\int_s^{\infty} \frac{\varphi(x)}{\Theta^c(x)}\Delta x\right)\Delta s,$$

we observe that

$$\int_{\varsigma_0}^{\infty}\psi(s)\big(\Upsilon^{\sigma}(s)\big)^{k-1}\left(\int_{s}^{\infty}\frac{\varphi(x)}{\Theta^{c}(x)}\Delta x\right)\Delta s$$
$$=\int_{\varsigma_0}^{\infty}\frac{\varphi(s)}{\Theta^{c}(s)}\left(\int_{\varsigma_0}^{\sigma(s)}\psi(x)\big[\Upsilon^{\sigma}(x)\big]^{k-1}\Delta x\right)\Delta s. \tag{73}$$

Substituting (50) into (73), we see that

$$\int_{\varsigma_0}^{\infty}\psi(s)\big(\Upsilon^{\sigma}(s)\big)^{k-1}\left(\int_{s}^{\infty}\frac{\varphi(x)}{\Theta^{c}(x)}\Delta x\right)\Delta s\le\int_{\varsigma_0}^{\infty}\frac{\varphi(s)}{\Theta^{c}(s)}\big[\Upsilon^{\sigma}(s)\big]^{k}\Delta s$$
$$\le(D+1)\int_{\varsigma_0}^{\infty}\frac{\varphi(s)}{\Theta^{c}(s)}\big[\Upsilon^{\sigma}(s)\big]^{k}\Delta s. \tag{74}$$

Since from Lemma 2.4 the function

$$\hbar(\varsigma)\left(\frac{1}{\Upsilon^{\sigma}(\varsigma)}\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}$$

is nonincreasing, then by applying Lemma 2.3 with

$$\upsilon(\varsigma)=\psi(\varsigma)\big(\Upsilon^{\sigma}(\varsigma)\big)^{k-1}\left[\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^{c}(x)}\Delta x\right],\qquad\phi(\varsigma)=(D+1)\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\big(\Upsilon^{\sigma}(\varsigma)\big)^{k},$$

and

$$z(\varsigma)=\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1},$$

we obtain from (72) and (74) that

$$\int_{\varsigma_0}^{\infty}\psi(\varsigma)\big(\Upsilon^{\sigma}(\varsigma)\big)^{k-1}\left(\int_{\varsigma}^{\infty}\frac{\varphi(x)}{\Theta^{c}(x)}\Delta x\right)\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1}\Delta\varsigma$$
$$\le(D+1)\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\big(\Upsilon^{\sigma}(\varsigma)\big)^{k}\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1}\Delta\varsigma. \tag{75}$$

Substituting (75) into (67), we have that

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma$$
$$\le k(D+1)\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\big(\Upsilon^{\sigma}(\varsigma)\big)^{k}\hbar(\varsigma)\left(\frac{\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau}{\Upsilon^{\sigma}(\varsigma)}\right)^{k-1}\Delta\varsigma$$
$$=k(D+1)\int_{\varsigma_0}^{\infty}\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{\Theta^{\frac{c}{k}}(\varsigma)}\Upsilon^{\sigma}(\varsigma)\hbar(\varsigma)\right]\left[\frac{\varphi^{\frac{k-1}{k}}(\varsigma)}{\Theta^{\frac{c(k-1)}{k}}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\right]\Delta\varsigma. \tag{76}$$

Applying Hölder's inequality (11) on the term

$$\int_{\varsigma_0}^{\infty}\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{\Theta^{\frac{c}{k}}(\varsigma)}\Upsilon^{\sigma}(\varsigma)\hbar(\varsigma)\right]\left[\frac{\varphi^{\frac{k-1}{k}}(\varsigma)}{\Theta^{\frac{c(k-1)}{k}}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\right]\Delta\varsigma,$$

with indices k and $k/(k-1)$, we see that

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{\Theta^{\frac{c}{k}}(\varsigma)}\Upsilon^{\sigma}(\varsigma)\hbar(\varsigma)\right]\left[\frac{\varphi^{\frac{k-1}{k}}(\varsigma)}{\Theta^{\frac{c(k-1)}{k}}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\right]\Delta\varsigma \\
&\quad\le\left(\int_{\varsigma_0}^{\infty}\left[\frac{\varphi^{\frac{1}{k}}(\varsigma)}{\Theta^{\frac{c}{k}}(\varsigma)}\Upsilon^{\sigma}(\varsigma)\hbar(\varsigma)\right]^{k}\Delta\varsigma\right)^{\frac{1}{k}} \\
&\qquad\times\left(\int_{\varsigma_0}^{\infty}\left[\frac{\varphi^{\frac{k-1}{k}}(\varsigma)}{\Theta^{\frac{c(k-1)}{k}}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k-1}\right]^{\frac{k}{k-1}}\Delta\varsigma\right)^{\frac{k-1}{k}} \\
&\quad=\left(\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\Upsilon^{\sigma}(\varsigma)\right)^{k}\hbar^{k}(\varsigma)\Delta\varsigma\right)^{\frac{1}{k}} \\
&\qquad\times\left(\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k}\Delta\varsigma\right)^{\frac{k-1}{k}}.
\end{aligned}\tag{77}$$

Finally, substituting (77) into (76), we obtain

$$\begin{aligned}
&\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma \\
&\quad\le k(D+1)\left(\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\Upsilon^{\sigma}(\varsigma)\right)^{k}\hbar^{k}(\varsigma)\Delta\varsigma\right)^{\frac{1}{k}} \\
&\qquad\times\left(\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(\tau)\hbar(\tau)\Delta\tau\right)^{k}\Delta\varsigma\right)^{\frac{k-1}{k}}.
\end{aligned}$$

This implies that

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma\le k^{k}(D+1)^{k}\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\Upsilon^{\sigma}(\varsigma)\right)^{k}\hbar^{k}(\varsigma)\Delta\varsigma,$$

which is (64). The proof is complete. □

Remark 3.9 In Theorem 3.4 we could replace $\hbar$ is rd-continuous with $\hbar$ is integrable.

Remark 3.10 Suppose that $\hbar$ is integrable, and also assume that

$$\int_{\varsigma_0}^{\infty}\frac{\varphi(\varsigma)}{\Theta^{c}(\varsigma)}\left(\int_{\varsigma_0}^{\sigma(\varsigma)}\psi(x)\hbar(x)\Delta x\right)^{k}\Delta\varsigma\le D\int_{\varsigma_0}^{\infty}\varphi(\varsigma)\frac{(\Upsilon^{\sigma}(\varsigma))^{k}}{\Theta^{c}(\varsigma)}\hbar^{k}(\varsigma)\Delta\varsigma\tag{78}$$

holds for some constant $D>0$. Then (78) holds when

$$\hbar(x)=\begin{cases}1, & x\in[\varsigma_0,\sigma(s)]_{\mathbb{T}},\\ 0, & x\notin[\varsigma_0,\sigma(s)]_{\mathbb{T}},\end{cases}$$

for any fixed $s \in (\varsigma_0, \infty)_{\mathbb{T}}$. For this $\hbar$, note

$$\int_{\varsigma_0}^{\infty} \frac{\varphi(\varsigma)}{\Theta^c(\varsigma)} \left(\int_{\varsigma_0}^{\sigma(s)} \psi(x)\Delta x \right)^k \Delta\varsigma \geq \left(\int_{\varsigma_0}^{\sigma(s)} \psi(x)\Delta x \right)^k \int_s^{\infty} \frac{\varphi(\varsigma)}{\Theta^c(\varsigma)} \Delta\varsigma$$
$$= \left(\Upsilon^{\sigma}(s) \right)^k \int_s^{\infty} \frac{\varphi(\varsigma)}{\Theta^c(\varsigma)} \Delta\varsigma,$$

so

$$\int_s^{\infty} \frac{\varphi(\varsigma)}{\Theta^c(\varsigma)} \Delta\varsigma \leq \frac{D}{(\Upsilon^{\sigma}(s))^k} \int_{\varsigma_0}^{\sigma(s)} \varphi(\varsigma) \frac{(\Upsilon^{\sigma}(\varsigma))^k}{\Theta^c(\varsigma)} \Delta\varsigma.$$

4 Conclusion

In this paper, sufficient conditions are established to prove the boundedness of Hardy's operator in a certain class of weights. The results are obtained in general platforms, so the obtained characterizations of the weighted functions contain the characterization of Ariňo and Muckenhoupt weights when $\mathbb{T} = \mathbb{R}$ and the characterizations of Bennett–Erdmann and Gao weights when $\mathbb{T} = \mathbb{N}$.

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