Edely and Mursaleen *Journal of Inequalities and Applications* (2021) 2021:34
https://doi.org/10.1186/s13660-021-02564-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On statistical $\mathfrak{A}$-Cauchy and statistical $\mathfrak{A}$-summability via ideal

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Abstract

The notion of statistical convergence was extended to $\mathfrak{I}$-convergence by (Kostyrko et al. in Real Anal. Exch. 26(2):669–686, 2000). In this paper we use such technique and introduce the notion of statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy and statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summability via the notion of ideal. We obtain some relations between them and prove that under certain conditions statistical $\mathfrak{A}^{\mathfrak{I}}$-Cauchy and statistical $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summability are equivalent. Moreover, we give some Tauberian theorems for statistical $\mathfrak{A}^{\mathfrak{I}}$-summability.

MSC: 40A35; 40G15; 40E05

Keywords: Statistical $\mathfrak{A}^{\mathfrak{I}}$-limit superior; Statistical $\mathfrak{A}^{\mathfrak{I}}$-limit inferior; Statistical $\mathfrak{A}^{\mathfrak{I}}$-bounded; Statistical $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summability; Statistical $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summability; Tauberian theorem

1 Introduction and preliminaries

Fast [10], introduced the notion of statistical convergence, which is an extension of convergence. A sequence $\eta = (\eta_k)$ in $\mathbb{R}$ is statistically convergent to the number $\mathfrak{s}$ if the set $K(\epsilon) = \{k \leq n : |\eta_k - \mathfrak{s}| \geq \epsilon, \forall \epsilon > 0\}$ has natural density 0; $\delta(K(\epsilon)) = \lim_n \frac{|K(\epsilon)|}{n} = 0$, where $|\cdot|$ indicates the number of elements in the set. We write $st\text{-}\lim \eta = \mathfrak{s}$. More generalization and application on this work can be found in ([1, 5, 8, 12, 14, 16, 23, 27]). One of such generalizations is the ideal (or $\mathfrak{I}$)-convergence [18] which generalizes the usual convergence as well as the statistical convergence.

A non-empty class $\mathfrak{I}$ ($\mathcal{F}$, *resp.*) $\subseteq \mathfrak{P}(\mathfrak{X})$ of subsets of $\mathfrak{X} \neq \varnothing$ is called ideal (filter, resp.) if (i) $\emptyset \in \mathfrak{I}$ ($\emptyset \notin \mathcal{F}$, resp.), (ii) ($\mathcal{D}_1 \cup \mathcal{D}_2$ for $\mathcal{D}_1, \mathcal{D}_2 \in \mathfrak{I}$) ($\mathcal{D}_1 \cap \mathcal{D}_2$ for $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{F}$, *resp.*) $\in \mathfrak{I}$ ($\in \mathcal{F}$, *resp.*), (iii) $\mathcal{D}_1 \in \mathfrak{I}$, $\mathcal{D}_2 \subseteq \mathcal{D}_1$ ($\mathcal{D}_1 \in \mathcal{F}, \mathcal{D}_2 \supseteq \mathcal{D}_1$, *resp.*) $\Longrightarrow \mathcal{D}_2 \in \mathfrak{I}$ ($\mathcal{D}_2 \in \mathcal{F}$, *resp.*). An ideal $\mathfrak{I}$ is called non-trivial if $\mathfrak{I} \neq \varnothing$, $\mathfrak{X} \notin \mathfrak{I}$, and is called admissible if $\{\mathfrak{a}\} \in \mathfrak{I}$, for each $\mathfrak{a} \in \mathfrak{X}$.

Let $\mathfrak{I}$ be a non-trivial ideal in $\mathfrak{X}$, the filter $\mathcal{F}_{\mathfrak{I}} = \{M = \mathfrak{X} \setminus \mathcal{A} : \mathcal{A} \in \mathfrak{I}\}$ is called the filter associated with the ideal $\mathfrak{I}$. Recall that a real sequence $\eta = (\eta_k)$ is said to be $\mathfrak{I}$-convergent to $\mathfrak{s} \in \mathbb{R}$ if $\{k : |\eta_k - \mathfrak{s}| \geq \epsilon$, for every $\epsilon > 0\} \in \mathfrak{I}$, and we write $\mathfrak{I}\text{-}\lim_k \eta_k = \mathfrak{s}$, [18]. More generalization and recent work can be found in ([3, 15, 17, 21, 22, 24, 25, 28, 29]).

Let $\mathfrak{A} = (\mathfrak{a}_{nk})$ be an infinite matrix and $\eta = (\eta_k)$ be a number sequence. By $\mathfrak{A}\eta = (\mathfrak{A}_n(\eta))$, we denote the $\mathfrak{A}$-transform of the sequence $\eta = (\eta_k)$, where $\mathfrak{A}_n(\eta) = \sum_{k=1}^{\infty} \mathfrak{a}_{nk}\eta_k$. A matrix

$\mathfrak{A}$ is regular if $\mathfrak{A}$-transforms c into c and $\lim_n \mathfrak{A}_n(\eta) = \lim_k \eta_k$ for all $\eta \in c$; the space of all convergent sequences. Let Ω denote the class of all nonnegative regular matrices. In [29], Savas et al. introduced the following definition. Let $\mathfrak{A} = (\mathfrak{a}_{nk}) \in \Omega$. A real sequence $\eta = (\eta_k)$ is $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s} \in \mathbb{R}$ if the sequence $(\mathfrak{A}_n(\eta))$ is $\mathfrak{I}$-convergent to $\mathfrak{s}$, which we write $\mathfrak{A}^{\mathfrak{I}}\text{-}\lim_k \eta_k = \mathfrak{s}$. Notice that, if $\mathfrak{I} = \mathfrak{I}_\delta = \{E \subseteq \mathbb{N} : \delta(E) = 0\}$, then $\mathfrak{A}^{\mathfrak{I}}$-summability becomes statistical $\mathfrak{A}$-summability due to [9].

Recently, Edely [6] introduced the notion of $\mathfrak{A}^{\mathfrak{I}^*}$-summability and gave some relations with $\mathfrak{A}^{\mathfrak{I}}$-summability.

Definition 1.1 ([6]) Let $\mathfrak{I}$ be a non-trivial admissible ideal in $\mathbb{N}$ and $\mathfrak{A} = (\mathfrak{a}_{nk}) \in \Omega$. We say that a sequence $\eta = (\eta_k)$ is $\mathfrak{A}^{\mathfrak{I}^*}$-summable to $\mathfrak{s}$ if there is a set $\mathfrak{H} \in \mathfrak{I}$ such that $\mathfrak{M} = \mathbb{N} \setminus \mathfrak{H} = \{m_1, m_2, \ldots\} \in \mathcal{F}_{\mathfrak{I}}$, and $\lim_i \sum_k \mathfrak{a}_{m_i k}\eta_k = \lim_i y_{m_i} = \mathfrak{s}$. In this case we write $\mathfrak{A}^{\mathfrak{I}^*}\text{-}\lim \eta_k = \mathfrak{s}$.

Theorem 1.1 ([6]) *Let $\mathfrak{I}$ be a non-trivial admissible ideal in $\mathbb{N}$.*
(a) *If $\mathfrak{A}^{\mathfrak{I}^*}\text{-}\lim \eta_k = \mathfrak{s}$ then $\mathfrak{A}^{\mathfrak{I}}\text{-}\lim \eta_k = \mathfrak{s}$.*
(b) *If $\mathfrak{I}$ satisfies the condition (AP) and $\mathfrak{A}^{\mathfrak{I}}\text{-}\lim \eta_k = \mathfrak{s}$, then $\mathfrak{A}^{\mathfrak{I}^*}\text{-}\lim \eta_k = \mathfrak{s}$.*

Definition 1.2 ([28]) A real sequence $\eta = (\eta_k)$ is $\mathfrak{I}$-statistically convergent to $\mathfrak{s} \in \mathbb{R}$ if $\forall \epsilon > 0$ and $\nu > 0$,

$$\left\{n : \frac{1}{n}\left|\left\{k \leq n : |\eta_k - \mathfrak{s}| \geq \epsilon\right\}\right| \geq \nu\right\} \in \mathfrak{I}$$

then we write $\mathfrak{I}\text{-}st \lim_k \eta_k = \mathfrak{s}$.

Remark 1.1 If $\mathfrak{I} = \mathfrak{I}_{\text{fin}} = \{E \subseteq \mathbb{N} : E \text{ is finite}\}$, then $\mathfrak{I}$-statistical convergence coincides with the statistical convergence due to Fast [10].

Recently, Edely [7] also introduced the notion of statistically $\mathfrak{A}^{\mathfrak{I}}$ and statistically $\mathfrak{A}^{\mathfrak{I}^*}$-summable and gave some relations.

Definition 1.3 ([7]) Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$. A sequence $\eta = (\eta_k)$ is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$ if $\forall \epsilon > 0$ and every $\nu > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{j \leq n : |y_j - \mathfrak{s}| \geq \epsilon\right\}\right| \geq \nu\right\} \in \mathfrak{I},$$

where $y_j = \mathfrak{A}_j(\eta)$. Thus η is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$ iff the sequence (y_j) is $\mathfrak{I}$-statistically convergent to $\mathfrak{s}$, then we write $(\mathfrak{A}^{\mathfrak{I}})_{st}\text{-}\lim \eta = \mathfrak{I}\text{-}st \lim A\eta$.

Remark 1.2 (a) If $\mathfrak{I} = \mathfrak{I}_{\text{fin}}$, then statistical $\mathfrak{A}^{\mathfrak{I}}$-summable coincides with the statistical $\mathfrak{A}$-summable due to Edely and Mursaleen [9].

(b) If $\mathfrak{A} = I$ the identity matrix, then statistical $\mathfrak{A}^{\mathfrak{I}}$-summable coincides with the $\mathfrak{I}$-statistical convergence due to Savas et al. [28]. If $\mathfrak{I} = \mathfrak{I}_\delta$ and $\mathfrak{A} = (C, 1)$ the Cesàro matrix of order 1, then it reduces to statistical summability $(C, 1)$ due to Móricz [20].

Definition 1.4 ([7]) Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$. A sequence $\eta = (\eta_k)$ is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-summable to $\mathfrak{s}$ if there is a set $M = \{m_i\}$, where $m_1 < m_2 < \cdots$ and $M \in \mathcal{F}_{\mathfrak{I}}$, $\delta(M) = 1$, such that

$$st - \lim_i \mathfrak{A}_{m_i}\eta = st - \lim_i y_{m_i} = \mathfrak{s},$$

where $y_{m_i} = \sum_k \mathfrak{a}_{m_i k}\eta_k$ i.e. $(\mathfrak{A}_{m_i}\eta)$ is statistically convergent to $\mathfrak{s}$, and we write $(\mathfrak{A}^{\mathfrak{I}^*})_{st}$-$\lim \eta = \mathfrak{I}^*$-$st \lim \mathfrak{A}\eta = \mathfrak{s}$.

Remark 1.3 If $\mathfrak{A} = I$, the identity matrix, then η is $\mathfrak{I}^*$-statistically convergent to the number $\mathfrak{s}$, and we write $\mathfrak{I}^* - st \lim \eta = \mathfrak{s}$.

Theorem 1.2 ([7]) (a) *If* $(\mathfrak{A}^{\mathfrak{I}^*})_{st}$-$\lim \eta_k = \mathfrak{s}$ *then* $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\lim \eta_k = \mathfrak{s}$.

(b) *If* $\mathfrak{I}$ *satisfies the condition* (APO), *then whenever* $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\lim \eta_k = \mathfrak{s}$ *we have* $(\mathfrak{A}^{\mathfrak{I}^*})_{st}$-$\lim_k \eta_k = \mathfrak{s}$.

Corollary 1.1 (a) *If* $\mathfrak{I}^*$-$st \lim \eta_k = \mathfrak{s}$ *then* $\mathfrak{I}$-$st \lim \eta_k = \mathfrak{s}$.

(b) *If* $\mathfrak{I}$ *satisfies the condition* (APO), *then whenever* $\mathfrak{I}$-$st \lim \eta_k = \mathfrak{s}$ *we have* $\mathfrak{I}^*$-$st \lim \eta_k = \mathfrak{s}$.

Recall that $\mathcal{I}$ satisfies the (APO) condition (cf. [2, 11]), if for every sequence $(\mathcal{C}_n)$ of (pairwise disjoint) sets from $\mathfrak{I}$ such that $\delta(\mathcal{C}_n) = 0$ for each n, then there exist sets $\mathcal{D}_n \in \mathfrak{I}$, $n \in \mathbb{N}$ such that the symmetric difference $\mathcal{C}_n \Delta \mathcal{D}_n$ is finite for every n, $\bigcup_n \mathcal{D}_n \in \mathfrak{I}$, $\delta(\bigcup_n \mathcal{D}_n) = 0$.

Remark 1.4 In what follows, $\mathfrak{I}$ will be a non-trivial admissible ideal in $\mathbb{N}$.

In this paper we use a technique and introduce the notion of statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy and statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summability via the notion of ideal. We obtain some relations between them and prove that under certain conditions statistical $\mathfrak{A}^{\mathfrak{I}}$-Cauchy and statistical $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summability are equivalent. Moreover, we give some Tauberian theorems for statistical $\mathfrak{A}^{\mathfrak{I}}$-summability.

2 Some related concepts

The concept of $\mathfrak{I}$-limit superior and inferior of a real sequence was given in [3], see also [17]. In this section we define and study some relations of statistically $\mathfrak{A}^{\mathfrak{I}}$-limit superior and statistically $\mathfrak{A}^{\mathfrak{I}}$-limit inferior of a real number sequence $\eta = (\eta_k)$.

Definition 2.1 Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$ and $\eta = (\eta_k)$ be a real sequence. Let us write G_η and F_η, for some $\upsilon > 0$, as

$$G_\eta = \left\{g \in \mathbb{R} : \left\{n \in \mathbb{N} : \frac{1}{n}\bigl|\{j \le n : y_j > g\}\bigr| > \upsilon\right\} \notin \mathfrak{I}\right\}$$

and

$$F_\eta = \left\{f \in \mathbb{R} : \left\{n \in \mathbb{N} : \frac{1}{n}\bigl|\{j \le n : y_j < f\}\bigr| > \upsilon\right\} \notin \mathfrak{I}\right\}.$$

Then we define

$$\left(\mathfrak{A}^{\mathfrak{I}}\right)_{st} - \limsup \eta = \mathfrak{I} - st \limsup \mathfrak{A}\eta = \begin{cases} \sup G_\eta & \text{if } G_\eta \neq \varnothing, \\ -\infty & \text{if } G_\eta = \varnothing, \end{cases}$$

and

$$(\mathfrak{A}^{\mathfrak{I}})_{st} - \liminf \eta = \mathfrak{I} - st \liminf \mathfrak{A}\eta = \begin{cases} \inf F_\eta & \text{if } F_\eta \neq \varnothing, \\ \infty & \text{if } F_\eta = \varnothing. \end{cases}$$

Remark 2.1 If $A = I$, then the statistical $\mathfrak{A}^{\mathfrak{I}}$-limit superior and statistical $\mathfrak{A}^{\mathfrak{I}}$-limit inferior of η reduced to $\mathfrak{I}$-statistical limit superior and inferior due to Mursaleen et al. [22]. Moreover if $\mathfrak{I} = \mathfrak{I}_{\text{fin}}$, then we have statistical limit superior and inferior cases due to [14].

The following result can be proved straightforward from Definition 2.1 and the least upper bound argument.

Theorem 2.1 (a) *If* $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup x = l_1$ *is finite, then* $\forall \epsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ j \leq n : y_j > l_1 - \epsilon \} \right| > \upsilon \right\} \notin \mathfrak{I} \tag{2.1}$$

for some $\upsilon > 0$, *and*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ j \leq n : y_j > l_1 + \epsilon \} \right| > \upsilon \right\} \in \mathfrak{I}, \tag{2.2}$$

for all $\upsilon > 0$. *Conversely If* (2.1) *and* (2.2) *hold* $\forall \epsilon > 0$, *then* $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta = l_1$.

(b) *If* $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta = l_2$ *is finite, then* $\forall \epsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ j \leq n : y_j < l_2 + \epsilon \} \right| > \upsilon \right\} \notin \mathfrak{I} \tag{2.3}$$

for some $\upsilon > 0$, *and*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ j \leq n : y_j < l_2 - \epsilon \} \right| > \upsilon \right\} \in \mathfrak{I} \tag{2.4}$$

for all $\upsilon > 0$. *Conversely If* (2.3) *and* (2.4) *hold for every* $\epsilon > 0$, *then* $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta = l_2$.

Definition 2.2 Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$. Then $\eta = (\eta_k)$ is said to be statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded if there is a number $t \in \mathbb{R}$ such that, for any $\upsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ j \leq n : |y_j| > t \right\} \right| > \upsilon \right\} \in \mathfrak{I}.$$

Remark 2.2 (a) If $\mathfrak{A} = I$, then the statistical $\mathfrak{A}^{\mathfrak{I}}$-boundedness reduces to $\mathfrak{I}$-statistical boundedness due to [22]. Moreover if $\mathfrak{I} = \mathfrak{I}_{\text{fin}}$, then we have the statistical bounded case of η due to [14].

(b) Statistical $\mathfrak{A}^{\mathfrak{I}}$-boundedness implies that $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta$ and $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta$ are finite.

(c) If $\eta \in \ell_\infty$, then η is statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded.

(d) If η is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable then η is statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded.

The following theorems can be directly obtained from Theorem 3.2 and Theorem 3.4 of [22].

Theorem 2.2 *Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$. Then, for any real sequence $\eta = (\eta_k)$,*

$$\left(\mathfrak{A}^{\mathfrak{I}}\right)_{st} - \liminf \eta \leq \left(\mathfrak{A}^{\mathfrak{I}}\right)_{st} - \limsup \eta.$$

Remark 2.3 From Definition 2.1 and Theorem 2.2, we have, for any real sequence η,

$$\liminf \eta \leq \left(\mathfrak{A}^{\mathfrak{I}}\right)_{st} - \liminf \eta \leq \left(\mathfrak{A}^{\mathfrak{I}}\right)_{st} - \limsup \eta \leq \limsup \eta.$$

Theorem 2.3 *Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$ and $\eta = (\eta_k)$ be statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded. Then η is statistically $\mathfrak{A}^{\mathfrak{I}}$-convergent iff $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta = (\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta$.*

Example 2.1 Let B_i be mutually disjoint infinite sets such that $\mathbb{N} = \bigcup_{i=1}^{\infty} B_i$. Let $\mathfrak{I}$ be the class defined as

$$\mathfrak{I} = \{B \subset \mathbb{N} : B \text{ intersects only finite numbers of } B_i\},$$

then $\mathfrak{I}$ is a non-trivial admissible ideal in $\mathbb{N}$. Define $\eta = (\eta_k)$ as

$$\eta_k = \begin{cases} 1 & \text{if } k \in B_i, k \text{ is odd}, \\ 0 & \text{otherwise}, \end{cases}$$

and let $\mathfrak{A} = (\mathfrak{a}_{jk})$ be the identity matrix.

Since η is bounded, η is statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded. Since $G_\eta = (-\infty, 1)$ and $F_\eta = (0, \infty)$, we have $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta = 0$, and $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta = 1$. Hence η is not statistically $\mathfrak{A}^{\mathfrak{I}}$-convergent.

Example 2.2 Let $\mathfrak{I}$ and $\mathfrak{A}$ be defined as in Example 2.1. Define $\eta = (\eta_k)$ as

$$\eta_k = \begin{cases} k & \text{if } k \in B_1, \\ 0 & \text{otherwise}. \end{cases}$$

Then, for any $\upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{j \leq n : |y_j| > 1\right\}\right| > \upsilon\right\} \in \mathfrak{I},$$

hence η is statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded. Since $G_\eta = (-\infty, 0)$ and $F_\eta = (0, \infty)$, we have $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta = 0$, and $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta = 0$. Hence η is statistically $\mathfrak{A}^{\mathfrak{I}}$-convergent to zero.

3 Statistical $\mathfrak{A}^{\mathfrak{I}}$-Cauchy and statistical $\mathfrak{A}^{\mathfrak{I}^*}$–Cauchy summability

Fridy [12], introduced the concept of Cauchy condition for statistical convergence for real sequences. In [4, 19] and [26] the notion of $\mathfrak{I}$-Cauchy sequence was studied which is a generalization of Cauchy condition for statistical convergence. Nabiev et al. [26] introduced the notion of a $\mathfrak{I}^*$-Cauchy sequence and proved that under certain conditions a $\mathfrak{I}^*$-Cauchy sequence is equivalent to a $\mathfrak{I}$-Cauchy sequence.

Definition 3.1 ([4, 26]) A real sequence $\eta = (\eta_n)$ is a $\mathfrak{I}$-Cauchy sequence if $\forall \epsilon > 0$ there exists $k = k(\epsilon) \in \mathbb{N}$ such that

$$\{n : |\eta_n - \eta_k| \geq \epsilon\} \in \mathfrak{I}.$$

Definition 3.2 ([26]) A real sequence $\eta = (\eta_n)$ is called an $\mathfrak{I}^*$-Cauchy sequence if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}_{\mathfrak{I}}$ such that the subsequence (η_{m_k}) is Cauchy in $\mathbb{R}$.

We introduce the notion of statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy and statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summability.

Definition 3.3 Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$. A real sequence $\eta = (\eta_k)$ is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable if for any $\epsilon > 0$ and $\forall \nu > 0$ there is $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left\{j \leq n : \frac{1}{n}\left|\left\{|y_j - y_N| \geq \epsilon\right\}\right| \geq \nu\right\} \in \mathfrak{I}.$$

Definition 3.4 Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \Omega$. A real sequence $\eta = (\eta_k)$ is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summable if there is a set $M = \{m_1, m_2, \ldots\}$, where $m_1 < m_2 < \cdots$, and $M \in \mathcal{F}(\mathfrak{I})$, $\delta(M) = 1$, such that the subsequence (y_{m_i}) is statistically Cauchy in $\mathbb{R}$.

Now, we give some relations between statistical $\mathfrak{A}^{\mathfrak{I}}$ (or statistical $\mathfrak{A}^{\mathfrak{I}^*}$)-summability and statistical $\mathfrak{A}^{\mathfrak{I}}$ (or statistical $\mathfrak{A}^{\mathfrak{I}^*}$)-Cauchy summability.

Theorem 3.1 *A real sequence η is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-summable to $\mathfrak{s}$ if and only if η is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summable.*

Proof The proof follows from Definition 1.4 and Definition 3.4 and using Theorem 1 of [12]; statistical convergence is equivalent to the statistical Cauchy for $\mathbb{R}$. □

Theorem 3.2 *A real sequence $\eta = (\eta_k)$ is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$ iff η is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable.*

Proof Let $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\lim \eta_k = \mathfrak{s}$, then, for any $\epsilon > 0$ and $\forall \nu > 0$, we have the set

$$B(\nu) = \left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j - \mathfrak{s}| \geq \frac{\epsilon}{2}\right\}\right| \geq \nu\right\} \in \mathfrak{I}.$$

Let us define B and C by

$$B = \left\{j \leq n : |y_j - \mathfrak{s}| \geq \frac{\epsilon}{2}\right\}$$

and

$$C = \left\{j \leq n : |y_j - y_N| \geq \epsilon\right\},$$

where $N \notin B$, such N exists as $\mathfrak{I}$ is an admissible ideal, otherwise the set $B(\frac{1}{2}) = \mathbb{N} \notin \mathfrak{I}$. We need first to show that $C \subseteq B$. Now for any $c \in C$, since

$$|y_c - y_N| \leq |y_c - \mathfrak{s}| + |y_N - \mathfrak{s}|,$$

we have

$$|y_c - \mathfrak{s}| + |y_N - \mathfrak{s}| \geq \epsilon.$$

Since $N \notin B$, we have

$$|y_N - \mathfrak{s}| < \frac{\epsilon}{2},$$

therefore

$$|y_c - \mathfrak{s}| > \frac{\epsilon}{2}.$$

Hence $c \in B$. So we have $C \subseteq B$, therefore

$$\frac{1}{n}|C| \leq \frac{1}{n}|B|.$$

Hence for any $\nu > 0$, we have

$$\left\{n : \frac{1}{n}|C| \geq \nu\right\} \subseteq \left\{n : \frac{1}{n}|B| \geq \nu\right\} = B(\nu) \in \mathfrak{I}.$$

Therefore $\{n : \frac{1}{n}|\{j \leq n : |y_j - y_N| \geq \epsilon\}| \geq \nu\} \in \mathfrak{I}$, hence η is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable.

Conversely, let η be statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable. Then, for any $\epsilon > 0$ and $\forall \nu > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$F(\upsilon) = \left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j - y_N| \geq \frac{\epsilon}{2}\right\}\right| \geq \nu\right\} \in \mathfrak{I},$$

therefore

$$G(\nu) = \left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j - y_N| \geq \frac{\epsilon}{2}\right\}\right| < \nu\right\} \in \mathcal{F}_{\mathfrak{I}}.$$

First, let us show that η is statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded. Let us define F and G by

$$F = \left\{j : |y_j - y_N| < \frac{\epsilon}{2}\right\}$$

and

$$G = \left\{j : |y_j| < \epsilon + |y_t|\right\},$$

where $t \in \mathbb{N}$ satisfied $|y_t - y_N| < \frac{\epsilon}{2}$, such t exists as I is an admissible ideal, otherwise, the set $F(\frac{1}{2}) = \mathbb{N} \notin I$. We need first to show that $F \subseteq G$. Now for any $a \in F$, since

$$|y_a - y_t| \leq |y_a - y_N| + |y_N - y_t| < \epsilon.$$

Therefore

$$|y_a| \leq |y_a - y_t| + |y_t| < \epsilon + |y_t|,$$

hence $a \in G$. So we have $F \subseteq G$, therefore

$$\frac{1}{n}|F| \leq \frac{1}{n}|G|.$$

Hence for any $\nu > 0$, we have

$$\left\{n : \frac{1}{n}|F| > \nu\right\} \subseteq \left\{n : \frac{1}{n}|G| > \nu\right\}.$$

Since $G(\nu) \in \mathcal{F}_{\mathfrak{I}}$, we have $\{n : \frac{1}{n}|F| > \nu\} \in \mathcal{F}_{\mathfrak{I}}$, therefore $\{n : \frac{1}{n}|G| > \nu\} \in \mathcal{F}_{\mathfrak{I}}$, so the set

$$\left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j| < \epsilon + |y_t|\right\}\right| > \nu\right\} \in \mathcal{F}_{\mathfrak{I}},$$

i.e.

$$\left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j| > \epsilon + |y_t|\right\}\right| < \nu\right\} \in \mathcal{F}_{\mathfrak{I}},$$

hence, the set

$$\left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j| > \epsilon + |y_t|\right\}\right| > \nu\right\} \in \mathfrak{I},$$

so η is statistically $\mathfrak{A}^{\mathfrak{I}}$-bounded. We use that statistical $\mathfrak{A}^{\mathfrak{I}}$-boundedness implies that $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta$ and $(\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta$ are finite. Using Theorem 2.2, we have $\alpha = (\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta \leq (\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta = \beta$. Given that η is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable, then, for any $\epsilon > 0$ and $\forall \nu > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left\{n : \frac{1}{n}\left|\left\{j \leq n : |y_j - y_{N(\frac{\epsilon}{2})}| \geq \frac{\epsilon}{2}\right\}\right| \geq \nu\right\} \in \mathfrak{I}.$$

Therefore

$$\left\{n : \frac{1}{n}\left|\left\{j \leq n : y_j > y_{N(\frac{\epsilon}{2})} + \frac{\epsilon}{2}\right\}\right| > \nu\right\} \in \mathfrak{I},$$

hence by Theorem 2.1(a), we have

$$\beta < y_{N(\frac{\epsilon}{2})} + \frac{\epsilon}{2}. \tag{3.1}$$

Also we have

$$\left\{n : \frac{1}{n}\left|\left\{j \le n : y_j < y_{N(\frac{\epsilon}{2})} - \frac{\epsilon}{2}\right\}\right| > \nu\right\} \in \mathfrak{I},$$

hence by Theorem 2.1(b), we have

$$y_{N(\frac{\epsilon}{2})} < \alpha + \frac{\epsilon}{2}. \tag{3.2}$$

Using (3.1) and (3.2), we have

$$\beta < \alpha + \epsilon.$$

Hence, for any $\vartheta > 0$, we always have $\beta < \alpha + \vartheta$, therefore $\beta \le \alpha$. Hence $\alpha = (\mathfrak{A}^{\mathfrak{I}})_{st}$-$\liminf \eta = (\mathfrak{A}^{\mathfrak{I}})_{st}$-$\limsup \eta = \beta$. Now by Theorem 2.3, η is statistically $\mathfrak{A}^{\mathfrak{I}}$-convergent. □

Theorem 3.3 (a) *If $\eta = (\eta_k)$ is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summable then η is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable.*

(b) *If $\mathfrak{I}$ satisfies the condition (APO), then η is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summable whenever η is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable.*

Proof (a) The proof follows from Theorem 3.1, Theorem 1.2(a) and Theorem 3.2.

(b) The proof follows from Theorem 3.2, Theorem 1.2(b) and Theorem 3.1. □

Remark 3.1 The converse of Theorem 3.3 (a) is not true in general.

Example 3.1 In [7] Example 2.9, the following example was given.

Let $B_i = \{2^{i-1}(2k-1) : k \in \mathbb{N}\}$ be mutually disjoint infinite sets such that $\mathbb{N} = \bigcup_{i=1}^{\infty} B_i$. Let $\mathfrak{I}$ be the class defined as

$$\mathfrak{I} = \left\{B \subset \mathbb{N} : B \text{ intersects only finite numbers of } B_i's\right\},$$

then $\mathfrak{I}$ is a non-trivial admissible ideal in $\mathbb{N}$. Define $\eta = (\eta_k)$ by

$$\eta_k = \frac{1}{i}, \quad k \in B_i,$$

and $\mathfrak{A} = (\mathfrak{a}_{jk})$ by

$$\mathfrak{a}_{jk} = \begin{cases} 1 & \text{if } k = j^2, \\ 0 & \text{otherwise.} \end{cases}$$

It is shown that η is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to zero but η is not statistically $\mathfrak{A}^{\mathfrak{I}^*}$-summable to any number. Hence from Theorem 3.1 and Theorem 3.2 we conclude that η is statistically $\mathfrak{A}^{\mathfrak{I}}$-Cauchy summable but η is not statistically $\mathfrak{A}^{\mathfrak{I}^*}$-Cauchy summable.

4 Some Tauberian theorems

In [12], a Tauberian theorem was given for statistical convergence. The next results are Tauberian theorems for statistical $\mathfrak{A}^{\mathfrak{I}}$-summability. Let τ denote the collection of lower triangular nonnegative summability matrices $\mathfrak{A}$ with (i) $\sum_{k=1}^{n} \mathfrak{a}_{nk} = 1$ and (ii) if $K \subseteq \mathbb{N}$ such that $\delta(K) = 0$, then $\lim_n \sum_{k\in K} \mathfrak{a}_{nk} = 0$, (cf. [13]). From these conditions any $\mathfrak{A} \in \tau$ is regular. Let us denote $\Delta\eta_k = \eta_k - \eta_{k+1}$.

Theorem 4.1 *Let $\mathfrak{I}$ be a non-trivial admissible ideal in $\mathbb{N}$ which satisfies the condition (APO). Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \tau$ and $\eta = (\eta_k)$ be a bounded sequence. If η is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$ and $\Delta A_{m_i}(\eta) = O(\frac{1}{m_i})$, where $M = \{m_i\} \in \mathcal{F}_{\mathfrak{I}}$, then η is $\mathfrak{I}$-statistically convergent to $\mathfrak{s}$.*

Proof Let η be statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$ and $\mathfrak{I}$ satisfy the condition (*APO*). From Theorem 1.2(b), η is statistically $\mathfrak{A}^{\mathfrak{I}^*}$-summable to $\mathfrak{s}$. Since $\Delta A_{m_i}(\eta) = O(\frac{1}{m_i})$, so by Theorem 3 of [12], η is $\mathfrak{A}^{\mathfrak{I}^*}$-summable to $\mathfrak{s}$. Since $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \tau$, we have $\mathfrak{A} = (\mathfrak{a}_{m_ik}) \in \tau$. Therefore by Theorem 1 of Fridy and Miller [13], η is $\mathfrak{I}^*$-statistically convergent to $\mathfrak{s}$. Hence by Corollary 1.1(a), η is $\mathfrak{I}$-statistically convergent to $\mathfrak{s}$. $\square$

Corollary 4.1 *Let $\mathfrak{I}$ be a non-trivial admissible ideal in $\mathbb{N}$ which satisfies the condition (APO). Let $\mathfrak{A} = (\mathfrak{a}_{jk}) \in \tau$ and $\eta = (\eta_k)$ be a bounded sequence. If η is statistically $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$ and $\Delta\mathfrak{A}_{m_i}(\eta) = O(\frac{1}{m_i})$, where $M = \{m_i\} \in \mathcal{F}_{\mathfrak{I}}$, then η is $\mathfrak{A}^{\mathfrak{I}}$-summable to $\mathfrak{s}$.*

Theorem 4.2 *Let $\mathfrak{I}$ be a non-trivial admissible ideal in $\mathbb{N}$ which satisfies the condition (APO). Let $\eta = (\eta_k)$ be a bounded sequence. If η is $\mathfrak{I}$-statistically convergent to $\mathfrak{s}$ and $\Delta\eta_{m_i} = O(\frac{1}{m_i})$, where $M = \{m_i\} \in \mathcal{F}(I)$, then η is $\mathfrak{I}$-convergent to $\mathfrak{s}$.*

Proof Let η be $\mathfrak{I}$-statistically convergent to $\mathfrak{s}$. Since $\mathfrak{I}$ satisfies the condition (*APO*), from Corollary 1.1(b), η is $\mathfrak{I}^*$-statistically convergent to $\mathfrak{s}$. Since $\Delta\eta_{m_i} = O(\frac{1}{m_i})$, by Theorem 3 of [12], η is $\mathfrak{I}^*$-convergent to $\mathfrak{s}$. Now by Proposition 3.2 of [18], η is $\mathfrak{I}$-convergent to $\mathfrak{s}$. $\square$

Acknowledgements
NA.

Funding
NA.

Availability of data and materials
NA.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 3 November 2020 Accepted: 3 February 2021 Published online: 17 February 2021

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