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(r_1, r_2)-Cesàro summable sequence space of non-absolute type and the involved pre-quasi ideal

Awad A. Bakery[1,2*] and OM Kalthum S.K. Mohamed[1,3]

*Correspondence:
aabhassan@uj.edu.sa;
awad_bakery@yahoo.com;
awad_bakry@hotmail.com
[1]Department of Mathematics, College of Science and Arts at Khulis, University of Jeddah, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Abbassia, Cairo, 11566, Egypt
Full list of author information is available at the end of the article

Abstract
We suggest a sufficient setting on any linear space of sequences $\mathcal{V}$ such that the class $\mathbb{B}^s_{\mathcal{V}}$ of all bounded linear mappings between two arbitrary Banach spaces with the sequence of s-numbers in $\mathcal{V}$ constructs a map ideal. We define a new sequence space $(ces^t_{r_1,r_2})_\upsilon$ for definite functional υ by the domain of (r_1, r_2)-Cesàro matrix in ℓ_t, where $r_1, r_2 \in (0, \infty)$ and $1 \leq t < \infty$. We examine some geometric and topological properties of the multiplication mappings on $(ces^t_{r_1,r_2})_\upsilon$ and the pre-quasi ideal $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$.

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1 Introduction

Finding out about (r_1, r_2)-mathematics or (r_1, r_2)-analogues of recognized consequences dates back to the time of Euler. It has several functions in the discipline of arithmetic specifically in the area of dynamical systems, combinatorics, special functions, quantum groups, learning about fractals and multi-fractal measures, and so forth. By (r_1, r_2)-analogue of a recognized expression, we suggest the generalization of that expression, the use of new parameters (r_1, r_2), which returns again to the authentic expression as $(r_1, r_2) \to (1, 1)$. In functional analysis, the multiplication mappings, and mapping ideals have an important role in spectrum theorem, fixed point theorem, the topological and geometric structure of Banach spaces, etc. We use the following conventions throughout the article; if others are used, we will state them.

Conventions 1.1 ([1, 2])

$\mathrm{N} = \{0, 1, 2, \dots\}$. $\mathcal{C}$: The complex numbers.
$\mathfrak{F}$: The space of all sets with a finite number of elements.
$\mathcal{C}^{\mathrm{N}}$: The space of all sequences of complex numbers.
ℓ_∞: The space of bounded sequences of complex numbers.
ℓ_r: The space of r-absolutely summable sequences of complex numbers.

c_0: The space of null sequences of complex numbers.
$e_l = (0, 0, \ldots, 1, 0, 0, \ldots)$, as 1 lies at the l^{th} coordinate for all $l \in \mathrm{N}$.
$\mathcal{F}$: The space of all sequences with infinite zero coordinates.
$\Im$: The space of all increasing sequences of real numbers.
$\mathbb{B}(\mathcal{P}, \mathcal{Q})$: The space of all bounded linear mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathbb{B}(\mathcal{P})$: The space of all bounded linear mappings from a Banach space $\mathcal{P}$ into itself.
$\mathbb{F}(\mathcal{P}, \mathcal{Q})$: The space of finite rank mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathbb{F}(\mathcal{P})$: The space of finite rank mappings from a Banach space $\mathcal{P}$ into itself.
$\mathcal{A}(\mathcal{P}, \mathcal{Q})$: The space of approximable mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathcal{A}(\mathcal{P})$: The space of approximable mappings from a Banach space $\mathcal{P}$ into itself.
$\mathcal{K}(\mathcal{P}, \mathcal{Q})$: The space of compact mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathcal{K}(\mathcal{P})$: The space of compact mappings from a Banach space $\mathcal{P}$ into itself.

Lemma 1.2 ([2]) *If $U \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $U \notin \mathcal{A}(\mathcal{P}, \mathcal{Q})$, then we have mappings $X \in \mathbb{B}(\mathcal{P})$ and $Y \in \mathbb{B}(\mathcal{Q})$ such that $YUXe_l = e_l$ with $l \in \mathrm{N}$.*

Definition 1.3 ([2]) A Banach space $\mathcal{V}$ is known as simple if the algebra $\mathbb{B}(\mathcal{V})$ contains one and only one nontrivial closed ideal.

Theorem 1.4 ([2]) *Assume that $\mathcal{V}$ is an infinite dimensional Banach space, then*

$$\mathbb{F}(\mathcal{V}) \subsetneqq \mathcal{A}(\mathcal{V}) \subsetneqq \mathcal{K}(\mathcal{V}) \subsetneqq \mathbb{B}(\mathcal{V}).$$

Definition 1.5 ([3]) A mapping $U \in \mathbb{B}(\mathcal{V})$ is known as Fredholm if $\dim(\mathrm{Range}(U))^c < \infty$, $\dim(\ker(U)) < \infty$, and $\mathrm{Range}(U)$ is closed, where $(\mathrm{Range}(U))^c$ describes the complement of $\mathrm{Range}(U)$.

Definition 1.6 ([4]) A subclass $\mathbb{W} \subseteq \mathbb{B}$ is known as an ideal if all elements $\mathbb{W}(\mathcal{P}, \mathcal{Q}) = \mathbb{W} \cap \mathbb{B}(\mathcal{P}, \mathcal{Q})$ satisfy the following conditions:

(i) $I_\Omega \in \mathbb{W}$ if Ω indicates a Banach space of one dimension;
(ii) $\mathbb{W}(\mathcal{P}, \mathcal{Q})$ is a linear space on $\mathcal{C}$;
(iii) If $X \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $Y \in \mathbb{W}(\mathcal{P}, \mathcal{Q})$, and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, then $ZYX \in \mathbb{W}(\mathcal{P}_0, \mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are normed spaces.

Definition 1.7 ([5]) A function $\Psi : \mathbb{W} \to [0, \infty)$ is known as a pre-quasi norm on the ideal $\mathbb{W}$ if the following setting is confirmed:

(1) For all $X \in \mathbb{W}(\mathcal{P}, \mathcal{Q})$, $\Psi(X) \geq 0$ and $\Psi(X) = 0 \Longleftrightarrow X = 0$;
(2) One has $E_0 \geq 1$ such that $\Psi(\kappa X) \leq E_0|\kappa|\Psi(X)$ for all $X \in \mathbb{W}(\mathcal{P}, \mathcal{Q})$ and $\kappa \in \mathcal{C}$;

(3) One has $G_0 \geq 1$ such that $\Psi(Z_1 + Z_2) \leq G_0[\Psi(Z_1) + \Psi(Z_2)]$ for every $Z_1, Z_2 \in \mathbb{W}(\mathcal{P}, \mathcal{Q})$;
(4) One has $D_0 \geq 1$ such that if $X \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $Y \in \mathbb{W}(\mathcal{P}, \mathcal{Q})$, and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, then $\Psi(ZYX) \leq D_0\|Z\|\Psi(Y)\|X\|$.

Theorem 1.8 ([5]) *If Ψ is a quasi norm on the ideal $\mathbb{W}$, then Ψ is a pre-quasi norm on the ideal $\mathbb{W}$.*

Definition 1.9 ([6]) An s-number function is a map acting on $\mathbb{B}(\mathcal{P}, \mathcal{Q})$, which gives to each map $X \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ a nonnegative scaler sequence $(s_l(X))_{l=0}^{\infty}$ satisfying the following set-up:
(a) $\|X\| = s_0(X) \geq s_1(X) \geq s_2(X) \geq \cdots \geq 0$ for every $X \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$;
(b) $s_{l+a-1}(X_1 + X_2) \leq s_l(X_1) + s_a(X_2)$ for each $X_1, X_2 \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $l, a \in \mathrm{N}$;
(c) Ideal property: $s_a(ZYX) \leq \|Z\|s_a(Y)\|X\|$ for all $X \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are any Banach spaces;
(d) If $G \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $\gamma \in \mathcal{C}$, then $s_a(\gamma G) = |\gamma|s_a(G)$;
(e) Rank property: Suppose $\operatorname{rank}(X) \leq a$, then $s_a(X) = 0$ for all $X \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$;
(f) Norming property: $s_{l\geq a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a denotes the unit mapping on the a-dimensional Hilbert space ℓ_2^a.

We mention here some examples of s-numbers:
(1) The ath Kolmogorov number, described by $d_a(X)$, is marked by

$$d_a(X) = \inf_{\dim J \leq a} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|.$$

(2) The ath approximation number, described by $\alpha_a(X)$, is marked by

$$\alpha_a(X) = \inf\bigl\{\|X - Y\| : Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \text{ and } \operatorname{rank}(Y) \leq a\bigr\}.$$

Notations 1.10 ([5])

$$\begin{aligned}
&\mathbb{B}^s_{\mathcal{V}} := \bigl\{\mathbb{B}^s_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}); \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces}\bigr\},\\
&\quad \text{where } \mathbb{B}^s_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) := \bigl\{X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : \bigl(\bigl(s_a(X)\bigr)_{a=0}^{\infty} \in \mathcal{V}\bigr\},\\
&\mathbb{B}^\alpha_{\mathcal{V}} := \bigl\{\mathbb{B}^\alpha_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}); \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces}\bigr\},\\
&\quad \text{where } \mathbb{B}^\alpha_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) := \bigl\{X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : \bigl(\bigl(\alpha_a(X)\bigr)_{a=0}^{\infty} \in \mathcal{V}\bigr\},\\
&\mathbb{B}^d_{\mathcal{V}} := \bigl\{\mathbb{B}^d_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}); \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces}\bigr\},\\
&\quad \text{where } \mathbb{B}^d_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) := \bigl\{X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : \bigl(\bigl(d_a(X)\bigr)_{a=0}^{\infty} \in \mathcal{V}\bigr\}.
\end{aligned}$$

Theorem 1.11 ([7]) *For s-type $\mathcal{V}_\upsilon := \{f = (s_r(X)) \in \mathcal{C}^{\mathrm{N}} : X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \text{ and } \upsilon(f) < \infty\}$. If $\mathbb{B}^s_{\mathcal{V}_\upsilon}$ is a map ideal, then the following conditions are verified:*
1. *$\mathcal{F} \subset$ s-type $\mathcal{V}_\upsilon$.*
2. *Assume $(s_r(X_1))_{r=0}^{\infty} \in$ s-type $\mathcal{V}_\upsilon$ and $(s_r(X_2))_{r=0}^{\infty} \in$ s-type $\mathcal{V}_\upsilon$, then $(s_r(X_1 + X_2))_{r=0}^{\infty} \in$ s-type $\mathcal{V}_\upsilon$.*
3. *If $\lambda \in \mathcal{C}$ and $(s_r(X))_{r=0}^{\infty} \in$ s-type $\mathcal{V}_\upsilon$, then $|\lambda|(s_r(X))_{r=0}^{\infty} \in$ s-type $\mathcal{V}_\upsilon$.*

4. *The sequence space* $\mathcal{V}_\upsilon$ *is solid, i.e., if* $(s_r(Y))_{r=0}^\infty \in s$*-type* $\mathcal{V}_\upsilon$ *and* $s_r(X) \leq s_r(Y)$ *for all* $r \in \mathrm{N}$ *and* $X, Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, *then* $(s_r(X))_{r=0}^\infty \in s$*-type* $\mathcal{V}_\upsilon$.

Some mapping ideals in the class of Banach spaces or Hilbert spaces are generated by sequence spaces of numbers. As the ideal of compact mappings is generated by c_0 and $d_a(X)$ with $X \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$. Pietsch [2] discussed the quasi-ideals $\mathbb{B}^\alpha_{\ell_b}$ when $0 < b < \infty$. He examined that the ideals of nuclear mappings and of Hilbert–Schmidt mappings between Hilbert spaces are constructed by ℓ_1 and ℓ_2, respectively. He showed that $\mathbb{F}(\ell_b)$ are dense in $\mathbb{B}(\ell_b)$, and the algebra $\mathbb{B}(\ell_b)$, where $(1 \leq b < \infty)$, produced simple Banach space. Pietsch [8] showed that $\mathbb{B}^\alpha_{\ell_b}$, with $0 < b < \infty$, is small. Makarov and Faried [9] proved that, for every infinite dimensional Banach space $\mathcal{P}$, $\mathcal{Q}$ and $r > b > 0$, then $\mathbb{B}^\alpha_{\ell_b}(\mathcal{P}, \mathcal{Q}) \subsetneqq \mathbb{B}^\alpha_{\ell_r}(\mathcal{P}, \mathcal{Q}) \subsetneqq \mathbb{B}(\mathcal{P}, \mathcal{Q})$. Yaying et al. [10] introduced the sequence space χ^t_r whose r-Cesàro matrix in ℓ_t with $r \in (0, 1]$ and $1 \leq t \leq \infty$. They investigated the quasi Banach ideal of type χ^t_r for $r \in (0, 1]$ and $1 < t < \infty$. They found its Schauder basis, α-, β-, and γ-duals, and determined certain matrix classes related to this sequence space. Başarir and Kara suggested the compact mappings on some Euler $B(m)$-difference sequence spaces [11], some difference sequence spaces of weighted means [12], the Riesz $B(m)$-difference sequence space [13], the B-difference sequence space derived by weighted mean [14], and the mth order difference sequence space of generalized weighted mean [15]. Mursaleen and Noman [16, 17] introduced the compact mappings on some difference sequence spaces. The multiplication maps on Cesàro sequence spaces with the Luxemburg norm were examined by Komal et al. [18]. İlkhan et al. [19] considered the multiplication maps on Cesàro second order function spaces. In the near past, several authors in the literature investigated some non-absolute type sequence spaces and introduced recent high quality papers; for example, Mursaleen and Noman [20] defined the sequence spaces ℓ^λ_p and ℓ^λ_∞ of non-absolute type and showed that the spaces ℓ^λ_p and ℓ^λ_p are linearly isomorphic for $0 < p \leq \infty$, ℓ^λ_p is a p-normed space and a BK-space in the cases for $0 < p < 1$ and $1 \leq p \leq \infty$, and formed the basis for the space ℓ^λ_p for $1 \leq p < \infty$. In [21], they studied the α-, β-, and γ-duals of ℓ^λ_p and ℓ^λ_∞ of non-absolute type for $1 \leq p < \infty$. They characterized some related matrix classes and derived the characterizations of some other classes by means of a given basic lemma. On Cesàro summable sequences, Mursaleen and Başar [22] defined some spaces of double sequences whose Cesàro transforms are bounded, convergent in the Pringsheim sense, null in the Pringsheim sense, both convergent in the Pringsheim sense and bounded, regularly convergent and absolutely q-summable, respectively, and examined some topological properties of those sequence spaces. The next inequality will be used in the sequel [23]: Suppose $1 \leq t < \infty$ and $x_a, z_a \in \mathcal{C}$, then

$$|x_a + z_a|^t \leq 2^{t-1}\big(|x_a|^t + |z_a|^t\big). \tag{1}$$

The design of this article is arranged as follows: In Sect. 2, we investigate sufficient conditions on any linear space of sequences $\mathcal{V}$ so that $\mathbb{B}^s_{\mathcal{V}}$ describes a mapping ideal. We apply this theorem on $(ces^t_{r_1,r_2})_\upsilon$ for definite functional υ. We examine the sufficient conditions on it to generate a pre-quasi Banach sequence space. In Sect. 3, we define a multiplication map on $(ces^t_{r_1,r_2})_\upsilon$ and introduce the necessity and sufficient conditions on this sequence space in order for the multiplication mapping to be bounded, approximable, invertible, Fredholm, and closed range. In Sect. 4, firstly, we give the sufficient conditions (not necessary) on $(ces^t_{r_1,r_2})_\upsilon$ so that $\overline{\mathbb{F}} = \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$. This gives a counter example of Rhoades [24] open problem about the linearity of s-type $(ces^t_{r_1,r_2})_\upsilon$ spaces. Secondly, we explore the set-up on $(ces^t_{r_1,r_2})_\upsilon$ so that $\mathbb{B}^s_{ces^t_{r_1,r_2}}$ is Banach and closed. Thirdly, we offer the sufficient set-up on $(ces^t_{r_1,r_2})_\upsilon$ in order for $\mathbb{B}^\alpha_{(ces^t_{r_1,r_2})_\upsilon}$ to be strictly confined for distinct powers. We advance the conditions so that $\mathbb{B}^\alpha_{(ces^t_{r_1,r_2})_\upsilon}$ is minimum. Fourthly, we make known the conditions in order that the $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$ is a simple Banach space. Fifthly, we declare the sufficient set-up on $(ces^t_{r_1,r_2})_\upsilon$ such that the class of all bounded linear mappings whose sequence of eigenvalues in $(ces^t_{r_1,r_2})_\upsilon$ is strictly contained in $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$. In Sect. 5, we give our conclusion.

2 The sequence space $(ces^t_{r_1,r_2})_\upsilon$

We introduce in this section the definition of the sequence space $(ces^t_{r_1,r_2})_\upsilon$ under the functional υ. We suggest a subspace of any linear space of sequences $\mathcal{V}$ (private sequence space ($\mathfrak{pss}$)) such that the class $\mathbb{B}^s_{\mathcal{V}}$ generates an ideal. We apply these conditions on $(ces^t_{r_1,r_2})_\upsilon$ equipped with definite functional υ to create a pre-modular $\mathfrak{pss}$ and a pre-quasi Banach $\mathfrak{pss}$.

Definition 2.1 For all $r_1, r_2 \in (0,\infty)$ and $1 \le t < \infty$, the sequence space $(ces^t_{r_1,r_2})_\upsilon$ under the functional υ is defined as follows:

$$\begin{aligned}
&\left(ces^t_{r_1,r_2}\right)_\upsilon = \left\{f = (f_k) \in \mathcal{C}^{\mathrm{N}} : \upsilon(\rho f) < \infty \text{ for every } \rho > 0\right\}, \\
&\quad \text{as } \upsilon(f) = \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} r_1^z r_2^{l-z} f_z|}{[l+1]_{r_1,r_2}}\right)^t \quad \text{and} \\
&[l]_{r_1,r_2} = \sum_{z=0}^{l-1} r_1^z r_2^{l-1-z} = \begin{cases} \frac{r_1^l - r_2^l}{r_1 - r_2}, & r_1 \neq r_2 \neq 1, \\ l r_1^{l-1}, & r_1 = r_2 \neq 1, \\ [l]_{r_2}, & r_1 = 1, \\ [l]_{r_1}, & r_2 = 1, \\ l, & r_1 = r_2 = 1. \end{cases}
\end{aligned}$$

Remark 2.2

(1) Assume $r_1 = r$ and $r_2 = 1$, the sequence space $ces^t_{r_1,r_2} = \chi^t_r$ was investigated by Yaying et al. [10].

(2) If $r_1 = r_2 = 1$, hence $ces^t_{r_1,r_2} = ces^t$, was made current and considered by Ng and Lee [25]. Distinctive classification of ces^t has been examined by many authors [21, 26–29].

Theorem 2.3 *If* $r_1, r_2 \in (0,\infty)$ *and* $1 \le t < \infty$, *then* $(ces^t_{r_1,r_2})_\upsilon$ *is of non-absolute type.*

Proof By taking $f = (-1, 1, 0, 0, 0, \dots)$, then $|f| = (1, 1, 0, 0, 0, \dots)$. We have

$$\begin{aligned}
\upsilon(f) &= 1 + \left(\frac{|-r_2 + r_1|}{[2]_{r_1,r_2}}\right)^t + \left(\frac{|-r_2^2 + r_1 r_2|}{[3]_{r_1,r_2}}\right)^t + \dots \\
&\neq 1 + \left(\frac{|r_2 + r_1|}{[2]_{r_1,r_2}}\right)^t + \left(\frac{|r_2^2 + r_1 r_2|}{[3]_{r_1,r_2}}\right)^t + \dots = \upsilon\big(|f|\big).
\end{aligned}$$

Therefore, the sequence space $(ces^t_{r_1,r_2})_\upsilon$ is of non-absolute type.

We name the sequence space $(ces^t_{r_1,r_2})_\upsilon$ as (r_1,r_2)-Cesàro summable sequence space of non-absolute type since it is constructed by the domain of (r_1,r_2)-Cesàro matrix in ℓ_t, where the (r_1,r_2)-Cesàro matrix $\Lambda(r_1,r_2) = (\lambda_{lz}(r_1,r_2))$ is defined as

$$\lambda_{lz}(r_1,r_2) = \begin{cases} \frac{r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}, & 0 \le z \le l, \\ 0, & z > l. \end{cases}$$ □

Definition 2.4 Pick up a linear space of sequences $\mathcal{V}$. The subspace $\mathcal{V}$ is known as a $\mathfrak{pss}$ if it supports the next set-up:

(1) $e_b \in \mathcal{V}$ for each $b \in \mathbb{N}$;

(2) If $f = (f_b) \in \mathcal{C}^{\mathbb{N}}$, $|g| = (|g_b|) \in \mathcal{V}$, and $|f_b| \le |g_b|$ for $b \in \mathbb{N}$, then $|f| \in \mathcal{V}$, i.e., $\mathcal{V}$ is solid;

(3) For $(|f_b|)_{b=0}^\infty \in \mathcal{V}$, we have $(|f_{[\frac{b}{2}]}|)_{b=0}^\infty \in \mathcal{V}$, where $[\frac{b}{2}]$ indicates the integral part of $\frac{b}{2}$.

Theorem 2.5 *Assume that the linear sequence space $\mathcal{V}$ is a $\mathfrak{pss}$, then $\mathbb{B}^s_{\mathcal{V}}$ is an ideal.*

Proof Similar to the proof of Theorem 3.2 in [5]. □

Theorem 2.6 *$ces^t_{r_1,r_2}$ is a $\mathfrak{pss}$ whenever $1 < t < \infty$ and $r_1 \le r_2$.*

Proof (1-i) Let $f, g \in ces^t_{r_1,r_2}$. Since $1 < t < \infty$, we obtain

$$\begin{aligned} &\sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z}(f_z+g_z)|}{[l+1]_{r_1,r_2}} \right)^t \\ &\quad \le 2^{t-1} \left(\sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} f_z|}{[l+1]_{r_1,r_2}} \right)^t + \sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} g_z|}{[l+1]_{r_1,r_2}} \right)^t \right) < \infty, \end{aligned}$$

hence $f + g \in ces^t_{r_1,r_2}$.

(1-ii) Assume $\rho \in \mathcal{C}, f \in ces^t_{r_1,r_2}$, and since $1 < t < \infty$, one has

$$\sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} \rho f_z|}{[l+1]_{r_1,r_2}} \right)^t = |\rho|^t \sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} f_z|}{[l+1]_{r_1,r_2}} \right)^t < \infty.$$

So $\rho f \in ces^t_{r_1,r_2}$. By using (1-i) and (1-ii), one has $ces^t_{r_1,r_2}$ is a linear space.

Besides, when $1 < t < \infty$, we have

$$\sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} (e_b)_z|}{[l+1]_{r_1,r_2}} \right)^t = r_1^{bt} r_2^{(l-b)t} \sum_{l=b}^\infty \left(\frac{1}{[l+1]_{r_1,r_2}} \right)^t < \infty.$$

Hence, $e_b \in ces^t_{r_1,r_2}$ for each $b \in \mathbb{N}$.

(2) Suppose $|f_b| \le |g_b|$ for all $b \in \mathbb{N}$ and $|g| \in ces^t_{r_1,r_2}$. We have

$$\sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} |f_z||}{[l+1]_{r_1,r_2}} \right)^t \le \sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l r_1^z r_2^{l-z} |g_z||}{[l+1]_{r_1,r_2}} \right)^t < \infty,$$

so $|f| \in ces^t_{r_1,r_2}$.

(3) If $(|f_z|) \in ces^t_{r_1,r_2}$, $1 < t < \infty$, and $r_1 \leq r_2$, one has

$$\begin{aligned}
&\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z}|f_{[\frac{z}{2}]}|}{[l+1]_{r_1,r_2}}\right)^t \\
&= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{2l} r_1^z r_2^{l-z}|f_{[\frac{z}{2}]}|}{[2l+1]_{r_1,r_2}}\right)^t + \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{2l+1} r_1^z r_2^{l-z}|f_{[\frac{z}{2}]}|}{[2l+2]_{r_1,r_2}}\right)^t \\
&\leq \sum_{l=0}^{\infty}\left(\frac{1}{[2l+1]_{r_1,r_2}}\left(r_1^{2l} r_2^{-l}|f_l| + \sum_{z=0}^{l}\left(r_1^{2z} r_2^{l-2z} + r_1^{2z+1} r_2^{l-2z-1}\right)|f_z|\right)\right)^t \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{1}{[2l+2]_{r_1,r_2}}\sum_{z=0}^{l}\left(r_1^{2z} r_2^{l-2z} + r_1^{2z+1} r_2^{l-2z-1}\right)|f_z|\right)^t \\
&\leq 2^{t-1}\left(\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_{r_1,r_2}}\sum_{z=0}^{l} r_1^z r_2^{l-z}|f_z|\right)^t + \sum_{l=0}^{\infty}\left(\frac{2}{[l+1]_{r_1,r_2}}\sum_{z=0}^{l} r_1^z r_2^{l-z}|f_z|\right)^t\right) \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{2}{[l+1]_{r_1,r_2}}\sum_{z=0}^{l} r_1^z r_2^{l-z}|f_z|\right)^t \\
&\leq \left(2^{2t-1} + 3 \times 2^{t-1}\right)\sum_{l=0}^{\infty}\left(\frac{1}{[l+1]_{r_1,r_2}}\sum_{z=0}^{l} r_1^z r_2^{l-z}|f_z|\right)^t < \infty,
\end{aligned}$$

so $(|f_{[\frac{z}{2}]}|) \in ces^t_{r_1,r_2}$. □

From Theorem 2.5, we conclude the following theorem.

Theorem 2.7 *Assume* $1 < t < \infty$ *and* $r_1 \leq r_2$, *then* $\mathbb{B}^s_{ces^t_{r_1,r_2}}$ *is an ideal.*

Definition 2.8 A subclass of the $\mathfrak{pss}$ is called a pre-modular $\mathfrak{pss}$ if the functional $\upsilon : \mathcal{V} \to [0,\infty)$ satisfies the next conditions:

(i) For $f \in \mathcal{V}, f = \theta \iff \upsilon(|f|) = 0$ for all $\upsilon(f) \geq 0$, with θ being the zero vector of $\mathcal{V}$;
(ii) For $f \in \mathcal{V}$ and $\rho \in \mathcal{C}$, one has $E_0 \geq 1$ so that $\upsilon(\rho f) \leq |\rho| E_0 \upsilon(f)$;
(iii) $\upsilon(f+g) \leq G_0(\upsilon(f) + \upsilon(g))$ verifies for some $G_0 \geq 1$, so that $f, g \in \mathcal{V}$;
(iv) If $b \in \mathrm{N}$, $|f_b| \leq |g_b|$, we have $\upsilon((|f_b|)) \leq \upsilon((|g_b|))$;
(v) The inequality $\upsilon((|f_b|)) \leq \upsilon((|f_{[\frac{b}{2}]}|)) \leq D_0 \upsilon((|f_b|))$ is satisfied for some $D_0 \geq 1$;
(vi) $\overline{\mathcal{F}} = \mathcal{V}_\upsilon$;
(vii) There is $\varpi > 0$ with $\upsilon(\rho, 0, 0, 0, \ldots) \geq \varpi |\rho| \upsilon(1, 0, 0, 0, \ldots)$ for all $\rho \in \mathcal{C}$.

Definition 2.9 The $\mathfrak{pss}$ $\mathcal{V}_\upsilon$ is called a pre-quasi normed $\mathfrak{pss}$ if υ satisfies conditions (i)–(iii) of Definition 2.8. When $\mathcal{V}$ is complete under υ, then $\mathcal{V}_\upsilon$ is called a pre-quasi Banach $\mathfrak{pss}$.

Theorem 2.10 *Every pre-modular* $\mathfrak{pss}$ *is a pre-quasi normed* $\mathfrak{pss}$ $\mathcal{V}_\upsilon$.

Theorem 2.11 $(ces^t_{r_1,r_2})_\upsilon$ *is a pre-modular* $\mathfrak{pss}$, *whenever* $1 < t < \infty$ *and* $r_1 \leq r_2$.

Proof (i) We have $\upsilon(f) \geq 0$ and $\upsilon(|f|) = 0 \Leftrightarrow f = \theta$.

(ii) One has $E_0 = \max\{1, |\rho|^{t-1}\} \geq 1$ with $\upsilon(\rho f) \leq E_0|\rho|\upsilon(f)$ for all $f \in ces^t_{r_1,r_2}$ and $\rho \in \mathcal{C}$.
(iii) We have $\upsilon(f+g) \leq 2^{t-1}(\upsilon(f) + \upsilon(g))$ for all $f, g \in ces^t_{r_1,r_2}$.
(iv) Obviously, from the proof part (2) of Theorem 2.6.
(v) Obviously, from the proof part (3) of Theorem 2.6 that $D_0 \geq 2^{2t-1} + 3 \times 2^{t-1} \geq 1$.
(vi) Definitely, $\overline{\mathcal{F}} = ces^t_{r_1,r_2}$.
(vii) One has $0 < \varpi \leq |\rho|^{t-1}$ for $\upsilon(\rho, 0, 0, 0, \ldots) \geq \varpi|\rho|\upsilon(1, 0, 0, 0, \ldots)$ when $\rho \neq 0$ and $\varpi > 0$ when $\rho = 0$. □

Theorem 2.12 *If $1 < t < \infty$ and $r_1 \leq r_2$, then $(ces^t_{r_1,r_2})_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$.*

Proof Let the set-up be satisfied, then from Theorem 2.11 the space $(ces^t_{r_1,r_2})_\upsilon$ is a pre-modular $\mathfrak{pss}$. By using Theorem 2.10, the space $(ces^t_{r_1,r_2})_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. To show that $(ces^t_{r_1,r_2})_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$, assume $f^a = (f^a_z)_{z=0}^\infty$ to be a Cauchy sequence in $(ces^t_{r_1,r_2})_\upsilon$, then for all $\varepsilon \in (0,1)$, there is $a_0 \in \mathrm{N}$ so that, for all $a, b \geq a_0$, one has

$$\upsilon\bigl(f^a - f^b\bigr) = \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} r_1^z r_2^{l-z}(f^a_z - f^b_z)|}{[l+1]_{r_1,r_2}}\right)^t < \varepsilon^t.$$

Hence, for $a, b \geq a_0$ and $z \in \mathrm{N}$, we get $|f^a_z - f^b_z| < \varepsilon$. So (f^b_z) is a Cauchy sequence in $\mathcal{C}$ for fixed $z \in \mathrm{N}$, this gives $\lim_{b\to\infty} f^b_z = f^0_z$ for fixed $z \in \mathrm{N}$. Hence $\upsilon(f^a - f^0) < \varepsilon^t$ for all $a \geq a_0$. Finally, to show that $f^0 \in (ces^t_{r_1,r_2})_\upsilon$, one has $\upsilon(f^0) \leq 2^{t-1}(\upsilon(f^a - f^0) + \upsilon(f^a)) < \infty$, so $f^0 \in (ces^t_{r_1,r_2})_\upsilon$. This means that $(ces^t_{r_1,r_2})_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$. □

Corollary 2.13 *If $1 < t < \infty$, then $(\chi^t_r)_\upsilon$ is a normed Banach $\mathfrak{pss}$, where $\upsilon(f) = [\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l} r^z f_z|}{[l+1]_r})^t]^{\frac{1}{t}}$ for all $f \in \chi^t_r$.*

By using Theorem 1.11, we conclude the following properties of the *s*-type $(ces^t_{r_1,r_2})_\upsilon$.

Theorem 2.14 *For s-type $(ces^t_{r_1,r_2})_\upsilon := \{f = (s_n(X)) \in \mathcal{C}^{\mathrm{N}} : X \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $\upsilon(f) < \infty\}$. The following settings are verified*:

1. *We have s-type $(ces^t_{r_1,r_2})_\upsilon \supset \mathcal{F}$.*
2. *If $(s_r(X_1))_{r=0}^\infty \in$ s-type $(ces^t_{r_1,r_2})_\upsilon$ and $(s_r(X_2))_{r=0}^\infty \in$ s-type $(ces^t_{r_1,r_2})_\upsilon$, then $(s_r(X_1 + X_2))_{r=0}^\infty \in$ s-type $(ces^t_{r_1,r_2})_\upsilon$.*
3. *For all $\lambda \in \mathcal{C}$ and $(s_r(X))_{r=0}^\infty \in$ s-type $(ces^t_{r_1,r_2})_\upsilon$, then $|\lambda|(s_r(X))_{r=0}^\infty \in$ s-type $(ces^t_{r_1,r_2})_\upsilon$.*
4. *The s-type $(ces^t_{r_1,r_2})_\upsilon$ is solid.*

3 Multiplication mappings on $(ces^t_{r_1,r_2})_\upsilon$

We introduce in this section a multiplication mapping on $(ces^t_{r_1,r_2})_\upsilon$. We examine the necessity and sufficient conditions on $(ces^t_{r_1,r_2})_\upsilon$ such that the multiplication mapping is invertible, bounded, Fredholm, approximable, and closed range.

Definition 3.1 Pick up $\omega = (\omega_k) \in \mathcal{C}^{\mathrm{N}}$ and $\mathcal{V}_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. The mapping $H_\omega : \mathcal{V}_\upsilon \to \mathcal{V}_\upsilon$ is named a multiplication mapping on $\mathcal{V}_\upsilon$ if $H_\omega f = (\omega_b f_b) \in \mathcal{V}_\upsilon$, so that $f \in \mathcal{V}_\upsilon$. The multiplication mapping is called produced by ω when $H_\omega \in \mathbb{B}(\mathcal{V}_\upsilon)$.

Theorem 3.2 *Assume $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, and $r_1 \leq r_2$, then $\omega \in \ell_\infty$ if and only if $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$.*

Proof Let the conditions be verified for $\omega \in \ell_\infty$. So there is $\nu > 0$ such that $|\omega_b| \leq \nu$ for all $b \in \mathrm{N}$. If $f \in (ces^t_{r_1,r_2})_\upsilon$, we have

$$\upsilon(H_\omega f) = \upsilon(\omega f) = \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} r_1^z r_2^{l-z}\omega_z f_z|}{[l+1]_{r_1,r_2}}\right)^t \leq \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} \nu r_1^z r_2^{l-z} f_z|}{[l+1]_{r_1,r_2}}\right)^t = \nu^t \upsilon(f).$$

Hence, $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$. However, suppose $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$ and $\omega \notin \ell_\infty$. So, for each $b \in \mathrm{N}$, one has $x_b \in \mathrm{N}$ such that $\omega_{x_b} > b$. We obtain

$$\begin{aligned}\upsilon(H_\omega e_{x_b}) = \upsilon(\omega e_{x_b}) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} r_1^z r_2^{l-z}\omega_z (e_{x_b})_z|}{[l+1]_{r_1,r_2}}\right)^t = \sum_{l=x_b}^{\infty}\left(\frac{r_1^{x_b} r_2^{l-x_b}|\omega_{x_b}|}{[l+1]_{r_1,r_2}}\right)^t \\ &> \sum_{l=x_b}^{\infty}\left(\frac{r_1^{x_b} r_2^{l-x_b} b}{[l+1]_{r_1,r_2}}\right)^t = b^t \upsilon(e_{x_b}).\end{aligned}$$

Therefore, $H_\omega \notin \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$. Hence $\omega \in \ell_\infty$. □

Theorem 3.3 *If* $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, *and* $r_1 \leq r_2$. *Then* $\omega_b = g$ *for all* $b \in \mathrm{N}$ *and* $g \in \mathcal{C}$ *so that* $|g| = 1$ *if and only if* H_ω *is an isometry.*

Proof Assume that the sufficient set-up is confirmed. We have

$$\upsilon(H_\omega f) = \upsilon(\omega f) = \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}\omega_k f_k|}{[l+1]_{r_1,r_2}}\right)^t = \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} |g| r_1^k r_2^{l-k} f_k|}{[l+1]_{r_1,r_2}}\right)^t = \upsilon(f)$$

for all $f \in (ces^t_{r_1,r_2})_\upsilon$. Hence H_ω is an isometry.

Suppose that the necessity set-up is verified and $|\omega_b| < 1$ for some $b = b_0$. One can see

$$\begin{aligned}\upsilon(H_\omega e_{b_0}) = \upsilon(\omega e_{b_0}) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}\omega_k (e_{b_0})_k|}{[l+1]_{r_1,r_2}}\right)^t = \sum_{l=b_0}^{\infty}\left(\frac{r_1^{b_0} r_2^{l-b_0}|\omega_{b_0}|}{[l+1]_{r_1,r_2}}\right)^t \\ &< \sum_{l=b_0}^{\infty}\left(\frac{r_1^{b_0} r_2^{l-b_0}}{[l+1]_{r_1,r_2}}\right)^t = \upsilon(e_{b_0}).\end{aligned}$$

Besides, if $|\omega_{b_0}| > 1$, one has $\upsilon(H_\omega e_{b_0}) > \upsilon(e_{b_0})$. For the two cases, we have a contradiction. So $|\omega_b| = 1$ with $b \in \mathrm{N}$. □

Theorem 3.4 *If* $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, *and* $r_1 \leq r_2$. *Then* $H_\omega \in \mathcal{A}((ces^t_{r_1,r_2})_\upsilon)$ *if and only if* $(\omega_b)_{b=0}^{\infty} \in c_0$.

Proof Suppose $H_\omega \in \mathcal{A}((ces^t_{r_1,r_2})_\upsilon)$, hence $H_\omega \in \mathcal{K}((ces^t_{r_1,r_2})_\upsilon)$. Assume $\lim_{b\to\infty}\omega_b \neq 0$. Hence, one has $\varrho > 0$ such that $K_\varrho = \{b \in \mathrm{N} : |\omega_b| \geq \varrho\} \not\subseteq \mathfrak{F}$ when $\{\alpha_b\}_{b\in\mathrm{N}} \subset K_\varrho$. Therefore, $\{e_{\alpha_b} : \alpha_b \in K_\varrho\} \in \ell_\infty$ is infinite in $(ces^t_{r_1,r_2})_\upsilon$. As

$$\begin{aligned}\upsilon(H_\omega e_{\alpha_a} - H_\omega e_{\alpha_b}) = \upsilon(\omega e_{\alpha_a} - \omega e_{\alpha_b}) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}\omega_k((e_{\alpha_a})_k - (e_{\alpha_b})_k)|}{[l+1]_{r_1,r_2}}\right)^t \\ &\geq \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}\varrho((e_{\alpha_a})_k - (e_{\alpha_b})_k)|}{[l+1]_{r_1,r_2}}\right)^t = \varrho^t \upsilon(e_{\alpha_a} - e_{\alpha_b})\end{aligned}$$

for all $\alpha_a, \alpha_b \in K_\varrho$. Hence, $\{e_{\alpha_b} : \alpha_b \in K_\varrho\} \in \ell_\infty$, which cannot have a convergent subsequence with H_ω. So $H_\omega \notin \mathcal{K}((ces^t_{r_1,r_2})_\upsilon)$. This gives $H_\omega \notin \mathcal{A}((ces^t_{r_1,r_2})_\upsilon)$, which is unreliability. Therefore, $\lim_{b\to\infty} \omega_b = 0$. Conversely, suppose $\lim_{b\to\infty} \omega_b = 0$. Therefore, for each $\varrho > 0$, we have $K_\varrho = \{b \in \mathrm{N} : |\omega_b| \geq \varrho\} \subset \mathfrak{F}$. So, for all $\varrho > 0$, one has $\dim(((ces^t_{r_1,r_2})_\upsilon)_{K_\varrho}) = \dim(\mathcal{C}^{K_\varrho}) < \infty$. Hence $H_\omega \in \mathbb{F}(((ces^t_{r_1,r_2})_\upsilon)_{K_\varrho})$. Assume $\omega_a \in \mathcal{C}^{\mathrm{N}}$, with $a \in \mathrm{N}$, by

$$(\omega_a)_b = \begin{cases} \omega_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise}. \end{cases}$$

Obviously, $H_{\omega_a} \in \mathbb{F}(((ces^t_{r_1,r_2})_\upsilon)_{B_{\frac{1}{a+1}}})$ since $\dim(((ces^t_{r_1,r_2})_\upsilon)_{B_{\frac{1}{a+1}}}) < \infty$ with $a \in \mathrm{N}$. From $1 < t < \infty$ and $r_1 \leq r_2$, we have

$$\begin{aligned}
\upsilon\big((H_\omega - H_{\omega_a})f\big) &= \upsilon\Big(\big((\omega_b - (\omega_a)_b)f_b\big)_{b=0}^\infty\Big) = \sum_{l=0}^\infty \left(\frac{|\sum_{b=0}^l r_1^b r_2^{l-b}(\omega_b - (\omega_a)_b)f_b|}{[l+1]_{r_1,r_2}}\right)^t \\
&= \sum_{l=0, l\in K_{\frac{1}{a+1}}}^\infty \left(\frac{|\sum_{b=0}^l r_1^b r_2^{l-b}(\omega_b - (\omega_a)_b)f_b|}{[l+1]_{r_1,r_2}}\right)^t \\
&\quad + \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^\infty \left(\frac{|\sum_{b=0}^l r_1^b r_2^{l-b}(\omega_b - (\omega_a)_b)f_b|}{[l+1]_{r_1,r_2}}\right)^t \\
&= \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^\infty \left(\frac{|\sum_{b=0}^l r_1^b r_2^{l-b}\omega_b f_b|}{[l+1]_{r_1,r_2}}\right)^t \\
&< \frac{1}{(a+1)^t} \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^\infty \left(\frac{|\sum_{b=0}^l r_1^b r_2^{l-b} f_b|}{[l+1]_{r_1,r_2}}\right)^t \\
&< \frac{1}{(a+1)^t} \sum_{l=0}^\infty \left(\frac{|\sum_{b=0}^l r_1^b r_2^{l-b} f_b|}{[l+1]_{r_1,r_2}}\right)^t = \frac{1}{(a+1)^t}\upsilon(f).
\end{aligned}$$

Therefore, $\|H_\omega - H_{\omega_a}\| \leq \frac{1}{(a+1)^t}$. Then $H_\omega = \lim_{a\to\infty} H_{\omega_a}$ and hence $H_\omega \in \mathcal{A}((ces^t_{r_1,r_2})_\upsilon)$. □

Theorem 3.5 *If $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, and $r_1 \leq r_2$, then $H_\omega \in \mathcal{K}((ces^t_{r_1,r_2})_\upsilon)$ if and only if $(\omega_b)_{b=0}^\infty \in c_0$.*

Proof Evidently, as $\mathcal{A}((ces^t_{r_1,r_2})_\upsilon) \subsetneqq \mathcal{K}((ces^t_{r_1,r_2})_\upsilon)$. □

Corollary 3.6 *If $1 < t < \infty$ and $r_1 \leq r_2$, then $\mathcal{K}((ces^t_{r_1,r_2})_\upsilon) \subsetneqq \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$.*

Proof Since the sequence $\omega = (1, 1, \dots)$ produces the multiplication mapping I on $(ces^t_{r_1,r_2})_\upsilon$. So, $I \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$ and $I \notin \mathcal{K}((ces^t_{r_1,r_2})_\upsilon)$. □

Theorem 3.7 *If $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, $r_1 \leq r_2$, and $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$. Then there exist $\alpha > 0$ and $\eta > 0$ such that $\alpha < |\omega_b| < \eta$ for all $b \in (\ker(\omega))^c$ if and only if $\mathrm{Range}(H_\omega)$ is closed.*

Proof Let the sufficient setting be verified. So there is $\varrho > 0$ such that $|\omega_b| \geq \varrho$ for all $b \in (\ker(\omega))^c$. To prove that $\mathrm{Range}(H_\omega)$ is closed, suppose that g is a limit point of $\mathrm{Range}(H_\omega)$.

One has $H_\omega f_b \in (ces^t_{r_1,r_2})_\upsilon$ for each $b \in \mathrm{N}$ so as to $\lim_{b\to\infty} H_\omega f_b = g$. Clearly, the sequence $H_\omega f_b$ is a Cauchy sequence. We have

$$\begin{aligned}
\upsilon(H_\omega f_a - H_\omega f_b) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}(\omega_k(f_a)_k - \omega_k(f_b)_k)|}{[l+1]_{r_1,r_2}}\right)^t \\
&= \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}(\omega_k(f_a)_k - \omega_k(f_b)_k)|}{[l+1]_{r_1,r_2}}\right)^t \\
&\quad + \sum_{l=0, l\notin(\ker(\omega))^c}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}(\omega_k(f_a)_k - \omega_k(f_b)_k)|}{[l+1]_{r_1,r_2}}\right)^t \\
&\geq \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}(\omega_k(f_a)_k - \omega_k(f_b)_k)|}{[l+1]_{r_1,r_2}}\right)^t \\
&= \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}(\omega_k(u_a)_k - \omega_k(u_b)_k)|}{[l+1]_{r_1,r_2}}\right)^t \\
&> \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l} r_1^k r_2^{l-k}\varrho((u_a)_k - (u_b)_k)|}{[l+1]_{r_1,r_2}}\right)^t = \varrho^t \upsilon(u_a - u_b),
\end{aligned}$$

where

$$(u_a)_k = \begin{cases} (f_a)_k, & k \in (\ker(\omega))^c, \\ 0, & k \notin (\ker(\omega))^c. \end{cases}$$

Then (u_a) is a Cauchy sequence in $(ces^t_{r_1,r_2})_\upsilon$. Since $(ces^t_{r_1,r_2})_\upsilon$ is complete, there is $f \in (ces^t_{r_1,r_2})_\upsilon$ so as to $\lim_{b\to\infty} u_b = f$. As $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$, one has $\lim_{b\to\infty} H_\omega u_b = H_\omega f$. From $\lim_{b\to\infty} H_\omega u_b = \lim_{b\to\infty} H_\omega f_b = g$. Therefore, $H_\omega f = g$. So $g \in \mathrm{Range}(H_\omega)$. Hence, $\mathrm{Range}(H_\omega)$ is closed. Then, let the necessity conditions be verified. Hence, there is $\varrho > 0$ such that $\upsilon(H_\omega f) \geq \varrho\upsilon(f)$ for all $f \in ((ces^t_{r_1,r_2})_\upsilon)_{(\ker(\omega))^c}$. Suppose $K = \{b \in (\ker(\omega))^c : |\omega_b| < \varrho\} \neq \emptyset$, hence if $a_0 \in K$, we have

$$\begin{aligned}
\upsilon(H_\omega e_{a_0}) = \upsilon\big(\big(\omega_b(e_{a_0})_b\big)_{b=0}^{\infty}\big) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{b=0}^{l} r_1^b r_2^{l-b}\omega_b(e_{a_0})_b)|}{[l+1]_{r_1,r_2}}\right)^t \\
&< \sum_{l=0}^{\infty}\left(\frac{|\sum_{b=0}^{l} r_1^b r_2^{l-b}(e_{a_0})_b\varrho|}{[l+1]_{r_1,r_2}}\right)^t = \varrho^t\upsilon(e_{a_0}),
\end{aligned}$$

this implies unreliability. Hence, $K = \phi$, one can see $|\omega_b| \geq \varrho$ for every $b \in (\ker(\omega))^c$. This shows the theorem. □

Theorem 3.8 *If* $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, *and* $r_1 \leq r_2$. *Then there are* $\alpha > 0$ *and* $\eta > 0$ *such that* $\alpha < |\omega_b| < \eta$ *for all* $b \in \mathrm{N}$ *if and only if* $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$ *is invertible.*

Proof Let the sufficient setting be confirmed. Assume $\kappa \in \mathcal{C}^{\mathrm{N}}$ with $\kappa_b = \frac{1}{\omega_b}$. By Theorem 3.2, one has $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$ and $H_\kappa \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$. Therefore, $H_\omega.H_\kappa = H_\kappa.H_\omega = I$. Therefore, $H_\kappa = H_\omega^{-1}$. Then, let H_ω be invertible. So $\mathrm{Range}(H_\omega) = ((ces^t_{r_1,r_2})_\upsilon)_{\mathrm{N}}$. Hence $\mathrm{Range}(H_\omega)$ is closed. By Theorem 3.7, one has $\alpha > 0$ such that $|\omega_b| \geq \alpha$ with $b \in (\ker(\omega))^c$.

We have $\ker(\omega) = \emptyset$, if $\omega_{b_0} = 0$ for $b_0 \in \mathrm{N}$, this gives $e_{b_0} \in \ker(H_\omega)$, which is unreliability, as $\ker(H_\omega)$ is trivial. Hence, $|\omega_b| \geq \alpha$ with $b \in \mathrm{N}$. As $H_\omega \in \ell_\infty$. From Theorem 3.2, we have $\eta > 0$ such that $|\omega_b| \leq \eta$ with $b \in \mathrm{N}$. Hence, we have $\alpha \leq |\omega_b| \leq \eta$ with $b \in \mathrm{N}$. □

Theorem 3.9 *If* $\omega \in \mathcal{C}^{\mathrm{N}}$, $1 < t < \infty$, $r_1 \leq r_2$, *and* $H_\omega \in \mathbb{B}((ces^t_{r_1,r_2})_\upsilon)$. *Then* H_ω *is a Fredholm mapping if and only if* (*i*) $\ker(\omega) \subsetneqq \mathrm{N}$ *is finite and* (*ii*) $|\omega_b| \geq \varrho$ *with* $b \in (\ker(\omega))^c$.

Proof Let the sufficient conditions be confirmed. Assume that $\ker(\omega) \subsetneqq \mathrm{N}$ is infinite, so $e_b \in \ker(H_\omega)$ for $b \in \ker(\omega)$. As e_bs are linearly independent, this implies that $\dim(\ker(H_\omega)) = \infty$, which is unreliability. As $\ker(\omega) \subsetneqq \mathrm{N}$ must be finite. Setting (ii) follows from Theorem 3.7. Next, suppose that conditions (i) and (ii) are satisfied. By Theorem 3.7, one has that setting (ii) gives that $\mathrm{Range}(H_\omega)$ is closed. Condition (i) implies that $\dim(\ker(H_\omega)) < \infty$ and $\dim((\mathrm{Range}(H_\omega))^c) < \infty$. This gives that H_ω is Fredholm. □

4 Configuration of pre-quasi ideal

In this section, firstly, we examine the sufficient conditions (not necessary) on $(ces^t_{r_1,r_2})_\upsilon$ so that $\overline{\mathbb{F}} = \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$, which implies a negative example of Rhoades [24] open problem about the linearity of s-type $(ces^t_{r_1,r_2})_\upsilon$ spaces. Secondly, we give the set-up on $(ces^t_{r_1,r_2})_\upsilon$ so as to $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$ is Banach and closed. Thirdly, we introduce the sufficient conditions on $(ces^t_{r_1,r_2})_\upsilon$ such that $\mathbb{B}^\alpha_{(ces^t_{r_1,r_2})_\upsilon}$ is closely included for distinct t and minimum. Fourthly, we explain the conditions so that the Banach pre-quasi ideal $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$ is simple. Fifthly, we give the sufficient conditions on $(ces^t_{r_1,r_2})_\upsilon$ such that the class $\mathbb{B}$ has its sequence of eigenvalues in $(ces^t_{r_1,r_2})_\upsilon$ closely included in $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$.

4.1 Denseness of finite rank mappings

Theorem 4.1 *The settings* $1 < t < \infty$ *and* $r_1 \leq r_2$ *are sufficient only for* $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P}, \mathcal{Q}) = \overline{\mathbb{F}(\mathcal{P}, \mathcal{Q})}$.

Proof Assume that the sufficient setting is confirmed. Since $e_l \in (ces^t_{r_1,r_2})_\upsilon$ for every $l \in \mathrm{N}$ and $(ces^t_{r_1,r_2})_\upsilon$ is a linear space. Assume $Z \in \mathbb{F}(\mathcal{P}, \mathcal{Q})$, we have $(s_l(Z))_{l=0}^\infty \in \mathcal{F}$. Therefore, $(s_l(Z))_{l=0}^\infty \in (ces^t_{r_1,r_2})_\upsilon$, and this implies $Z \in \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P}, \mathcal{Q})$. So $\overline{\mathbb{F}(\mathcal{P}, \mathcal{Q})} \subseteq \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P}, \mathcal{Q})$. To prove that $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P}, \mathcal{Q}) \subseteq \overline{\mathbb{F}(\mathcal{P}, \mathcal{Q})}$. Since $1 < t < \infty$, we get $\sum_{l=0}^\infty (\frac{1}{[l+1]_{r_1,r_2}})^t < \infty$. Suppose $Z \in \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P}, \mathcal{Q})$, one has $(s_l(Z))_{l=0}^\infty \in (ces^t_{r_1,r_2})_\upsilon$. Since $\upsilon(s_l(Z))_{l=0}^\infty < \infty$, assume $\rho \in (0,1)$, then there exists $l_0 \in \mathrm{N} - \{0\}$ such that $\upsilon((s_l(Z))_{l=l_0}^\infty) < \frac{\rho}{2^{t+3}\eta d}$ for some $d \geq 1$, where $\eta = \max\{1, \sum_{l=l_0}^\infty (\frac{1}{[l+1]_{r_1,r_2}})^t\}$. Since $s_l(Z)$ is decreasing, we have

$$\begin{aligned}
\sum_{l=l_0+1}^{2l_0} \left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j} s_{2l_0}(Z)}{[l+1]_{r_1,r_2}}\right)^t &\leq \sum_{l=l_0+1}^{2l_0} \left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j} s_j(Z)}{[l+1]_{r_1,r_2}}\right)^t \leq \sum_{l=l_0}^{\infty} \left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j} s_j(Z)}{[l+1]_{r_1,r_2}}\right)^t \\
&< \frac{\rho}{2^{t+3}\eta d}. \qquad (2)
\end{aligned}$$

Hence, there exists $Y \in \mathbb{F}_{2l_0}(\mathcal{P}, \mathcal{Q})$ with $\mathrm{rank}(Y) \leq 2l_0$ and

$$\sum_{l=2l_0+1}^{3l_0} \left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j} \|Z - Y\|}{[l+1]_{r_1,r_2}}\right)^t \leq \sum_{l=l_0+1}^{2l_0} \left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j} \|Z - Y\|}{[l+1]_{r_1,r_2}}\right)^t < \frac{\rho}{2^{t+3}\eta d}, \qquad (3)$$

as $1 < t < \infty$, one has

$$\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{l_0} r_1^j r_2^{l-j}\|Z-Y\|\right)^t < \frac{\rho}{2^{2t+2}\eta}. \tag{4}$$

Hence, we have

$$\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t < \frac{\rho}{2^{t+3}\eta d}. \tag{5}$$

Since $1 < t < \infty$ and by using inequalities (1)–(5), we obtain

$$\begin{aligned}
d(Z,Y) &= \upsilon\big(s_l(Z-Y)\big)_{l=0}^{\infty} = \sum_{l=0}^{3l_0-1}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t + \sum_{l=3l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t \\
&\leq \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+2l_0+1]_{r_1,r_2}}\right)^t \\
&\leq \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{i} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t \\
&\quad + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_1^j r_2^{l-j}s_j(Z-Y) + \sum_{j=2l_0}^{l+2l_0} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t + 2^{t-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t\right. \\
&\quad \left. + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=2l_0}^{l+2l_0} r_1^j r_2^{l-j}s_j(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t\right] \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t + 2^{t-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t\right. \\
&\quad \left. + \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_1^{j+2l_0} r_2^{l-j-2l_0}s_{j+2l_0}(Z-Y)}{[l+1]_{r_1,r_2}}\right)^t\right] \\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}\|Z-Y\|}{[l+1]_{r_1,r_2}}\right)^t \\
&\quad + 2^{t-1}\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{2l_0-1} r_1^j r_2^{l-j}\|Z-Y\|\right)^t \sum_{l=l_0}^{\infty}\big([l+1]_{r_1,r_2}\big)^{-t} \\
&\quad + 2^{t-1}\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} r_1^j r_2^{l-j}s_j(Z)}{[l+1]_{r_1,r_2}}\right)^t < \rho.
\end{aligned}$$

On the other hand, since $I_5 \in \mathbb{B}^s_{(ces^1_{0,1})_\upsilon}(\mathcal{P},\mathcal{Q})$ but $t>1$ is not confirmed. This completes the proof. □

Corollary 4.2 *Pick up* $1<t<\infty$ *and* $0<r\leq 1$, *then* $\mathbb{B}^s_{(\chi^t_r)_\upsilon}(\mathcal{P},\mathcal{Q}) = \overline{\mathbb{F}(\mathcal{P},\mathcal{Q})}$.

4.2 Banach and closed pre-quasi ideal

Theorem 4.3 *Assume that* $(\mathcal{V})_\upsilon$ *is a pre-modular* $\mathfrak{pss}$, *then the functional* Ψ *is a pre-quasi norm on* $\mathbb{B}^s_{(\mathcal{V})_\upsilon}$ *with* $\Psi(Z)=\upsilon(s_b(Z))_{b=0}^\infty$ *and* $Z\in\mathbb{B}^s_{(\mathcal{V})_\upsilon}(\mathcal{P},\mathcal{Q})$.

Proof Suppose that $(\mathcal{V})_\upsilon$ is a pre-modular $\mathfrak{pss}$, so Ψ verifies the next set-up:

1. When $X\in\mathbb{B}^s_{(\mathcal{V})_\upsilon}(\mathcal{P},\mathcal{Q})$, $\Psi(X)=\upsilon(s_b(X))_{b=0}^\infty\geq 0$ and $\Psi(X)=\upsilon(s_b(X))_{b=0}^\infty=0$ if and only if $s_b(X)=0$ for all $b\in\mathrm{N}$ if and only if $X=0$;
2. We have $E_0\geq 1$ with $\Psi(\rho X)=\upsilon(s_b(\rho X))_{b=0}^\infty\leq E_0|\rho|\Psi(X)$ for every $X\in\mathbb{B}^s_{(\mathcal{V})_\upsilon}(\mathcal{P},\mathcal{Q})$ and $\rho\in\mathcal{C}$;
3. One has $D\geq 1$ so that, for $X_1,X_2\in\mathbb{B}^s_{(\mathcal{V})_\upsilon}(\mathcal{P},\mathcal{Q})$, one can see

$$\begin{aligned}\Psi(X_1+X_2)=\upsilon\big(s_b(X_1+X_2)\big)_{b=0}^\infty &\leq G_0\big(\upsilon\big(s_{[\frac{b}{2}]}(X_1)\big)_{b=0}^\infty+\upsilon\big(s_{[\frac{b}{2}]}(X_2)\big)_{b=0}^\infty\big)\\ &\leq G_0D_0\big(\upsilon\big(s_b(X_1)\big)_{b=0}^\infty+\upsilon\big(s_b(X_2)\big)_{b=0}^\infty\big)\\ &\leq D\big[\Psi(X_1)+\Psi(X_2)\big];\end{aligned}$$

4. We have $\varrho\geq 1$ if $X\in\mathbb{B}(\mathcal{P}_0,\mathcal{P})$, $Y\in\mathbb{B}^s_{(\mathcal{V})_\upsilon}(\mathcal{P},\mathcal{Q})$, and $Z\in\mathbb{B}(\mathcal{Q},\mathcal{Q}_0)$, then $\Psi(ZYX)=\upsilon(s_b(ZYX))_{b=0}^\infty\leq\upsilon(\|X\|\|Z\|s_b(Y))_{b=0}^\infty\leq\varrho\|X\|\Psi(Y)\|Z\|$. □

Theorem 4.4 *If* $1<t<\infty$ *and* $r_1\leq r_2$, *then* $(\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon},\Psi)$ *is a pre-quasi Banach ideal.*

Proof Since $(ces^t_{r_1,r_2})_\upsilon$ is a pre-modular $\mathfrak{pss}$, by using Theorem 4.3, Ψ is a pre-quasi norm on $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$. If $(X_b)_{b\in\mathrm{N}}$ is a Cauchy sequence in $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. As $\mathbb{B}(\mathcal{P},\mathcal{Q})\supseteq\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$, we have

$$\Psi(X_a-X_b)=\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}s_z(X_a-X_b)}{[l+1]_{r_1,r_2}}\right)^t\geq\sum_{l=0}^\infty\left(\frac{r_2^l\|X_a-X_b\|}{[l+1]_{r_1,r_2}}\right)^t\geq\|X_a-X_b\|^t,$$

hence $(X_b)_{b\in\mathrm{N}}$ is a Cauchy sequence in $\mathbb{B}(\mathcal{P},\mathcal{Q})$. As $\mathbb{B}(\mathcal{P},\mathcal{Q})$ is a Banach space, there exists $X\in\mathbb{B}(\mathcal{P},\mathcal{Q})$ such that $\lim_{b\to\infty}\|X_b-X\|=0$. As $(s_l(X_b))_{l=0}^\infty\in(ces^t_{r_1,r_2})_\upsilon$ with $b\in\mathrm{N}$. Hence, by using Definition 2.8 parts (ii), (iii), and (v), we get

$$\begin{aligned}\Psi(X)&=\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}s_z(X)}{[l+1]_{r_1,r_2}}\right)^t\\ &\leq 2^{t-1}\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}s_{[\frac{z}{2}]}(X-X_b)}{[l+1]_{r_1,r_2}}\right)^t+2^{t-1}\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}s_{[\frac{z}{2}]}(X_b)}{[l+1]_{r_1,r_2}}\right)^t\\ &\leq 2^{t-1}\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}\|X-X_b\|}{[l+1]_{r_1,r_2}}\right)^t+2^{t-1}D_0\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}s_z(X_b)}{[l+1]_{r_1,r_2}}\right)^t<\infty.\end{aligned}$$

Therefore, $(s_l(X))_{l=0}^\infty\in(ces^t_{r_1,r_2})_\upsilon$, hence $X\in\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. □

Theorem 4.5 *If $\mathcal{P}$, $\mathcal{Q}$ are normed spaces, $1 < t < \infty$, and $r_1 \leq r_2$, then $(\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}, \Psi)$ is a pre-quasi closed ideal.*

Proof Since $(ces^t_{r_1,r_2})_\upsilon$ is a pre-modular $\mathfrak{pss}$, by following Theorem 4.3, we have Ψ is a pre-quasi norm on $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$. Let $X_b \in \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$ with $b \in \mathrm{N}$ and $\lim_{b\to\infty}\Psi(X_b - X) = 0$. Since $\mathbb{B}(\mathcal{P},\mathcal{Q}) \supseteq \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$, one has

$$\Psi(X - X_b) = \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(X - X_b)}{[l+1]_{r_1,r_2}}\right)^t \geq \sum_{l=0}^{\infty}\left(\frac{r_2^l \|X - X_b\|}{[l+1]_{r_1,r_2}}\right)^t \geq \|X - X_b\|^t,$$

so $(X_b)_{b\in\mathrm{N}}$ is a convergent sequence in $\mathbb{B}(\mathcal{P},\mathcal{Q})$. As $(s_l(X_b))_{l=0}^{\infty} \in (ces^t_{r_1,r_2})_\upsilon$ with $b \in \mathrm{N}$. From Definition 2.8 parts (ii), (iii), and (v), we have

$$\begin{aligned}\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(X)}{[l+1]_{r_1,r_2}}\right)^t \\ &\leq 2^{t-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_{[\frac{z}{2}]}(X - X_b)}{[l+1]_{r_1,r_2}}\right)^t + 2^{t-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_{[\frac{z}{2}]}(X_b)}{[l+1]_{r_1,r_2}}\right)^t \\ &\leq 2^{t-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} \|X - X_b\|}{[l+1]_{r_1,r_2}}\right)^t + 2^{t-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(X_b)}{[l+1]_{r_1,r_2}}\right)^t < \infty.\end{aligned}$$

One can see $(s_l(X))_{l=0}^{\infty} \in (ces^t_{r_1,r_2})_\upsilon$, hence $X \in \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. □

4.3 Minimum pre-quasi ideal

Theorem 4.6 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t_1 < t_2 < \infty$, and $r_1 \leq r_2$, then*

$$\mathbb{B}^s_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}^s_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}(\mathcal{P},\mathcal{Q}).$$

Proof Assume $Z \in \mathbb{B}^s_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$, then $(s_l(Z)) \in (ces^{t_1}_{r_1,r_2})_\upsilon$. We have

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(Z)}{[l+1]_{r_1,r_2}}\right)^{t_2} < \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(Z)}{[l+1]_{r_1,r_2}}\right)^{t_1} < \infty,$$

then $Z \in \mathbb{B}^s_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. Next, by taking $(s_l(Z))_{l=0}^{\infty}$ such that $\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(Z) = \frac{[l+1]_{r_1,r_2}}{\sqrt[t_1]{l+1}}$, one has $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ with

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(Z)}{[l+1]_{r_1,r_2}}\right)^{t_1} = \sum_{l=0}^{\infty}\frac{1}{l+1} = \infty$$

and

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(Z)}{[l+1]_{r_1,r_2}}\right)^{t_2} = \sum_{l=0}^{\infty}\left(\frac{1}{l+1}\right)^{\frac{t_2}{t_1}} < \infty.$$

Therefore, $Z \notin \mathbb{B}^s_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$ and $Z \in \mathbb{B}^s_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. Evidently, $\mathbb{B}^s_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}) \subset \mathbb{B}(\mathcal{P},\mathcal{Q})$. Then, by choosing $(s_l(Z))_{l=0}^\infty$ with $\sum_{z=0}^l r_1^z r_2^{l-z} s_z(Z) = \frac{[l+1]_{r_1,r_2}}{\sqrt[t_2]{l+1}}$, one can conclude $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ and $Z \notin \mathbb{B}^s_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. □

Corollary 4.7 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < q < t < \infty$, and $0 < r \leq 1$, then*

$$\mathbb{B}^s_{(\chi_r^q)_\upsilon}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}^s_{(\chi_r^t)_\upsilon}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}(\mathcal{P},\mathcal{Q}).$$

Theorem 4.8 $\mathbb{B}^\alpha_{(ces^t_{r_1,r_2})_\upsilon}$ *is minimum, when $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t < \infty$, and $r_1 \leq r_2$.*

Proof Suppose that the sufficient set-up is verified. Hence $(\mathbb{B}^\alpha_{ces^t_{r_1,r_2}}, \Psi)$ is a pre-quasi Banach ideal with $\Psi(Z) = \sum_{l=0}^\infty (\frac{\sum_{z=0}^l r_1^z r_2^{l-z} \alpha_z(Z)}{[l+1]_{r_1,r_2}})^t$. Assume $\mathbb{B}^\alpha_{ces^t_{r_1,r_2}}(\mathcal{P},\mathcal{Q}) = \mathbb{B}(\mathcal{P},\mathcal{Q})$, so we have $\eta > 0$ with $\Psi(Z) \leq \eta \|Z\|$ for $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$. From Dvoretzky's theorem [30], for $b \in \mathrm{N}$, we have quotient spaces $\mathcal{P}/Y_b$ and subspaces M_b of $\mathcal{Q}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \leq 2$ and $\|X_b\|\|X_b^{-1}\| \leq 2$. Let I_b be the identity map on ℓ_2^b, T_b be the quotient map from $\mathcal{P}$ onto $\mathcal{P}/Y_b$, and J_b be the natural embedding map from M_b into $\mathcal{Q}$. Suppose that m_z is the Bernstein numbers [31], we have

$$\begin{aligned}
1 = m_z(I_b) &= m_z\big(X_b X_b^{-1} I_b V_b V_b^{-1}\big) \leq \|X_b\| m_z\big(X_b^{-1} I_b V_b\big) \big\|V_b^{-1}\big\| \\
&= \|X_b\| m_z\big(J_b X_b^{-1} I_b V_b\big) \big\|V_b^{-1}\big\| \leq \|X_b\| d_z\big(J_b X_b^{-1} I_b V_b\big) \big\|V_b^{-1}\big\| \\
&= \|X_b\| d_z\big(J_b X_b^{-1} I_b V_b T_b\big) \big\|V_b^{-1}\big\| \leq \|X_b\| \alpha_z\big(J_b X_b^{-1} I_b V_b T_b\big) \big\|V_b^{-1}\big\|
\end{aligned}$$

if $0 \leq l \leq b$. One can conclude

$$\begin{aligned}
&\sum_{z=0}^l r_1^z r_2^{l-z} \leq \sum_{z=0}^l \|X_b\| r_1^z r_2^{l-z} \alpha_z\big(J_b X_b^{-1} I_b V_b T_b\big) \big\|V_b^{-1}\big\| \quad \Rightarrow \\
&\left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}\right)^t \leq \big(\|X_b\| \big\|V_b^{-1}\big\|\big)^t \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z} \alpha_z(J_b X_b^{-1} I_b V_b T_b)}{[l+1]_{r_1,r_2}}\right)^t.
\end{aligned}$$

Therefore, for some $\varrho \geq 1$, we have

$$\begin{aligned}
&\sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}\right)^t \leq \varrho \|X_b\| \big\|V_b^{-1}\big\| \sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z} \alpha_z(J_b X_b^{-1} I_b V_b T_b)}{[l+1]_{r_1,r_2}}\right)^t \quad \Rightarrow \\
&\sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}\right)^t \leq \varrho \|X_b\| \big\|V_b^{-1}\big\| \Psi\big(J_b X_b^{-1} I_b V_b T_b\big) \quad \Rightarrow \\
&\sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}\right)^t \leq \varrho\eta \|X_b\| \big\|V_b^{-1}\big\| \big\|J_b X_b^{-1} I_b V_b T_b\big\| \quad \Rightarrow \\
&\sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}\right)^t \\
&\quad \leq \varrho\eta \|X_b\| \big\|V_b^{-1}\big\| \big\|J_b X_b^{-1}\big\| \|I_b\| \|V_b T_b\| = \varrho\eta \|X_b\| \big\|V_b^{-1}\big\| \big\|X_b^{-1}\big\| \|I_b\| \|V_b\| \quad \Rightarrow \\
&\sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}}\right)^t \leq 4\varrho\eta.
\end{aligned}$$

For $b \to \infty$ and since $\sum_{l=0}^{\infty}(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z}}{[l+1]_{r_1,r_2}})^t = \sum_{l=0}^{\infty}(\frac{r_2^l + r_2^{l-1} r_1 + \ldots + r_2 r_1^{l-1} + r_1^l}{[l+1]_{r_1,r_2}})^t = \sum_{l=0}^{\infty} 1 = \infty$. This implies unreliability. Hence $\mathcal{P}$ and $\mathcal{Q}$ both cannot be infinite dimensional if $\mathbb{B}^{\alpha}_{ces^t_{r_1,r_2}}(\mathcal{P},\mathcal{Q}) = \mathbb{B}(\mathcal{P},\mathcal{Q})$. This confirms the proof. □

Theorem 4.9 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t < \infty$, and $r_1 \leq r_2$, then $\mathbb{B}^{d}_{ces^t_{r_1,r_2}}$ is minimum.*

Corollary 4.10 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t < \infty$, and $0 < r \leq 1$, then $\mathbb{B}^{\alpha}_{\chi^t_r}$ is minimum.*

Corollary 4.11 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t < \infty$, and $0 < r \leq 1$, then $\mathbb{B}^{d}_{\chi^t_r}$ is minimum.*

4.4 Non-trivial closed pre-quasi ideal

Theorem 4.12 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $r_1 \leq r_2$, and $1 < t_1 < t_2 < \infty$, then*

$$\mathbb{B}\big(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})\big) = \mathcal{A}\big(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})\big).$$

Proof Assume $X \in \mathbb{B}(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$ and $X \notin \mathcal{A}(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$. By using Lemma 1.2, we have $Y \in \mathbb{B}(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$ and $Z \in \mathbb{B}(\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$ so that $ZXYI_b = I_b$. Hence, for all $b \in \mathrm{N}$, one has

$$\begin{aligned}\|I_b\|_{\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})} &= \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} r_1^z r_2^{l-z} s_z(I_b)}{[l+1]_{r_1,r_2}}\right)^{t_1} \\ &\leq \|ZXY\|\|I_b\|_{\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})} \leq \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{i} r_1^z r_2^{l-z} s_z(I_b)}{[l+1]_{r_1,r_2}}\right)^{t_2}.\end{aligned}$$

This contradicts Theorem 4.6. Therefore, $X \in \mathcal{A}(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$, which completes the proof. □

Corollary 4.13 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $r_1 \leq r_2$, and $1 < t_1 < t_2 < \infty$, then*

$$\mathbb{B}\big(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})\big) = \mathcal{K}\big(\mathbb{B}^{s}_{(ces^{t_2}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(ces^{t_1}_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})\big).$$

Proof Definitely, since $\mathcal{A} \subset \mathcal{K}$. □

Theorem 4.14 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t < \infty$, and $r_1 \leq r_2$, then $\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}$ is simple.*

Proof Assume that the closed ideal $\mathcal{K}(\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$ contains a mapping $X \notin \mathcal{A}(\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$. By using Lemma 1.2, we have $Y, Z \in \mathbb{B}(\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$ with $ZXYI_b = I_b$. This implies that $I_{\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})} \in \mathcal{K}(\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$. Therefore, $\mathbb{B}(\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})) = \mathcal{K}(\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}))$. So $\mathbb{B}^{s}_{(ces^t_{r_1,r_2})_\upsilon}$ is simple. □

4.5 Spectrum of pre-quasi ideal

Notation 4.15

$$\begin{aligned}&\big(\mathbb{B}^s_{\mathcal{V}}\big)^\rho := \big\{\big(\mathbb{B}^s_{\mathcal{V}}\big)^\rho(\mathcal{P},\mathcal{Q}); \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces}\big\}, \quad \text{where}\\ &\big(\mathbb{B}^s_{\mathcal{V}}\big)^\rho(\mathcal{P},\mathcal{Q}) := \big\{X \in \mathbb{B}(\mathcal{P},\mathcal{Q}) : \big(\big(\rho_l(X)\big)_{l=0}^\infty \in \mathcal{V} \text{ and } \big\|X - \rho_l(X)I\big\|^{-1}\\ &\qquad \text{does not exist for every } l \in \mathrm{N}\big\}.\end{aligned}$$

Theorem 4.16 *If $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces, $1 < t < \infty$, and $r_1 \le r_2$, then*

$$\big(\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}\big)^\rho(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q}).$$

Proof Assume $X \in (\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon})^\rho(\mathcal{P},\mathcal{Q})$, so $(\rho_l(X))_{l=0}^\infty \in (ces^t_{r_1,r_2})_\upsilon$ and $\|X - \rho_l(X)I\| = 0$ for every $l \in \mathrm{N}$. One has $X = \rho_l(X)I$ for all $l \in \mathrm{N}$, hence $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$ for each $l \in \mathrm{N}$. Hence $(s_l(X))_{l=0}^\infty \in (ces^t_{r_1,r_2})_\upsilon$, which implies $X \in \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. Next, by putting $(\rho_l(X))_{l=0}^\infty$ so that $\sum_{z=0}^l r_1^z r_2^{l-z}\rho_z(X) = \frac{[l+1]_{r_1,r_2}}{\sqrt[t]{l+1}}$, we have $X \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ so that

$$\sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l r_1^z r_2^{l-z}\rho_z(X)}{[l+1]_{r_1,r_2}}\right)^t = \sum_{l=0}^\infty \frac{1}{l+1} = \infty,$$

and by choosing $(s_l(X))_{l=0}^\infty$ with $\sum_{z=0}^l r_1^z r_2^{l-z} s_z(X) = \frac{[l+1]_{r_1,r_2}}{l+1}$. Therefore, $X \notin (\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon})^\rho(\mathcal{P},\mathcal{Q})$ and $X \in \mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}(\mathcal{P},\mathcal{Q})$. This confirms the proof. □

5 Conclusion

Many authors in the near past investigated and studied the r-Cesàro matrix and the linked summability methods [32–35]. In this paper, we explain some topological and geometric structure of the class $\mathbb{B}^s_{(ces^t_{r_1,r_2})_\upsilon}$ and the multiplication mappings defined on $(ces^t_{r_1,r_2})_\upsilon$. When $r_1 = r$ and $r_2 = 1$, we have $ces^t_{r_1,r_2} = \chi^r_t$. Some new properties to the sequence space χ^r_t have been added. This article has many benefits for researchers such as studying the fixed points of any contraction maps on this pre-quasi normed sequence space, which is more general than the quasi normed sequence spaces, a new general space of solutions for many difference equations, the spectrum of any bounded linear operators between any two Banach spaces with s-numbers in this sequence space, and noting that the operator ideals are the prime structural components of a vector lattice; consequently, closed ideals are bound to play a positive role in the theory of Banach lattices. We open the way for many authors to generalize the results by a sequence $t = (t_l)_{l=0}^\infty$ and build $(ces^{(t)}_{r_1,r_2})_\upsilon$ of non-absolute type.

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Author details
[1]Department of Mathematics, College of Science and Arts at Khulis, University of Jeddah, Jeddah, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Abbassia, Cairo, 11566, Egypt.
[3]Department of Mathematics, Academy of Engineering and Medical Sciences, Khartoum, Sudan.

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References
1. Lima, Å., Oja, E.: Ideals of finite rank operators, intersection properties of balls, and the approximation property. Stud. Math. **133**, 175–186 (1999)
2. Pietsch, A.: Operator Ideals. North-Holland, Amsterdam (1980)
3. Mrowka, T.: A brief introduction to linear analysis: Fredholm operators, geometry of manifolds, Fall 2004. Massachusetts Institute of Technology, MIT OpenCouseWare (2004)
4. Kalton, N.J.: Spaces of compact operators. Math. Ann. **208**, 267–278 (1974)
5. Faried, N., Bakery, A.A.: Small operator ideals formed by *s* numbers on generalized Cesàro and Orlicz sequence spaces. J. Inequal. Appl. **2018**(1), Article ID 357 (2018). https://doi.org/10.1186/s13660-018-1945-y
6. Pietsch, A.: Eigenvalues and *s*-Numbers. Cambridge University Press, New York (1986)
7. Bakery, A.A., Elmatty, A.R.A.: A note on Nakano generalized difference sequence space. Adv. Differ. Equ. **2020**, 620 (2020). https://doi.org/10.1186/s13662-020-03082-1
8. Pietsch, A.: Small ideals of operators. Stud. Math. **51**, 265–267 (1974)
9. Makarov, B.M., Faried, N.: Some properties of operator ideals constructed by *s* numbers. In: Theory of Operators in Functional Spaces, pp. 206–211. Academy of Science. Siberian section, Novosibirsk (1977) (in Russian)
10. Yaying, T., Hazarika, B., Mursaleen, M.: On sequence space derived by the domain of *q*-Cesàro matrix in ℓ_p space and the associated operator ideal. J. Math. Anal. Appl. **493**, 124453 (2021). https://doi.org/10.1016/j.jmaa.2020.124453
11. Kara, E.E., Başarir, M.: On compact operators and some Euler $B(m)$-difference sequence spaces. J. Math. Anal. Appl. **379**(2), 499–511 (2011). https://doi.org/10.1016/j.jmaa.2011.01.028
12. Başarir, M., Kara, E.E.: On some difference sequence spaces of weighted means and compact operators. Ann. Funct. Anal. **2**(2), 116–131 (2011)
13. Başarir, M., Kara, E.E.: On compact operators on the Riesz $B(m)$-difference sequence space. Iran. J. Sci. Technol. Trans. A, Sci. **35**(A4), 279–285 (2011)
14. Başarir, M., Kara, E.E.: On the *B*-difference sequence space derived by weighted mean and compact operators. J. Math. Anal. Appl. **391**(1), 67–81 (2012). https://doi.org/10.1016/j.jmaa.2012.02.031
15. Başarir, M., Kara, E.E.: On the *m*th order difference sequence space of generalized weighted mean and compact operators. Acta Math. Sci. **33**(3), 797–813 (2013)
16. Mursaleen, M., Noman, A.K.: Compactness by the Hausdorff measure of noncompactness. Nonlinear Anal. **73**, 2541–2557 (2010)
17. Mursaleen, M., Noman, A.K.: Compactness of matrix operators on some new difference sequence spaces. Linear Algebra Appl. **436**, 41–52 (2012)
18. Komal, B.S., Pandoh, S., Raj, K.: Multiplication operators on Cesàro sequence spaces. Demonstr. Math. **49**(4), 430–436 (2016)
19. İlkhan, M., Demiriz, S., Kara, E.E.: Multiplication operators on Cesàro second order function spaces. Positivity **24**, 605–614 (2020)
20. Mursaleen, M., Noman, A.K.: On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I. Filomat **25**(2), 33–51 (2011)
21. Mursaleen, M., Noman, A.K.: On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ II. Math. Commun. **16**(2), 383–398 (2011)
22. Mursaleen, M., Başar, F.: Domain of Cesàro mean of order one in some spaces of double sequences. Studia Sci. Math. Hung. **51**(3), 335–356 (2014)
23. Altay, B., Başar, F.: Generalization of the sequence space $\ell(p)$ derived by weighted means. J. Math. Anal. Appl. **330**(1), 147–185 (2007)
24. Rhoades, B.E.: Operators of $A - p$ type. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8) **59**(3–4), 238–241 (1975)
25. Ng, P.-N., Lee, P.-Y.: Cesàro sequence spaces of non-absolute type. Comment. Math. Prace Mat. **20**(2), 429–433 (1978)
26. Aydin, C., Başar, F.: Some new sequence spaces which include the spaces ℓ_p and ℓ_∞. Demonstr. Math. **XXXVIII**(3), 641–656 (2005)
27. Başar, F.: Infinite matrices and Cesàro sequence spaces of non-absolute type. Math. J. Ibaraki Univ. **31**, 1–12 (1999)
28. Şengönül, M., Başar, F.: Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c. Soochow J. Math. **31**(1), 107–119 (2005)
29. Tripathy, B.C., Esi, A., Tripathy, B.: On a new type of generalized difference Cesàro sequence spaces. Soochow J. Math. **31**(3), 333–340 (2005)

30. Pietsch, A.: Operator Ideals. VEB, Berlin (1978)
31. Pietsch, A.: s-Numbers of operators in Banach spaces. Stud. Math. **51**, 201–223 (1974)
32. Aktuglu, H., Bekar, Ş.: q-Cesàro matrix and q-statistical convergence. J. Comput. Appl. Math. **235**(16), 4717–4723 (2011)
33. Bustoz, J., Gordillo, L., Luis, F.: q-Hausdorff summability. J. Comput. Anal. Appl. **7**(1), 35–48 (2005)
34. Carl, B.: On s-numbers, quasi s-number, s-moduli and Weyl inequalities of operators in Banach spaces. Rev. Mat. Complut. **23**(2), 467–487 (2010)
35. Selmanogullari, T., Savaş, E., Rhoades, B.E.: On q-Hausdorff matrices. Taiwan. J. Math. **15**(6), 2429–2437 (2011)