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Linear combinations of composition operators on H^∞ spaces over the unit ball and polydisk

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Abstract

In this paper, we characterize completely the compactness of linear combinations of composition operators acting on the space $H^\infty(\mathbb{B}_N)$ of bounded holomorphic functions over the unit ball $\mathbb{B}_N$ from two different aspects. The same problems are also investigated on the space $H^\infty(\mathbb{D}^N)$ over the unit polydisk $\mathbb{D}^N$.

Keywords: Linear combination; Composition operator; Compactness

1 Introduction

Let H be a Banach space of holomorphic functions on a domain G of $\mathbb{C}^N$ and φ be a holomorphic map of G into itself. The composition operator C_φ is a linear operator defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H.$$

Such operators have been investigated mainly in various Banach spaces of holomorphic functions to characterize the operator theoretic behavior of C_φ by the function theoretic properties of φ, see the books [3, 15, 22].

An area of considerable interest is the topological structure of the set of composition operators acting on a given function space. That work was originally investigated by Berkson [2] in the setting of Hardy–Hilbert space $H^2(\mathbb{D})$ on the open unit disk $\mathbb{D}$, and then generalized by MacCluer [12] and Shapiro and Sundberg [16]. On the space of all bounded holomorphic functions on $\mathbb{D}$, denoted by $H^\infty(\mathbb{D})$, MacCluer, Ohno, and Zhao [13] studied the topological structure of the set of composition operators and characterized completely compact differences of composition operators. Hosokawa, Izuchi, and Zheng continued this investigation in [7], where they showed that a composition operator that is isolated in the norm topology is also isolated in the essential norm topology. Furthermore, Toews [19] generalized those results to the H^∞ space over the unit ball, and Wolf [21] characterized the boundedness and compactness of differences of composition operators between weighted Banach spaces of holomorphic functions on the unit polydisk.

After these works, many authors contributed to exploring norms and essential norms of differences of composition operators on $H^\infty(\mathbb{D})$, see for example [1, 5, 6]. Moreover, Gorkin and Mortini [4] estimated norms and essential norms of linear sums of endomorphisms on uniform algebras. Following that, Izuchi and Ohno [9] characterized the compactness of linear combinations of composition operators on $H^\infty(\mathbb{D})$ and computed norms and essential norms of them. In this paper, quite influenced by [9], we investigate to extend those results just mentioned to H^∞ spaces over the unit ball and polydisk. We want also to mention that some related results on difference of composition and weighted composition operators to weighted type spaces can be found in [11] and [18] and in the related references therein.

Recall that the unit ball $\mathbb{B}_N$ resp. polydisk $\mathbb{D}^N$ is defined as

$$\mathbb{B}_N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z| := (|z_1|^2 + \dots + |z_N|^2)^{1/2} < 1\} \quad \text{resp.,}$$
$$\mathbb{D}^N = \Big\{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z|_{\max} := \max_{1 \le j \le N} |z_j| < 1\Big\}.$$

Let $H^\infty(\mathbb{B}_N)$ resp. $H^\infty(\mathbb{D}^N)$ be the Banach space of all bounded holomorphic functions with the supremum norms over the unit ball $\mathbb{B}_N$ resp. the unit polydisk $\mathbb{D}^N$. Throughout this work, we use the same confusing notation $\|\cdot\|_\infty$ standing for the supremum norms over $\mathbb{B}_N$ or $\mathbb{D}^N$, according to the context.

This paper is organized as follows. Section 2 includes some background materials needed in the sequel. In Sect. 3 we determine conditions under which linear combinations of composition operators are compact on $H^\infty(\mathbb{B}_N)$ and $H^\infty(\mathbb{D}^N)$, respectively. One of the main difficult problems of our proof is how to construct suitable test functions.

2 Preliminaries and definitions

In order to handle linear combinations of composition operators, we need some auxiliary results. For $z = (z_1, \dots, z_N)$ and $w = (w_1, \dots, w_N)$ in $\mathbb{C}^N$, the inner product of z and w is defined by

$$\langle z, w\rangle := z_1\overline{w_1} + \dots + z_N\overline{w_N},$$

and then $|z| = \langle z, z\rangle^{1/2}$. For each $z \in \mathbb{C}^N$, denote by $[z]$ the complex subspace spanned by z. The involutive automorphism of $\mathbb{B}_N$ that interchanges a and 0 is given by

$$\Phi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B}_N,$$

where P_a is the projection onto $[a]$ (that is, $P_0 = 0$, $P_a(z) := \frac{\langle z,a\rangle}{\langle a,a\rangle} a$ if $a \ne 0$), $Q_a(z) = (I - P_a)(z)$ is the projection onto $[z]^\perp$, and $s_a := (1 - \langle a, a\rangle)^{1/2}$. Clearly, $\langle P_a(z), a\rangle = \langle z, a\rangle$. For z and w in $\mathbb{B}_N$, the pseudo-hyperbolic distance $\beta(z, w)$ is defined by

$$\beta(z, w) := \sup\{|f(z)| : f \in H^\infty(\mathbb{B}_N), \|f\|_\infty \le 1, f(w) = 0\};$$

and the induced distance $d_\infty(z, w)$ is defined by

$$d_\infty(z, w) := \sup\{|f(z) - f(w)| : f \in H^\infty(\mathbb{B}_N), \|f\|_\infty \le 1\}.$$

We also recall the following relations, for example, see [10, 19].

Lemma 2.1 *For any z and w in $\mathbb{B}_N$, we have*
(a) $d_\infty(z,w) = \frac{2-2\sqrt{1-\beta(z,w)^2}}{\beta(z,w)}$;
(b) $\beta(z,w) = |\Phi_z(w)|$;
(c) $\{p : \beta(z,p) < \lambda\} = \Phi_z(\lambda\mathbb{B}_N)$ *for* $0 \le \lambda \le 1$.

For $0 < \lambda < 1$, $\Phi_z(\lambda\mathbb{B}_N)$ is the set of all points $w \in \mathbb{B}_N$ satisfying

$$\frac{|P_z(w) - C_{z\lambda}|^2}{\lambda^2\rho_{z\lambda}^2} + \frac{|Q_z(w)|^2}{\lambda^2\rho_{z\lambda}} < 1,$$

an ellipsoid with center $C_{z\lambda}$, where

$$C_{z\lambda} := \frac{(1-\lambda^2)z}{1-\lambda^2|z|^2} \quad \text{and} \quad \rho_{z\lambda} := \frac{1-|z|^2}{1-\lambda^2|z|^2}$$

(see [14, page 10] for details). As in [19], observe that $\Phi_z(\lambda\mathbb{B}_N) \cap [z]^\perp$ is a ball of radius $\lambda\sqrt{\rho_{z\lambda}}$, while $\Phi_z(\lambda\mathbb{B}_N) \cap [z]$ is a disk centered at $C_{z\lambda}$ of radius $\lambda\rho_{z\lambda}$ as follows:

$$\Phi_z(\lambda\mathbb{B}_N) \cap [z] = \left\{ w \in [z] : \left| \frac{z-w}{1-\langle z,w\rangle} \right| < \lambda \right\}. \tag{2.1}$$

The pseudo-hyperbolic distance between two points z and w in $\mathbb{D}^N$ is defined by

$$\rho_N(z,w) := \max_{1\le j\le N} \rho(z_j,w_j),$$

where we denote by ρ the pseudo-hyperbolic distance on $\mathbb{D}$, i.e., $\rho(a,b) = |\frac{a-b}{1-\overline{a}b}|$ for $a,b \in \mathbb{D}$. We also define the induced distance for any z and w in $\mathbb{D}^N$:

$$d_{\max}(z,w) := \sup\{ |f(z) - f(w)| : f \in H^\infty(\mathbb{D}^N), \|f\|_\infty \le 1 \}.$$

The following relation can be obtained by using Schwarz's lemma for the polydisk and the argument in the proof of Lemma 2.2 in [20].

Lemma 2.2 $d_{\max}(z,w) \le 2\rho_N(z,w)$ *for any* $z,w \in \mathbb{D}^N$.

At the end of this section, we give compactness criterions for linear combinations of composition operators. Let $X = H^\infty(\mathbb{B}_N)$ (resp. $H^\infty(\mathbb{D}^N)$) and $G = \mathbb{B}_N$ (resp. $\mathbb{D}^N$). Let $B(X)$ be the space of all bounded linear operators from X to X. Then an operator $T \in B(X)$ is said to be compact if $\overline{T(S)}$ is compact in the norm topology in X, where S is the unit sphere of X.

Proposition 2.3 *Let $\varphi_1,\dots,\varphi_M$ be holomorphic self-maps of G. Then, for $a_1,\dots,a_M \in \mathbb{C}$, $\sum_{i=1}^M a_i C_{\varphi_i}$ is compact on X if and only if $\|\sum_{i=1}^M a_i C_{\varphi_i} f_j\|_\infty \to 0$ for any bounded sequence $\{f_j\} \subseteq X$ with $f_j \to 0$ uniformly on each compact subset of G, as $j \to \infty$.*

Throughout this paper we use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative functions X and Y to mean that there exists $C > 0$ such that $X \le CY$, where C does not depend on the associated variables. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

3 Linear combinations of composition operators

3.1 Compactness on $H^\infty(\mathbb{B}_N)$

As is well known, every holomorphic self-map φ of $\mathbb{B}_N$ induces a bounded composition operator C_φ on $H^\infty(\mathbb{B}_N)$. Given $M \geq 2$, let $a_1, \dots, a_M \in \mathbb{C} \setminus \{0\}$ and $\varphi_1, \dots, \varphi_M$ be holomorphic self-maps of $\mathbb{B}_N$. If $\varphi_i(\mathbb{B}_N) \subseteq r\mathbb{B}_N$ for some positive number $r < 1$, then C_{φ_i} is compact on $H^\infty(\mathbb{B}_N)$. We may exclude such trivial ones from our linear sums and assume that $\sup_{z\in\mathbb{B}_N} |\varphi_i(z)| = 1$ for each i throughout this subsection. Denote by $\mathcal{Z} := \mathcal{Z}(\varphi_1, \dots, \varphi_M)$ the family of sequences $\{z^{(j)}\}$ in $\mathbb{B}_N$ satisfying the following conditions:

(a) $|\varphi_i(z^{(j)})| \to 1$ as $j \to \infty$ for some i;

(b) $\{\varphi_i(z^{(j)})\}$ converges for each i;

(c) $\{\beta(\varphi_i(z^{(j)}), \varphi_k(z^{(j)})\}$ is a convergent sequence for every i, k.

By our hypothesis, there is $\{z^{(j)}\} \in \mathcal{Z}$, and then we write

$$I(\{z^{(j)}\}) = \{i : 1 \leq i \leq N, |\varphi_i(z^{(j)})| \to 1 \text{ as } j \to \infty\}.$$

Clearly $I(\{z^{(j)}\}) \neq \emptyset$ by (a). By (b), for each $m \notin I(\{z^{(j)}\})$, there exists a positive constant $\delta < 1$ such that $|\varphi_m(z^{(j)})| \leq \delta$. Given $t \in I(\{z^{(j)}\})$, we write

$$I_0(\{z^{(j)}\}, t) = \{i \in I(\{z^{(j)}\}) : \beta(\varphi_t(z^{(j)}), \varphi_i(z^{(j)})) \to 0 \text{ as } j \to \infty\}.$$

Then it is easy to see that

$$I_0(\{z^{(j)}\}, t_1) = I_0(\{z^{(j)}\}, t_2) \quad \text{or} \quad I_0(\{z^{(j)}\}, t_1) \cap I_0(\{z^{(j)}\}, t_2) = \emptyset$$

for $t_1, t_2 \in I(\{z^{(j)}\})$. Hence there is a subset $\{t_1, \dots, t_\ell\} \subseteq I(\{z^{(j)}\})$ such that

$$I(\{z^{(j)}\}) = \bigcup_{k=1}^{\ell} I_0(\{z^{(j)}\}, t_k)$$

and $I_0(\{z^{(j)}\}, t_i) \cap I_0(\{z^{(j)}\}, t_m) = \emptyset$ for $i \neq m$. Under these notations, we can characterize completely the compactness of linear sums of composition operators on $H^\infty(\mathbb{B}_N)$ as follows.

Theorem 3.1 *Under the above notation and definitions,* $\sum_{i=1}^M a_i C_{\varphi_i}$ *is compact on* $H^\infty(\mathbb{B}_N)$ *if and only if* $\sum_{i\in I_0(\{z^{(j)}\},t)} a_i = 0$ *for every* $\{z^{(j)}\} \in \mathcal{Z}$ *and* $t \in I(\{z^{(j)}\})$.

Proof Suppose that $\sum_{i=1}^M a_i C_{\varphi_i}$ is compact on $H^\infty(\mathbb{B}_N)$. Let $\{z^{(j)}\} \in \mathcal{Z}$ and $t \in I(\{z^{(j)}\})$. Then we set $\{k_1, \dots, k_\ell\} = \{1, \dots, M\} \setminus I_0(\{z^{(j)}\}, t)$. So, for each $1 \leq m \leq \ell$, considering subsequences of $\{z^{(j)}\}$, there is some $\delta > 0$ such that $\beta(\varphi_{k_m}(z^{(j)}), \varphi_i(z^{(j)})) \geq \delta$ for any $i \in I_0(\{z^{(j)}\}, t)$ and j large enough. Meanwhile, for every $i \in I_0(\{z^{(j)}\}, t)$, then $\beta(\varphi_t(z^{(j)}), \varphi_i(z^{(j)})) \to 0$, and thus $d_\infty(\varphi_t(z^{(j)}), \varphi_i(z^{(j)})) \to 0$ according to Lemma 2.1(a) and $\lim_{t\to 0} \frac{2-2\sqrt{1-t^2}}{t} = 0$.

For each j and $k_m \notin I_0(\{z^{(j)}\}, t)$, let

$$\epsilon_j^{(k_m)} := \left| \frac{\varphi_{k_m}(z^{(j)}) - P_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)}))}{1 - \langle P_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)})), \varphi_{k_m}(z^{(j)})\rangle} \right|.$$

Considering subsequences of $\{z^{(j)}\}$, we may assume that $\{\epsilon_j^{(k_m)}\}$ converges in the sequel. By (2.1), $P_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)})) \in \Phi_{\varphi_{k_m}(z^{(j)})}(\epsilon_j^{(k_m)}\mathbb{B}_N) \cap [\varphi_{k_m}(z^{(j)})]$. Thus

$$\left|P_{\varphi_{k_m}(z^{(j)})}\left(\varphi_t\left(z^{(j)}\right)\right) - \varphi_{k_m}\left(z^{(j)}\right)\right| \leq 2\epsilon_j^{(k_m)}\rho_{\varphi_{k_m}(z^{(j)})\epsilon_j^{(k_m)}};$$

and it is on the order of $\epsilon_j^{(k_m)}(1 - |\varphi_{k_m}(z^{(j)})|^2)$. Define the functions

$$g_{k_m}^{(j)}(z) := \frac{\langle z, Q^e_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)}))\rangle^2}{1 - \langle z, \varphi_{k_m}(z^{(j)})\rangle},$$

where we set the unit vector

$$Q^e_{\varphi_{k_m}(z^{(j)})}\left(\varphi_t\left(z^{(j)}\right)\right) := \frac{Q_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)}))}{|Q_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)}))|}.$$

If $\{\epsilon_j^{(k_m)}\}$ tends to zero for some $k_m \notin I_0(\{z^{(j)}\}, t)$, then it follows from the proof of [19, Lemma 10] that the following statements hold:

(i) $\|g_{k_m}^{(j)}\|_\infty \leq 2$;

(ii) $g_{k_m}^{(j)}(\varphi_{k_m}(z^{(j)})) = 0$ and

$$\liminf_{j\to\infty}\left|g_{k_m}^{(j)}\left(\varphi_t\left(z^{(j)}\right)\right)\right| = \liminf_{j\to\infty}\frac{|Q_{\varphi_{k_m}(z^{(j)})}(\varphi_t(z^{(j)}))|^2}{|1 - \langle\varphi_t(z^{(j)}), \varphi_{k_m}(z^{(j)})\rangle|} \geq \delta^2.$$

Furthermore,

(iii) for every $i \in I_0(\{z^{(j)}\}, t)$,

$$\lim_{j\to\infty} g_{k_m}^{(j)}\left(\varphi_i\left(z^{(j)}\right)\right) = \lim_{j\to\infty} g_{k_m}^{(j)}\left(\varphi_t\left(z^{(j)}\right)\right) := \alpha_{k_m,t} \neq 0.$$

That is deduced by the fact that

$$\left|g_{k_m}^{(j)}\left(\varphi_i\left(z^{(j)}\right)\right) - g_{k_m}^{(j)}\left(\varphi_t\left(z^{(j)}\right)\right)\right| \leq d_\infty\left(\varphi_{k_m}\left(z^{(j)}\right), \varphi_i\left(z^{(j)}\right)\right) \to 0,$$

and the limit $\alpha_{k_m,t} \neq 0$ by (ii).

On the other hand, if $\{\epsilon_j^{(k'_m)}\}$ does not tend to zero for some $k'_m \notin I_0(\{z^{(j)}\}, t)$, then there is some positive constant σ such that

$$\left|\frac{\varphi_{k'_m}(z^{(j)}) - P_{\varphi_{k'_m}(z^{(j)})}(\varphi_t(z^{(j)}))}{1 - \langle\varphi_t(z^{(j)}), \varphi_{k'_m}(z^{(j)})\rangle}\right| > \sigma$$

for j large enough. Now we define the functions

$$h_{k'_m}^{(j)}(z) := \frac{\langle\varphi_{k'_m}(z^{(j)}) - P_{\varphi_{k'_m}(z^{(j)})}(z), \frac{\varphi_{k'_m}(z^{(j)})}{|\varphi_{k'_m}(z^{(j)})|}\rangle}{1 - \langle z, \varphi_{k'_m}(z^{(j)})\rangle}.$$

Similarly, the following statements can be obtained:

(i′) $\|h^{(j)}_{k'_m}\|_\infty \le 2$;
(ii′) $h^{(j)}_{k'_m}(\varphi_{k'_m}(z^{(j)})) = 0$ and

$$\liminf_{j\to\infty}\left|h^{(j)}_{k'_m}\left(\varphi_t\left(z^{(j)}\right)\right)\right| = \liminf_{j\to\infty}\left|\frac{\varphi_{k'_m}(z^{(j)}) - P_{\varphi_{k'_m}(z^{(j)})}(\varphi_t(z^{(j)}))}{1-\langle\varphi_t(z^{(j)}),\varphi_{k'_m}(z^{(j)})\rangle}\right| \ge \sigma;$$

(iii′) for every $i \in I_0(\{z^{(j)}\},t)$,

$$\lim_{j\to\infty} h^{(j)}_{k'_m}\left(\varphi_i\left(z^{(j)}\right)\right) = \lim_{j\to\infty} h^{(j)}_{k'_m}\left(\varphi_t\left(z^{(j)}\right)\right) := \beta_{k'_m,t} \ne 0.$$

Without loss of generality, let $\{k_p : 1 \le p \le \ell_0 \le \ell\}$ be the set of all $k_p \notin I_0(\{z^{(j)}\},t)$ such that $\epsilon_j^{(k_p)} \to 0$ as $j \to \infty$. For an arbitrary positive integer j, define

$$f_j(z) := \frac{1-|\varphi_t(z^{(j)})|^2}{1-\langle z,\varphi_t(z^{(j)})\rangle}\prod_{p=1}^{\ell_0} g^{(j)}_{k_p}(z)\cdot\prod_{q=\ell_0+1}^{\ell} h^{(j)}_{k_q}(z). \tag{3.1}$$

Then $f_j \in H^\infty(\mathbb{B}_N)$ with $\|f_j\|_\infty \le 2^M$, and $\{f_j\}$ converges uniformly to zero on compact subsets of $\mathbb{B}_N$. Also $f_j(\varphi_{k_m}(z^{(j)})) = 0$ for every $1 \le m \le \ell$. Therefore

$$\begin{aligned}\left\|\sum_{i=1}^M a_i C_{\varphi_i} f_j\right\|_\infty &\ge \left|\sum_{i=1}^M a_i f_j\left(\varphi_i\left(z^{(j)}\right)\right)\right|\\ &= \left|\sum_{i\in I_0(\{s_j\},t)} a_i\frac{1-|\varphi_t(z^{(j)})|^2}{1-\langle\varphi_i(z^{(j)}),\varphi_t(z^{(j)})\rangle}\prod_{p=1}^{\ell_0} g^{(j)}_{k_p}\left(\varphi_i\left(z^{(j)}\right)\right)\cdot\prod_{q=\ell_0+1}^{\ell} h^{(j)}_{k_q}\left(\varphi_i\left(z^{(j)}\right)\right)\right|.\end{aligned}$$

Note that

$$\frac{1-|\varphi_t(z^{(j)})|^2}{1-\langle\varphi_i(z^{(j)}),\varphi_t(z^{(j)})\rangle} = 1 + \frac{\langle\varphi_i(z^{(j)})-\varphi_t(z^{(j)}),\varphi_t(z^{(j)})\rangle}{1-\langle\varphi_i(z^{(j)}),\varphi_t(z^{(j)})\rangle}$$

and

$$\begin{aligned}\frac{|\langle\varphi_i(z^{(j)})-\varphi_t(z^{(j)}),\varphi_t(z^{(j)})\rangle|}{|1-\langle\varphi_i(z^{(j)}),\varphi_t(z^{(j)})\rangle|} &= \left|\varphi_t\left(z^{(j)}\right)\right|\left|\frac{P_{\varphi_t(z^{(j)})}(\varphi_i(z^{(j)}))-\varphi_t(z^{(j)})}{1-\langle\varphi_i(z^{(j)}),\varphi_t(z^{(j)})\rangle}\right|\\ &\approx \left|\frac{P_{\varphi_t(z^{(j)})}(\varphi_i(z^{(j)}))-\varphi_t(z^{(j)})}{1-\langle P_{\varphi_t(z^{(j)})}(\varphi_i(z^{(j)})),\varphi_t(z^{(j)})\rangle}\right|\end{aligned}$$

because of $|\varphi_t(z^{(j)})| \to 1$ as $j \to \infty$. By (2.1) and Lemma 2.1(b), for arbitrary $\varepsilon > 0$,

$$\begin{aligned}&\left\{\varphi_i\left(z^{(j)}\right) : \beta\left(\varphi_i\left(z^{(j)}\right),\varphi_t\left(z^{(j)}\right)\right) < \varepsilon\right\} \cap \left[\varphi_t\left(z^{(j)}\right)\right]\\ &\quad= \left\{P_{\varphi_t(z^{(j)})}\left(\varphi_i\left(z^{(j)}\right)\right) : \left|\frac{P_{\varphi_t(z^{(j)})}(\varphi_i(z^{(j)}))-\varphi_t(z^{(j)})}{1-\langle P_{\varphi_t(z^{(j)})}(\varphi_i(z^{(j)})),\varphi_t(z^{(j)})\rangle}\right| < \varepsilon\right\}.\end{aligned}$$

Replacing ε by positive numbers ε_j tending to zero, we have

$$\lim_{j\to\infty}\frac{1-|\varphi_t(z^{(j)})|^2}{1-\langle\varphi_i(z^{(j)}),\varphi_t(z^{(j)})\rangle} = 1.$$

Therefore

$$\begin{aligned}\lim_{j\to\infty}\left\|\sum_{i=1}^{M}a_iC_{\varphi_i}f_j\right\|_\infty &\geq \left|\sum_{i\in I_0(\{s_j\},t)} a_i\prod_{p=1}^{\ell_0}\alpha_{k_p,t}\cdot\prod_{q=\ell_0+1}^{\ell}\beta_{k_q,t}\right| \\ &= \prod_{p=1}^{\ell_0}|\alpha_{k_p,t}|\cdot\prod_{q=\ell_0+1}^{\ell}|\beta_{k_q,t}|\cdot\left|\sum_{i\in I_0(\{z^{(j)}\},t)} a_i\right|,\end{aligned}$$

which indicates that $\sum_{i\in I_0(\{z^{(j)}\},t)} a_i = 0$.

Conversely, suppose that $\sum_{i=1}^{M} a_iC_{\varphi_i}$ is not compact on $H^\infty(\mathbb{B}_N)$. Then there exists a sequence $\{g_j\}\subseteq H^\infty(\mathbb{B}_N)$ with $\|g_j\|_\infty\leq 1$ such that it converges uniformly to zero on every compact subset of $\mathbb{B}_N$, and

$$\left\|\sum_{i=1}^{M} a_ig_j\circ\varphi_i\right\|_\infty \nrightarrow 0 \quad \text{as } j\to\infty.$$

Then, for some constant $\varepsilon_0>0$, we can take $z^{(j)}\in\mathbb{B}_N$ such that $|z^{(j)}|\to 1$, and

$$\left|\sum_{i=1}^{M} a_ig_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr)\right|>\varepsilon_0.$$

Considering subsequences of $\{z^{(j)}\}$, we may assume that $|\varphi_i(z^{(j)})|\to\alpha_i$ with $\alpha_i\geq 0$, as $j\to\infty$ for every i. Also $\{g_j\}$ converges uniformly to zero on every compact subset of $\mathbb{B}_N$, so $\alpha_i=1$ for some i. Now we can assume that $\{z^{(j)}\}\in\mathcal{Z}$. And we get

$$\liminf_{j\to\infty}\left|\sum_{i\in I(\{z^{(j)}\})} a_ig_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr)\right|\geq\varepsilon_0. \tag{3.2}$$

Note that $\{g_j(\varphi_i(z^{(j)}))\}$ is bounded, and then considering a subsequence of $\{z^{(j)}\}$, we may assume that $g_j(\varphi_i(z^{(j)}))\to\xi_i$ as $j\to\infty$ for every i. Recall that there is a subset $\{t_1,\dots,t_\ell\}\subseteq I(\{z^{(j)}\})$ such that

$$I\bigl(\bigl\{z^{(j)}\bigr\}\bigr)=\bigcup_{k=1}^{\ell} I_0\bigl(\bigl\{z^{(j)}\bigr\},t_k\bigr)$$

and $I_0(\{z^{(j)}\},t_q)\cap I_0(\{z^{(j)}\},t_m)=\emptyset$ for $q\neq m$. For $i\in I_0(\{z^{(j)}\},t_m)$, due to the part (a) of Lemma 2.1 and $\lim_{t\to 0}\frac{2-2\sqrt{1-t^2}}{t}=0$, we have

$$\bigl|g_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr)-g_j\bigl(\varphi_{t_m}\bigl(z^{(j)}\bigr)\bigr)\bigr|\leq d_\infty\bigl(\varphi_i\bigl(z^{(j)}\bigr),\varphi_{t_m}\bigl(z^{(j)}\bigr)\bigr)\to 0$$

as $j\to\infty$, which shows that $\xi_i=\xi_{t_m}$. Hence

$$\begin{aligned}\lim_{j\to\infty}\sum_{i\in I(\{z^{(j)}\})} a_ig_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr) &= \lim_{j\to\infty}\sum_{m=1}^{\ell}\sum_{i\in I_0(\{z^{(j)}\},t_m)} a_ig_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr) \\ &= \sum_{m=1}^{\ell}\xi_{t_m}\Biggl(\sum_{i\in I_0(\{z^{(j)}\},t_m)} a_i\Biggr)=0\end{aligned}$$

by the hypothesis $\sum_{i\in I_0(\{z^{(j)}\},t)} a_i = 0$. This contradicts (3.2). Hence the proof is complete. □

The following corollaries can be obtained immediately from Theorem 3.1.

Corollary 3.2 *If $\sum_{i\in J} a_i \neq 0$ for every subset J of $\{1,2,\dots,M\}$, then $\sum_{i=1}^M a_iC_{\varphi_i}$ is not compact on $H^\infty(\mathbb{B}_N)$.*

Corollary 3.3 *Suppose that $\sum_{i=1}^M a_i = 0$ and $\sum_{i\in J} a_i \neq 0$ for every nonempty proper subset J of $\{1,2,\dots,M\}$. Then $\sum_{i=1}^M a_iC_{\varphi_i}$ is compact on $H^\infty(\mathbb{B}_N)$ if and only if $C_{\varphi_i} - C_{\varphi_j}$ is compact on $H^\infty(\mathbb{B}_N)$ for every i,j with $i\neq j$.*

As is well known, C_φ is compact on $H^\infty(\mathbb{B}_N)$ if and only if

$$\lim_{j\to\infty}\sup_{\xi\in\partial\mathbb{B}_N}\bigl\|\langle\varphi,\xi\rangle^j\bigr\|_\infty = 0.$$

Motivated by [17], we give another criterion for the compactness of linear combination of composition operators $\sum_{i=1}^M a_iC_{\varphi_i}$ on $H^\infty(\mathbb{B}_N)$ by polynomials.

Theorem 3.4 *For arbitrary $a_1,\dots,a_M \in \mathbb{C}\backslash\{0\}$ and holomorphic self-maps $\varphi_1,\dots,\varphi_M$ of $\mathbb{B}_N$, $\sum_{i=1}^M a_iC_{\varphi_i}$ is compact on $H^\infty(\mathbb{B}_N)$ if and only if*

$$\sup_{0\le\ell\le N}\ \sup_{\frac{1}{\ell+1}\sum_{m=0}^{\ell}\sigma_m>L}\ \sup_{\xi,\eta_i\in\partial\mathbb{B}_N}\left\|\sum_{i=1}^M a_i\left(\langle\varphi_i,\xi\rangle^{\sigma_0}\prod_{p=1}^{\ell_0}\langle\varphi_i,\eta_p\rangle^{\sigma_p}\prod_{q=\ell_0+1}^{\ell}\langle\varphi_i,\eta_q\rangle^{\sigma_q}\right)\right\|_\infty$$

tends to zero, as $L\to\infty$.

Proof For simplicity, denote $T := \sum_{i=1}^M a_iC_{\varphi_i}$. For arbitrary positive integers σ_m $(0\le m\le N)$, $0\le\ell_0\le\ell\le N$, and $\xi,\eta_i\in\partial\mathbb{B}_N$, we denote $\overline{\sigma}_\ell := \frac{1}{\ell+1}\sum_{m=0}^{\ell}\sigma_m$, $\eta := (\eta_1,\dots,\eta_\ell)$. Then we define the family of functions:

$$F_{\overline{\sigma}_\ell}^{(\xi,\eta)}(z) := \langle z,\xi\rangle^{\sigma_0}\prod_{p=1}^{\ell_0}\langle z,\eta_p\rangle^{\sigma_p}\prod_{q=\ell_0+1}^{\ell}\langle z,\eta_q\rangle^{\sigma_q}.$$

Note that $\|F_{\overline{\sigma}_\ell}^{(\xi,\eta)}\|_\infty\le 1$ and $F_{\overline{\sigma}_\ell}^{(\xi,\eta)}$ converges uniformly to zero on every compact subset of $\mathbb{B}_N$, as $\overline{\sigma}_\ell\to\infty$. If T is compact on $H^\infty(\mathbb{B}_N)$, then we have

$$\lim_{L\to\infty}\sup_{0\le\ell\le N}\sup_{\overline{\sigma}_\ell>L}\sup_{\xi,\eta_i\in\partial\mathbb{B}_N}\bigl\|TF_{\overline{\sigma}_\ell}^{(\xi,\eta)}\bigr\|_\infty = 0,$$

following from an argument similar to the proof of Theorem 3.2 in [8].

To prove the sufficiency, we continue to use the same notation in the proof of Theorem 3.1. A brief retrospective analysis has proved that the following statements are equivalent:

(1) $\sum_{i=1}^M a_iC_{\varphi_i}$ is compact;
(2) $\lim_{j\to\infty}\|Tf_j\|_\infty = 0$ for the functions f_j given by (3.1);
(3) $\sum_{i\in I_0(\{z^{(j)}\},t)} a_i = 0$ for every $\{z^{(j)}\}\in\mathcal{Z}$ and $t\in I(\{z^{(j)}\})$.

To end the proof, we first compute

$$
\begin{aligned}
Tf_j(z) = {} & \sum_{m=1}^{M} a_m \bigl(1 - \bigl|\varphi_t\bigl(z^{(j)}\bigr)\bigr|^2\bigr) \sum \bigl|\varphi_t\bigl(z^{(j)}\bigr)\bigr|^k C_{\varphi_m}\biggl(\biggl\langle z, \frac{\varphi_t(z^{(j)})}{|\varphi_t(z^{(j)})|}\biggr\rangle^k\biggr) \\
& \times \prod_{p=1}^{\ell_0} \sum \bigl|\varphi_{k_p}\bigl(z^{(j)}\bigr)\bigr|^k C_{\varphi_m}\biggl(\bigl\langle z, Q^e_{\varphi_{k_p}(z^{(j)})}\bigl(\varphi_t\bigl(z^{(j)}\bigr)\bigr)\bigr\rangle^2 \biggl\langle z, \frac{\varphi_{k_p}(z^{(j)})}{|\varphi_{k_p}(z^{(j)})|}\biggr\rangle^k\biggr) \\
& \times \prod_{q=\ell_0+1}^{\ell} \sum \bigl|\varphi_{k_q}\bigl(z^{(j)}\bigr)\bigr|^k C_{\varphi_m}\biggl(\biggl(\bigl|\varphi_{k_q}\bigl(z^{(j)}\bigr)\bigr| - \biggl\langle z, \frac{\varphi_{k_q}(z^{(j)})}{|\varphi_{k_q}(z^{(j)})|}\biggr\rangle\biggr)\biggl\langle z, \frac{\varphi_{k_q}(z^{(j)})}{|\varphi_{k_q}(z^{(j)})|}\biggr\rangle^k\biggr).
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \bigl\| \langle \varphi_m, \zeta \rangle^k \bigr\|_\infty \le 1, \\
& \bigl\| C_{\varphi_m} \bigl\langle z, Q^e_{\varphi_{k_p}(z^{(j)})}\bigl(\varphi_t\bigl(z^{(j)}\bigr)\bigr)\bigr\rangle^2 \bigr\|_\infty \le 1, \\
& \biggl\| \bigl|\varphi_{k_q}\bigl(z^{(j)}\bigr)\bigr| - C_{\varphi_m}\biggl\langle z, \frac{\varphi_{k_q}(z^{(j)})}{|\varphi_{k_q}(z^{(j)})|}\biggr\rangle \biggr\|_\infty \le 2,
\end{aligned}
$$

for each $\zeta \in \partial\mathbb{B}_N$, m, k, j, and $1 \le p \le \ell_0 < q \le \ell \le N$. For each $L \in \mathbb{N}$, we have

$$
\varepsilon_j^{(L)} := \bigl(1 - \bigl|\varphi_t\bigl(z^{(j)}\bigr)\bigr|^2\bigr) \sum_{k=0}^{L} \bigl|\varphi_t\bigl(z^{(j)}\bigr)\bigr|^k \times \prod_{p=1}^{\ell_0} \sum_{k=0}^{L} \bigl|\varphi_{k_p}\bigl(z^{(j)}\bigr)\bigr|^k \times \prod_{q=\ell_0+1}^{\ell} \sum_{k=0}^{L} \bigl|\varphi_{k_q}\bigl(z^{(j)}\bigr)\bigr|^k
$$

goes to 0, because of $|\varphi_t(z^{(j)})| \to 1$, as $j \to \infty$. Clearly, there exists a positive constant $\delta < 1$ such that $|\varphi_{k_m}(z^{(j)})| \le \delta$ for each $k_m \in \{1, \ldots, M\} \setminus I_0(\{z^{(j)}\}, t)$, so for j large enough, we have

$$
\begin{aligned}
\|Tf_j\|_\infty \lesssim {} & \varepsilon_j^{(L)} + \bigl(1 - \bigl|\varphi_t\bigl(z^{(j)}\bigr)\bigr|^2\bigr) \sum \bigl|\varphi_t\bigl(z^{(j)}\bigr)\bigr|^k \times \prod_{p=1}^{\ell_0} \sum \bigl|\varphi_{k_p}\bigl(z^{(j)}\bigr)\bigr|^k \\
& \times \prod_{q=\ell_0+1}^{\ell} \sum \bigl|\varphi_{k_q}\bigl(z^{(j)}\bigr)\bigr|^k \times \sup_{\overline{\sigma}_\ell > L} \sup_{\xi, \eta_i \in \partial\mathbb{B}_N} \bigl\| TF^{(\xi,\eta)}_{\overline{\sigma}_\ell} \bigr\|_\infty \\
\le {} & \varepsilon_j^{(L)} + \frac{2}{(1-\delta)^\ell} \times \sup_{0 \le \ell \le N} \sup_{\overline{\sigma}_\ell > L} \sup_{\xi, \eta_i \in \partial\mathbb{B}_N} \bigl\| TF^{(\xi,\eta)}_{\overline{\sigma}_\ell} \bigr\|_\infty \\
\lesssim {} & \varepsilon_j^{(L)} + \sup_{0 \le \ell \le N} \sup_{\overline{\sigma}_\ell > L} \sup_{\xi, \eta_i \in \partial\mathbb{B}_N} \bigl\| TF^{(\xi,\eta)}_{\overline{\sigma}_\ell} \bigr\|_\infty,
\end{aligned}
$$

where $\overline{\sigma}_\ell := \frac{1}{\ell+1} \sum_{m=0}^{\ell} \sigma_m$. Therefore, we get

$$
\limsup_{j\to\infty} \|Tf_j\|_\infty \lesssim \limsup_{L\to\infty} \sup_{0 \le \ell \le N} \sup_{\overline{\sigma}_\ell > L} \sup_{\xi, \eta_i \in \partial\mathbb{B}_N} \bigl\| TF^{(\xi,\eta)}_{\overline{\sigma}_\ell} \bigr\|_\infty = 0.
$$

So T is compact, which completes the proof. □

3.2 Compactness on $H^\infty(\mathbb{D}^N)$

Our aim of this subsection is to extend the above results to the space $H^\infty(\mathbb{D}^N)$. Let $M \ge 2$, $a_1, \ldots, a_M \in \mathbb{C} \setminus \{0\}$, and let $\{\varphi_j = (\varphi_1^{(j)}, \ldots, \varphi_N^{(j)})\}_{j=1}^M$ be holomorphic self-maps of $\mathbb{D}^N$. For given j, if $\|\varphi_k^{(j)}\|_\infty < 1$ for all k, then C_{φ_j} is compact on $H^\infty(\mathbb{D}^N)$. We may exclude such trivial ones from our linear combinations, and assume that $\max\{\|\varphi_1^{(j)}\|_\infty, \dots, \|\varphi_N^{(j)}\|_\infty\} = 1$ for each j, unless otherwise specified in this subsection. Given $\lambda \in \{1, \dots, N\}$, denote by $\mathcal{Z}_\lambda := \mathcal{Z}_\lambda(\varphi_1, \dots, \varphi_M)$ the family of sequences $\{z^{(j)}\}$ in $\mathbb{D}^N$ satisfying the following conditions:

(a) $|\varphi_\lambda^{(i)}(z^{(j)})| \to 1$ as $j \to \infty$ for some i;
(b) $\{\varphi_\lambda^{(i)}(z^{(j)})\}$ converges for each i;
(c) $\{\rho(\varphi_\lambda^{(i)}(z^{(j)}), \varphi_\lambda^{(k)}(z^{(j)}))\}$ is a convergent sequence for every i, k.

Note that there exists some λ such that $\mathcal{Z}_\lambda \neq \emptyset$. For $\{z^{(j)}\} \in \mathcal{Z}_\lambda$, we define

$$I^{(\lambda)}(\{z^{(j)}\}) = \{i : 1 \le i \le M, |\varphi_\lambda^{(i)}(z^{(j)})| \to 1 \text{ as } j \to \infty\}.$$

By (b), for each $m \notin I^{(\lambda)}(\{z^{(j)}\})$, there exists a positive constant $\delta < 1$ such that $|\varphi_\lambda^{(m)}(z^{(j)})| \le \delta$. Given $t \in I^{(\lambda)}(\{z^{(j)}\})$, we write

$$I_0^{(\lambda)}(\{z^{(j)}\}, t) = \{i \in I^{(\lambda)}(\{z^{(j)}\}) : \rho(\varphi_\lambda^{(t)}(z^{(j)}), \varphi_\lambda^{(i)}(z^{(j)})) \to 0 \text{ as } j \to \infty\}.$$

Also there is a subset $\{t_1, \dots, t_\ell\} \subseteq I^{(\lambda)}(\{z^{(j)}\})$ such that

$$I^{(\lambda)}(\{z^{(j)}\}) = \bigcup_{k=1}^{\ell} I_0^{(\lambda)}(\{z^{(j)}\}, t_k)$$

and $I_0^{(\lambda)}(\{z^{(j)}\}, t_i) \cap I_0^{(\lambda)}(\{z^{(j)}\}, t_m) = \emptyset$ for $i \neq m$. Furthermore, we set

$$J_0^{(\lambda)}(\{z^{(j)}\}, t) = \Big\{i \in I_0^{(\lambda)}(\{z^{(j)}\}, t) : \lim_{j\to\infty} \rho_N(\varphi_t(z^{(j)}), \varphi_i(z^{(j)})) \to 0\Big\}.$$

Now we determine when linear combinations of composition operators are compact on the space $H^\infty(\mathbb{D}^N)$.

Theorem 3.5 *Under the notation above, the following statements hold*:

(1) *If* $\sum_{i=1}^M a_i C_{\varphi_i}$ *is compact on* $H^\infty(\mathbb{D}^N)$, *then* $\sum_{i \in I_0^{(\lambda)}(\{z^{(j)}\}, t)} a_i = 0$, *whenever* $\{z^{(j)}\} \in \mathcal{Z}_\lambda$ *and* $t \in I^{(\lambda)}(\{z^{(j)}\})$ *for each* $\lambda \in \{1, \dots, N\}$.
(2) *If* $J_0^{(\lambda)}(\{z^{(j)}\}, t) = I_0^{(\lambda)}(\{z^{(j)}\}, t)$ *and* $\sum_{i \in I_0^{(\lambda)}(\{z^{(j)}\}, t)} a_i = 0$ *whenever* $\{z^{(j)}\} \in \mathcal{Z}_\lambda$ *and* $t \in I^{(\lambda)}(\{z^{(j)}\})$ *for each* $\lambda \in \{1, \dots, N\}$, *then* $\sum_{i=1}^M a_i C_{\varphi_i}$ *is compact on* $H^\infty(\mathbb{D}^N)$.

Proof (1) Given $\lambda \in \{1, \dots, N\}$, let $\{z^{(j)}\} \in \mathcal{Z}_\lambda$ and $t \in I^{(\lambda)}(\{z^{(j)}\})$. Then we can assume that $\{k_1, \dots, k_\ell\} = \{1, \dots, M\} \setminus I_0^{(\lambda)}(\{z^{(j)}\}, t)$. So, for each $1 \le m \le \ell$, considering subsequences of $\{z^{(j)}\}$, we may say that

$$\lim_{j\to\infty} \rho(\varphi_\lambda^{(k_m)}(z^{(j)}), \varphi_\lambda^{(t)}(z^{(j)})) := \beta_{k_m,t} \neq 0.$$

For every $i \in I_0^{(\lambda)}(\{z^{(j)}\}, t)$, it is easy to see that

$$\lim_{j\to\infty} \rho(\varphi_\lambda^{(k_m)}(z^{(j)}), \varphi_\lambda^{(i)}(z^{(j)})) = \lim_{j\to\infty} \rho(\varphi_\lambda^{(k_m)}(z^{(j)}), \varphi_\lambda^{(t)}(z^{(j)}))$$

by the well-known triangle inequality

$$\left|\rho\left(\varphi_\lambda^{(k_m)}\left(z^{(j)}\right),\varphi_\lambda^{(i)}\left(z^{(j)}\right)\right)-\rho\left(\varphi_\lambda^{(k_m)}\left(z^{(j)}\right),\varphi_\lambda^{(t)}\left(z^{(j)}\right)\right)\right|\le\rho\left(\varphi_\lambda^{(t)}\left(z^{(j)}\right),\varphi_\lambda^{(i)}\left(z^{(j)}\right)\right).$$

For an arbitrary positive integer j, define

$$f_j^{(\lambda)}(z)=\frac{1-|\varphi_\lambda^{(t)}(z^{(j)})|^2}{1-\overline{\varphi_\lambda^{(t)}(z^{(j)})}z_\lambda}\prod_{m=1}^{\ell}\frac{\varphi_\lambda^{(k_m)}(z^{(j)})-z_\lambda}{1-\overline{\varphi_\lambda^{(k_m)}(z^{(j)})}z_\lambda},$$

where $k_m\notin I_0^{(\lambda)}(\{z^{(j)}\},t)$. Then clearly $f_j^{(\lambda)}\in H^\infty(\mathbb{D})\subseteq H^\infty(\mathbb{D}^N)$ with $\|f_j^{(\lambda)}\|_\infty\le 2$, and $\{f_j^{(\lambda)}\}$ converges uniformly to zero on every compact subset of $\mathbb{D}$. Therefore

$$\begin{aligned}\left\|\sum_{i=1}^{M}a_iC_{\varphi_i}f_j^{(\lambda)}\right\|_\infty&\ge\left|\sum_{i=1}^{M}a_if_j^{(\lambda)}\left(\varphi_i\left(z^{(j)}\right)\right)\right|\\&=\left|\sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\},t)}a_i\frac{1-|\varphi_\lambda^{(t)}(z^{(j)})|^2}{1-\overline{\varphi_\lambda^{(t)}(z^{(j)})}\varphi_\lambda^{(i)}(z^{(j)})}\prod_{m=1}^{\ell}\frac{\varphi_\lambda^{(k_m)}(z^{(j)})-\varphi_\lambda^{(i)}(z^{(j)})}{1-\overline{\varphi_\lambda^{(k_m)}(z^{(j)})}\varphi_\lambda^{(i)}(z^{(j)})}\right|.\end{aligned}$$

Clearly

$$\lim_{j\to\infty}\frac{1-|\varphi_\lambda^{(t)}(z^{(j)})|^2}{1-\overline{\varphi_\lambda^{(t)}(z^{(j)})}\varphi_\lambda^{(i)}(z^{(j)})}=1$$

for each $i\in I_0^{(\lambda)}(\{z^{(j)}\},t)$. Now we get

$$\begin{aligned}\lim_{j\to\infty}\left\|\sum_{i=1}^{M}a_iC_{\varphi_i}f_j^{(\lambda)}\right\|_\infty&\ge\left|\sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\},t)}a_i\prod_{m=1}^{\ell}\beta_{k_m,t}\right|\\&=\prod_{m=1}^{\ell}|\beta_{k_m,t}|\left|\sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\},t)}a_i\right|,\end{aligned}$$

and then $\sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\},t)}a_i=0$ due to the compactness of $\sum_{i=1}^{M}a_iC_{\varphi_i}$. Thus statement (1) is obtained.

(2) Here we argue by contradiction, and suppose that $\sum_{i=1}^{M}a_iC_{\varphi_i}$ is not compact on $H^\infty(\mathbb{D}^N)$. Then there exists a sequence $\{g_j\}\subseteq H^\infty(\mathbb{D}^N)$ with $\|g_j\|_\infty\le 1$ such that it converges uniformly to zero on each compact subset of $\mathbb{D}^N$, whereas

$$\left\|\sum_{i=1}^{M}a_ig_j\circ\varphi_i\right\|_\infty\nrightarrow 0\quad\text{as } j\to\infty.$$

For some constant $\varepsilon_0>0$, then we can take $z^{(j)}\in\mathbb{D}^N$ with $|z^{(j)}|_{\max}\to 1$ and

$$\left|\sum_{i=1}^{M}a_ig_j\left(\varphi_i\left(z^{(j)}\right)\right)\right|>\varepsilon_0.$$

Considering subsequences of $\{z^{(j)}\}$, we may assume that $\varphi_i(z^{(j)}) \to \alpha_i$ as $j \to \infty$ for every i. Also $\{g_j\}$ converges uniformly to zero on each compact subset of $\mathbb{D}^N$, so $|\alpha_i|_{\max} = 1$ for some i. Now we may say that $\{z^{(j)}\} \in \bigcup_{\lambda=1}^{N} \mathcal{Z}_\lambda$. And we get

$$\liminf_{j\to\infty} \left| \sum_{i\in\bigcup_{\lambda=1}^{N} I^{(\lambda)}(\{z^{(j)}\})} a_i g_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr) \right| \geq \varepsilon_0. \tag{3.3}$$

Note that $\{g_j(\varphi_i(z^{(j)}))\}_{j=1}^{\infty}$ is bounded, and then considering a subsequence of $\{z^{(j)}\}$, we may assume that $g_j(\varphi_i(z^{(j)})) \to \xi^{(i)}$ as $j \to \infty$ for every i. Without loss of generality, we may assume that $\{z^{(j)}\} \in \mathcal{Z}_1$. Recall that there is a subset $\{t_1^{(1)}, \ldots, t_{\ell_1}^{(1)}\} \subseteq I^{(1)}(\{z^{(j)}\})$ such that

$$E_1\bigl(\bigl\{z^{(j)}\bigr\}\bigr) := I^{(1)}\bigl(\bigl\{z^{(j)}\bigr\}\bigr) = \bigcup_{k=1}^{\ell} I_0^{(1)}\bigl(\bigl\{z^{(j)}\bigr\}, t_k^{(1)}\bigr)$$

and $I_0^{(1)}(\{z^{(j)}\}, t_q^{(1)}) \cap I_0^{(1)}(\{z^{(j)}\}, t_m^{(1)}) = \emptyset$ for $q \neq m$. If $I^{(2)}(\{z^{(j)}\}) \setminus I^{(1)}(\{z^{(j)}\}) \neq \emptyset$, then there exists a subset $\{t_1^{(2)}, \ldots, t_{\ell_2}^{(2)}\} \subseteq I^{(2)}(\{z^{(j)}\})$ such that

$$E_2\bigl(\bigl\{z^{(j)}\bigr\}\bigr) := I^{(2)}\bigl(\bigl\{z^{(j)}\bigr\}\bigr) \setminus I^{(1)}\bigl(\bigl\{z^{(j)}\bigr\}\bigr) = \bigcup_{k=1}^{\ell_2} I_0^{(2)}\bigl(\bigl\{z^{(j)}\bigr\}, t_k^{(2)}\bigr),$$

and $I_0^{(2)}(\{z^{(j)}\}, t_q^{(2)}) \cap I_0^{(2)}(\{z^{(j)}\}, t_m^{(2)}) = \emptyset$ for $q \neq m$. And so on, if $I^{(N)}(\{z^{(j)}\}) \setminus \bigcup_{\lambda=1}^{N-1} I^{(\lambda)}(\{z^{(j)}\}) \neq \emptyset$, then there exists a subset $\{t_1^{(N)}, \ldots, t_{\ell_N}^{(N)}\} \subseteq I^{(N)}(\{z^{(j)}\})$ such that

$$E_N\bigl(\bigl\{z^{(j)}\bigr\}\bigr) := I^{(N)}\bigl(\bigl\{z^{(j)}\bigr\}\bigr) \setminus \bigcup_{\lambda=1}^{N-1} I^{(\lambda)}\bigl(\bigl\{z^{(j)}\bigr\}\bigr) = \bigcup_{k=1}^{\ell_N} I_0^{(N)}\bigl(\bigl\{z^{(j)}\bigr\}, t_k^{(N)}\bigr),$$

and $I_0^{(N)}(\{z^{(j)}\}, t_q^{(N)}) \cap I_0^{(N)}(\{z^{(j)}\}, t_m^{(N)}) = \emptyset$ for $q \neq m$.

For each λ and $i \in I_0^{(\lambda)}(\{z^{(j)}\}, t_m^{(\lambda)})$, we have $\rho_N(\varphi_i(z^{(j)}), \varphi_{t_m^{(\lambda)}}(z^{(j)})) \to 0$ as $j \to \infty$, by the hypothesis $I_0^{(\lambda)}(\{z^{(j)}\}, t_m^{(\lambda)}) = J_0^{(\lambda)}(\{z^{(j)}\}, t_m^{(\lambda)})$. Then it follows from Lemma 2.2 d) that

$$\bigl|g_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr) - g_j\bigl(\varphi_{t_m^{(\lambda)}}\bigl(z^{(j)}\bigr)\bigr)\bigr| \leq 2\rho_N\bigl(\varphi_i\bigl(z^{(j)}\bigr), \varphi_{t_m^{(\lambda)}}\bigl(z^{(j)}\bigr)\bigr) \to 0$$

as $j \to \infty$, which implies that $\xi^{(i)} = \xi^{(t_m^{(\lambda)})}$. Hence

$$\begin{aligned}
&\lim_{j\to\infty} \sum_{i\in\bigcup_{\lambda=1}^{N} I^{(\lambda)}(\{z^{(j)}\})} a_i g_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr) \\
&\quad = \lim_{j\to\infty} \sum_{\lambda=1}^{N} \sum_{i\in E_\lambda(\{z^{(j)}\})} a_i g_j\bigl(\varphi_i\bigl(z^{(j)}\bigr)\bigr) \\
&\quad = \sum_{\lambda=1}^{N} \sum_{k=1}^{\ell_\lambda} \sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\}, t_k^{(\lambda)})} a_i \xi^{(t_k^{(\lambda)})} \\
&\quad = \sum_{\lambda=1}^{N} \left(\sum_{k=1}^{\ell_\lambda} \xi^{(t_k^{(\lambda)})} \right) \left(\sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\}, t_k^{(\lambda)})} a_i \right) = 0
\end{aligned}$$

by the hypothesis $\sum_{i\in I_0^{(\lambda)}(\{z^{(j)}\},t)} a_i = 0$. This contradicts (3.3), which completes the proof. □

As an application, the following characterizes compact differences of composition operators.

Corollary 3.6 *Let φ and ψ be arbitrary holomorphic self-maps of $\mathbb{D}^N$ with*

$$\max\{\|\varphi_1\|_\infty,\dots,\|\varphi_N\|_\infty\} = \max\{\|\psi_1\|_\infty,\dots,\|\psi_N\|_\infty\} = 1.$$

Then $C_\varphi - C_\psi$ is compact on $H^\infty(\mathbb{D}^N)$ if and only if

$$\begin{aligned}&\lim_{|\varphi_i(z)|\to 1} \rho_N\big(\varphi(z),\psi(z)\big) = 0 \quad \textit{for every } 1\le i\le N \quad \textit{and}\\ &\lim_{|\psi_i(z)|\to 1} \rho_N\big(\varphi(z),\psi(z)\big) = 0 \quad \textit{for every } 1\le i\le N.\end{aligned} \tag{3.4}$$

Proof Let (3.4) hold. Then it is only needed to see that

$$\lim_{j\to\infty}\big\|(C_\varphi - C_\psi)g_j\big\|_\infty = 0$$

for any sequence $\{g_j\}$ in the unit sphere of $H^\infty(\mathbb{D}^N)$ converging uniformly to zero on each compact subset of $\mathbb{D}^N$. By Lemma 2.2, we get

$$\big|g_j\big(\varphi(z)\big) - g_j\big(\psi(z)\big)\big| \le 2\rho_N\big(\varphi(z),\psi(z)\big) \quad \text{for each } z\in\mathbb{D}^N.$$

Under this hypothesis, given $\epsilon > 0$, we may choose $0 < \delta < 1$ such that

$$\rho_N\big(\varphi(z),\psi(z)\big) < \epsilon/4,$$

whenever $|\varphi_i(z)| > \delta$ or $|\psi_j(z)| > \delta$ for each i, j. Since $\{g_j\}$ converges uniformly to zero on each compact subset of $\mathbb{D}^N$, for j large enough, we have

$$\big|g_j\big(\varphi(z)\big) - g_j\big(\psi(z)\big)\big| \le \epsilon/2$$

when $|\varphi_i(z)| \le \delta$ and $|\psi_i(z)| \le \delta$ for each i. Hence $|g_j(\varphi(z)) - g_j(\psi(z))| < \epsilon$ for j large enough, which implies the compactness of $C_\varphi - C_\psi$.

Conversely, suppose that $C_\varphi - C_\psi$ is compact on $H^\infty(\mathbb{D}^N)$, whereas (3.4) does not hold. Without loss of generality, we may assume that there is $\{z^{(j)}\} \in \mathcal{Z}_1$. By Theorem 3.5, we have $I^{(1)}(\{z^{(j)}\}) = \{1,2\}$ and $I_0^{(1)}(\{z^{(j)}\},t) = \{1,2\}$ for every $t \in I^{(1)}(\{z^{(j)}\})$. Hence

$$\lim_{|\varphi_1(z)|\to 1}\rho\big(\varphi_1(z),\psi_1(z)\big) = \lim_{|\psi_1(z)|\to 1}\rho\big(\varphi_1(z),\psi_1(z)\big) = 0.$$

So, for some $m \ge 2$, $\rho(\varphi_m(z^{(j)}),\psi_m(z^{(j)})) \nrightarrow 0$ as $j\to\infty$. Then we may assume that, considering subsequences of $\{z^{(j)}\}$,

$$\lim_{j\to\infty}\rho\big(\varphi_m(z^{(j)}),\psi_m(z^{(j)})\big) := \alpha_m \ne 0.$$

For an arbitrary positive integer j, we define the functions

$$f_j(z) = \frac{1 - |\varphi_1(z^{(j)})|^2}{1 - \overline{\varphi_1(z^{(j)})} z_1} \cdot \frac{\varphi_m(z^{(j)}) - z_m}{1 - \overline{\varphi_m(z^{(j)})} z_m}.$$

Clearly $f_j \in H^\infty(\mathbb{D}^N)$ with $\|f_j\|_\infty \le 2$, and $\{f_j\}$ converges uniformly to zero on every compact subset of $\mathbb{D}^N$. Thus

$$\begin{aligned}\|(C_\varphi - C_\psi)f_j\|_\infty &\ge \left|f_j(\varphi(z^{(j)})) - f_j(\psi(z^{(j)}))\right| \\ &= \left|\frac{1 - |\varphi_1(z^{(j)})|^2}{1 - \overline{\varphi_1(z^{(j)})}\psi_1(z^{(j)})} \cdot \frac{\varphi_m(z^{(j)}) - \psi_m(z^{(j)})}{1 - \overline{\varphi_m(z^{(j)})}\psi_m(z^{(j)})}\right|.\end{aligned}$$

Note that

$$\lim_{m\to\infty} \frac{1 - |\varphi_1(z^{(j)})|^2}{1 - \overline{\varphi_1(z^{(j)})}\psi_1(z^{(j)})} = 1.$$

Therefore

$$\liminf_{m\to\infty} \|(C_\varphi - C_\psi)f_j\|_\infty \ge |\alpha_m| > 0,$$

which leads to a contradiction with the compactness of $C_\varphi - C_\psi$. So

$$\lim_{|\varphi_1(z)|\to 1} \rho_N(\varphi(z), \psi(z)) = \lim_{|\psi_1(z)|\to 1} \rho_N(\varphi(z), \psi(z)) = 0,$$

which implies the desired result (3.4). □

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