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Integral transforms for logharmonic mappings

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Abstract

Bieberbach's conjecture was very important in the development of geometric function theory, not only because of the result itself, but also due to the large amount of methods that have been developed in search of its proof. It is in this context that the integral transformations of the type $f_\alpha(z)=\int_0^z(f(\zeta)/\zeta)^\alpha\,d\zeta$ or $F_\alpha(z)=\int_0^z(f'(\zeta))^\alpha\,d\zeta$ appear. In this note we extend the classical problem of finding the values of $\alpha\in\mathbb{C}$ for which either f_α or F_α are univalent, whenever f belongs to some subclasses of univalent mappings in $\mathbb{D}$, to the case of logharmonic mappings by considering the extension of the *shear construction* introduced by Clunie and Sheil-Small in (Clunie and Sheil-Small in Ann. Acad. Sci. Fenn., Ser. A I 9:3–25, 1984) to this new scenario.

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1 Introduction

One of the most studied classes of functions in the context of geometric function theory is the well known class S of univalent functions f defined in the unit disk, normalized by $f(0)=1-f'(0)=0$. The investigations of problems associated with the class S and some of its subclasses go back to the works of Koebe, developed at beginning of the last century. Undoubtedly, one of the most important problems in this field was the Bieberbach conjecture, presented in 1916 and solved by de Brange in 1984. Solving this problem had a profound influence in the development of the theory of univalent functions, providing, among other things, powerful methods which have been used to study other problems in geometric function theory. One important question arising in this context is determining the values of $\alpha\in\mathbb{C}$ for which the functions

$$\varphi_\alpha(z)=\int_0^z\left(\frac{\varphi(\zeta)}{\zeta}\right)^\alpha d\zeta, \tag{1}$$

or

$$\Phi_\alpha(z)=\int_0^z\left(\varphi'(\zeta)\right)^\alpha d\zeta, \tag{2}$$

belong to the class S, when φ is a function of this class. The univalence of these integral operators was first studied by Royster [28], and although some partial results have been obtained, in the general case, this question remains open. It is known, for example, that if $|\alpha| \leq 1/4$ then both transforms are univalent, see [17, 25]. However, there are functions $\varphi \in S$ for which Φ_α is not univalent for any $|\alpha| > 1/3$ with $\alpha \neq 1$. An analogous result was obtained in [17] for the transform of type (1); in that paper it is proved that, for all $|\alpha| > 1/2$, there are functions $\varphi \in S$ such that the corresponding φ_α is not univalent. For a summary of these results, we refer the reader to the classical book by Goodman [15]. Some recent results and other problems related to transforms (1) and (2), in the context of analytic and meromorphic functions, can be found in [18, 19, 24, 27].

Since the work by Clunie and Sheil-Small [13], many problems of geometric function theory have been extended from the setting of holomorphic functions to the wider class of harmonic mappings in the plane. In this direction, in [11] and subsequently in [8], the authors proposed an extension of the integral transforms (1) and (2) to the setting of sense-preserving harmonic mapping, see also Theorems 2 and 3 in [6]. The definitions given in [8] make use of the *shear construction* introduced by Clunie and Sheil-Small in [13] as follows: let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the usual normalization $g(0) = h(0) = 1 - h'(0) = 0$ and dilatation $\omega = g'/h'$. Given $\alpha \in \overline{\mathbb{D}}$, when $\varphi = h - g$ is zero only at $z = 0$, we define F_α as the horizontal shear of φ_α defined by (1) with dilatation $\omega_\alpha = \alpha\omega$. This is, $F_\alpha = H + \overline{G}$, where H, G satisfy

$$H - G = \varphi_\alpha \quad \text{and} \quad \frac{G'}{H'} = \alpha\omega$$

with $H(0) = G(0) = 0$. In a similar way we extend the integral transform (2) to sense-preserving harmonic mappings in $\mathbb{D}$. The reader can find interesting results, some of which are related to the question of knowing for which values of α the integral transforms lead functions belonging to the class S (or to some of its subclasses) to the class S in [8]. It is important to point out that the analog of the Bieberbach conjecture for sense-preserving univalent harmonic mappings f, proposed by Clunie and Sheil-Small [13], remains open even for the second Taylor coefficient a_2 of the analytic part h of f. The best known estimate for $|a_2|$ is obtained in [7].

The main objective of the present paper is to investigate this type of problems, but in the context of logharmonic mappings defined in the unit disk. To this end, we extend the integral transforms defined by (1) and (2) to the case of logharmonic mappings. Analogous to the extension of (1) and (2) previously described for sense-preserving harmonic mappings, we use a method similar to the shear construction by Clunie and Sheil-Small, introduced in [2, 22] for constructing univalent logharmonic mappings. The manuscript is organized as follows. In Sect. 2 some preliminaries concerning logharmonic mappings are introduced, in Sect. 3 we introduce the integral transform of the first type for logharmonic mappings, in particular, we generalize a result obtained by Pfaltzgraff in [25]. The extension of the integral transform of type (2) to logharmonic mappings is given in Sect. 4, in which we obtain similar results to those obtained in Sect. 3. Finally, Sect. 5 is devoted to extending the integral transforms to the special case of non-vanishing logharmonic mappings.

2 Some preliminaries about logharmonic mappings

A logharmonic mapping defined in the unit disk $\mathbb{D}$ is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}(z)} = \omega(z)\left(\frac{\overline{f(z)}}{f(z)}\right)f_z(z),$$

where ω is an analytic function from $\mathbb{D}$ into itself, which is called second complex dilatation of f. The study of logharmonic functions was initiated mainly by the works of Abdulhadi, Bshouty, and Hengartner [3, 5], and later continued in a series of papers in which the basic theory of logharmonic functions has been developed. Since then many papers have been published dealing with this subject, see for example [1, 4, 9, 22, 23].

Because of the condition on ω, the Jacobian J_f of f, given by

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2\left(1 - |\omega|^2\right),$$

is nonnegative, and therefore, every nonconstant logharmonic mapping is sense-preserving and open in $\mathbb{D}$. If f is a nonconstant logharmonic mapping defined in $\mathbb{D}$ and vanishes only at the origin, then f has the representation

$$f(z) = z^m|z|^{2\beta m}h(z)\overline{g(z)},$$

where m is a nonnegative integer, $\mathrm{Re}\{\beta\} > -m/2$, and h and g are analytic mappings in the unit disk, such that $g(0) = 1$ and $h(0) \neq 0$ (see [3]). In particular, when f is a univalent logharmonic mapping defined in $\mathbb{D}$ and $f(0) = 0$, we can represent f in the form

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}, \quad z \in \mathbb{D},$$

where $\mathrm{Re}\{\beta\} > -1/2$, and h and g are analytic mappings in $\mathbb{D}$ such that $g(0) = 1$ and $0 \notin hg(\mathbb{D})$. This class of logharmonic mappings has been widely studied. For more details the reader can read the summary paper in [2] and the references given there.

In the first part of this study, we consider univalent logharmonic mappings in $\mathbb{D}$ with $\omega(0) = 0$, the case in which f has the form

$$f(z) = zh(z)\overline{g(z)}, \quad z \in \mathbb{D},$$

which implies that its dilatation ω is given by

$$\omega(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)}.$$

In relation to this class of functions, denoted by S_{Lh}, the following result, which is one of the tools that we shall use in this manuscript, was proved in [1], see also [5]; it asserts the following.

Theorem A *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping defined in $\mathbb{D}$ such that $0 \notin hg(\mathbb{D})$. Then $\varphi(z) = zh(z)/g(z)$ is a starlike analytic function if and only if $f(z)$ is a starlike logharmonic mapping.*

Also we will consider non-vanishing logharmonic mappings in $\mathbb{D}$. It is well known that such mappings can be expressed in the form

$$f(z) = h(z)\overline{g(z)},$$

where h and g are non-vanishing analytic functions in $\mathbb{D}$. In terms of h and g, when f is locally univalent, the dilatation ω of f is given by

$$\omega = \frac{g'h}{gh'}.$$

Note that in this case h is locally univalent.

Throughout this paper, we mainly study integral transforms of type (1) and (2) for two classes of logharmonic functions: first we consider univalent logharmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$ normalized as $h(0) = g(0) = 1$ and its several subfamilies as starlike and convex mappings normalized as before. In a similar way we study the case when f is a non-vanishing univalent logharmonic function of the form $f = h\overline{g}$ such that $h(0) = g(0) = 1$.

3 Integral transform of the first type

Let $f = zh(z)\overline{g(z)}$ be a locally univalent logharmonic mapping defined in the unit disk with the normalization described above. Then $\varphi(z) = zh(z)/g(z)$ is an analytic function in $\mathbb{D}$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$, and $\varphi(z) \neq 0$ if $z \neq 0$. So, h, g are solution of the system of nonlinear differential equations

$$\frac{zh(z)}{g(z)} = \varphi(z) \quad \text{and} \quad \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = \omega(z), \quad z \in \mathbb{D}.$$

This fact was used in [22] to establish a method of construction of starlike (univalent) logharmonic mappings satisfying that φ is an analytic starlike function and ω is a Schwarz map. Here we use it to extend the integral transform (1) to the case of logharmonic mappings, with the above notation, as follows: given $\alpha \in \overline{\mathbb{D}}$, we define f_α to be the logharmonic mapping, with dilatation $\omega_\alpha = \alpha\omega$, given by $f_\alpha(z) = zH(z)\overline{G(z)}$, where H, G satisfy the system

$$\frac{zH(z)}{G(z)} = \varphi_\alpha(z) = \int_0^z \left(\frac{\varphi(\xi)}{\xi}\right)^\alpha d\xi \quad \text{and} \quad \frac{zG'(z)/G(z)}{1 + zH'(z)/H(z)} = \omega_\alpha(z), \tag{3}$$

with the initial conditions $H(0) = G(0) = 1$.

Here, we consider $\alpha \in \mathbb{D}$ since ω_α must be a Schwarz map. However, we can consider any complex number α such that $|\alpha\omega| < 1$. Note that if $\alpha = 0$, then $H/G \equiv 1$ and $\omega_\alpha = 0$, whence it follows that G is constant, say k, which leads to $f_\alpha(z) = |k|^2 z$. On the other hand, if $\alpha = 1$ we have that $\varphi_1(z) = zH(z)/G(z)$ satisfies

$$1 + z\frac{\varphi_1''(z)}{\varphi_1'(z)} = z\frac{\varphi'(z)}{\varphi(z)},$$

which implies that when φ is starlike, φ_1 is convex (in particular starlike). Consequently, according to [22], f_1 is starlike. However, we know from the analytic case that, for α of module greater than 1/3, there are univalent functions whose respective functions f_α are not univalent in $\mathbb{D}$. Because of this and other similar problems that appear both in the context of analytic functions and in the context of harmonic mappings, we are interested in studying the values of α for which we can ensure that f_α belongs to S_{Lh} or to some of its subclasses. A first result in this direction is given in the following proposition.

Proposition 3.1 *Let $f(z) = zh(z)\overline{g(z)}$ be a univalent logharmonic mapping defined in $\mathbb{D}$ with dilatation ω. Let $zh(z)/g(z) = \varphi(z)$ and the integral transformation f_α be defined by* (3).

(i) *If φ is a convex mapping, then f_α is a starlike logharmonic mapping when $\alpha \in [0,2]$ and $|\alpha\omega| < 1$.*
(ii) *If φ is a starlike mapping, then f_α is a starlike logharmonic mapping when $\alpha \in [0,1]$.*

Proof We note first that if $f_z(z_0) = 0$ for some $z_0 \in \mathbb{D}$, then $z_0 \neq 0$,

$$z_0 h'(z_0) + h(z_0) = 0 \quad \text{and} \quad g'(z_0) = 0,$$

which implies $\varphi'(z_0) = 0$. Hence, J_f is positive in $\mathbb{D}$.

By the definition of $f_\alpha(z) = zH(z)\overline{G(z)}$, we have that $zH(z)/G(z) = \varphi_\alpha(z)$ satisfies

$$1 + \operatorname{Re}\left\{z\frac{\varphi_\alpha''(z)}{\varphi_\alpha'(z)}\right\} = 1 + \alpha \operatorname{Re}\left\{z\frac{\varphi'(z)}{\varphi(z)} - 1\right\}, \tag{4}$$

$\alpha \in \mathbb{R}$. To prove (i) we argue as follows: Since φ is convex, it follows that $\operatorname{Re}\{z\varphi'(z)/\varphi(z)\} > 1/2$, which implies that

$$1 + \operatorname{Re}\left\{z\frac{\varphi_\alpha''(z)}{\varphi_\alpha'(z)}\right\} > 1 - \frac{\alpha}{2} > 0 \quad \text{for all } \alpha \in [0,2).$$

Hence, φ_α is convex and, in particular, is a starlike mapping. Using inequality (4) we conclude that

$$1 + \operatorname{Re}\left\{z\frac{\varphi_\alpha''(z)}{\varphi_\alpha'(z)}\right\} > 1 - \alpha > 0 \quad \text{for all } \alpha \in [0,1).$$

Analogously, φ_α is a starlike mapping when φ satisfies the hypothesis in (ii). Thus the proofs of (i) and (ii) are concluded by applying Theorem A. □

In [4] the authors extend to the case of logharmonic mappings the concept of stable univalent function (resp. starlike, convex, close-to-convex, etc.) studied in [16] for harmonic and analytic functions, see also [14, 20]. More precisely, they give the following definition.

Definition 3.1 A logharmonic function $f(z) = zh(z)\overline{g(z)}$, normalized by $h(0) = g(0) = 1$, is called stable univalent logharmonic or SU_{Lh} if, for all $|\lambda| = 1$, $f_\lambda(z) = zh(z)\overline{g(z)}^\lambda$ is univalent logharmonic.

Next, we present a criterion to establish the stable univalence of f_α.

Theorem 3.1 *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping defined in the unit disk with dilatation ω, normalized by $h(0) = g(0) = 1$, and such that $zh(z)/g(z) = \varphi(z)$ is a univalent mapping. Then f_α defined by (3) is a stable univalent logharmonic mapping if α satisfies the inequality*

$$|\alpha| \leq \frac{1}{4(2\delta + 1 + 8\|\omega^*\|/15)}, \tag{5}$$

where

$$\delta = \frac{4\|\omega\|}{4 - \|\omega\|}, \qquad \|\omega\| = \sup\{|\omega(z)| : z \in \mathbb{D}\} \quad \text{and} \quad \|\omega^*\| = \sup_{z\in\mathbb{D}} \frac{|\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2}.$$

Proof As in the previous proposition, $J_f > 0$ in $\mathbb{D}$ and therefore f is locally univalent. For $|\lambda| = 1$, we define

$$\psi_\lambda(z) = z\frac{H(z)}{G(z)^\lambda}, \quad z \in \mathbb{D},$$

from where

$$\begin{aligned}\psi_\lambda'(z) &= \psi_\lambda(z)\left(\frac{1}{z} + \frac{H'(z)}{H(z)} - \lambda\frac{G'(z)}{G(z)}\right) \\ &= \psi_\lambda(z)\frac{1}{z}\left(1 + \frac{zH'(z)}{H(z)}\right)\big(1 - \lambda\omega_\alpha(z)\big).\end{aligned}$$

On the other hand,

$$\frac{z\varphi_\alpha'(z)}{\varphi_\alpha(z)} = 1 + z\frac{H'(z)}{H(z)} - z\frac{G'(z)}{G(z)} = \left(1 + z\frac{H'(z)}{H(z)}\right)\big(1 - \omega_\alpha(z)\big),$$

from which we get

$$\psi_\lambda'(z) = \psi_\lambda(z)\frac{1-\lambda\omega_\alpha(z)}{1-\omega_\alpha(z)}\frac{\varphi_\alpha'(z)}{\varphi_\alpha(z)} \tag{6}$$

and

$$\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)} = \frac{\psi_\lambda'(z)}{\psi_\lambda(z)} + \frac{\varphi_\alpha''(z)}{\varphi_\alpha'(z)} - \frac{\varphi_\alpha'(z)}{\varphi_\alpha(z)} + \frac{(1-\lambda)\omega_\alpha'(z)}{(1-\lambda\omega_\alpha(z))(1-\omega_\alpha(z))}.$$

Replacing (6) in the last equality, we get

$$\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)} = \frac{(1-\lambda)\omega_\alpha(z)}{1-\omega_\alpha(z)}\frac{\varphi_\alpha'(z)}{\varphi_\alpha(z)} + \frac{\varphi_\alpha''(z)}{\varphi_\alpha'(z)} + \frac{(1-\lambda)\omega_\alpha'(z)}{(1-\lambda\omega_\alpha(z))(1-\omega_\alpha(z))}, \tag{7}$$

hence

$$\begin{aligned}\big(1-|z|^2\big)\left|z\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| = \big(1-|z|^2\big)|\alpha|\bigg|&\frac{(1-\lambda)\omega(z)}{1-\omega_\alpha(z)}\frac{z\varphi_\alpha'(z)}{\varphi_\alpha(z)} \\ &+ \frac{z\varphi'(z)}{\varphi(z)} - 1 + \frac{z(1-\lambda)\omega'(z)}{(1-\lambda\omega_\alpha(z))(1-\omega_\alpha(z))}\bigg|\end{aligned}$$

and, in consequence,

$$\left(1-|z|^2\right)\left|z\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \le \left(1-|z|^2\right)|\alpha|\left(\left|\frac{(1-\lambda)\omega(z)}{1-\omega_\alpha(z)}\frac{z\varphi_\alpha'(z)}{\varphi_\alpha(z)}\right| + \left|\frac{z\varphi'(z)}{\varphi(z)}-1\right| + \left|\frac{z(1-\lambda)\omega'(z)}{(1-\lambda\omega_\alpha(z))(1-\omega_\alpha(z))}\right|\right).$$

Since $|\alpha| \le 1/4$, then $\varphi_\alpha \in S$, therefore

$$\left|\frac{z\varphi_\alpha'(z)}{\varphi_\alpha(z)}\right| \le \frac{1+|z|}{1-|z|} \quad \text{and} \quad \left|\frac{z\varphi'(z)}{\varphi(z)}-1\right| \le \frac{2}{1-|z|}, \tag{8}$$

the second inequality being a consequence of $\varphi \in S$. Moreover, it is easy to see that

$$\frac{|\omega(z)|}{|1-\omega_\alpha(z)|} \le \frac{4\|\omega\|}{4-\|\omega\|} \tag{9}$$

and

$$\begin{aligned}\frac{|\omega'(z)|(1-|z|^2)}{|(1-\lambda\omega_\alpha(z))(1-\omega_\alpha(z))|} &\le \frac{\|\omega^*\|(1-|\omega(z)|^2)}{(1-|\alpha||\omega(z)|)^2} \\ &\le \frac{\|\omega^*\|(1-|\omega(z)|^2)}{(1-|\omega(z)|/4)^2} \\ &\le \frac{16}{15}\|\omega^*\|\end{aligned} \tag{10}$$

for all $z \in \mathbb{D}$. Thus, using inequalities (8), (9), and (10), it follows that

$$\left(1-|z|^2\right)\left|z\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \le 2|\alpha|\left(\frac{16\|\omega\|}{4-\|\omega\|} + 2 + \frac{16}{15}\|\omega^*\|\right).$$

Inequality (5) and Becker's criterion (see [10]) imply that ψ_λ is univalent for all λ in the unit circle, which completes the proof. □

Note that in the case when f is an analytic mapping, $\omega \equiv 0$ and consequently $\delta = 0$ and $\|\omega^*\| = 0$. Then inequality (5) becomes in $|\alpha| \le 1/4$, which is the best known bound of the values of α such that the corresponding f_α is univalent in $\mathbb{D}$.

For the following corollary, we recall that a family $\mathcal{F}$ of normalized locally univalent functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots, \quad z \in \mathbb{D},$$

is said to be linear invariant (LIF) if, for all $f \in \mathcal{F}$, we have $(f\circ\varphi_a - f(a))/((1-|a|^2)f'(a)) \in \mathcal{F}$ for all automorphism $\varphi_a(z) = (z+a)/(1+\overline{a}z)$ of $\mathbb{D}$. The order of an LIF $\mathcal{F}$ is the supremum of the modulus of the second Taylor coefficient $|a_2(f)|$ of the functions $f \in \mathcal{F}$. The notion of a linear invariant family of holomorphic functions was introduced by Pommerenke [26], and it has been extended in subsequent studies in various directions; we refer the reader to [21, 29] for an extension of LIF to the setting of harmonic mappings.

The authors proved in [8, Lemma 3] that if φ is a univalent analytic function in a linear invariant family $\mathcal{F}$ with order β, then

$$\left(1-|z|^2\right)\left|z\frac{\varphi'(z)}{\varphi(z)}\right| \leq 2\beta \quad \text{for all } z\in\mathbb{D}. \tag{11}$$

By using this inequality, we can improve the last theorem.

Corollary 3.1 *Let f, φ be as in Theorem 3.1. Suppose, moreover, that φ is univalent in a linear invariant family $\mathcal{F}$ with order β. Then f_α is stable univalent logharmonic if α satisfies the condition*

$$|\alpha| \leq \min\left\{\frac{1}{2(4\delta+\beta+\frac{1}{2}+16\|\omega^*\|/15)}, \frac{1}{4}\right\}.$$

Proof The proof is completely analogous to that of the previous theorem; it is enough to replace the second condition in (8) with the inequality

$$\left|\frac{z\varphi'(z)}{\varphi(z)}-1\right| \leq \frac{2\beta}{1-|z|^2}+1,$$

which is an immediate consequence of (11). □

4 Integral transform of the second type

Following the same procedure as in the previous section, we extend the integral transform (2) to the case of logharmonic mappings as follows: given $\alpha\in\overline{\mathbb{D}}$ and $f(z)=zh(z)\overline{g(z)}$ a locally univalent logharmonic mapping defined in the unit disk, with the normalization $h(0)=g(0)=1$ and $0\notin hg(\mathbb{D})$, we define F_α to be the logharmonic mapping, with dilatation $\omega_\alpha=\alpha\omega$, given by $F_\alpha=zH(z)\overline{G(z)}$, where H, G satisfy the system

$$z\frac{H(z)}{G(z)}=\Phi_\alpha(z)=\int_0^z\left(\varphi'(\xi)\right)^\alpha d\xi \quad \text{and} \quad \omega_\alpha=\alpha\omega \tag{12}$$

with the initial conditions $H(0)=G(0)=1$. As in the previous case, $\varphi(z)=zh(z)/g(z)$, $z\in\mathbb{D}$.

The following two results are dual to Proposition 3.1 and Theorem 3.1, respectively. In both cases we assume the above normalization.

Proposition 4.1 *Let $f(z)=zh(z)\overline{g(z)}$ be a logharmonic mapping defined in $\mathbb{D}$ with dilatation ω. If $zh(z)/g(z)=\varphi(z)$ is a convex mapping, then F_α defined by (12) is a starlike logharmonic mapping in the unit disk for $\alpha\in[0,1]$.*

Proof As in Proposition 3.1, the condition on φ implies that J_f is positive, so f is locally univalent. Since φ is a convex mapping, we have that, for $\alpha\in[0,1]$,

$$\operatorname{Re}\left\{1+z\frac{\Phi_\alpha''(z)}{\Phi_\alpha'(z)}\right\}=1+\alpha\operatorname{Re}\left\{z\frac{\varphi''(z)}{\varphi'(z)}\right\}>1-\alpha>0,$$

from which it concludes that $\Phi_\alpha(z)$ is a convex mapping and in particular starlike. Using Theorem A, we have that F_α is a starlike logharmonic mapping. □

Theorem 4.1 *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping defined in $\mathbb{D}$ with dilatation ω and such that $zh(z)/g(z) = \varphi(z)$ is a univalent mapping. Then F_α defined as in (12) is a stable univalent logharmonic mapping if α satisfies the inequality*

$$|\alpha| \leq \frac{1}{2(4\delta + 3 + 16\|\omega^*\|/15)}$$

with δ and $\|\omega^*\|$ as in Theorem 3.1.

Proof Since φ is univalent, we have $J_f > 0$ and

$$\sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)\left|\frac{\Phi''_\alpha(z)}{\Phi'_\alpha(z)}\right| \leq |\alpha| \sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)\left|\frac{\varphi''(z)}{\varphi'(z)}\right| \leq 6|\alpha|. \tag{13}$$

It follows from equation (7) and inequalities (8), (9), and (10) that

$$\bigl(1-|z|^2\bigr)\left|z\frac{\psi''_\lambda(z)}{\psi'_\lambda(z)}\right| \leq 2|\alpha|\bigl(4\delta + 3 + 16\|\omega^*\|/15\bigr),$$

from which we get, by virtue of the classical univalence criterion of Becker, that F_α is a stable univalent logharmonic mapping. □

Note that if φ is a convex mapping, then the right-hand side of inequality (13) can be replaced with $4|\alpha|$, from where the range of the values of α, for which the corresponding mapping F_α is stable univalent, is determined by the inequality

$$|\alpha| \leq \frac{1}{4(2\delta + 1 + 8\|\omega^*\|/15)}.$$

5 Integral transforms of non-vanishing logharmonic mappings

In this section, we propose an extension of the integral transforms (1) and (2) to non-vanishing logharmonic mappings in $\mathbb{D}$. To this end, we will use the fact that if f is a non-vanishing logharmonic mapping in $\mathbb{D}$, there is a branch of $\log f$, which is harmonic in $\mathbb{D}$ with dilatation $\omega_{\log f} = \omega_f$, and apply to it the theory that we developed in [8]. Note that $\omega_{\log f} = \omega_f$ implies that $\log f$ is a sense-preserving harmonic mapping and therefore J_f is positive in $\mathbb{D}$.

As was mentioned in Sect. 2, if f is a non-vanishing logharmonic mapping in $\mathbb{D}$ with dilatation ω, then $f = h\overline{g}$, where h, g are non-vanishing analytic functions in $\mathbb{D}$. In this case $\omega = g'h/gh'$, and if we assume the normalization $h(0) = g(0) = 1$, we can choose branches of $\log f$, $\log h$, and $\log g$ satisfying

$$\log f(0) = \log h(0) = \log g(0) = 0 \quad \text{and} \quad \log f = \log h + \overline{\log g}.$$

The following lemma is a general tool that appears as a natural version of Theorem 1 in [12] for logharmonic mappings. We will use this lemma in the next subsections.

Lemma 5.1 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping in $\mathbb{D}$ with dilatation ω, and suppose that $\psi = \log h/g$ is a univalent mapping such that $\Omega := \psi(\mathbb{D})$ is M-linearly connected. Then f is univalent in $\mathbb{D}$ if $\|\omega\| < 1/(2M+1)$.*

Proof Suppose that there are $z_1 \neq z_2$ in $\mathbb{D}$ such that $f(z_1) = f(z_2)$, and let S be a path in Ω joining $\psi(z_1)$ to $\psi(z_2)$ such that $\ell(S) \leq M|\psi(z_1) - \psi(z_2)|$. So,

$$e^{\psi(z_1)}\left|g(z_1)\right|^2 = e^{\psi(z_2)}\left|g(z_2)\right|^2,$$

and in consequence,

$$\left|\psi(z_1) - \psi(z_2)\right| \leq \left|2\log g(z_1) - 2\log g(z_2)\right| \leq 2\int_\gamma \left|\frac{g'(\xi)}{g(\xi)}\right| |d\xi|,$$

where $\gamma = \psi^{-1}(S)$. From here and the equality

$$\frac{g'}{g} = \omega\frac{h'}{h} = \frac{\omega}{1-\omega}\psi',$$

it follows that

$$\left|\psi(z_1) - \psi(z_2)\right| \leq 2\frac{\|\omega\|}{1-\|\omega\|}\int_\gamma \left|\psi'(\xi)\right| |d\xi| \leq 2\frac{\|\omega\|}{1-\|\omega\|}\ell(S) < \left|\psi(z_1) - \psi(z_2)\right|,$$

if ω satisfies $\|\omega\| < 1/(2M+1)$. This contradiction ends the proof. □

5.1 Integral transform of the first type for non-vanishing logharmonic mappings

We consider a non-vanishing logharmonic mapping $f = h\overline{g}$ defined in $\mathbb{D}$, with dilatation $\omega = g'h/h'g$, and normalized by $h(0) = g(0) = 1$. We suppose, moreover, that $\varphi = \log h - \log g$ is zero only at $z = 0$. Given $\alpha \in \overline{\mathbb{D}}$, we define the integral transform of the first type of f by

$$f_\alpha = e^H\overline{e^G} = \exp\{H + \overline{G}\}, \tag{14}$$

where H and G satisfy the system

$$H(z) - G(z) = \varphi_\alpha(z) = \int_0^z \left(\frac{\varphi(\zeta)}{\zeta}\right)^\alpha d\zeta \quad \text{and} \quad \frac{G'}{H'} = \alpha\omega,$$

with the initial conditions $H(0) = G(0) = 0$. In other words, f_α is defined in such a way that $\log f_\alpha = H + \overline{G}$ is a harmonic branch of the logarithm of f_α in $\mathbb{D}$, which is the horizontal shear of φ_α with dilatation $\alpha\omega$. The reader can find the details of the shear construction of harmonic mappings in [13].

Theorem 5.1 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping in $\mathbb{D}$ with dilatation ω, and let f_α be defined by equation* (14). *If $\varphi = \log h/g$ is a univalent function and $|\alpha| \leq 0.165$, then f_α is univalent.*

Proof For $|\lambda| = 1$, we define $\Psi_\lambda = H + \lambda G$. A direct calculation shows that

$$\frac{H''(z)}{H'(z)} = \alpha\left(\frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z}\right) + \frac{\omega_\alpha'(z)}{1-\omega_\alpha(z)}$$

and

$$\begin{aligned}\frac{z\Psi_\lambda''(z)}{\Psi_\lambda'(z)} &= \frac{zH''(z)}{H'(z)} + \frac{\lambda z\omega_\alpha'(z)}{1+\lambda\omega_\alpha(z)} \\ &= \alpha\left(\frac{z\varphi'(z)}{\varphi(z)} - 1\right) + \frac{(1+\lambda)z\omega_\alpha'(z)}{(1-\omega_\alpha(z))(1+\lambda\omega_\alpha(z))},\end{aligned} \tag{15}$$

from which we see that

$$\begin{aligned}(1-|z|^2)\left|\frac{z\Psi_\lambda''(z)}{\Psi_\lambda'(z)}\right| &\le |\alpha|\left[(1-|z|^2)\left|\frac{z\varphi'(z)}{\varphi(z)} - 1\right| + 2\frac{(1-|z|^2)|\omega'(z)|}{(1-|\alpha||\omega(z)|)^2}\right] \\ &\le |\alpha|\left[4 + 2\frac{(1-|\omega(z)|^2)\|\omega^*\|}{(1-|\alpha||\omega(z)|)^2}\right] \\ &\le 2|\alpha|\left(2+\frac{1}{1-|\alpha|^2}\right),\end{aligned}$$

being the last inequality a consequence of

$$\max_{z\in\mathbb{D}} \frac{(1-|\omega(z)|^2)}{(1-|\alpha||\omega(z)|)^2} \le \frac{1}{1-|\alpha|^2} \quad \text{and} \quad \|\omega^*\| \le 1.$$

It follows from Becker's criterion that Ψ_λ is univalent if $2|\alpha|(2+\frac{1}{1-|\alpha|^2}) \le 1$, whence $H+\overline{G}$ is stable harmonic univalent if $|\alpha| \le 0.165$. In consequence, f_α is univalent for these values of α. □

Proposition 5.1 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping in $\mathbb{D}$ with dilatation ω, and let f_α be defined by equation (14). If $\varphi = \log h/g$ is a starlike function and $\alpha \in (-0.303, 0.707)$, then f_α is a univalent logharmonic mapping in $\mathbb{D}$.*

Proof Since φ is starlike, we have $\operatorname{Re}\{z\varphi'(z)/\varphi(z)\} > 0$ for all $z \in \mathbb{D}$. So, for $\alpha > 0$ and Ψ_λ defined as in the previous theorem, we get from (15) that

$$\begin{aligned}&\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + z\frac{\Psi_\lambda''(z)}{\Psi_\lambda'(z)}\right\} d\theta \\ &= \int_{\theta_1}^{\theta_2} \left[1-\alpha+\alpha\operatorname{Re}\left\{z\frac{\varphi'(z)}{\varphi(z)} + \frac{\lambda z\omega'(z)}{1+\lambda\omega_\alpha(z)} + \frac{z\omega'}{1-\omega_\alpha(z)}\right\}\right] d\theta \\ &\ge (1-\alpha)(\theta_2-\theta_1) + \operatorname{Arg}\left\{\frac{1+\lambda\alpha\omega(re^{i\theta_2})}{1+\lambda\alpha\omega(re^{i\theta_1})} \cdot \frac{1-\alpha\omega(re^{i\theta_2})}{1-\alpha\omega(re^{i\theta_1})}\right\} \\ &> -4\arcsin(\alpha)\end{aligned}$$

for all $0 \le \theta_2 - \theta_1 \le 2\pi$. Therefore,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + z\frac{\Psi_\lambda''(z)}{\Psi_\lambda'(z)}\right\} d\theta > -\pi \quad \text{if } 0 \le \alpha \le \sqrt{2}/2,$$

from where Ψ_λ is a close to convex mapping in $\mathbb{D}$ if $0 \le \alpha \le \sqrt{2}/2$.

On the other hand, since φ is starlike, then $|\operatorname{Arg}\{\varphi(z)/z\}| \leq 2\arcsin(|z|)$ and, in consequence,

$$\begin{aligned}\left|\operatorname{Arg}\left\{\Psi'_\lambda(z)\right\}\right| &= \left|\operatorname{Arg}\left\{\left(\frac{\varphi(z)}{z}\right)^\alpha \frac{1+\lambda\alpha\omega(z)}{1-\alpha\omega(z)}\right\}\right| \\ &\leq 2|\alpha|\arcsin(r) + 2\arcsin\bigl(r|\alpha|\bigr), \quad r = |z|.\end{aligned}$$

Hence, by a straightforward calculation, we have $|\operatorname{Arg}\{\Psi'_\lambda(z)\}| < \pi/2$ if $|\alpha| < 0.303$, which implies that $\operatorname{Re}\{\Psi'_\lambda(z)\} > 0$ for these values of α and hence Ψ_λ is a close to convex mapping in $\mathbb{D}$, when $|\alpha| < 0.303$. From this and the discussion above, it follows that $\log f_\alpha = H + \overline{G}$ is stable harmonic close-to-convex if $\alpha \in (-0.303, 0.707)$, whence f_α is univalent in $\mathbb{D}$. □

Proposition 5.2 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping defined in $\mathbb{D}$ with $\|\omega\| < 1/3$, and let f_α be defined by equation (14). If $\varphi = \log h/g$ is a convex function and $\alpha \in [0,2]$, then f_α is univalent.*

Proof The proof follows as a direct application of Lemma 5.1 and the fact that φ_α is a convex mapping for $\alpha \in [0,2]$, the case in which $\varphi_\alpha(\mathbb{D})$ is a M-linearly connected domain with $M = 1$. □

5.2 Integral transform of the second type for non-vanishing logharmonic mappings

The definition of the integral transform of the second type for non-vanishing logharmonic mappings is completely analogous to that given in the previous subsection: let $f = h\overline{g}$ be a non-vanishing logharmonic mapping in $\mathbb{D}$ with dilatation $\omega = g'h/h'g$ and normalized by $h(0) = g(0) = 1$. Note that from the condition $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, it follows that $\varphi = \log h - \log g$ is locally univalent in $\mathbb{D}$. We define the logharmonic mapping $F_\alpha = e^H\overline{e^G}$, where H, G satisfy the system

$$H(z) - G(z) = \Phi_\alpha(z) = \int_0^z \bigl(\varphi'(\zeta)\bigr)^\alpha \, d\zeta \quad \text{and} \quad \omega_{F_\alpha} = \alpha\omega \tag{16}$$

with the initial conditions $H(0) = G(0) = 0$.

Theorem 5.2 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping in $\mathbb{D}$ with dilatation ω, and let F_α be defined by equation (16). If φ is a univalent function and $|\alpha| \leq 0.125$, then F_α is univalent.*

Proof For $|\lambda| = 1$, we define $\Psi_\lambda = H + \lambda G$. Using (16), we obtain by a direct calculation

$$\frac{H''(z)}{H'(z)} = \alpha\left(\frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z}\right) + \frac{\omega'_\alpha(z)}{1-\omega_\alpha(z)}$$

and

$$\frac{z\Psi''_\lambda(z)}{\Psi'_\lambda(z)} = \frac{zH''(z)}{H'(z)} + \frac{\lambda z\omega'_\alpha(z)}{1+\lambda\omega_\alpha(z)} = \alpha\left(\frac{z\varphi'(z)}{\varphi(z)} - 1\right) + \frac{(1+\lambda)z\omega'_\alpha(z)}{(1-\omega_\alpha(z))(1+\lambda\omega_\alpha(z))}.$$

It follows from the univalence of φ and

$$\max_{z\in\mathbb{D}} \frac{(1-|\omega(z)|^2)}{(1-|\alpha||\omega(z)|)^2} \le \frac{1}{1-|\alpha|^2}$$

that

$$\begin{aligned}
&(1-|z|^2)\left|\frac{z\Psi_\lambda''(z)}{\Psi_\lambda'(z)}\right| \\
&\quad\le |\alpha|\left(\left|(1-|z|^2)\frac{\varphi''(z)}{\varphi'(z)} - 2\overline{z}\right| + 2|z|^2 + 2\frac{(1-|\omega(z)|^2)\|\omega^*\|}{(1-|\alpha||\omega(z)|)^2}\right) \\
&\quad\le 2|\alpha|\left(3 + \frac{1}{1-|\alpha|^2}\right).
\end{aligned}$$

Consequently, we conclude from Becker's criterion that Ψ_λ is univalent if $|\alpha| \le 0.125$, and therefore $\log F_\alpha$ is stable harmonic univalent for these values of α. This completes the proof of the theorem. □

Proposition 5.3 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping defined in $\mathbb{D}$ with dilatation ω, and let F_α be defined by equation (16). If φ is a convex function and $\alpha \in (-0.303, 0.5605)$, then F_α is a univalent logharmonic mapping in $\mathbb{D}$.*

Proof The proof is almost the same as that of Proposition 5.1. Indeed, with the same notation, one sees that

$$\begin{aligned}
&\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + z\frac{\Psi_\lambda''(z)}{\Psi_\lambda'(z)}\right\} d\theta \\
&\quad= \int_{\theta_1}^{\theta_2} \left[1-\alpha + \alpha \operatorname{Re}\left\{z\frac{\varphi''(z)}{\varphi'(z)} + \frac{\lambda z\omega'(z)}{1+\lambda\omega_\alpha(z)} + \frac{z\omega'}{1-\omega_\alpha(z)}\right\}\right] d\theta \\
&\quad\ge (1-2\alpha)(\theta_2-\theta_1) + \operatorname{Arg}\left\{\frac{1+\lambda\alpha\omega(re^{i\theta_2})}{1+\lambda\alpha\omega(re^{i\theta_1})}\cdot\frac{1-\alpha\omega(re^{i\theta_2})}{1-\alpha\omega(re^{i\theta_1})}\right\} \\
&\quad> -4\arcsin(\alpha) > -\pi
\end{aligned}$$

for all $0 \le \theta_2 - \theta_1 \le 2\pi$ and $0 \le \alpha < 1/2$, which implies that Ψ_λ is a close to convex mapping in the unit disk if $0 \le \alpha < 1/2$. The same conclusion is obtained if we assume $1/2 \le \alpha < 0.5605$ since in this case

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + z\frac{\Psi_\lambda''(z)}{\Psi_\lambda'(z)}\right\} d\theta \ge 2\pi(1-2\alpha) - 4\arcsin(\alpha) > -\pi$$

for all $0 \le \theta_2 - \theta_1 \le 2\pi$. On the other hand,

$$\begin{aligned}
\left|\operatorname{Arg}\{\Psi_\lambda'(z)\}\right| &= \left|\operatorname{Arg}\left\{(\varphi'(z))^\alpha \frac{1+\lambda\alpha\omega(z)}{1-\alpha\omega(z)}\right\}\right| \\
&\le \pi|\alpha| + 2\arcsin(r|\alpha|), \quad r = |z| \\
&< \pi/2
\end{aligned}$$

if $|\alpha| < 0.303$. The proof is completed by proceeding as at the end of the proof of Proposition 5.1.

Here we have used the fact that 0.5605 and 0.303 are the approximate roots of $3\pi - 4\pi x - 4\arcsin(x) = 0$ and $\pi x - \pi/2 + 2\arcsin(x) = 0$, respectively. □

The proof of the following proposition is essentially the same as that of Proposition 5.2; so we omit its proof.

Proposition 5.4 *Let $f = h\overline{g}$ be a non-vanishing logharmonic mapping defined in the unit disk with $\|\omega\| < 1/3$, and let F_α be defined by equation (16). If φ is a convex function, then F_α is univalent for $\alpha \in [0,1]$.*

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