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Hyponormality of Toeplitz operators with non-harmonic symbols on the Bergman spaces

Sumin Kim[1] and Jongrak Lee[2*]

*Correspondence: jrlee@jejunu.ac.kr
[2]Department of Mathematics, Jeju National University, Jeju, 63243, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we present some necessary and sufficient conditions for the hyponormality of Toeplitz operator T_φ on the Bergman space $A^2(\mathbb{D})$ with non-harmonic symbols under certain assumptions.

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Keywords: Toeplitz operator; Hyponormal operator; Bergman space

1 Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is *hyponormal* if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive semidefinite. Let dA be the normalized area measure on the open unit disk $\mathbb{D}$ in $\mathbb{C}$ and $L^2(\mathbb{D})$ be a Hilbert space of square-integrable measurable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z).$$

The *Bergman space* $A^2(\mathbb{D})$ is the space of analytic functions in $L^2(\mathbb{D})$. The *multiplication operator* M_ψ with symbol $\psi \in L^\infty(\mathbb{D})$ is defined by $M_\psi f = \psi f$ for $f \in A^2(\mathbb{D})$. For any $\varphi \in L^\infty(\mathbb{D})$, the *Toeplitz operator* T_φ on the Bergman space is defined by $T_\varphi f = P(\varphi f)$ for $f \in A^2(\mathbb{D})$ and P is the orthogonal projection that maps $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. Recall that the power series representation of $f \in A^2(\mathbb{D})$ is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } \sum_{n=0}^{\infty} \frac{1}{n+1}|a_n|^2 < \infty.$$

In [1, 4, 5], and [7], the basic properties of the Bergman space and the Hardy space are well known. The hyponormality of Toeplitz operators on the Hardy space has been developed in [2, 3, 10], and [12]. In [2], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by the properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. Cowen's method is to reconstruct the operator-theoretic problem of hyponormal Toeplitz operator into the problem of finding a solution of equations of functionals. Recently, in [8, 9], the authors characterized the hyponormality of Toeplitz operators on the Bergman space with harmonic symbols.

Proposition 1.1 ([8]) *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$ and $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$. If $a_m \overline{a_N} = a_{-m} \overline{a_{-N}}$, then T_φ is hyponormal*

$$\iff \quad \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \leq |a_N|, \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}|. \end{cases}$$

Proposition 1.2 ([9]) *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$ and $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$. If T_φ is hyponormal and $|a_N| \leq |a_{-N}|$, then we have*

$$N^2\left(|a_{-N}|^2 - |a_N|^2\right) \leq m^2\left(|a_m|^2 - |a_{-m}|^2\right).$$

Since the hyponormality of operators is translation invariant, we may assume that constant term is zero. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman space. Let f, g be in $L^\infty(\mathbb{D})$ and $\alpha, \beta \in \mathbb{C}$, then we can easily check that $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$, $T_f^* = T_{\overline{f}}$, and $T_{\overline{f}} T_g = T_{\overline{f} g}$ if f or g is analytic.

We briefly summarize a number of partial results relating to the hyponormality of Toeplitz operator with non-harmonic symbols, which have been recently developed in [6] and [14].

Proposition 1.3 ([6])

(i) *Suppose $f = a_{m,n} z^m \overline{z}^n$ and $g = a_{i,j} z^i \overline{z}^j$ with $m > n$, $i > j$ and $m - n > i - j$. Then T_{f+g} is hyponormal if, for each $k \geq 0$, the term*

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m-n+k+1}{(m+k+1)^2} + \left|\frac{a_{i,j}}{a_{m,n}}\right| \frac{i-j+k+1}{(i+k+1)^2}$$

is sufficiently large.

(ii) *Suppose $f = a_{m,n} z^m \overline{z}^n$ and $g = a_{i,j} z^i \overline{z}^j$ with $m > n$ and $i > j$. Then T_{f+g} is hyponormal if, for each $k \geq 0$,*

$$\left|\frac{a_{m,n}}{a_{i,j}}\right| \frac{m-n+k+1}{(m+k+1)^2} - \left|\frac{a_{i,j}}{a_{m,n}}\right| \frac{i-j+k+1}{(i+k+1)^2}$$

is sufficiently large.

Proposition 1.4 ([14]) *Suppose $c \in \mathbb{C}$, $s \in (0, \infty)$ and $n \in \mathbb{N}$. If $T_{z^n + C|z|^s}$ is hyponormal, then $|C| \leq \frac{n}{s}$. If $s \geq 2n$, then the converse is also true (i.e., $T_{z + C|z|^2}$ is hyponormal $\iff |C| \leq \frac{1}{2}$).*

Furthermore, in [11], the authors extended Proposition 1.4 to the weighted Bergman spaces. The purpose of this paper is to characterize the hyponormal Toeplitz operators T_φ with non-harmonic symbols acting on $A^2(\mathbb{D})$.

2 Toeplitz operators with non-harmonic symbols

We need several auxiliary lemmas to prove the main theorem in this section. We begin with the following.

Lemma 2.1 ([8]) *For any* $s,t \in \mathbb{N}$,

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t, \\ 0 & \text{if } s < t. \end{cases}$$

The proof for Lemma 2.2 follows the proof of Lemma 2.1 in [8].

Lemma 2.2 *For* $0 \leq m \leq N$, *we deduce that*

(i) $\|\overline{z}^m \sum_{i=0}^{\infty} c_i z^i\|^2 = \sum_{i=0}^{\infty} \frac{1}{i+m+1}|c_i|^2$,

(ii) $\|P(\overline{z}^m \sum_{i=0}^{\infty} c_i z^i)\|^2 = \sum_{i=m}^{\infty} \frac{i-m+1}{(i+1)^2}|c_i|^2$.

In [13], the author characterized the hyponormality of Toeplitz operators $T_{\overline{g}+f}$ with bounded and analytic functions f and g by $\|(I-P)(\overline{g}k)\| \leq \|(I-P)(\overline{f}k)\|$ for every k in $A^2(\mathbb{D})$. Furthermore, many authors have used the inequality to study the hyponormal Toeplitz operators. However, we consider the hyponormality of T_φ on $A^2(\mathbb{D})$ with the non-analytic symbol φ. So, in our case, we cannot apply that inequality to φ since we cannot separate φ to analytic and coanalytic parts. Therefore we directly calculate the self-commutator of T_φ. First, we consider the symbol φ of the form $\varphi(z) = a_{m,n}z^m\overline{z}^n$ with $a_{m,n} \in \mathbb{C}$.

Theorem 2.3 *Let* $\varphi(z) = a_{m,n}z^m\overline{z}^n$ *with* $a_{m,n} \in \mathbb{C}$. *Then* T_φ *on* $A^2(\mathbb{D})$ *is hyponormal if and only if* $m \geq n$.

Proof If $m \geq n$, then the authors as in [6] proved that T_φ is hyponormal. Suppose that T_φ is hyponormal. By the definition of hyponormal Toeplitz operators, T_φ is hyponormal if and only if

$$\left\langle \left(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*\right) \sum_{i=0}^{\infty} c_i z^i, \sum_{i=0}^{\infty} c_i z^i \right\rangle \geq 0$$

for all $c_i \in \mathbb{C}$. Using Lemmas 2.1 and 2.2, we have that

$$\begin{aligned}
&\left\| T_\varphi \sum_{i=0}^{\infty} c_i z^i \right\|^2 - \left\| T_\varphi^* \sum_{i=0}^{\infty} c_i z^i \right\|^2 \\
&= \left\| T_{a_{m,n}z^m\overline{z}^n} \sum_{i=0}^{\infty} c_i z^i \right\|^2 - \left\| T_{\overline{a}_{m,n}\overline{z}^m z^n} \sum_{i=0}^{\infty} c_i z^i \right\|^2 \\
&= \left\| P\left(a_{m,n}z^m\overline{z}^n \sum_{i=0}^{\infty} c_i z^i \right) \right\|^2 - \left\| P\left(\overline{a}_{m,n}\overline{z}^m z^n \sum_{i=0}^{\infty} c_i z^i \right) \right\|^2 \\
&= |a_{m,n}|^2 \sum_{i=\max\{n-m,0\}}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 - |a_{m,n}|^2 \sum_{i=\max\{m-n,0\}}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 \geq 0.
\end{aligned}$$

Hence T_φ is hyponormal if and only if

$$\sum_{i=\max\{n-m,0\}}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 \geq \sum_{i=\max\{m-n,0\}}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2$$

for all $c_i \in \mathbb{C}$. Since c_is are arbitrary, we have that T_φ is hyponormal if and only if $m \geq n$. This completes the proof. □

We now consider the hyponormality of Toeplitz operators with two terms non-harmonic symbols.

Theorem 2.4 *Let $\varphi(z) = az^m\overline{z}^n + bz^n\overline{z}^m$ with nonnegative integers m, n with $m \geq n$ and nonzeros $a, b \in \mathbb{C}$. Then T_φ on $A^2(\mathbb{D})$ is hyponormal if and only if $|a| \geq |b|$.*

Proof In a similar way to the proof of Theorem 2.3, T_φ is hyponormal if and only if

$$\begin{aligned}
&\left\|T_\varphi \sum_{i=0}^{\infty} c_i z^i\right\|^2 - \left\|T_\varphi^* \sum_{i=0}^{\infty} c_i z^i\right\|^2 \\
&= \left\|P\left(a\overline{z}^n \sum_{i=0}^{\infty} c_i z^{i+m}\right) + P\left(b\overline{z}^m \sum_{i=0}^{\infty} c_i z^{i+n}\right)\right\|^2 \\
&\quad - \left\|P\left(\overline{a}\overline{z}^m \sum_{i=0}^{\infty} c_i z^{i+n}\right) + P\left(\overline{b}\overline{z}^n \sum_{i=0}^{\infty} c_i z^{i+m}\right)\right\|^2 \\
&= |a|^2 \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 + |b|^2 \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 \\
&\quad - |a|^2 \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 - |b|^2 \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 \\
&= \left(|a|^2 - |b|^2\right)\left[\sum_{i=0}^{m-n-1} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2\right. \\
&\quad \left. + \sum_{i=m-n}^{\infty} \left(\frac{m+i-n+1}{(m+i+1)^2} - \frac{n+i-m+1}{(n+i+1)^2}\right)|c_i|^2\right] \geq 0.
\end{aligned}$$

Since $\frac{m+i-n+1}{(m+i+1)^2}$ and $\frac{m+i-n+1}{(m+i+1)^2} - \frac{n+i-m+1}{(n+i+1)^2}$ are positive for all $i \geq 0$ and $i \geq m-n$, respectively, T_φ is hyponormal if and only if $|a| \geq |b|$. □

The following theorem gives a general characterization of hyponormal Toeplitz operators with the symbols of the form $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ ($m \geq n \geq 0$, $t \geq s \geq 0$) with some conditions.

Theorem 2.5 *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ with nonnegative integers m, n, s, t with $m \geq n$, $t \geq s$, $m \neq t$, $m - n = t - s$ and nonzeros $a, b \in \mathbb{C}$. If T_φ on $A^2(\mathbb{D})$ is hyponormal,* *then*

$$\begin{cases} |a|^2 \geq \max\{\frac{(2m-n)^2}{(t+m-n)^2}, \Lambda(m,n,t,s)\}|b|^2 & \text{if } t > m, \\ |a|^2 \geq \max\{\frac{(m+1)^2}{(t+1)^2}, \Lambda(m,n,t,s)\}|b|^2 & \text{if } t < m, \end{cases}$$

where $\Lambda(m,n,t,s) = \max_{i\in[m-n,\infty)} \frac{\frac{(t+i-s+1)}{(t+i+1)^2} - \frac{(s+i-t+1)}{(s+i+1)^2}}{\frac{(m+i-n+1)}{(m+i+1)^2} - \frac{(n+i-m+1)}{(n+i+1)^2}}$.

Proof In a similar way to the proof of Theorem 2.4, T_φ is hyponormal if and only if

$$\begin{aligned} &\left\| T_\varphi \sum_{i=0}^{\infty} c_i z^i \right\|^2 - \left\| T_\varphi^* \sum_{i=0}^{\infty} c_i z^i \right\|^2 \\ &\quad = |a|^2 \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 + |b|^2 \sum_{i=t-s}^{\infty} \frac{s+i-t+1}{(s+i+1)^2}|c_i|^2 \\ &\qquad - |a|^2 \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 - |b|^2 \sum_{i=0}^{\infty} \frac{t+i-s+1}{(t+i+1)^2}|c_i|^2 \\ &\qquad + 2\operatorname{Re}\left(a\overline{b} \sum_{i=m-n}^{\infty} \frac{i+1}{(n+i+1)(t+i+1)} c_{i-m+n}\overline{c}_{t-s+i} \right) \\ &\qquad - 2\operatorname{Re}\left(a\overline{b} \sum_{i=t-s}^{\infty} \frac{i+1}{(m+i+1)(s+i+1)} \overline{c}_{i+m-n}c_{i-t+s} \right) \geq 0 \end{aligned} \tag{2.1}$$

for any $c_i \in \mathbb{C}$ $(i = 0,1,2,\ldots)$.

Since $m-n = t-s$ and $m \neq t$, from (2.1), T_φ is hyponormal if and only if

$$\begin{aligned} &|a|^2 \left\{ \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 - \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 \right\} \\ &\quad \geq |b|^2 \left\{ \sum_{i=0}^{\infty} \frac{m+i-n+1}{(t+i+1)^2}|c_i|^2 - \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(s+i+1)^2}|c_i|^2 \right\} \\ &\qquad + 2\operatorname{Re}\left(a\overline{b} \sum_{i=m-n}^{\infty} \left(\frac{i+1}{(m+i+1)(s+i+1)} - \frac{i+1}{(n+i+1)(t+i+1)} \right) \overline{c}_{i+m-n}c_{i-m+n} \right) \end{aligned} \tag{2.2}$$

for any $c_i \in \mathbb{C}$ $(i = 0,1,2,\ldots)$. Since c_is are arbitrary, set $\operatorname{Re}(a\overline{b}\overline{c}_{i+m-n}c_{i-m+n}) = 0$ for any i, $i \geq m-n$. If $0 \leq i < m-n$, then (2.2) implies

$$|a|^2 \geq \frac{(m+i+1)^2}{(t+i+1)^2}|b|^2.$$

There are two cases to consider. If $t > m$, then $\frac{(m+i+1)^2}{(t+i+1)^2}$ is increasing in i, and hence $|a|^2 \geq \frac{(2m-n)^2}{(t+m-n)^2}|b|^2$. If $t < m$, then $\frac{(m+i+1)^2}{(t+i+1)^2}$ is decreasing in i and hence

$$|a|^2 \geq \frac{(m+1)^2}{(t+1)^2}|b|^2.$$

For $i \geq m - n = t - s$,

$$|a|^2 \geq \max_{i\in[m-n,\infty)} \frac{\frac{t+i-s+1}{(t+i+1)^2} - \frac{s+i-t+1}{(s+i+1)^2}}{\frac{m+i-n+1}{(m+i+1)^2} - \frac{n+i-m+1}{(n+i+1)^2}} |b|^2.$$

Hence, if T_φ is hyponormal, then

$$\begin{cases} |a|^2 \geq \max\{\frac{(2m-n)^2}{(t+m-n)^2}, \Lambda(m,n,t,s)\}|b|^2 & \text{if } t > m, \\ |a|^2 \geq \max\{\frac{(m+1)^2}{(t+1)^2}, \Lambda(m,n,t,s)\}|b|^2 & \text{if } t < m, \end{cases}$$

where $\Lambda(m,n,t,s) = \max_{i\in[m-n,\infty)} \frac{\frac{(t+i-s+1)}{(t+i+1)^2} - \frac{(s+i-t+1)}{(s+i+1)^2}}{\frac{(m+i-n+1)}{(m+i+1)^2} - \frac{(n+i-m+1)}{(n+i+1)^2}}$. □

Corollary 2.6 *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^t$ with nonnegative integers m, n, s, t with $m \geq n$, $t \geq s$, $m > t$, $m - n = t - s$ and nonzeros $a, b \in \mathbb{C}$. If*

$$|a|^2 < \max\left\{\frac{(m+1)^2}{(t+1)^2}, \frac{(m+1)^2(s+t)(2m-n+1)^2}{(t+1)^2(m+n)(2t-s+1)^2}\right\}|b|^2,$$

then T_φ on $A^2(\mathbb{D})$ is never hyponormal.

Proof By a direct calculation,

$$\frac{\frac{(t+i-s+1)}{(t+i+1)^2} - \frac{(s+i-t+1)}{(s+i+1)^2}}{\frac{(m+i-n+1)}{(m+i+1)^2} - \frac{(n+i-m+1)}{(n+i+1)^2}} = \frac{(m+i+1)^2(n+i+1)^2\{(s+t)i + (t^2+s^2+t+s)\}}{(t+i+1)^2(s+i+1)^2\{(m+n)i + (n^2+m^2+n+m)\}}.$$

For convenience, we set

$$G(i) = \frac{(m+i+1)(n+i+1)}{(t+i+1)(s+i+1)} \quad \text{and} \quad H(i) = \frac{(s+t)i + (t^2+s^2+t+s)}{(m+n)i + (n^2+m^2+n+m)},$$

then

$$\Lambda(m,n,t,s) = \max_{i\in[m-n,\infty)} G^2(i)H(i).$$

By direct calculations,

$$\begin{aligned} G'(i) = {} & \frac{(s+t-m-n)i^2 + 2\{(t+1)(s+1) - (m+1)(n+1)\}i}{(t+i+1)^2(s+i+1)^2} \\ & + \frac{(m+n+2)(t+1)(s+1) - (s+t+2)(m+1)(n+1)}{(t+i+1)^2(s+i+1)^2}. \end{aligned}$$

Write $G'(i) = \frac{P(i)}{Q(i)}$. Since $s + t - m - n < 0$, $P(i)$ has a maximum at $i = -\frac{(t+1)(s+1)-(m+1)(n+1)}{s+t-m-n} < 0$, and since

$$\begin{aligned} P(0) &= (m+n+2)(t+1)(s+1) - (s+t+2)(m+1)(n+1) \\ &= (m+1)(t+1)(s-n) + (n+1)(s+1)(t-m) < 0, \end{aligned}$$

$P(i) < 0$ in $i \geq m-n$ and $Q(i) > 0$ in $i \geq m-n$. Hence $G(i)$ is decreasing in $i \geq m-n$. Similarly,

$$H'(i) = \frac{ms(m-s) + nt(n-t) + mt(m-t) + ns(n-s)}{\{(m+n)i + (n^2+m^2+n+m)\}^2},$$

and since $m > s$, $m > t$, $n > s$, and

$$nt(n-t) + mt(m-t) > mt\bigl(m-t-|n-t|\bigr) > 0,$$

$H'(i) > 0$ and so $H(i)$ is increasing in $i, i \geq m-n$. Furthermore, $\lim_{i\to\infty} H(i) = \frac{s+t}{m+n}$, we have that

$$\max_{i\in[m-n,\infty)} G^2(i)H(i) \leq \frac{s+t}{m+n} \max_{i\in[m-n,\infty)} G^2(i) \leq \frac{(m+1)^2(s+t)(2m-n+1)^2}{(t+1)^2(m+n)(2t-s+1)^2}.$$

Hence, by Theorem 2.5, we have the results. □

Theorem 2.7 *Let $\varphi(z) = az^m\overline{z}^{m-1} + b\overline{z}^{m-1}z^{m-2}$ with $m > 0$ and nonzeros $a, b \in \mathbb{C}$. If T_φ on $A^2(\mathbb{D})$ is hyponormal, then*

$$|a|^2 \geq \frac{(m+1)^2}{m^2}|b|^2.$$

Proof Let $\varphi(z) = az^m\overline{z}^{m-1} + b\overline{z}^{m-1}z^{m-2}$. From (2.1), if T_φ is hyponormal, then

$$\begin{aligned} &|a|^2\left\{\sum_{i=0}^{\infty}\frac{i+2}{(m+i+1)^2}|c_i|^2 - \sum_{i=1}^{\infty}\frac{i}{(m+i)^2}|c_i|^2\right\} \\ &\quad\geq |b|^2\left\{\sum_{i=0}^{\infty}\frac{i+2}{(m+i)^2}|c_i|^2 - \sum_{i=1}^{\infty}\frac{i}{(m+i-1)^2}|c_i|^2\right\} \end{aligned}$$

for any $c_i \in \mathbb{C}$ with $\operatorname{Re}(a\overline{b}\overline{c}_{i+2}c_i) = 0$ $(i = 0,1,2,\ldots)$. If $c_0 \neq 0$ and $c_i = 0$ for $i \geq 0$, then $|a|^2 \geq \frac{(m+1)^2}{m^2}|b|^2$ and if $c_0 = 0$ and $c_i \neq 0$ for $i \geq 1$,

$$|a|^2 \geq \max_{i\in[1,\infty)} \frac{\frac{i+2}{(m+i)^2} - \frac{i}{(m+i-1)^2}}{\frac{i+2}{(m+i+1)^2} - \frac{i}{(m+i)^2}}|b|^2.$$

If $i \geq 1$, then we can easily check that $\frac{\frac{i+2}{(m+i)^2} - \frac{i}{(m+i-1)^2}}{\frac{i+2}{(m+i+1)^2} - \frac{i}{(m+i)^2}}$ is decreasing in i. Hence, if T_φ is hyponormal, then

$$|a|^2 \geq \max\left\{\frac{(m+1)^2}{m^2}, \frac{(m+2)^2(2m^2-2m-1)}{m^2(2m^2+2m-1)}\right\}|b|^2.$$

Since for every nonnegative integer m,

$$\frac{(m+1)^2}{m^2} > \frac{(m+2)^2(2m^2-2m-1)}{m^2(2m^2+2m-1)},$$

this completes the proof. □

Now we give the example mentioned above.

Example 2.8 Let $\varphi(z) = az^3\overline{z}^2 + bz\overline{z}^2$ with nonzeros $a, b \in \mathbb{C}$. Then, by Theorem 2.7,

$$\frac{(m+1)^2}{m^2} = \frac{16}{9},$$

and so if T_φ is hyponormal, then

$$|a|^2 \geq \frac{16}{9}|b|^2.$$

Theorem 2.9 *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^m$ with nonnegative integers m, n, s with $m \geq s > n$ and nonzeros $a, b \in \mathbb{C}$. If T_φ on $A^2(\mathbb{D})$ is hyponormal, then*

$$|a|^2 \geq \max\left\{\frac{2m-2s}{2m-n-s}, \frac{\frac{(2m-n-s)}{(2m-n)^2} - \frac{(s-n)}{(s+m-n)^2}}{\frac{2(m-n)}{(2m-n)^2}}, \Lambda(m,n,m,s)\right\}|b|^2,$$

where $\Lambda(m,n,m,s)$ is given in Theorem 2.5.

Proof In a similar way to the proof of Theorem 2.5, if T_φ is hyponormal, then

$$\begin{aligned}
&\left\|T_\varphi \sum_{i=0}^{\infty} c_i z^i\right\|^2 - \left\|T_\varphi^* \sum_{i=0}^{\infty} c_i z^i\right\|^2 \\
&\quad = |a|^2 \sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 + |b|^2 \sum_{i=m-s}^{\infty} \frac{s+i-m+1}{(s+i+1)^2}|c_i|^2 \\
&\qquad - |a|^2 \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 - |b|^2 \sum_{i=0}^{\infty} \frac{m+i-s+1}{(m+i+1)^2}|c_i|^2 \geq 0
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&|a|^2\left(\sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 - \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2\right) \\
&\quad \geq |b|^2\left(\sum_{i=0}^{\infty} \frac{m+i-s+1}{(m+i+1)^2}|c_i|^2 - \sum_{i=m-s}^{\infty} \frac{s+i-m+1}{(s+i+1)^2}|c_i|^2\right)
\end{aligned} \tag{2.3}$$

for any $c_i \in \mathbb{C}$ with $\operatorname{Re}(a\overline{b}c_{i-m+n}\overline{c}_{i-m-s}) = 0$ and $\operatorname{Re}(a\overline{b}\overline{c}_{i+m-n}c_{i-m+s}) = 0$ ($i = 0, 1, 2, \ldots$). If $c_i \neq 0$ for $0 \leq i < m-s$ and $c_i = 0$ for $i \geq m-s$, then (2.3) implies

$$|a|^2 \geq \frac{m+i-s+1}{m+i-n+1}|b|^2,$$

and since $\frac{m+i-s+1}{m+i-n+1}$ is increasing in i, we have that

$$|a|^2 \geq \frac{2m-2s}{2m-n-s}|b|^2.$$

If $c_i \neq 0$ for $m-s \leq i < m-n$ and $c_i = 0$ for $i < m-s$ or $i \geq m-n$, then

$$|a|^2 \geq \frac{\frac{m+i-s+1}{(m+i+1)^2} - \frac{s+i-m+1}{(s+i+1)^2}}{\frac{m+i-n+1}{(m+i+1)^2}} |b|^2.$$

By direct calculations, $\frac{\frac{m+i-s+1}{(m+i+1)^2} - \frac{s+i-m+1}{(s+i+1)^2}}{\frac{m+i-n+1}{(m+i+1)^2}}$ is increasing and hence

$$|a|^2 \geq \frac{\frac{(2m-n-s)}{(2m-n)^2} - \frac{(s-n)}{(s+m-n)^2}}{\frac{2(m-n)}{(2m-n)^2}} |b|^2.$$

If $c_i \neq 0$ for $i \geq m-n$ and $c_i = 0$ for $i < m-n$, then

$$|a|^2 \geq \Lambda(m,n,m,s)|b|^2,$$

where $\Lambda(m,n,m,s)$ is given in Theorem 2.5. □

Corollary 2.10 *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^m$ with nonnegative integers m, n, s with $m \geq s > n$ and nonzeros $a, b \in \mathbb{C}$. If $|a|^2 < C|b|^2$, where*

$$C = \max\left\{ \frac{2m-2s}{2m-n-s}, \frac{\frac{(2m-n-s)}{(2m-n)^2} - \frac{(s-n)}{(s+m-n)^2}}{\frac{2(m-n)}{(2m-n)^2}}, \right.$$
$$\left. \frac{(m-s)\{2m^2+(s-n+1)m+s^2+s-sn\}}{(m-n)(2m^2+m+n)}, \frac{m^2-s^2}{m^2-n^2} \right\},$$

then T_φ on $A^2(\mathbb{D})$ is never hyponormal.

Proof By a direct calculation,

$$\frac{\frac{(m+i-s+1)}{(m+i+1)^2} - \frac{(s+i-m+1)}{(s+i+1)^2}}{\frac{(m+i-n+1)}{(m+i+1)^2} - \frac{(n+i-m+1)}{(n+i+1)^2}} = \frac{(n+i+1)^2\{(m+s)(m-s)i + (m-s)(m^2+s^2+m+s)\}}{(s+i+1)^2\{(m+n)(m-n)i + (m-n)(n^2+m^2+n+m)\}}.$$

For convenience, we set

$$G(i) = \frac{n+i+1}{s+i+1} \quad \text{and} \quad H(i) = \frac{(m+s)(m-s)i + (m-s)(m^2+s^2+m+s)}{(m+n)(m-n)i + (m-n)(n^2+m^2+n+m)},$$

then

$$\Lambda(m,n,m,s) = \max_{i \in [m-n,\infty)} G^2(i)H(i).$$

Since

$$G'(i) = \frac{s-n}{(s+i+1)^2},$$

$G(i)$ is increasing and $\lim_{i\to\infty} G(i) = 1$. Similarly,

$$H'(i) = \frac{(m-s)\{(m+s)(m^2+n^2+m+n) - (m+n)(m^2+s^2+m+s)\}}{(m-n)\{(m+n)i + (m^2+n^2+m+n)\}^2},$$

and thus $H(i)$ is monotone for $i \geq m-n$. Therefore

$$\max_{i\in[m-n,\infty)} H(i) \leq \max\left\{\frac{(m-s)\{2m^2+(s-n+1)m+s^2+s-sn\}}{(m-n)(2m^2+m+n)}, \frac{m^2-s^2}{m^2-n^2}\right\}.$$

Furthermore, we have that

$$\begin{aligned}\max_{i\in[m-n,\infty)} G^2(i)H(i) &\leq \max_{i\in[m-n,\infty)} H(i)\\ &\leq \max\left\{\frac{(m-s)\{2m^2+(s-n+1)m+s^2+s-sn\}}{(m-n)(2m^2+m+n)}, \frac{m^2-s^2}{m^2-n^2}\right\}.\end{aligned}$$

Hence, by Theorem 2.9, we have the results. □

Example 2.11 Let $\varphi(z) = az^3\overline{z} + bz^2\overline{z}^3$ with nonzeros $a, b \in \mathbb{C}$. Then, by Theorem 2.9,

$$\frac{2m-2s}{2m-n-s} = \frac{2}{3}, \qquad \frac{\frac{(2m-n-s)}{(2m-n)^2} - \frac{(s-n)}{(s+m-n)^2}}{\frac{2(m-n)}{(2m-n)^2}} = \frac{23}{64},$$

$$\Lambda(m,n,t,s) = \max_{i\in[2,\infty)} \frac{(i+2)^2(5i+18)}{4(i+3)^2(2i+7)}.$$

Since $\frac{(i+2)^2(5i+18)}{4(i+3)^2(2i+7)}$ is increasing for $i \geq 2$, we have that

$$\Lambda(m,n,t,s) = \lim_{i\to\infty} \frac{(i+2)^2(5i+18)}{4(i+3)^2(2i+7)} = \frac{5}{8}.$$

Therefore, if T_φ is hyponormal, then

$$|a|^2 \geq \frac{2}{3}|b|^2.$$

Theorem 2.12 *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^m$ with nonnegative integers m, n, s with $m \geq n > s$ and nonzeros $a, b \in \mathbb{C}$. If T_φ on $A^2(\mathbb{D})$ is hyponormal, then*

$$|a|^2 \geq \max\left\{\frac{m-s+1}{m-n+1}, \frac{\frac{2m-2s}{(2m-s)^2}}{\frac{2m-s-n}{(2m-s)^2} - \frac{n-s}{(n+m-s)^2}}, \Lambda(m,n,m,s)\right\}|b|^2,$$

where $\Lambda(m,n,m,s)$ is given in Theorem 2.5.

Proof In a similar way to the proof of Theorem 2.4, T_φ is hyponormal if and only if

$$\begin{aligned}&|a|^2\left(\sum_{i=0}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 - \sum_{i=m-n}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2\right)\\ &\quad\geq |b|^2\left(\sum_{i=0}^{\infty} \frac{m+i-s+1}{(m+i+1)^2}|c_i|^2 - \sum_{i=m-s}^{\infty} \frac{s+i-m+1}{(s+i+1)^2}|c_i|^2\right)\end{aligned} \tag{2.4}$$

for any $c_i \in \mathbb{C}$ with $\mathrm{Re}(a\overline{b}c_{i-m+n}\overline{c}_{i-m-s}) = 0$ and $\mathrm{Re}(a\overline{b}\overline{c}_{i+m-n}c_{i-m+s}) = 0$ $(i = 0, 1, 2, \ldots)$. If $c_i \neq 0$ for $0 \leq i < m-n$ and $c_i = 0$ for $i \geq m-n$, then (2.4) implies

$$|a|^2 \geq \frac{m+i-s+1}{m+i-n+1}|b|^2,$$

and since $\frac{m+i-s+1}{m+i-n+1}$ is decreasing in i, we have that

$$|a|^2 \geq \frac{m-s+1}{m-n+1}|b|^2.$$

If $c_i \neq 0$ for $m-n \leq i < m-s$ and $c_i = 0$ for $i < m-n$ or $i \geq m-s$, then

$$|a|^2 \geq \frac{\frac{m+i-s+1}{(m+i+1)^2}}{\frac{m+i-n+1}{(m+i+1)^2} - \frac{n+i-m+1}{(n+i+1)^2}}|b|^2.$$

By direct calculations, $\frac{\frac{m+i-s+1}{(m+i+1)^2}}{\frac{m+i-n+1}{(m+i+1)^2} - \frac{n+i-m+1}{(n+i+1)^2}}$ is increasing and hence

$$|a|^2 \geq \frac{\frac{2m-2s}{(2m-s)^2}}{\frac{2m-s-n}{(2m-s)^2} - \frac{n-s}{(n+m-s)^2}}|b|^2.$$

If $c_i \neq 0$ for $i \geq m-s$ and $c_i = 0$ for $i < m-s$, then

$$|a|^2 \geq \Lambda(m,n,m,s)|b|^2,$$

where $\Lambda(m,n,m,s)$ is given in Theorem 2.5. □

Corollary 2.13 *Let $\varphi(z) = az^m\overline{z}^n + bz^s\overline{z}^m$ with nonnegative integers m, n, s with $m \geq n > s$ and nonzeros $a, b \in \mathbb{C}$. If*

$$|a|^2 < \max\left\{\frac{m-s+1}{m-n+1}, \frac{\frac{2m-2s}{(2m-s)^2}}{\frac{2m-s-n}{(2m-s)^2} - \frac{n-s}{(n+m-s)^2}}, \frac{C_1(m+1)^2}{(s+m-n+1)^2}\right\}|b|^2,$$

where $C_1 = \max\{\frac{(m-s)\{2m^2+(s-n+1)m+s^2+s-sn\}}{(m-n)(2m^2+m+n)}, \frac{m^2-s^2}{m^2-n^2}\}$, then T_φ on $A^2(\mathbb{D})$ is never hyponormal.

Proof

$$\frac{\frac{(m+i-s+1)}{(m+i+1)^2} - \frac{(s+i-m+1)}{(s+i+1)^2}}{\frac{(m+i-n+1)}{(m+i+1)^2} - \frac{(n+i-m+1)}{(n+i+1)^2}} = \frac{(n+i+1)^2\{(m+s)(m-s)i + (m-s)(m^2+s^2+m+s)\}}{(s+i+1)^2\{(m+n)(m-n)i + (m-n)(n^2+m^2+n+m)\}}.$$

For convenience, we set

$$G(i) = \frac{n+i+1}{s+i+1} \quad \text{and} \quad H(i) = \frac{(m+s)(m-s)i + (m-s)(m^2+s^2+m+s)}{(m+n)(m-n)i + (m-n)(n^2+m^2+n+m)},$$

then

$$\Lambda(m,n,m,s) = \max_{i \in [m-n,\infty)} G^2(i)H(i).$$

Since

$$G'(i) = \frac{s-n}{(s+i+1)^2},$$

$G(i)$ is decreasing. Similarly,

$$H'(i) = \frac{(m-s)\{(m+s)(m^2+n^2+m+n)-(m+n)(m^2+s^2+m+s)\}}{(m-n)\{(m+n)i+(m^2+n^2+m+n)\}^2},$$

and thus $H(i)$ is monotone for $i \geq m-n$. Therefore

$$\max_{i\in[m-n,\infty)} H(i) \leq \max\left\{\frac{(m-s)\{2m^2+(s-n+1)m+s^2+s-sn\}}{(m-n)(2m^2+m+n)}, \frac{m^2-s^2}{m^2-n^2}\right\}.$$

Hence

$$\max_{i\in[m-n,\infty)} G^2(i)H(i) \leq C_1 \max_{i\in[m-n,\infty)} G^2(i) \leq \frac{C_1(m+1)^2}{(s+m-n+1)^2},$$

where $C_1 = \max\{\frac{(m-s)\{2m^2+(s-n+1)m+s^2+s-sn\}}{(m-n)(2m^2+m+n)}, \frac{m^2-s^2}{m^2-n^2}\}$. Hence, by Theorem 2.12, we have the results. □

Example 2.14 Let $\varphi(z) = az^3\overline{z}^2 + bz\overline{z}^3$ with nonzeros $a, b \in \mathbb{C}$. Then, by Theorem 2.12,

$$\frac{m-s+1}{m-n+1} = \frac{3}{2}, \qquad \frac{\frac{(2m-n-s+1)}{(2m-n+1)^2}}{\frac{(2m-2n+1)}{(2m-n+1)^2} - \frac{1}{(m+1)^2}} = \frac{64}{23},$$

$$\Lambda(m,n,t,s) = \max_{i\in[1,\infty)} \frac{4(i+3)^2(2i+7)}{(i+2)^2(5i+18)}.$$

Since $\frac{4(i+3)^2(2i+7)}{(i+2)^2(5i+18)}$ is decreasing for $i \geq 1$, we have that

$$\Lambda(m,n,t,s) = \frac{64}{23}.$$

Therefore, if T_φ is hyponormal, then

$$|a|^2 \geq \frac{64}{23}|b|^2.$$

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Authors' contributions
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Author details
[1]Department of Mathematics, Sungkyunkwan University, Suwon, 440-746, Korea. [2]Department of Mathematics, Jeju National University, Jeju, 63243, Korea.

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