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# Modified subgradient extragradient method for system of variational inclusion problem and finite family of variational inequalities problem in real Hilbert space

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**Abstract**

For the purpose of this article, we introduce a modified form of a generalized system of variational inclusions, called the generalized system of modified variational inclusion problems (GSMVIP). This problem reduces to the classical variational inclusion and variational inequalities problems. Motivated by several recent results related to the subgradient extragradient method, we propose a new subgradient extragradient method for finding a common element of the set of solutions of GSMVIP and the set of a finite family of variational inequalities problems. Under suitable assumptions, strong convergence theorems have been proved in the framework of a Hilbert space. In addition, some numerical results indicate that the proposed method is effective.

**Keywords:** System of variational inclusions problem; Variational inequalities problem; Half-space; Nonexpansive mapping

## 1 Introduction

Throughout this paper, let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$ with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $T:C\to C$ be a mapping. Then $T$ is called nonexpansive if $\|Tx-Ty\|\le\|x-y\|$, for all $x,y\in C$. We denote by $F(T)$ the set of fixed points of $T$, that is, $F(T)=\{x\in C: Tx=x\}$. It is well known that $F(T)$ is closed convex and also nonempty.

Let $B:H\to H$ be a mapping and $M:H\to 2^H$ be a multi-valued mapping. The variational inclusion problem is to find $x\in H$ such that

$$\theta\in Bx+Mx, \tag{1}$$

where $\theta$ is the zero vector in $H$. The set of solutions of (1) is denoted by $VI(H,B,M)$. This problem has received much attention due to its applications in large variety of problems arising in convex programming, variational inequalities, split feasibility problems, and minimization problems. To be more precise, some concrete problems in machine learning, image processing, and linear inverse problems can be modeled mathematically by this formulation.

The variational inequality problem (VIP) is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{2}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. This problem is an important tool in economics, engineering and mathematics. It includes, as special cases, many problems of nonlinear analysis such as optimization, optimal control problems, saddle point problems and mathematical programming; see, for example, [1–4].

It is well known that one of the most popular methods for solving the problem (VIP) is the extragradient method proposed by Korpelevich [5]. The extragradient method is needed to calculate two projections onto the feasible set $C$ in each iteration. So, in the case that the set $C$ is not simple to project on to it, as analyzed in some remarks of the authors in [6], when the subset is a closed expression as in the case of a ball or a half-space, the projection onto the feasible subset $C$ can be computed easily. This can affect the efficiency of the used method. In recent years, the extragradient method has received great attention by many authors, who improved it in various ways; see, e.g. [7–13] and the references therein.

In 2011, Censor et al. [12] proposed the subgradient extragradient method for solving variational inequality problems as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ T_n = \{x \in H : \langle x_n - \lambda Ax_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), \end{cases} \tag{3}$$

for each $n \geq 1$, where $\lambda \in (0, 1/L)$. In this method, they have replaced the second projection in Korpelevich's extragradient method by a projection on to a half-space, which is computed explicitly.

Motivated by the problem (1), in this paper, we introduce a new problem of the system of variational inclusions in a real Hilbert space as follows:

Let $H$ be a real Hilbert space and let $A : H \to H$ be mapping and $M_A, M_B : H \to 2^H$ be set value mapping. We consider the problem for finding $x^* \in H$ such that

$$\theta \in Ax^* + M_A x^* \quad \text{and} \quad \theta \in Ax^* + M_B x^*, \tag{4}$$

where $\theta$ is the zero mapping in $H$, which is called a generalized system of modified variational inclusion problems (in short, GSMVIP). The set of solutions of (4) is denoted by $\Omega$,i.e., $\Omega = \{x^* \in H : \theta \in Ax^* + M_A x^* \text{ and } \theta \in Ax^* + M_B x^*\}$. In particular, if $M_A = M_B$, then the problem (4) reduces to the problem (1) and if $J_{M_A,\lambda_A} = J_{M_B,\lambda_B} = P_C$, then the problem (4) reduces to VIP.

In 2012, Kangtunyakarn [14] modified the set of variational inequality problems as follows:

$$VI\big(C, aA + (1 - a)B\big)$$

$$= \{x \in C : \langle y - x, (aA + (1-a)B)x \rangle \geq 0, \forall y \in C, a \in (0,1)\}, \tag{5}$$

where $A$ and $B$ are the mappings of C into $H$.

In order to develop efficient algorithms for finding solution of a finite family variational inequalities problem, inspired by problem (5), we define the new half-space $Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^{N} a_i A_i)x_n - y_n, y_n - z \rangle \geq 0\}$, which as a tool to prove the strong convergence theorem. In particular, if we put $i = 1$, then $Q_n$ reduces to $T_n$ in subgradient extragradient method (3). However, the sequence $\{x_n\}$ generated by (3) converges weakly to a solution of the variational inequality problem.

In this paper, motivated by recent research [7, 12] and [14], we introduce a new problem (4) and the new iterative scheme for finding a common element of the set of a finite family of variational inequalities problems and the set of solutions of the proposed problem (4) in a real Hilbert space. Then we establish and prove the strong convergence theorem under some proper conditions. Furthermore, we also give some various examples to support our main result.

## 2 Preliminaries

In this section, we give some useful lemmas that will be needed to prove our main result.

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. We denote strong convergence and weak convergence by the notations $\rightarrow$ and $\rightharpoonup$, respectively. For every $x \in H$, there exists a unique nearest point $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

$P_C$ is called a metric projection of $H$ onto $C$. It follows that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \text{for all } x \in H, y \in C. \tag{6}$$

**Lemma 2.1** ([15]) *Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if we have the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

**Definition 2.2** Let $M : H \rightarrow 2^H$ be a multi-valued mapping.

(i) The graph $G(M)$ of $M$ is defined by

$$G(M) := \{(x, u) \in H \times H : u \in M(x)\},$$

(ii) the operator $M$ is called *a maximal monotone operator* if $M$ is monotone, i.e.

$$\langle u - v, x - y \rangle \geq 0 \quad \forall u \in M(x), v \in M(y),$$

and the graph $G(M)$ of $M$ is not properly contained in the graph of any other monotone operator. It is clear that a monotone mapping $M$ is maximal if and only if for any $(x, u) \in H \times H$, $\langle u - v, x - y \rangle \geq 0$ for every $(y, v) \in G(M)$ implies that $u \in M(x)$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the resolvent operator associated with $M$ where $\lambda$ is positive number and $I$ is an identity mapping; see [16]. Note that $J_{M,\lambda}$ is a nonexpansive mapping.

**Definition 2.3** Let $A : C \to H$ be a mapping.

(i) $A$ is called *$\mu$-Lipschitz continuous* if there exists a nonnegative real number $\mu \geq 0$ such that

$$\|Ax - Ay\| \leq \mu \|x - y\|, \quad \forall x, y \in C.$$

(ii) $A$ is called *$\alpha$-inverse strongly monotone* if there exists a nonnegative real number $\alpha \geq 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

**Lemma 2.4** ([14]) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $A$,$B$: $C \to H$ be $\alpha$- and $\beta$-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \emptyset$. Then*

$$VI\bigl(C, aA + (1 - a)B\bigr) = VI(C, A) \cap VI(C, B), \quad \forall a \in (0, 1).$$

*Furthermore, if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we find that $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.*

*Remark* 2.5 For every $i = 1, 2, \dots, N$ the mapping $A_i : C \to H$ be $\alpha_i$-inverse strongly monotone mappings with $\eta = \min_{1,2,\dots,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$. Then

$$VI\left(C, \sum_{i=1}^N a_i A_i\right) = \bigcap_{i=1}^N VI(C, A_i), \tag{7}$$

where $\sum_{i=1}^N a_i = 1$ and $0 < a_i < 1$ for every $i = 1, 2, \dots, N$. Moreover, we find that $\sum_{i=1}^N a_i A_i$ is monotone and is a $\mu$-Lipschitz continuous mapping.

*Proof* It easy to see that $\sum_{i=k+1}^N \frac{a_i}{\prod_{j=1}^k (1-a_j)} A_i$ is $\eta$-inverse strongly monotone mappings with $\eta = \min\{\beta_i\}$ for each $i = 2, \dots, N$ and $k = 1, 2, \dots, N - 1$.

Take $N = 3$ and let $VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3) \neq \emptyset$. By using Lemma 2.4, we have

$$\begin{aligned} VI(C, a_1 A_1 + a_2 A_2 + a_3 A_3) &= VI\left(C, a_1 A_1 + (1 - a_1)\left(\frac{a_2}{1 - a_1} A_2 + \frac{a_3}{1 - a_1} A_3\right)\right) \\ &= VI(C, A_1) \cap VI\left(C, \frac{a_2}{1 - a_1} A_2 + \frac{a_3}{1 - a_1} A_3\right) \\ &= VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3), \end{aligned} \tag{8}$$

where $a_1, a_2, a_3 \in (0, 1)$ and $\sum_{i=1}^3 a_i = 1$.

Take $N = 4$ and let $\bigcap_{i=1}^{4} VI(C, A_i) \neq \emptyset$. By using Lemma 2.4 and (8), we have

$$\begin{aligned} &VI(C, a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4) \\ &\quad = VI\left(C, (1-a_4)\left(\frac{a_1}{1-a_4}A_1 + \frac{a_2}{1-a_4}A_2 + \frac{a_3}{1-a_4}A_3\right) + a_4A_4\right) \\ &\quad = VI\left(C, \frac{a_1}{1-a_4}A_1 + \frac{a_2}{1-a_4}A_2 + \frac{a_3}{1-a_4}A_3\right) \cap VI(C, A_4) \\ &\quad = VI(C, A_1) \cap VI(C, A_2) \cap VI(C, A_3) \cap VI(C, A_4), \end{aligned} \tag{9}$$

where $a_1, a_2, a_3, a_4 \in (0, 1)$ and $\sum_{i=1}^{4} a_i = 1$.

In the same way, if $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$, we obtain

$$VI\left(C, \sum_{i=1}^{N} a_iA_i\right) = \bigcap_{i=1}^{N} VI(C, A_i), \tag{10}$$

where $a_i \in (0, 1)$, for each $i = 1, 2, \dots, N$, and $\sum_{i=1}^{N} a_i = 1$. □

**Lemma 2.6** *In real Hilbert spaces $H$, the following well-known results hold*:

(*i*) *For all $x, y \in H$ and $\alpha \in [0, 1]$,*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

(*ii*) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all $x, y \in H$.*

**Lemma 2.7** ([17]) *Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$. If $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, then the mapping $I - T$ is demiclosed at* 0, *i.e., if $\{x_n\}$ is a sequence in $C$ weakly converging to $x \in C$ and if $\{x_n - Tx_n\}$ converges strongly to* 0, *then $x \in F(T)$.*

**Lemma 2.8** ([17]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall \geq 0,$$

*where $\{\alpha_n\}$ is a sequence in* (0,1) *and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to 0} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| = \infty$.

*Then* $\lim_{n\to 0} s_n = 0$.

**Lemma 2.9** ([17]) *Each Hilbert space $H$ satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

*holds for every $y \in H$ with $x \neq y$.*

**Lemma 2.10** ([16]) *$u \in H$ is a solution of variational inclusion (1) if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, $\forall \lambda > 0$, i.e.,*

$$VI(H,B,M) = F\big(J_{M,\lambda}(I - \lambda B)\big), \quad \forall \lambda > 0.$$

*If $\lambda \in (0, 2\alpha]$, then $VI(H,B,M)$ is a closed convex subset in $H$.*

The next lemma presents the association of the fixed point of a nonlinear mapping and the solution of GSMVIP under suitable conditions on the parameters.

**Lemma 2.11** *Let $H$ be a real Hilbert space and let $A_G : H \to H$ be an $\alpha$-inverse strongly monotone mapping. Let $M_A, M_B : H \to 2^H$ be multi-value maximum monotone mappings with $\Omega \neq \emptyset$. $x^* \in \Omega$ if and only if $x^* = Gx^*$, where $G : H \to H$ is a mapping defined by*

$$G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x\big),$$

*for all $x \in H$, $b \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha)$. Moreover, we see that $G$ is a nonexpansive mapping.*

*Proof* Let the conditions hold.

($\Rightarrow$) Let $x^* \in \Omega$, we have $x \in H$ such that $\theta \in A_G x^* + M_A x^*$ and $\theta \in A_G x^* + M_B x^*$, that is, $x^* \in VI(H, A_G, M_A)$ and $x^* \in VI(H, A_G, M_B)$.

From Lemma 2.10, we have $x^* \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G))$ and $x^* \in F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$.

It implies that

$$x^* = J_{M_A,\lambda_A}(I - \lambda_A A_G)x^* \tag{11}$$

and

$$x^* = J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*. \tag{12}$$

By the definition of $G$, (11) and (12), we have

$$\begin{aligned} G\big(x^*\big) &= J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*\big) \\ &= x^*. \end{aligned}$$

($\Leftarrow$) Let $x^* = G(x^*)$. Applying the same method of Lemma 2.1 (2) in [16], we find that $J_{M_A,\lambda_A}(I - \lambda_A A_G)$ and $J_{M_B,\lambda_B}(I - \lambda_B A_G)$ are nonexpansive mappings.

Since $x^* = G(x^*)$, we have

$$x^* = G\big(x^*\big) = J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*\big).$$

Let $y \in \Omega$, we have $\theta \in A_G y + M_A y$ and $\theta \in A_G y + M_B y$.

From Lemma 2.10, it implies that

$y \in F(J_{M_A,\lambda_A}(I - \lambda_A A_G)) \cap F(J_{M_B,\lambda_B}(I - \lambda_B A_G))$. Then

$$\big\|x^* - y\big\|^2 = \big\|J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x^*\big) - y\big\|^2$$

$$
\begin{aligned}
&= \big\| J_{M_A,\lambda_A}(I-\lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big) \\
&\quad - J_{M_A,\lambda_A}(I-\lambda_A A_G)y\big\|^2 \\
&\le \big\| \big(bx^* + (1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big) - y\big\|^2 \\
&= \big\| b\big(x^*-y\big) + (1-b)\big(J_{M_B,\lambda_B}(I-\lambda_B A_G)x^* - y\big)\big\|^2 \\
&= b\big\|x^*-y\big\|^2 + (1-b)\big\|J_{M_B,\lambda_B}(I-\lambda_B A_G)x^* - y\big\|^2 \\
&\quad - b(1-b)\big\|x^* - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big\|^2 \\
&\le b\big\|x^*-y\big\|^2 + (1-b)\big\|x^*-y\big\|^2 - b(1-b)\big\|x^* \\
&\quad - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big\|^2 \\
&= \big\|x^*-y\big\|^2 - b(1-b)\big\|x^* - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big\|^2.
\end{aligned}
\tag{13}
$$

It implies that $\|x^* - J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\| = 0$.

That is, $x^* \in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$.

Since $x^* = G(x^*)$ and $x^* \in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$, we have

$$
\begin{aligned}
x^* &= J_{M_A,\lambda_A}(I-\lambda_A A_G)\big(bx^* + (1-b)J_{M_B,\lambda_B}(I-\lambda_B A_G)x^*\big) \\
&= J_{M_A,\lambda_A}(I-\lambda_A A_G)\big(bx^* + (1-b)x^*\big) \\
&= J_{M_A,\lambda_A}(I-\lambda_A A_G)x^*.
\end{aligned}
$$

Therefore $x^* \in F(J_{M_A,\lambda_A}(I-\lambda_A A_G))$.

From Lemma 2.10, $x^* \in F(J_{M_A,\lambda_A}(I-\lambda_A A_G))$ and $x^* \in F(J_{M_B,\lambda_B}(I-\lambda_B A_G))$, we have $\theta \in A_G x^* + M_A x^*$ and $\theta \in A_G x^* + M_B x^*$. Then $x^* \in \Omega$.

Applying (13), we can conclude that $G$ is a nonexpansive mapping. □

We give some examples to support Lemma 2.11 and show that Lemma 2.11 is not true if some condition fails.

*Example* 2.12 Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle\cdot,\cdot\rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle \mathbf{x},\mathbf{y}\rangle = x\cdot y = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1,x_2) \in \mathcal{R}^2, \mathbf{y} = (y_1,y_2) \in \mathcal{R}^2$ and a usual norm $\|\cdot\| : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ give by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ for all $\mathbf{x} = (x_1,x_2) \in \mathcal{R}^2$ and $A_G: \mathcal{R}^2 \to \mathcal{R}^2$ defined by $A_G((x_1,x_2)) = (x_1-5, x_2-5)$. Let $M_A : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(2x_1-1, 2x_2-1)\}$ and $M_B : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(\frac{x_1}{2}+2, \frac{x_2}{2}+2)\}$. Show that $(2,2) \in F(G)$.

*Solution*. It is obvious that $\Omega = \{(2,2)\}$. Choose $\lambda_A = \frac{1}{2}$. From $M_A(x_1,x_2) = \{(2x_1-1, 2x_2-1)\}$ and the resolvent of $M_A$, $J_{M_A,\lambda_A}x = (I+\lambda_A M_A)^{-1}x$ for all $x = (x_1,x_2) \in \mathcal{R}^2$, we have

$$
J_{M_A,\lambda_A}(x) = \frac{x}{2} + \frac{1}{4},
\tag{14}
$$

for all $x = (x_1,x_2) \in \mathcal{R}^2$. Choose $\lambda_B = 1$. From $M_B(x_1,x_2) = \{(\frac{x_1}{2}+2, \frac{x_2}{2}+2)\}$ and the resolvent of $M_B$, $J_{M_B,\lambda_B}x = (I+\lambda_B M_B)^{-1}x$ for all $x = (x_1,x_2) \in \mathcal{R}^2$, we have

$$
J_{M_B,\lambda_B}(x) = \frac{2x}{3} - \frac{4}{3},
\tag{15}
$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. It is easy to see that $A_G$ is 1-inverse strongly monotone. Choose $b = \frac{1}{4}$. From (14) and (15), we have

$$G(x) = J_{M_A,\frac{1}{2}}\left(I - \frac{1}{2}A_G\right)\left(\frac{1}{4}x + \frac{3}{4}J_{M_B,1}(I - 1A_G)x\right)$$
$$= \frac{x}{16} + \frac{30}{16},$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. By Lemma 2.11, we have $(2,2) \in F(G)$.

*Example* 2.13 Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle \mathbf{x}, \mathbf{y} \rangle = x \cdot y = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1, x_2) \in \mathcal{R}^2, \mathbf{y} = (y_1, y_2) \in \mathcal{R}^2$ and a usual norm $\| \cdot \| : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ give by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ for all $\mathbf{x} = (x_1, x_2) \in \mathcal{R}^2$ and $A_G: \mathcal{R}^2 \to \mathcal{R}^2$ defined by $A_G((x_1, x_2)) = (x_1 - 5, x_2 - 5)$. Let $M_A : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(2x_1 - 1, 2x_2 - 1)\}$ and $M_B : \mathcal{R}^2 \to 2^{\mathcal{R}^2}$ be defined by $\{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$. Show that $(2,2) \notin F(G)$.

*Solution*. It is obvious that $\Omega = \{(2,2)\}$. Choose $\lambda_A = 2$. From $M_A(x_1, x_2) = \{(2x_1 - 1, 2x_2 - 1)\}$ and the resolvent of $M_A$, $J_{M_A,\lambda_A}x = (I + \lambda_A M_A)^{-1}x$ for all $x = (x_1, x_2) \in \mathcal{R}^2$, we have

$$J_{M_A,\lambda_A}(x) = \frac{x}{5} + \frac{2}{5}, \tag{16}$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. Choose $\lambda_B = 4$. From $M_B(x_1, x_2) = \{(\frac{x_1}{2} + 2, \frac{x_2}{2} + 2)\}$ and the resolvent of $M_B$, $J_{M_B,\lambda_B}x = (I + \lambda_B M_B)^{-1}x$ for all $x = (x_1, x_2) \in \mathcal{R}^2$, we have

$$J_{M_B,\lambda_B}(x) = \frac{x}{3} - \frac{8}{3}, \tag{17}$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. Choose $b = \frac{1}{4}$. From (16), (17) and $A_G$ being 1-inverse strongly monotone, we have

$$G(x) = J_{M_A,2}(I - 2A_G)\left(\frac{1}{4}x + \frac{3}{4}J_{M_B,4}(I - 4A_G)x\right)$$
$$= \frac{x}{10} + \frac{9}{5},$$

for all $x = (x_1, x_2) \in \mathcal{R}^2$. By Lemma 2.11, we have $(2,2) \notin F(G)$.

**Lemma 2.14** ([18]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that do not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also we consider the sequence of integers $\{\tau(n)\}_{n \geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

*Then $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$ and, for all $n \geq n_0$,*

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

**Lemma 2.15** *Let $H$ be a real Hilbert space, for every $i = 1,2,\dots,N$, let $A_i : H \to H$ be $\alpha_i$-inverse strongly monotone mappings with $\eta = \min\{\alpha_i\}$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be a sequence generated by $y_n = P_C(I - \lambda \sum_{i=1}^{N} a_iA_i)x_n$, $Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^{N} a_iA_i)x_n - y_n, y_n - z\rangle \geq 0\}$ and $x^* \in \bigcap_{i=1}^{N} VI(C,A_i)$ for all $i = 1,2,\dots,N$. Then the following inequality is fulfilled*:

$$\begin{aligned}
&\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right) - x^*\right\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \left(1 - \frac{\lambda}{\eta}\right)\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right) - y_n\right\|^2 \\
&\qquad - \left(1 - \frac{\lambda}{\eta}\right)\|x_n - y_n\|^2,
\end{aligned}$$

*where $\sum_{i=1}^{N} a_i = 1, 0 < a_i < 1$ and $\lambda \in (0,\eta)$ with $\eta = \min_{i=1,2,\dots,N}\{\alpha_i\}$ for every $i = 1,2,\dots,N$.*

*Proof* Since $x^* \in \bigcap_{i=1}^{N} VI(C,A_i)$, we have $x^* \in VI(C,A_i)$ for every $i = 1,2,\dots,N$ and (6), we obtain

$$\begin{aligned}
&\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right) - x^*\right\|^2 \\
&\quad \leq \left\| x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n - x^*\right\|^2 \\
&\qquad - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right) - \left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right)\right\|^2 \\
&\quad = \|x_n - x^*\|^2 \\
&\qquad - 2\lambda\left\langle P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right) - x^*, \sum_{i=1}^{N} a_iA_iy_n\right\rangle \\
&\qquad - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right) - x_n\right\|^2.
\end{aligned} \tag{18}$$

From the monotonicity of $\sum_{i=1}^{N} a_iA_i$, we have

$$\begin{aligned}
0 &\leq \left\langle \sum_{i=1}^{N} a_iA_iy_n - \sum_{i=1}^{N} a_iA_ix^*, y_n - x^*\right\rangle \\
&= \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - x^*\right\rangle - \left\langle \sum_{i=1}^{N} a_iA_ix^*, y_n - x^*\right\rangle \\
&\leq \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - x^*\right\rangle \\
&= \left\langle \sum_{i=1}^{N} a_iA_iy_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_iA_iy_n\right)\right\rangle
\end{aligned}$$

$$+\left\langle \sum_{i=1}^{N} a_i A_i y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - x^*\right\rangle.$$

It implies that

$$\begin{aligned}
&\left\langle x^* - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right), \sum_{i=1}^{N} a_i A_i y_n\right\rangle \\
&\quad\le \left\langle \sum_{i=1}^{N} a_i A_i y_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right)\right\rangle.
\end{aligned} \tag{19}$$

From (18) and (19), we have

$$\begin{aligned}
&\left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - x^*\right\|^2 \\
&\quad\le \|x_n - x^*\|^2 + 2\lambda\left\langle \sum_{i=1}^{N} a_i A_i y_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right)\right\rangle \\
&\qquad - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - x_n\right\|^2 \\
&\quad= \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2 \\
&\qquad - 2\left\langle P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n, y_n - x_n\right\rangle \\
&\qquad + 2\lambda\left\langle \sum_{i=1}^{N} a_i A_i y_n, y_n - P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right)\right\rangle \\
&\quad= \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2 \\
&\qquad + 2\left\langle x_n - y_n - \lambda \sum_{i=1}^{N} a_i A_i y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\rangle \\
&\quad= \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2 \\
&\qquad + 2\left\langle \left(I - \lambda \sum_{i=1}^{N} a_i A_i\right)x_n - y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\rangle \\
&\qquad + 2\left\langle \lambda \sum_{i=1}^{N} a_i A_i x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n, P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\rangle \\
&\quad\le \|x_n - x^*\|^2 - \left\| P_{Q_n}\left(x_n - \lambda \sum_{i=1}^{N} a_i A_i y_n\right) - y_n\right\|^2 - \|y_n - x_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&+2\lambda\left\|\sum_{i=1}^{N}a_iA_ix_n-\sum_{i=1}^{N}a_iA_iy_n\right\|\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\| \\
&\leq\left\|x_n-x^*\right\|^2-\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2-\|y_n-x_n\|^2 \\
&+2\frac{\lambda}{\eta}\|x_n-y_n\|\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\| \\
&=\left\|x_n-x^*\right\|^2-\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2-\|y_n-x_n\|^2 \\
&+\frac{\lambda}{\eta}\left(\|x_n-y_n\|^2+\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2\right) \\
&=\left\|x_n-x^*\right\|^2-\left(1-\frac{\lambda}{\eta}\right)\left\|P_{Q_n}\left(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n\right)-y_n\right\|^2 \\
&-\left(1-\frac{\lambda}{\eta}\right)\|y_n-x_n\|^2. \qquad (20)
\end{aligned}
$$

□

## 3 Main result

In this section, we prove the strong convergence of the sequence acquired from the proposed iterative methods for finding a common element of the set of finite family variational inequalities problems and the set of solutions of the proposed problem.

**Theorem 3.1** *Let $H$ be a real Hilbert space. For $i = 1,2,\dots,N$, let $A_i : H \to H$ be $\alpha_i$-inverse strongly monotone mappings and let $A_G : H \to H$ be $\alpha_G$-inverse strongly monotone mappings. Define the mapping $G : H \to H$ by $G(x) = J_{M_A,\lambda_A}(I-\lambda_AA_G)(bx+(1-b)J_{M_B,\lambda_B}(I-\lambda_BA_G)x)$ for all $x \in H$, $b \in (0,1)$ and $\lambda_A,\lambda_B \in (0,2\alpha_G)$. Assume that $\Gamma = \bigcap_{i=1}^{N} VI(C,A_i) \cap F(G) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1,u \in H$ and*

$$
\begin{cases}
y_n = P_C(I-\lambda\sum_{i=1}^{N}a_iA_i)x_n, \\
Q_n = \{z \in H : \langle (I-\lambda\sum_{i=1}^{N}a_iA_i)x_n-y_n, y_n-z\rangle \geq 0\}, \\
x_{n+1} = \alpha_nu+\beta_nP_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)+\gamma_nGx_n,
\end{cases}
\qquad (21)
$$

*where $\sum_{i=1}^{N}a_i = 1, 0 < a_i < 1, \{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset [0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$, $\lambda \in (0,\eta)$ with $\eta = \min_{i=1,2,\dots,N}\{\alpha_i\}$.*

*Suppose the following conditions hold*:

*(i)* $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$,

*(ii)* $0 < c < \beta_n, \gamma_n \leq d < 1$.

*Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.*

*Proof* We must show that $\{x_n\}$ is bounded. Let $z_n = P_{Q_n}(x_n-\lambda\sum_{i=1}^{N}a_iA_iy_n)$.

We consider

$$x_{n+1} = \alpha_nu+\beta_nz_n+\gamma_nGx_n$$

$$
\begin{aligned}
&= \alpha_n u + (1-\alpha_n)\left(\frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n}\right) \\
&= \alpha_n u + (1-\alpha_n) t_n,
\end{aligned}
$$

where $t_n = \frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n}$. Letting $x^* \in \Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap F(G)$, we have

$$
\begin{aligned}
\|t_n - x^*\|^2 &= \left\| \frac{\beta_n z_n + \gamma_n G x_n}{1-\alpha_n} - x^* \right\|^2 \\
&= \left\| \frac{\beta_n z_n + \gamma_n G x_n - (1-\alpha_n) x^*}{1-\alpha_n} \right\|^2 \\
&= \frac{\beta_n}{1-\alpha_n} \|z_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n} \|G x_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} \|z_n - G x_n\|^2. 
\end{aligned} \tag{22}
$$

From definition of $x_{n+1}$ and (22), we consider

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1-\alpha_n) t_n - x^*\|^2 \\
&= \|\alpha_n (u - x^*) - (1-\alpha_n)(t_n - x^*)\|^2 \\
&= \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \|t_n - x^*\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\
&= \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \left[ \frac{\beta_n}{1-\alpha_n} \|z_n - x^*\|^2 \right. \\
&\quad \left. + \frac{\gamma_n}{1-\alpha_n} \|G x_n - x^*\|^2 - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} \|z_n - G x_n\|^2 \right] \\
&\quad - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|G x_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - G x_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - G x_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2. 
\end{aligned} \tag{23}
$$

By Lemma 2.15 and $\lambda \in (0,1)$, we have

$$
\|z_n - x^*\|^2 \le \|x_n - x^*\|^2. \tag{24}
$$

From (23) and (24), we get

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - G x_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2
\end{aligned}
$$

$$\begin{aligned}
&\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\
&= \alpha_n \|u - x^*\|^2 + (1-\beta_n) \|x_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2 - \alpha_n (1-\alpha_n) \|u - t_n\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + (1-\beta_n) \|x_n - x^*\|^2 \\
&\quad \vdots \\
&\leq \max\{ \|u - x^*\|^2 + \|x_1 - x^*\|^2 \}.
\end{aligned} \tag{25}$$

By induction,

$$\|x_{n+1} - x^*\|^2 \leq \max\{ \|u - x^*\|^2 + \|x_1 - x^*\|^2 \},$$

then $\{x_n\}$ is a bounded sequence.

We use

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \left[ \|x_n - x^*\|^2 - \left(1 - \frac{\lambda}{\eta}\right) \|z_n - y_n\|^2 \right. \\
&\quad \left. - \left(1 - \frac{\lambda}{\eta}\right) \|x_n - y_n\|^2 \right] + \gamma_n \|x_n - x^*\|^2 - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2 \\
&= \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \|x_n - x^*\|^2 - \beta_n \left(1 - \frac{\lambda}{\eta}\right) \|z_n - y_n\|^2 \\
&\quad - \beta_n \left(1 - \frac{\lambda}{\eta}\right) \|x_n - y_n\|^2 - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \beta_n \left(1 - \frac{\lambda}{\eta}\right) \|z_n - y_n\|^2 \\
&\quad - \beta_n \left(1 - \frac{\lambda}{\eta}\right) \|x_n - y_n\|^2 - \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\beta_n \left(1 - \frac{\lambda}{\eta}\right) \|z_n - y_n\|^2 + \beta_n \left(1 - \frac{\lambda}{\eta}\right) \|x_n - y_n\|^2 + \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2 \\
&\quad \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.
\end{aligned} \tag{26}$$

Let $S_n := \beta_n (1 - \frac{\lambda}{\eta}) \|z_n - y_n\|^2 + \beta_n (1 - \frac{\lambda}{\eta}) \|x_n - y_n\|^2 + \frac{\beta_n \gamma_n}{1-\alpha_n} \|z_n - Gx_n\|^2$.

Then we have

$$S_n \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{27}$$

Now, we consider two possible cases:

*Case* 1. Put $\Gamma_n := \|x_n - x^*\|^2$ for all $n \in \mathcal{N}$.

Assume that there is $n_0 \geq 0$ such that, for each $n \geq n_0$, $\Gamma_{n+1} \leq \Gamma_n$.

In this case, $\lim_{n\to\infty} \Gamma_n$ exists and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$.

Since $\lim_{n\to\infty} \alpha_n = 0$, it follows from (27) that $\lim_{n\to\infty} S_n = 0$.

Therefore, we have $\lim_{n\to\infty} \beta_n(1-\frac{\lambda}{\eta})\|z_n - y_n\|^2 = 0$, $\lim_{n\to\infty} \beta_n(1-\frac{\lambda}{\eta})\|x_n - y_n\|^2 = 0$ and $\lim_{n\to\infty} \frac{\beta_n\gamma_n}{1-\alpha_n}\|z_n - Gx_n\|^2 = 0$.

From the assumptions i), ii), we obtain

$$\lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - Gx_n\| = 0. \tag{28}$$

Hence, we obtain

$$\|x_n - Gx_n\| \leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - Gx_n\|.$$

From (28), we have

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{29}$$

We now show that $\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle \leq 0$.

We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle. \tag{30}$$

Because $\{x_n\}$ is a bounded sequence in $H$, there exists a subsequence of $\{x_n\}$ that converges weakly to an element in $H$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$ where $w \in H$. Since $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we have $z_{n_i} \rightharpoonup w$.

Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $y_{n_i} \rightharpoonup w$.

Assume that $w \notin \bigcap_{i=1}^N VI(C, A_i)$. So, we have $w \notin F(P_C(I - \lambda\sum_{i=1}^N a_iA_i))$.

Then we have $w \neq P_C(I - \lambda\sum_{i=1}^N a_iA_i)w$. By the nonexpansiveness of $P_C(I - \lambda\sum_{i=1}^N a_iA_i)$, (28) and Opial's property, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|x_{n_i} - w\| \\
&\quad < \liminf_{n\to\infty} \left\| x_{n_i} - P_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)w \right\| \\
&\quad \leq \liminf_{n\to\infty} \left( \|x_{n_i} - y_{n_i}\| + \left\| y_{n_i} - P_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)w \right\| \right) \\
&\quad \leq \liminf_{n\to\infty} \left( \|x_{n_i} - y_{n_i}\| \right. \\
&\qquad \left. + \left\| P_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)x_{n_i} - P_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)w \right\| \right) \\
&\quad \leq \liminf_{n\to\infty} \|x_{n_i} - w\|.
\end{aligned}$$

This is a contradiction; we have $w \in VI(C, \sum_{i=1}^{N} a_i A_i)$. From Remark 2.5, we have

$$w \in \bigcap_{i=1}^{N} VI(C, A_i). \tag{31}$$

Assume that $w \notin F(G)$. Then we have $w \neq Gw$. From (29) and Opial's property, we have

$$\begin{aligned}
\liminf_{n\to\infty} \|x_{n_i} - w\| &< \liminf_{n\to\infty} \|x_{n_i} - Gw\| \\
&\le \liminf_{n\to\infty} \big(\|x_{n_i} - Gx_{n_i}\| + \|Gx_{n_i} - Gw\|\big) \\
&\le \liminf_{n\to\infty} \big(\|x_{n_i} - Gx_{n_i}\| + \|x_{n_i} - w\|\big) \\
&\le \liminf_{n\to\infty} \|x_{n_i} - w\|.
\end{aligned}$$

This is a contradiction; we have

$$w \in F(G). \tag{32}$$

From (31) and (32), we have $w \in \bigcap_{i=1}^{N} VI(C, A_i) \cap F(G)$.

Therefore, we get

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle u - x^*, x_{n_i} - x^* \rangle = \langle u - x^*, w - x^* \rangle \le 0, \tag{33}$$

where $x^* = P_\Gamma u$.

Next, we show that $\{x_n\}$ converges strongly to $x^*$, where $x^* = P_\Gamma u$.

From the nonexpansiveness of $G$, (22) and (24), we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \frac{\beta_n}{1-\alpha_n} \|z_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n} \|Gx_n - x^*\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} \|z_n - Gx_n\|^2 \\
&\le \frac{\beta_n}{1-\alpha_n} \|z_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n} \|Gx_n - x^*\|^2 \\
&\le \frac{\beta_n}{1-\alpha_n} \|x_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n} \|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2.
\end{aligned} \tag{34}$$

From the definition of $x_n$, (34) and $x^* = P_\Gamma u$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \big\|\alpha_n (u - x^*) - (1-\alpha_n)(t_n - x^*)\big\|^2 \\
&\le (1-\alpha_n) \|t_n - x^*\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\
&\le (1-\alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle.
\end{aligned} \tag{35}$$

By applying Lemma 2.8 to (35), we find that the sequence $\{x_n\}$ converges strongly to $x^*$.

*Case* 2. Assume that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} \leq \Gamma_{n_i+1}$ for all $i \in \mathcal{N}$. In this case, we can define $\tau : \mathcal{N} \to \mathcal{N}$ by $\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$.

Then we have $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. So, we have from (26)

$$\begin{aligned}
&\beta_{\tau(n)}\left(1 - \frac{\lambda}{\eta}\right)\|z_{\tau(n)} - y_{\tau(n)}\|^2 + \beta_{\tau(n)}\left(1 - \frac{\lambda}{\eta}\right)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\
&\quad + \frac{\beta_{\tau(n)}\gamma_{\tau(n)}}{1 - \alpha_{\tau(n)}}\|z_{\tau(n)} - Gx_{\tau(n)}\|^2 \\
&\quad \leq \alpha_{\tau(n)}\|u - x^*\|^2 + \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2.
\end{aligned}$$

Arguing as in Case 1, we have

$$\lim_{n\to\infty}\|z_{\tau(n)} - y_{\tau(n)}\| = \lim_{n\to\infty}\|x_{\tau(n)} - y_{\tau(n)}\| = \lim_{n\to\infty}\|z_{\tau(n)} - Gx_{\tau(n)}\| = 0. \tag{36}$$

Because $\{x_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{x_{\tau(n_j)}\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)} - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle.$$

Following the same argument as the proof of Case 1 for $\{x_{\tau(n_j)}\}$, we have

$$\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle \leq 0$$

and

$$\|x_{\tau(n)+1} - x^*\|^2 \leq (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle,$$

where $\alpha_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty}\alpha_{\tau(n)} = \infty$ and $\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle \leq 0$.

Hence, by Lemma 2.8, we have $\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0$ and $\lim_{n\to\infty}\|x_{\tau(n)+1} - x^*\| = 0$

Therefore, by Lemma 2.14, we have

$$0 \leq \|x_n - x^*\| \leq \max\{\|x_{\tau(n)} - x^*\|, \|x_n - x^*\|\} \leq \|x_{\tau(n)+1} - x^*\|.$$

Hence, $\{x_n\}$ converge strongly to $x^* = P_\Gamma u$. This completes the proof of the main theorem. □

## 4 Application

In 2013, Kangtunyakarn [14] introduced a modification of the system of variational inequalities as follows: finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)z^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{37}$$

where $D_1, D_2 : C \to H$ be two mappings, for every $\lambda_1, \lambda_2 \geq 0$ and $a \in [0,1]$.

Let $h$ be a proper lower semicontinuous convex function of $H$ into $(-\infty, +\infty]$. The subdifferential $\partial h$ of $h$ is defined by

$$\partial h(x) = \{z \in H : h(x) + \langle z, u - x\rangle \leq h(u), \forall u \in H\}$$

for all $x \in H$. From Rockafellar [19], we find that $\partial h$ is a maximal monotone operator. Let $C$ be a nonempty closed convex subset of $H$ and $i_C$ be the indicator function of $C$, i.e.,

$$i_C = \begin{cases} 0; & \text{if } x \in C, \\ +\infty; & \text{if } x \notin C, \end{cases}$$

Then $i_C$ is a proper, lower semicontinuous and convex function on $H$ and so the subdifferential $\partial i_C$ of $i_C$ is a maximal monotone operator. The resolvent operator $J_{\partial i_C, r}$ of $i_C$ for $\lambda > 0$, can be defined by $J_{\partial i_C, r}(x) = (I + \lambda \partial i_C)^{-1}(x), x \in H$. We have $J_{\partial i_C, r}(x) = P_C x$, for all $x \in H$ and $\lambda > 0$. As a special case, if $M_A = M_B = \partial i_C$ in Lemma 2.11, we find that $J_{M_A, \lambda_A} = J_{M_B, \lambda_B} = P_C$. So we obtain the following result.

**Lemma 4.1** ([14]) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $b \in [0,1]$, the following statements are equivalent*:

(*a*) $(x^*, z^*) \in C \times C$ *is a solution of problem* (37),

(*b*) $x^*$ *is a fixed point of the mapping* $\widehat{G} : C \to C$, *i.e.*, $x^* \in F(T)$, *defined by*

$$\widehat{G}(x) = P_C(I - \lambda_1 D_1)\bigl(bx + (1-b)P_C(I - \lambda_2 D_2)x\bigr), \tag{38}$$

*where* $z^* = P_C(I - \lambda_2 D_2)x^*$

**Theorem 4.2** *Let $H$ be a real Hilbert space. For $i = 1, 2, \dots, N$, let $A_i : H \to H$ be $\alpha_i$-inverse strongly monotone mappings and let $A_G : H \to H$ be $\alpha_G$-inverse strongly monotone mappings. Define the mapping $\widehat{G} : H \to H$ by* (38). *Assume that $\Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap F(T) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1, u \in H$ and*

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^N a_i A_i)x_n, \\ Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^N a_i A_i)x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^N a_i A_i y_n) + \gamma_n T x_n, \end{cases} \tag{39}$$

*where $\sum_{i=1}^N a_i = 1, 0 < a_i < 1, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, \eta)$ with $\eta = \min_{i=1,2,\dots,N}\{\alpha_i\}$.*

*Suppose the following conditions hold*:

(*i*) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

(*ii*) $0 < c < \beta_n, \gamma_n \leq d < 1$.

*Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.*

*Proof* Taking $J_{M_A, \lambda_A} = J_{M_B, \lambda_B} = P_C$ in Theorem 3.1, we obtain the desired conclusion. □

In order to apply our main result, we give the following lemma.

**Lemma 4.3** ([14]) *Let $C$ be a nonempty closed convex subset of real Hilbert space $H$. Let $T, S : C \to C$ be nonexpansive mappings. Define a mapping $B^A : C \to C$ by $B^A xj = T(aI + (1a)S)x$ for every $x \in C$ and $a \in (0,1)$. Then $F(B^A) = F(T) \cap F(S)$ and $B^A$ is a nonexpansive mapping.*

We apply our Theorem 3.1, by using with Lemma 4.3 ([14]), to find a solution of the variational inclusion problem.

**Lemma 4.4** *Let $H$ be a real Hilbert space and let $A_G : H \to H$ be $\alpha_G$-inverse strongly monotone mappings. Let $M_A, M_B : H \to 2^H$ be a multi-value maximum monotone mapping with $VI(H, A_G, M_A) \cap VI(H, A_G, M_B) \neq \emptyset$. Define a mapping $G : H \to H$ as in Lemma 2.11 for all $x \in H$, $a \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha_G)$. Then $F(G) = VI(H, A_G, M_A) \cap VI(H, A_G, M_B)$.*

*Proof* Let $x, y \in C$. From Lemma 2.11, we find that $G$ is nonexpansive and $J_{M_A,\lambda_A}(I - \lambda_A A_G)$ and $J_{M_B,\lambda_B}(I - \lambda_B A_G)$ are nonexpansive. Since

$$G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)\big(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x\big)$$

and Lemma 4.3, we have

$$F(G) = F\big(J_{M_A,\lambda_A}(I - \lambda_A A_G)\big) \cap F\big(J_{M_B,\lambda_B}(I - \lambda_B A_G)\big).$$

By Lemma 2.10, we have

$$F(G) = VI(H, A_G, M_A) \cap VI(H, A_G, M_B).$$ □

**Theorem 4.5** *Let $H$ be a real Hilbert space. For $i = 1, 2, \dots, N$, let $A_i : H \to H$ be $\alpha_i$-inverse strongly monotone mappings and let $A_G : H \to H$ be $\alpha_G$-inverse strongly monotone mappings. Define the mapping $G : H \to H$ by $G(x) = J_{M_A,\lambda_A}(I - \lambda_A A_G)(bx + (1-b)J_{M_B,\lambda_B}(I - \lambda_B A_G)x)$ for all $x \in H$, $b \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\alpha_G)$. Assume that $\Gamma = \bigcap_{i=1}^N VI(C, A_i) \cap VI(H, A_G, M_A) \cap VI(H, A_G, M_B) \neq \emptyset$. Let the sequence $\{y_n\}$ and $\{x_n\}$ be generated by $x_1, u \in H$ and*

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^N a_i A_i)x_n, \\ Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^N a_i A_i)x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^N a_i A_i y_n) + \gamma_n G x_n, \end{cases} \tag{40}$$

*where $\sum_{i=1}^N a_i = 1, 0 < a_i < 1, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, \eta)$ with $\eta = \min_{i=1,2,\dots,N}\{\alpha_i\}$.*

*Suppose the following conditions hold*:

*(i)* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

*(ii)* $0 < c < \beta_n, \gamma_n \leq d < 1$.

*Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma u$.*

*Proof* From Lemma 4.4, and Theorem 3.1, we obtain the desired conclusion. □

*Remark* 4.6 if $VI(H, A_G, M_A) \cap VI(H, A_G, M_B) \neq \emptyset$, then observe that $VI(H, A_G, M_A) \cap VI(H, A_G, M_B) = \Omega$.

## 5 Example and numerical results

In this section, we give an example supporting Theorem 3.1.

*Example* 5.1 Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle\cdot,\cdot\rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle x,y\rangle = x\cdot y = x_1y_1 + x_2y_2$ and the usual norm $\|\cdot\| : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ given by $\|x\| = \sqrt{x_1^2 + x_2^2}$ for all $x = (x_1,x_2) \in \mathcal{R}^2$. Let $C_1 = \{(x_1,x_2) \in H| -2x_1 + x_2 \le 1\}$ and $C_2 = \{(x_1,x_2) \in H|4x_1 - 2x_2 \le 3\}$. Define the mapping $A_1 : C_1 \to \mathcal{R}^2$ by $A_1(x_1,x_2) = (\frac{3x_1}{2}, \frac{3x_2}{2})$. Define the mapping $A_2 : C_2 \to \mathcal{R}^2$ by $A_2(x_1,x_2) = (2x_1,2x_2)$. Let the mapping $A_G : \mathcal{R}^2 \to \mathcal{R}^2$ be defined by $A_G(x_1,x_2) = (x_1 + 1,x_2 + 1)$. Let $C = C_1 \cap C_2$. We have

$$P_C(x_1,x_2) = \begin{cases} (-1999x_1 + 1000x_2 + 750, 4000x_1 - 1999x_2 - 1500); \\ \quad \text{if } -40x_1 + 20x_2 < -15, \\ (x_1,x_2); \\ \quad \text{if } -15 \le -40x_1 + 20x_2 \le 5, \\ (-1999x_1 + 1000x_2 - 250, 4000x_1 - 1999x_2 - 500); \\ \quad \text{if } -40x_1 + 20x_2 > 5. \end{cases}$$

Let $x_1, u \in \mathcal{R}^2$, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by

$$\begin{cases} y_n = P_C(I - \lambda \sum_{i=1}^2 a_iA_i)x_n, \\ Q_n = \{z \in H : \langle (I - \lambda \sum_{i=1}^2 a_iA_i)x_n - y_n, y_n - z\rangle \ge 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda \sum_{i=1}^2 a_iA_iy_n) + \gamma_n Gx_n, \end{cases} \tag{41}$$

where $\{\alpha_n\} = \frac{1}{12n}, \{\beta_n\} = \frac{5n-2}{12n}, \{\gamma_n\} = \frac{7n+1}{12n} \subset [0,1]$ and $a = 0.5 \in (0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0,0)$.

*Solution.* Since $A_1, A_2$ and $A_G$ are $\frac{2}{3}, \frac{1}{2}$ and 1-inverse strongly monotone mappings, respectively, $\eta = \frac{1}{2}$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(x_1,x_2) = (\frac{x_1}{16}, \frac{x_2}{16})$. Choose $\lambda = \frac{1}{4} \in (0,\eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $(0,0) \in VI(C,A_1) \cap VI(C,A_2) \cap F(G)$. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0,0)$.

*Example* 5.2 Let $H = L_2([-1,1])$ with product $\langle f,g\rangle = \int_{-1}^1 f(t)g(t)\,dt$ and the associated norm given as $\|f\| := \sqrt{\int_{-1}^1 f(t)g(t)\,dt}$ for all $f,g \in L_2([-1,1])$. Take $C = \{x \in H : \|x\| \le 2\}$. Define the mapping $A_1 : L_2([-1,1]) \to L_2([-1,1])$ by $A_1(h(t)) = h(t) - 2t$ for all $t \in [-1,1]$. Define the mapping $A_2 : L_2([-1,1]) \to L_2([-1,1])$ by $A_2(h(t)) = \frac{3}{2}h(t) - 3t$ for all $t \in [-1,1]$. Let the mapping $A_G : L_2([-1,1]) \to L_2([-1,1])$ be defined by $A_G(h(t)) = h(t) - 5t$ for all $t \in [-1,1]$. We have

$$P_C\big(f(t)\big) = \begin{cases} f(t); & \text{if } \|f(t)\| \le 2, \\ \frac{2f(t)}{\|f(t)\|}; & \text{if } \|f(t)\| > 2. \end{cases}$$

Let $i = 1,2, x_1, u \in \mathcal{R}^2$, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated by (21) where $\{\alpha_n\} = \frac{1}{12n}, \{\beta_n\} = \frac{5n-2}{12n}, \{\gamma_n\} = \frac{7n+1}{12n} \subset [0,1]$ and $a = 0.4 \in (0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $2t$.

**Table 1** Detailed analysis of computational methods (21) and (3) for Example 1 with $\mathbf{u} = (5,5)$, $N = 15$, $E(x_1^n) = \|x_1^{n+1} - x_1^n\|, n \in N_0$ and $E(x_2^n) = \|x_2^{n+1} - x_2^n\|, n \in N_0$

| | Iterative (21) | | Iterative (3) | |
|---|---|---|---|---|
| $n$ | $E(\mathbf{x}_1^n)$ | $E(\mathbf{x}_2^n)$ | $E(\mathbf{x}_1^n)$ | $E(\mathbf{x}_2^n)$ |
| 1 | 1.0000 | 2.0000 | 1.0000 | 2.0000 |
| 2 | 0.6468 | 0.8770 | 0.6497 | 0.8828 |
| 3 | 0.3961 | 0.4630 | 0.3995 | 0.4681 |
| 4 | 0.2619 | 0.2826 | 0.2646 | 0.2862 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 0.0472 | 0.0457 | 0.0476 | 0.0476 |

**Table 2** Detailed analysis of computational methods (21) and (3) for Example 1 with $\mathbf{u} = 3t$, $N = 15$ and $E(x_n) = \|x_{n+1} - x_n\|, n \in N_0$

| $n$ | $E(\mathbf{x}_n)$: Algorithm (21) | $E(\mathbf{x}_n)$: Algorithm (3) |
|---|---|---|
| 1 | 0.7626 | 0.7626 |
| 2 | 0.1291 | 0.1221 |
| 3 | 0.0480 | 0.0492 |
| 4 | 0.0208 | 0.0226 |
| ⋮ | ⋮ | ⋮ |
| 15 | 0.0006 | 0.0007 |

*Solution.* Since $A_1, A_2$ and $A_G$ are $\frac{1}{2}, \frac{1}{3}$ and 1-inverse strongly monotone mappings, respectively, $\eta = \frac{1}{2}$. Choose $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(h(t)) = \frac{h(t)}{16}$. Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $2t \in VI(C, A_1) \cap VI(C, A_2) \cap F(G)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $2I$.

*Example* 5.3 Let $f : H \to \mathcal{R}$ be a convex function. Consider the following convex optimization problem:

$$\min_{x \in H} f(x) \tag{42}$$

and

$$\min_{x \in H} g(x) \tag{43}$$

It is well known that $x^* \in C$ solves (42) and (43) if and only if $x^* \in C$ satisfies the following variational inequalities:

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{44}$$

and

$$\langle \nabla g(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{45}$$

that is, $x^* \in VI(C, \nabla f) \cap VI(C, \nabla g)$. Let $H = \mathcal{R}$. Take $C = [1, 10]$. Define the mapping $f : [1, 10] \to \mathcal{R}$ by $f(x) = \frac{(x-1)^2}{3} + 1$. Define the mapping $g : [1, 10] \to \mathcal{R}$ by $g(x) = \frac{x^2}{2} - \ln x - \frac{1}{2}$.

**Table 3** Detailed analysis of computational methods (21) and (3) for Example 1 with $\mathbf{u} = 3$, $N = 15$ and $E(x_n) = \|x_{n+1} - x_n\|, n \in N_0$

| $n$ | $E(\mathbf{x}_n)$: Algorithm (21) | $E(\mathbf{x}_n)$: Algorithm (3) |
|---|---|---|
| 1 | 2.9044 | 2.7500 |
| 2 | 0.7088 | 0.7428 |
| 3 | 0.2200 | 0.2681 |
| 4 | 0.0762 | 0.1082 |
| ⋮ | ⋮ | ⋮ |
| 15 | 0.0012 | 0.0015 |

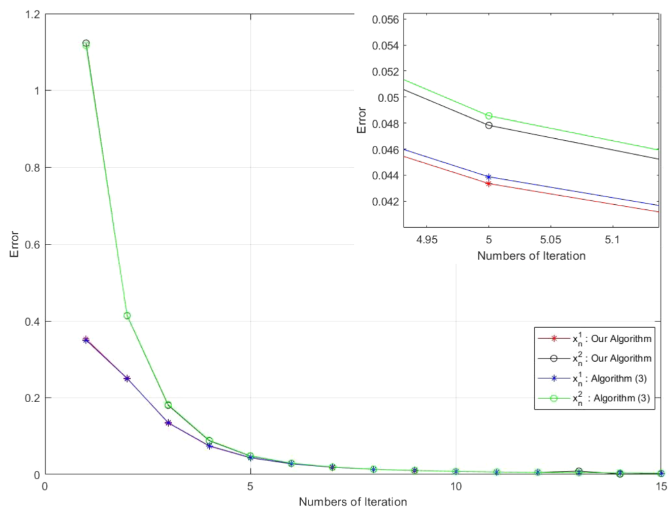


**Figure 1** Comparison between Algorithms (21) and (3) for Example 1 with $\mathbf{u} = (5, 5)$ and $N = 15$

Let $x_1, u \in \mathcal{R}^2$. From (21), we find that $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are generated by

$$\begin{cases} y_n = P_C(I - \lambda(a_1\nabla f + a_2\nabla g))x_n, \\ Q_n = \{z \in H : \langle (I - \lambda(a_1\nabla f + a_2\nabla g))x_n - y_n, y_n - z\rangle \geq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{Q_n}(x_n - \lambda(a_1\nabla f + a_2\nabla g)y_n) + \gamma_n G x_n, \end{cases} \tag{46}$$

where $\{\alpha_n\} = \frac{1}{12n}, \{\beta_n\} = \frac{5n-2}{12n}, \{\gamma_n\} = \frac{7n+1}{12n} \subset [0, 1]$ and $a = 0.5 \in (0, 1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

*Solution*. Since $f$ and $g$ are convex and differentiable with $f'(x) = \frac{2(x-1)}{3}$ and $g'(x) = x - \frac{1}{x}$. It implies that $\nabla f$ and $\nabla g$ are $\frac{2}{3}$ and 1-inverse strongly monotone mappings, respectively. Choose $\eta = \frac{1}{2}$, $\lambda_A = \frac{1}{2}, \lambda_B = 1 \in (0, 2\alpha_G)$ and $b = \frac{1}{4}$, we obtain $G(x) = \frac{x}{12} + \frac{11}{12}$. Choose $\lambda = \frac{1}{4} \in (0, \eta)$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1 and $1 \in VI(C, \nabla f) \cap VI(C, \nabla g) \cap F(G)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

*Remark* 5.4 According to Tables 1–3 and Figs. 1–3, it is shown that our Algorithm (21) converges to an element of the set $\bigcap_{i=1}^{N} VI(C, A_i) \cap F(G)$ at a faster rate than Algorithm (3). Therefore, our algorithm is more efficient.

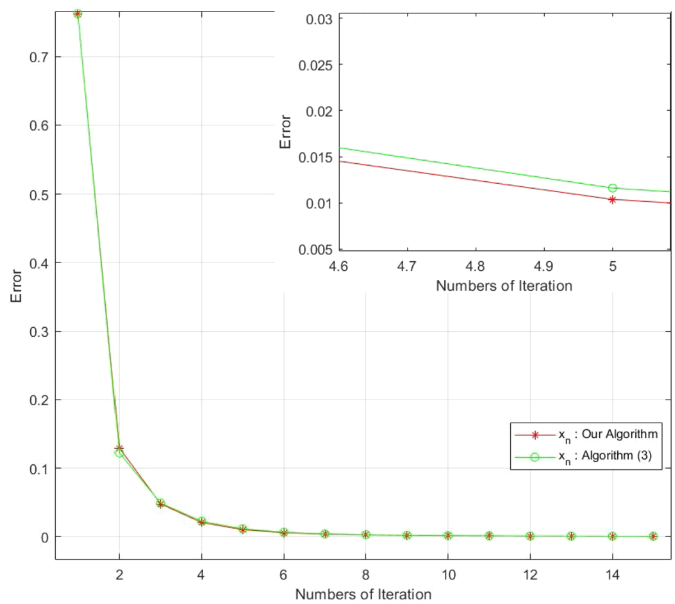


**Figure 2** Comparison between Algorithms (21) and (3) for Example 2 with $\mathbf{u} = 3t$ and $N = 15$

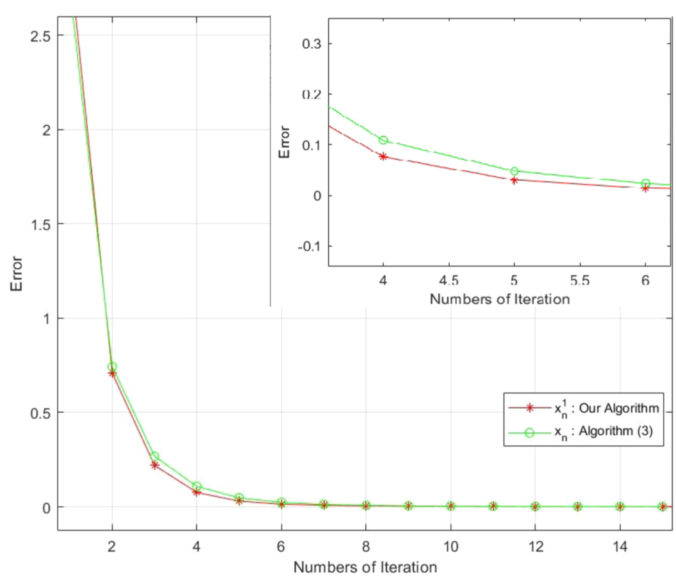


**Figure 3** Comparison between Algorithms (21) and (3) for Example 3 with $\mathbf{u} = 3$ and $N = 15$

## 6 Conclusion

In this paper, we have proposed a new problem, called a generalized system of modified variational inclusion problems (GSMVIP). This problem can be reduced to a classical variational inclusion problem and a classical variational inequalities problem. Moreover, we study the half-space

$$Q_n = \left\{ z \in H : \left\langle \left( I - \lambda \sum_{i=1}^{N} a_i A_i \right) x_n - y_n, y_n - z \right\rangle \geq 0 \right\},$$

which can be reduced to $T_n$ in Algorithm (3). In order to solve the GSMVIP and the set of a finite family of variational inequalities problem, we have presented a new subgradient extragradient algorithm which uses $Q_n$ and show that it converges to a solution of the GSMVIP and the set of a finite family of variational inequalities problem under suitable conditions. Therefore, our algorithm improves the algorithm proposed by Censor et al. [12]. The efficiency of the proposed algorithm has also been illustrated by several numerical experiments.

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**Authors' contributions**
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