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On a class of Hilbert-type inequalities in the whole plane related to exponent function

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Abstract

By introducing a kernel involving an exponent function with multiple parameters, we establish a new Hilbert-type inequality and its equivalent Hardy form. We also prove that the constant factors of the obtained inequalities are the best possible. Furthermore, by introducing the Bernoulli number, Euler number, and the partial fraction expansion of cotangent function and cosecant function, we get some special and interesting cases of the newly obtained inequality.

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Keywords: Hilbert-type inequality; Exponent function; Partial fraction expansion; Bernoulli number; Euler number

1 Introduction

Suppose that $f(x)$ and $\mu(x)$ (>0) are two measurable functions defined on a measurable set Ω, and $p>1$. Define

$$L_{\mu}^{p}(\Omega):=\left\{f:\|f\|_{p,\mu}:=\left(\int_{\Omega}\mu(x)\left|f(x)\right|^{p}\,\mathrm{d}m\right)^{\frac{1}{p}}<\infty\right\}. \tag{1.1}$$

Specially, for $\mu(x)=1$, we have the abbreviated forms: $\|f\|_{p,\mu}:=\|f\|_{p}$ and $L_{\mu}^{p}(\Omega):=L^{p}(\Omega)$.

Consider two real-valued functions $f,g\geq 0$ and $f,g\in L^{p}(\mathbb{R}_{+})$. Suppose that $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have the following two classical Hilbert-type inequalities [1]:

$$\int_{\mathbb{R}_{+}}\int_{\mathbb{R}_{+}}\frac{f(x)g(y)}{x+y}\,\mathrm{d}x\,\mathrm{d}y<\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_{p}\|g\|_{q}, \tag{1.2}$$

$$\int_{\mathbb{R}_{+}}\int_{\mathbb{R}_{+}}\frac{f(x)g(y)}{\max\{x,y\}}\,\mathrm{d}x\,\mathrm{d}y<pq\|f\|_{p}\|g\|_{q}, \tag{1.3}$$

where the constant factors $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq in (1.2) and (1.3) are the best possible.

In the past 100 years, especially after the 1990s, by the introduction of several parameters and special functions such as β-function and Γ-function, some classical Hilbert-type integral inequalities like (1.2) and (1.3) as well as their discrete forms were extended to more general forms (see [2–12]). The inequality below is a typical extension of (1.2) which was established by Yang [13] in 2004:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\beta + y^\beta}\,\mathrm{d}x\,\mathrm{d}y < \frac{\pi}{\beta \sin\frac{\pi}{r}} \|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.4}$$

where $\rho > 0$, $\mu(x) = x^{p(1-\frac{\beta}{r})-1}$, $\nu(x) = x^{q(1-\frac{\beta}{s})-1}$, $\frac{1}{r} + \frac{1}{s} = 1$, and the constant factor is the best possible.

In recent years, by constructing new kernel functions and studying their discrete form, half-discrete form, reverse form, multi-dimensional extension and coefficient refinement, researchers have established a large number of new Hilbert-type inequalities (see [14–25]). It should be noted that the establishment of these inequalities fully demonstrates the techniques of modern analysis and proves to be critical in the development of modern analysis [26].

In these numerous publications related to the Hilbert inequality, we will present some results with the kernels involving exponent function, and the motivation of this work is precisely from these results. The first result presented below was established by Yang [27] in 2012, that is,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} e^{-\frac{ax}{y}} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < a^{-\beta}\Gamma(\beta)\|f\|_{2,\mu}\|g\|_{2,\mu}, \tag{1.5}$$

where $a > 0$, $\beta > 0$, $\mu(x) = x^{-2\beta+1}$ and $\nu(y) = y^{2\beta+1}$.

In addition, Liu [28] established an inequality with the kernel involving hyperbolic secant function in 2013, and Yang [29] established an inequality with the kernel involving hyperbolic cosecant function in 2014. The two inequalities can be written as follows:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \operatorname{sech}(xy) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < 2c_0\|f\|_{2,\mu}\|g\|_{2,\mu}, \tag{1.6}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \operatorname{csch}(xy) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \frac{\pi^2}{4}\|f\|_{2,\mu}\|g\|_{2,\mu}, \tag{1.7}$$

where $\operatorname{sech}(t) = \frac{2}{e^t+e^{-t}}$, $\operatorname{csch}(t) = \frac{2}{e^t-e^{-t}}$, $\mu(x) = x^{-3}$ and $c_0 = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = 0.91596559^+$, which is the Catalan constant.

In this work, the integral interval of inequality (1.6) and (1.7) will be extended to the whole plane, and the following new inequalities will be established:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sech}(xy) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \frac{E_n}{4^n}\pi^{2n+1}\|f\|_{2,\mu}\|g\|_{2,\mu}, \tag{1.8}$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\operatorname{csch}(xy)\right| f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \frac{B_n}{n}\left(2^{2n}-1\right)\pi^{2n}\|f\|_{2,\nu}\|g\|_{2,\nu}, \tag{1.9}$$

where $\mu(x) = |x|^{-4n-1}$, $\nu(x) = |x|^{-4n+1}$, E_n is an Euler number [30, 31] and B_n is a Bernoulli number [30, 31].

Furthermore, we also present some interesting inequalities involving other hyperbolic functions in this paper. More generally, a new kernel function including both the homogeneous case and the non-homogeneous case is constructed, and a Hilbert-type inequality involving this new kernel is established. It will be shown that the newly obtained inequality is a unified extension of (1.8), (1.9) and some other special Hilbert-type inequalities.

2 Some lemmas

Lemma 2.1 *Let* $r,s>0$, $r+s=1$, $\varphi(x)=\cot x$. *Then*

$$\varphi^{(2n-1)}(r\pi)=-\frac{(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\left\{\frac{1}{(k+r)^{2n}}+\frac{1}{(k+s)^{2n}}\right\},\quad n\in\mathbb{N}^{+}; \tag{2.1}$$

$$\varphi^{(2n)}(r\pi)=\frac{(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\left\{\frac{1}{(k+r)^{2n+1}}-\frac{1}{(k+s)^{2n+1}}\right\},\quad n\in\mathbb{N}. \tag{2.2}$$

Proof Consider the rational fraction expansion of $\varphi(x)=\cot x$ [30]:

$$\varphi(x)=\frac{1}{x}+\sum_{k=1}^{\infty}\left\{\frac{1}{x+k\pi}+\frac{1}{x-k\pi}\right\},$$

and find the $(2n-1)$th derivative of $\varphi(x)$, then we obtain

$$\varphi^{(2n-1)}(x)=-(2n-1)!\left\{\sum_{k=0}^{\infty}\frac{1}{(x+k\pi)^{2n}}+\sum_{k=1}^{\infty}\frac{1}{(x-k\pi)^{2n}}\right\}. \tag{2.3}$$

Set $x=r\pi$ in (2.3). Since $r+s=1$, it follows that

$$\begin{aligned}\varphi^{(2n-1)}(r\pi)&=-\frac{(2n-1)!}{\pi^{2n}}\left\{\sum_{k=0}^{\infty}\frac{1}{(k+r)^{2n}}+\sum_{k=1}^{\infty}\frac{1}{(k-r)^{2n}}\right\}\\&=-\frac{(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\left\{\frac{1}{(k+r)^{2n}}+\frac{1}{(k+s)^{2n}}\right\}.\end{aligned}$$

Therefore, (2.1) is proved, and similar computation yields (2.2). □

Furthermore, by considering the following rational fraction expansion of $\psi(x)=\csc x$ [30]:

$$\psi(x)=\frac{1}{x}+\sum_{k=1}^{\infty}(-1)^{k}\left(\frac{1}{x+k\pi}+\frac{1}{x-k\pi}\right),$$

we can obtain Lemma 2.2.

Lemma 2.2 *Let* $r,s>0$, $r+s=1$ *and* $\psi(x)=\csc x$. *Then*

$$\psi^{(2n-1)}(r\pi)=-\frac{(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}(-1)^{k}\left\{\frac{1}{(k+r)^{2n}}-\frac{1}{(k+s)^{2n}}\right\},\quad n\in\mathbb{N}^{+}; \tag{2.4}$$

$$\psi^{(2n)}(r\pi)=\frac{(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}(-1)^{k}\left\{\frac{1}{(k+r)^{2n+1}}+\frac{1}{(k+s)^{2n+1}}\right\},\quad n\in\mathbb{N}. \tag{2.5}$$

Remark 2.3 Let $r=\frac{1}{2}$ in (2.1). For $n\in\mathbb{N}^{+}$, we have

$$\varphi^{(2n-1)}\left(\frac{\pi}{2}\right)=-\frac{2^{2n+1}}{\pi^{2n}}(2n-1)!\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n}}. \tag{2.6}$$

By using the equality [30, 31] $\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=\frac{(2\pi)^{2n}}{2(2n)!}B_n$, where B_n is a Bernoulli number, we have

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n}}=\sum_{k=1}^{\infty}\frac{1}{k^{2n}}-\sum_{k=1}^{\infty}\frac{1}{(2k)^{2n}}=\frac{2^{2n}-1}{2(2n)!}B_n\pi^{2n}. \tag{2.7}$$

Applying (2.7) to (2.6), we obtain

$$\varphi^{(2n-1)}\left(\frac{\pi}{2}\right)=-\frac{B_n}{n}2^{2n-1}\left(2^{2n}-1\right). \tag{2.8}$$

In addition, letting $r=\frac{1}{4}$ in (2.1), we can also obtain

$$\varphi^{(2n-1)}\left(\frac{\pi}{4}\right)=-\frac{B_n}{n}4^{2n-1}\left(2^{2n}-1\right). \tag{2.9}$$

Furthermore, let $r=\frac{1}{4}$ in (2.2) and $r=\frac{1}{2}$ in (2.5). In view of [30] $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}=\frac{\pi^{2n+1}}{2^{2n+2}(2n)!}E_n$, where E_n is an Euler number, we obtain

$$E_n=\frac{1}{4^n}\varphi^{(2n)}\left(\frac{\pi}{4}\right)=\psi^{(2n)}\left(\frac{\pi}{2}\right). \tag{2.10}$$

Lemma 2.4 *Let $\eta_1,\eta_2,\delta\in\{1,-1\}$, $a>c\geq d>b>0$ and $c\neq d$ for $\eta_2=-1$. Let β be such that $\beta\geq 1$, and $\beta\neq 1$ for $\eta_1=-1$, $\eta_2=1$. Define*

$$K(x,y):=\frac{|c^{xy^\delta}+\eta_2 d^{xy^\delta}|}{|a^{xy^\delta}+\eta_1 b^{xy^\delta}|} \tag{2.11}$$

and

$$\begin{aligned}C_{\eta_1,\eta_2}(a,b,c,d,\beta):={}&\sum_{k=0}^{\infty}\left(\frac{(-\eta_1)^k}{(k\ln\frac{a}{b}+\ln\frac{a}{c})^\beta}+\frac{\eta_2(-\eta_1)^k}{(k\ln\frac{a}{b}+\ln\frac{a}{d})^\beta}\right)\\&+\sum_{k=0}^{\infty}\left(\frac{(-\eta_1)^k}{(k\ln\frac{a}{b}+\ln\frac{d}{b})^\beta}+\frac{\eta_2(-\eta_1)^k}{(k\ln\frac{a}{b}+\ln\frac{c}{b})^\beta}\right).\end{aligned} \tag{2.12}$$

Then

$$\int_{\mathbb{R}}K(t,1)|t|^{\beta-1}\,\mathrm{d}t=\Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta), \tag{2.13}$$

where $\Gamma(\beta)=\int_{\mathbb{R}_+}x^{\beta-1}e^{-x}\,\mathrm{d}x$ $(\beta>0)$ is the second type of Euler integral (Γ-function) [30, 31], and $\Gamma(\beta)=(\beta-1)!$ for $\beta\in\mathbb{N}^+$.

Proof

$$\begin{aligned}&\int_{\mathbb{R}}K(t,1)|t|^{\beta-1}\,\mathrm{d}t\\&\quad=\int_{\mathbb{R}_+}K(t,1)|t|^{\beta-1}\,\mathrm{d}t+\int_{\mathbb{R}_-}K(t,1)|t|^{\beta-1}\,\mathrm{d}t:=I_1+I_2.\end{aligned} \tag{2.14}$$

Observing that $a > b > 0$ and $\eta_1 \in \{1, -1\}$, we obtain

$$\frac{1}{a^t + \eta_1 b^t} = \frac{a^{-t}}{1 + \eta_1 (a^{-1} b)^t} = \sum_{k=0}^{\infty} (-\eta_1)^k \left\{ \left(\frac{b}{a}\right)^k \frac{1}{a} \right\}^t. \tag{2.15}$$

Hence

$$\begin{aligned} I_1 &= \int_{\mathbb{R}_+} K(t,1)|t|^{\beta-1}\,\mathrm{d}t \\ &= \sum_{k=0}^{\infty} (-\eta_1)^k \int_{\mathbb{R}_+} \left\{ \left(\frac{b}{a}\right)^k \frac{c}{a} \right\}^t t^{\beta-1}\,\mathrm{d}t + \sum_{k=0}^{\infty} (-\eta_1)^k \eta_2 \int_{\mathbb{R}_+} \left\{ \left(\frac{b}{a}\right)^k \frac{d}{a} \right\}^t t^{\beta-1}\,\mathrm{d}t \\ &:= \sum_{k=0}^{\infty} \left\{ (-\eta_1)^k I_{11} + (-\eta_1)^k \eta_2 I_{12} \right\}. \end{aligned} \tag{2.16}$$

Setting $t = \frac{u}{k \ln \frac{a}{b} + \ln \frac{a}{c}}$, we have

$$I_{11} = \frac{1}{(k \ln \frac{a}{b} + \ln \frac{a}{c})^\beta} \int_{\mathbb{R}_+} e^{-u} u^{\beta-1}\,\mathrm{d}u = \frac{\Gamma(\beta)}{(k \ln \frac{a}{b} + \ln \frac{a}{c})^\beta}. \tag{2.17}$$

Similarly, we can obtain

$$I_{12} = \frac{1}{(k \ln \frac{a}{b} + \ln \frac{a}{d})^\beta} \int_{\mathbb{R}_+} e^{-u} u^{\beta-1}\,\mathrm{d}u = \frac{\Gamma(\beta)}{(k \ln \frac{a}{b} + \ln \frac{a}{d})^\beta}. \tag{2.18}$$

Applying (2.17) and (2.18) to (2.16), we obtain

$$I_1 = \Gamma(\beta) \sum_{k=0}^{\infty} \left(\frac{(-\eta_1)^k}{(k \ln \frac{a}{b} + \ln \frac{a}{c})^\beta} + \frac{\eta_2 (-\eta_1)^k}{(k \ln \frac{a}{b} + \ln \frac{a}{d})^\beta} \right). \tag{2.19}$$

Similarly, we can obtain

$$\begin{aligned} I_2 &= \int_{\mathbb{R}_-} K(t,1)|t|^{\beta-1}\,\mathrm{d}t = \int_{\mathbb{R}_+} K(-t,1) t^{\beta-1}\,\mathrm{d}t \\ &= \Gamma(\beta) \sum_{k=0}^{\infty} \left(\frac{(-\eta_1)^k}{(k \ln \frac{a}{b} + \ln \frac{d}{b})^\beta} + \frac{\eta_2 (-\eta_1)^k}{(k \ln \frac{a}{b} + \ln \frac{c}{b})^\beta} \right). \end{aligned} \tag{2.20}$$

Plugging (2.19) and (2.20) into (2.14), and using (2.12), we have (2.13). □

Lemma 2.5 *Let $\eta_1, \eta_2, \delta \in \{1, -1\}$, $a > c \ge d > b > 0$ and $c \ne d$ for $\eta_2 = -1$. Let β be such that $\beta \ge 1$, and $\beta \ne 1$ for $\eta_1 = -1$, $\eta_2 = 1$. $C_{\eta_1,\eta_2}(a,b,c,d,\beta)$ is defined by Lemma 2.4. Suppose $D_\delta = \{y : |y|^\delta < 1\}$, and, for arbitrary natural number n which is large enough, set*

$$f_n(x) = \begin{cases} 0, & x \in [-1,1], \\ |x|^{\beta-1-\frac{2}{np}}, & x \in \mathbb{R} \setminus [-1,1], \end{cases}$$

$$g_n(y) = \begin{cases} |y|^{\delta\beta-1+\frac{2\delta}{nq}}, & y \in D_\delta, \\ 0, & y \in \mathbb{R} \setminus D_\delta. \end{cases}$$

Then

$$\frac{1}{n}\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f_n(x)g_n(y)\,\mathrm{d}x\,\mathrm{d}y = C_{\eta_1,\eta_2}(a,b,c,d,\beta) + o(1). \tag{2.21}$$

Proof Setting $D_\delta^+ = \{y : y > 0, y \in D_\delta\}$, $D_\delta^- = \{y : y < 0, y \in D_\delta\}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f_n(x)g_n(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_1^\infty x^{\beta-1-\frac{2}{np}} \int_{D_\delta^+} K(x,y)y^{\delta\beta-1+\frac{2\delta}{nq}}\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad+ \int_1^\infty x^{\beta-1-\frac{2}{np}} \int_{D_\delta^-} K(x,y)|y|^{\delta\beta-1+\frac{2\delta}{nq}}\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad+ \int_{-\infty}^{-1} |x|^{\beta-1-\frac{2}{np}} \int_{D_\delta^+} K(x,y)y^{\delta\beta-1+\frac{2\delta}{nq}}\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad+ \int_{-\infty}^{-1} |x|^{\beta-1-\frac{2}{np}} \int_{D_\delta^-} K(x,y)|y|^{\delta\beta-1+\frac{2\delta}{nq}}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{2.22}$$

Setting $xy^\delta = t$, and using Fubini's theorem, where δ is equal to 1 or -1, we can get

$$\begin{aligned}
J_1 = J_4 &= \int_1^\infty x^{-1-\frac{2}{n}} \left(\int_0^x K(t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_1^\infty x^{-1-\frac{2}{n}} \int_0^1 K(t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad+ \int_1^\infty x^{-1-\frac{2}{n}} \int_1^x K(t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t\,\mathrm{d}x \\
&= \frac{n}{2}\int_0^1 K(t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t + \int_1^\infty K(t,1)t^{\beta-1+\frac{2}{nq}} \int_t^\infty x^{-1-\frac{2}{n}}\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{n}{2}\int_0^1 K(t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t + \frac{n}{2}\int_1^\infty K(t,1)t^{\beta-1-\frac{2}{np}}\,\mathrm{d}t.
\end{aligned} \tag{2.23}$$

Similarly, setting $xy^\delta = -t$, we can also obtain

$$\begin{aligned}
J_2 = J_3 &= \int_1^\infty x^{-1-\frac{2}{n}} \int_0^x K(-t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t\,\mathrm{d}x \\
&= \frac{n}{2}\int_0^1 K(-t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t + \frac{n}{2}\int_1^\infty K(-t,1)t^{\beta-1-\frac{2}{np}}\,\mathrm{d}t.
\end{aligned} \tag{2.24}$$

Applying (2.23) and (2.24) to (2.22), we have

$$\begin{aligned}
&\frac{1}{n}\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f_n(x)g_n(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_0^1 K(t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t + \int_1^\infty K(t,1)t^{\beta-1-\frac{2}{np}}\,\mathrm{d}t \\
&\qquad+ \int_0^1 K(-t,1)t^{\beta-1+\frac{2}{nq}}\,\mathrm{d}t + \int_1^\infty K(-t,1)t^{\beta-1-\frac{2}{np}}\,\mathrm{d}t.
\end{aligned} \tag{2.25}$$

Letting $n \to \infty$ in (2.25), and using (2.13), we get

$$\begin{aligned}
&\frac{1}{n}\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f_n(x)g_n(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad= \int_{\mathbb{R}^+} K(t,1)t^{\beta-1}\,\mathrm{d}t + \int_{\mathbb{R}^+} K(-t,1)t^{\beta-1}\,\mathrm{d}t + o(1)\\
&\quad= \int_{\mathbb{R}} K(t,1)t^{\beta-1}\,\mathrm{d}t + o(1) = C_{\eta_1,\eta_2}(a,b,c,d,\beta) + o(1).
\end{aligned}$$

The proof of Lemma 2.5 is completed. □

3 Main results

Theorem 3.1 *Let $\eta_1,\eta_2,\delta \in \{1,-1\}$, $a > c \geq d > b > 0$ and $c \neq d$ for $\eta_2 = -1$. Let β be such that $\beta \geq 1$, and $\beta \neq 1$ for $\eta_1 = -1$, $\eta_2 = 1$. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu(x) = |x|^{p(1-\beta)-1}$ and $\nu(y) = |y|^{q(1-\delta\beta)-1}$. Let $f(x), g(y) \geq 0$ with $f(x) \in L^p_\mu(\mathbb{R})$ and $g(y) \in L^q_\nu(\mathbb{R})$. $K(x,y)$ and $C_{\eta_1,\eta_2}(a,b,c,d,\beta)$ are defined via Lemma 2.4. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{3.1}$$

where the constant factor $\Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)$ is the best possible.

Proof By Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \Bigl(\bigl(K(x,y)\bigr)^{\frac{1}{p}}|y|^{\frac{\delta\beta-1}{p}}|x|^{\frac{1-\beta}{q}}f(x)\Bigr)\\
&\qquad\times \Bigl(\bigl(K(x,y)\bigr)^{\frac{1}{q}}|x|^{\frac{\beta-1}{q}}|y|^{\frac{1-\delta\beta}{p}}g(y)\Bigr)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq \biggl(\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)|y|^{\delta\beta-1}|x|^{\frac{p(1-\beta)}{q}}f^p(x)\,\mathrm{d}x\,\mathrm{d}y\biggr)^{\frac{1}{p}}\\
&\qquad\times \biggl(\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)|x|^{\beta-1}|y|^{\frac{q(1-\delta\beta)}{p}}g^q(y)\,\mathrm{d}x\,\mathrm{d}y\biggr)^{\frac{1}{q}}\\
&\quad= \biggl(\int_{\mathbb{R}} \omega(x)|x|^{\frac{p(1-\beta)}{q}}f^p(x)\,\mathrm{d}x\biggr)^{\frac{1}{p}}\biggl(\int_{\mathbb{R}} \varpi(y)|y|^{\frac{q(1-\delta\beta)}{p}}g^q(y)dy\biggr)^{\frac{1}{q}},
\end{aligned} \tag{3.2}$$

where $\omega(x) = \int_{\mathbb{R}} K(x,y)|y|^{\delta\beta-1}\,\mathrm{d}y$, and $\varpi(y) = \int_{\mathbb{R}} K(x,y)|x|^{\beta-1}\,\mathrm{d}x$.

Setting $xy^\eta = t$, and using (2.13), we have

$$\begin{aligned}
\omega(x) &= |x|^{-\beta}\int_{\mathbb{R}} K(t,1)|t|^{\beta-1}\,\mathrm{d}t\\
&= \Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)|x|^{-\beta} \quad (x \neq 0)
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\varpi(y) &= |y|^{-\delta\beta}\int_{\mathbb{R}} K(t,1)|t|^{\beta-1}\,\mathrm{d}t \\ &= \Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)|y|^{-\delta\beta} \quad (y\neq 0).\end{aligned} \tag{3.4}$$

Plugging (3.3) and (3.4) back to (3.2), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \le \Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{3.5}$$

If (3.5) takes the form of an equality, then there must be two constants A_1 and A_2 that are not both equal to zero, such that

$$A_1K(x,y)|y|^{\delta\beta-1}|x|^{\frac{p(1-\beta)}{q}}f^p(x) = A_2K(x,y)|x|^{\beta-1}|y|^{\frac{q(1-\delta\beta)}{p}}g^q(y),$$

a.e. in $\mathbb{R}\times\mathbb{R}$, that is,

$$A_1|x|^{p(1-\beta)}f^p(x) = A_2|y|^{q(1-\delta\beta)}g^q(y),$$

a.e. in $\mathbb{R}\times\mathbb{R}$. Therefore, there exists a constant A such that $A_1|x|^{p(1-\beta)}f^p(x) = A$, a.e. in $\mathbb{R}$, and $A_2|y|^{q(1-\delta\beta)}g^q(y) = A$, a.e. in $\mathbb{R}$. Without loss of generality, we assume that $A_1\neq 0$, then we can obtain $x^{p(1-\beta)-1}f^p(x) = \frac{A}{A_1x}$ a.e. in $\mathbb{R}$, which contradicts the fact $f(x)\in L^p_\mu(\mathbb{R})$. Hence, (3.5) keeps the form of a strict inequality, and (3.1) is obtained.

What we need to prove next is that the constant factor $\Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)$ in inequality (3.1) is the best possible. Suppose that there exists a positive constant $k < \Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta)$, such that (3.1) still holds if $C_{\eta_1,\eta_2}(a,b,c,d,\beta)$ is replaced by k. That is,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < k\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{3.6}$$

Replacing f and g in (3.6) with f_n and g_n defined in Lemma 2.5, respectively, we obtain

$$\begin{aligned}&\frac{1}{n}\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f_n(x)g_n(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad< \frac{k}{n}\left(\int_1^\infty x^{-\frac{2}{n}-1}\,\mathrm{d}x + \int_{-\infty}^{-1}|x|^{-\frac{2}{n}-1}\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\qquad\times\left(\int_{D_\delta}|x|^{\frac{2\delta}{n}-1}\,\mathrm{d}x\right)^{\frac{1}{q}} = k.\end{aligned} \tag{3.7}$$

Combining (3.7) and (2.21), we have $C_{\eta_1,\eta_2}(a,b,c,d,\beta)+o(1) < k$. Let $n\to\infty$, then we have $C_{\eta_1,\eta_2}(a,b,c,d,\beta)\le k$. This contradicts the hypothesis obviously. Therefore, the constant factor in (3.1) is the best possible. □

Theorem 3.2 *Let $\eta_1,\eta_2,\delta\in\{1,-1\}$, $a>c\ge d>b>0$ and $c\neq d$ for $\eta_2=-1$. Let β be such that $\beta\ge 1$, and $\beta\neq 1$ for $\eta_1=-1$, $\eta_2=1$. Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu(x)=|x|^{p(1-\beta)-1}$*

and $\nu(y) = |y|^{q(1-\delta\beta)-1}$. *Let* $f(x) \geq 0$ *with* $f(x) \in L^p_\mu(0,\infty)$. $K(x,y)$ *and* $C_{\eta_1,\eta_2}(a,b,c,d,\beta)$ *be defined via Lemma* 2.4. *Then*

$$\int_{\mathbb{R}} |y|^{p\delta\beta-1}\left(\int_{\mathbb{R}} K(x,y)f(x)\,\mathrm{d}x\right)^p \mathrm{d}y < \left(C_{\eta_1,\eta_2}(a,b,c,d,\beta)\|f\|_{p,\mu}\right)^p, \tag{3.8}$$

where the constant factor $(\Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta))^p$ *is the best possible, and* (3.8) *is equivalent to* (3.1).

Proof Consider $g(y) := |y|^{p\delta\beta-1}(\int_{\mathbb{R}} K(x,y)f(x)\,\mathrm{d}x)^{p-1}$. By Theorem 3.1, we can get

$$\begin{aligned}
0 &< \left(\|g\|_{q,\nu}\right)^{pq} = \left(\int_{\mathbb{R}} |y|^{q(1-\delta\beta)-1} g^q(y)\,\mathrm{d}y\right)^p \\
&= \left(\int_{\mathbb{R}} |y|^{p\delta\beta-1}\left(\int_{\mathbb{R}} K(x,y)f(x)\,\mathrm{d}x\right)^p \mathrm{d}y\right)^p \\
&= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\right)^p \leq \left(C_{\eta_1,\eta_2}(a,b,c,d,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}\right)^p.
\end{aligned} \tag{3.9}$$

Therefore

$$\begin{aligned}
0 &< \int_{\mathbb{R}} |y|^{p\delta\beta-1}\left(\int_{\mathbb{R}} K(x,y)f(x)\,\mathrm{d}x\right)^p \mathrm{d}y \\
&= \left(\|g\|_{q,\nu}\right)^q \leq \left(C_{\eta_1,\eta_2}(a,b,c,d,\beta)\|f\|_{p,\mu}\right)^p.
\end{aligned} \tag{3.10}$$

Since $f(x) \in L^p_\mu(\mathbb{R})$, by using (3.10), we obtain $g(y) \in L^q_\nu(\mathbb{R})$. By using Theorem 3.1 again, we can deduce that both (3.9) and (3.1) take the form of a strict inequality, and (3.8) is proved. On the other hand, if (3.8) is valid, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}} \left(|y|^{-(1-\delta\beta-\frac{1}{q})}\int_{\mathbb{R}} K(x,y)f(x)\,\mathrm{d}x\right)\left(|y|^{1-\delta\beta-\frac{1}{q}}g(y)\right)\mathrm{d}y \\
&\quad \leq \left(\int_{\mathbb{R}} |y|^{p\delta\beta-1}\left(\int_{\mathbb{R}} K(x,y)f(x)\,\mathrm{d}x\right)^p \mathrm{d}y\right)^{\frac{1}{p}} \|g\|_{q,\nu}.
\end{aligned} \tag{3.11}$$

Combining (3.8) and (3.11), we can get (3.1). Therefore, (3.1) is equivalent to (3.8). From the equivalence of (3.1) and (3.8), we can easily show that the constant factor $(\Gamma(\beta)C_{\eta_1,\eta_2}(a,b,c,d,\beta))^p$ in (3.8) is the best possible. Theorem 3.2 is proved. □

4 Applications

Setting $\eta_1 = -1$, $\eta_2 = 1$, $c = d = 1$ and $\beta = 2n$ ($n \in \mathbb{N}^+$) in Theorem 3.1, and using (2.1), we can obtain

$$C_{\eta_1,\eta_2}(a,b,c,d,\beta) = \frac{-2}{(2n-1)!}\left(\frac{\pi}{\ln\frac{a}{b}}\right)^{2n} \varphi^{(2n-1)}\left(\frac{\pi\ln a}{\ln\frac{a}{b}}\right).$$

Therefore, the following corollary holds.

Corollary 4.1 *Let $\delta \in \{1,-1\}$, $a > 1 > b > 0$. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\varphi(x) = \cot x$, $\mu(x) = |x|^{p(1-2n)-1}$ and $\nu(y) = |y|^{q(1-2\delta n)-1}$, where $n \in \mathbb{N}^{+}$. Let $f(x), g(y) \ge 0$ with $f(x) \in L_{\mu}^{p}(\mathbb{R})$ and $g(y) \in L_{\nu}^{q}(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(x)g(y)}{|a^{xy^{\delta}} - b^{xy^{\delta}}|} \,\mathrm{d}x\,\mathrm{d}y < -\left(\frac{\pi}{\ln \frac{a}{b}}\right)^{2n} \varphi^{(2n-1)}\left(\frac{\pi \ln a}{\ln \frac{a}{b}}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.1}$$

Particularly, let $a = b^{-1} = e$ in (4.1), by virtue of (2.8), then (4.1) is transformed to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|\operatorname{csch}\left(xy^{\delta}\right)\right| f(x)g(y) \,\mathrm{d}x\,\mathrm{d}y < \frac{B_n}{n}\left(2^{2n} - 1\right)\pi^{2n} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.2}$$

Let $p = q = 2$, $\delta = 1$ in (4.2), then we have (1.9).

Setting $\eta_1 = 1$, $\eta_2 = 1$, $c = d = 1$ and $\beta = 2n + 1$ ($n \in \mathbb{N}$) in Theorem 3.1, and using (2.5), we have

$$C_{\eta_1,\eta_2}(a,b,c,d,\beta) = \frac{2}{(2n)!}\left(\frac{\pi}{\ln \frac{a}{b}}\right)^{2n+1} \psi^{(2n)}\left(\frac{\pi \ln a}{\ln \frac{a}{b}}\right).$$

Therefore, we obtain the following corollary.

Corollary 4.2 *Let $\delta \in \{1,-1\}$ and $a > 1 > b > 0$. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\psi(x) = \csc x$, $\mu(x) = |x|^{-2np-1}$ and $\nu(y) = |y|^{-2\delta nq-1}$, where $n \in \mathbb{N}$. Let $f(x), g(y) \ge 0$ with $f(x) \in L_{\mu}^{p}(\mathbb{R})$ and $g(y) \in L_{\nu}^{q}(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(x)g(y)}{a^{xy^{\delta}} + b^{xy^{\delta}}} \,\mathrm{d}x\,\mathrm{d}y < \left(\frac{\pi}{\ln \frac{a}{b}}\right)^{2n+1} \psi^{(2n)}\left(\frac{\pi \ln a}{\ln \frac{a}{b}}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.3}$$

Particularly, let $a = b^{-1} = e$ in (4.3), by virtue of (2.10), then (4.3) is transformed to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{sech}\left(xy^{\delta}\right) f(x)g(y) \,\mathrm{d}x\,\mathrm{d}y < \frac{E_n}{4^n}\pi^{2n+1} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.4}$$

Let $p = q = 2$ and $\delta = 1$ in (4.4), then we get (1.8).

Setting $\eta_1 = -1$, $\eta_2 = 1$, $ab = cd$ and $\beta = 2n$ ($n \in \mathbb{N}^{+}$) in Theorem 3.1, and using (2.1), we obtain

$$C_{\eta_1,\eta_2}(a,b,c,d,\beta) = \frac{-2}{(2n-1)!}\left(\frac{\pi}{\ln \frac{a}{b}}\right)^{2n} \varphi^{(2n-1)}\left(\frac{\pi \ln(a/c)}{\ln(a/b)}\right).$$

Hence, we can obtain another corollary as follows.

Corollary 4.3 *Let $\delta \in \{1,-1\}$, $a > c \ge d > b > 0$ and $ab = cd$. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\varphi(x) = \cot x$, $\mu(x) = |x|^{p(1-2n)-1}$ and $\nu(y) = |y|^{q(1-2\delta n)-1}$, where $n \in \mathbb{N}^{+}$. Let $f(x)$, $g(y) \ge 0$ with $f(x) \in L_{\mu}^{p}(\mathbb{R})$ and $g(y) \in L_{\nu}^{q}(\mathbb{R})$. Then*

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{c^{xy^{\delta}} + d^{xy^{\delta}}}{|a^{xy^{\delta}} - b^{xy^{\delta}}|} f(x)g(y) \,\mathrm{d}x\,\mathrm{d}y \\ &\quad < -2\left(\frac{\pi}{\ln \frac{a}{b}}\right)^{2n} \varphi^{(2n-1)}\left(\frac{\pi \ln(a/c)}{\ln(a/b)}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \end{aligned} \tag{4.5}$$

Let $a = e^{\lambda_1}$, $b = e^{-\lambda_1}$, $c = e^{\lambda_2}$ and $d = e^{-\lambda_2}$ in (4.5), where $\lambda_1 > \lambda_2 > 0$, then we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\bigl|\operatorname{csch}(\lambda_1 xy^{\delta})\bigr|\cosh(\lambda_2 xy^{\delta})f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad< -2\left(\frac{\pi}{2\lambda_1}\right)^{2n}\varphi^{(2n-1)}\left(\frac{(\lambda_1-\lambda_2)\pi}{2\lambda_1}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.6}$$

Letting $\lambda_1 = 2$ and $\lambda_2 = 1$ in (4.6), in view of (2.9), we can also obtain (4.2).

Letting $\lambda_1 = 4$ and $\lambda_2 = 1$ in (4.6), we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sech}(2xy^{\delta})\bigl|\operatorname{csch}(xy^{\delta})\bigr|f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad< \frac{-\pi^{2n}}{8^{2n-1}}\varphi^{(2n-1)}\left(\frac{3\pi}{8}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.7}$$

Setting $\eta_1 = -1$, $\eta_2 = -1$, $ab = cd$ and $\beta = 2n+1$ ($n \in \mathbb{N}$) in Theorem 3.1, and using (2.2), we have

$$C_{\eta_1,\eta_2}(a,b,c,d,\beta) = \frac{2}{(2n)!}\left(\frac{\pi}{\ln\frac{a}{b}}\right)^{2n+1}\varphi^{(2n)}\left(\frac{\pi\ln(a/c)}{\ln(a/b)}\right).$$

Therefore, the following corollary holds obviously.

Corollary 4.4 *Let $\delta \in \{1,-1\}$, $a > c \geq d > b > 0$ and $ab = cd$. Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. Suppose that $\varphi(x) = \cot x$, $\mu(x) = |x|^{-2np-1}$ and $\nu(y) = |y|^{-2\delta nq-1}$, where $n \in \mathbb{N}$. Let $f(x), g(y) \geq 0$ with $f(x) \in L^p_\mu(\mathbb{R})$ and $g(y) \in L^q_\nu(\mathbb{R})$. Then*

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{c^{xy^{\delta}}-d^{xy^{\delta}}}{a^{xy^{\delta}}-b^{xy^{\delta}}}f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad< 2\left(\frac{\pi}{\ln\frac{a}{b}}\right)^{2n+1}\varphi^{(2n)}\left(\frac{\pi\ln(a/c)}{\ln(a/b)}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.8}$$

Let $a = e^{\lambda_1}$, $b = e^{-\lambda_1}$, $c = e^{\lambda_2}$ and $d = e^{-\lambda_2}$ in (4.8), where $\lambda_1 > \lambda_2 > 0$, then we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{csch}(\lambda_1 xy^{\delta})\sinh(\lambda_2 xy^{\delta})f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad< 2\left(\frac{\pi}{2\lambda_1}\right)^{2n+1}\varphi^{(2n)}\left(\frac{(\lambda_1-\lambda_2)\pi}{2\lambda_1}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.9}$$

Let $\lambda_1 = 2$ and $\lambda_2 = 1$ in (4.9), then it follows from (2.10) that we also get (4.4).

Let $\lambda_1 = 4$ and $\lambda_2 = 1$ in (4.9), then we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sech}(xy^{\delta})\operatorname{sech}(2xy^{\delta})f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \frac{\pi^{2n+1}}{8^{2n}}\varphi^{(2n)}\left(\frac{3\pi}{8}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.10}$$

Setting $\eta_1 = 1$, $\eta_2 = -1$, $ab = cd$ and $\beta = 2n$ ($n \in \mathbb{N}^+$) in Theorem 3.1, and using (2.4), we have

$$C_{\eta_1,\eta_2}(a,b,c,d,\beta) = \frac{-2}{(2n-1)!}\left(\frac{\pi}{\ln\frac{a}{b}}\right)^{2n}\psi^{(2n-1)}\left(\frac{\pi\ln(a/c)}{\ln(a/b)}\right).$$

Therefore, we obtain the following corollary.

Corollary 4.5 *Let $\delta \in \{1,-1\}$, $a > c \geq d > b > 0$ and $ab = cd$. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\psi(x) = \csc x$, $\mu(x) = |x|^{p(1-2n)-1}$ and $\nu(y) = |y|^{q(1-2\delta n)-1}$, where $n \in \mathbb{N}^{+}$. Let $f(x)$, $g(y) \geq 0$ with $f(x) \in L^p_\mu(\mathbb{R})$ and $g(y) \in L^q_\nu(\mathbb{R})$. Then*

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|c^{xy^\delta} - d^{xy^\delta}|}{a^{xy^\delta} + b^{xy^\delta}} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad < -2\left(\frac{\pi}{\ln\frac{a}{b}}\right)^{2n} \psi^{(2n-1)}\left(\frac{\pi\ln(a/c)}{\ln(a/b)}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.11}$$

Let $a = e^{\lambda_1}$, $b = e^{-\lambda_1}$, $c = e^{\lambda_2}$ and $d = e^{-\lambda_2}$ in (4.11), where $\lambda_1 > \lambda_2 > 0$, then we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sech}(\lambda_1 xy^\delta)\left|\sinh(\lambda_2 xy^\delta)\right| f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad < -2\left(\frac{\pi}{2\lambda_1}\right)^{2n} \psi^{(2n-1)}\left(\frac{(\lambda_1-\lambda_2)\pi}{2\lambda_1}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.12}$$

Let $\lambda_1 = 2$ and $\lambda_2 = 1$ in (4.12), then we can have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sech}(xy^\delta)\left|\tanh(2xy^\delta)\right| f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad < \frac{-\pi^{2n}}{4^{2n-1}} \psi^{(2n-1)}\left(\frac{\pi}{4}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.13}$$

At last, setting $\eta_1 = 1$, $\eta_2 = 1$, $ab = cd$ and $\beta = 2n+1$ ($n \in \mathbb{N}$) in Theorem 3.1, by virtue of (2.5), then the following corollary holds.

Corollary 4.6 *Let $\delta \in \{1,-1\}$, $a > c \geq d > b > 0$ and $ab = cd$. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\psi(x) = \csc x$, $\mu(x) = |x|^{-2np-1}$ and $\nu(y) = |y|^{-2\delta nq-1}$, where $n \in \mathbb{N}$. Let $f(x)$, $g(y) \geq 0$ with $f(x) \in L^p_\mu(\mathbb{R})$ and $g(y) \in L^q_\nu(\mathbb{R})$. Then*

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{c^{xy^\delta} + d^{xy^\delta}}{a^{xy^\delta} + b^{xy^\delta}} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad < 2\left(\frac{\pi}{\ln\frac{a}{b}}\right)^{2n+1} \psi^{(2n)}\left(\frac{\pi\ln(a/c)}{\ln(a/b)}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.14}$$

Let $a = e^{\lambda_1}$, $b = e^{-\lambda_1}$, $c = e^{\lambda_2}$ and $d = e^{-\lambda_2}$ in (4.14), where $\lambda_1 > \lambda_2 > 0$, then we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sech}(\lambda_1 xy^\delta)\cosh(\lambda_2 xy^\delta) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad < 2\left(\frac{\pi}{2\lambda_1}\right)^{2n+1} \psi^{(2n)}\left(\frac{(\lambda_1-\lambda_2)\pi}{2\lambda_1}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.\end{aligned} \tag{4.15}$$

Letting $\lambda_1 = 2$, $\lambda_2 = 1$ in (4.15), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{csch}(xy^\delta)\tanh(2xy^\delta) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \frac{\pi^{2n+1}}{4^{2n}} \psi^{(2n)}\left(\frac{\pi}{4}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}.$$

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