Zhao and Zhou *Journal of Inequalities and Applications* (2021) 2021:29
https://doi.org/10.1186/s13660-021-02560-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

New weighted norm inequalities for multilinear Calderón–Zygmund operators with kernels of Dini's type and their commutators

Yichun Zhao[1] and Jiang Zhou[1*]

*Correspondence:
zhoujiang@xju.edu.cn
[1]College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, P.R. China

Abstract

In this paper, we introduce certain classes of multilinear Calderón–Zygmund operators with kernels of Dini's type. Applying the sharp method and $A_{\vec{p}}^{\infty}(\varphi)$ functions, we first establish some weighted norm inequalities for multilinear Calderón–Zygmund operators with kernels of Dini's type, including pointwise estimates, strong type, and weak endpoint estimates. Furthermore, similar weighted norm inequalities for commutators with $\mathrm{BMO}_{\theta}(\varphi)$ functions are also obtained, but the weak endpoint estimate is of $L(\log L)$ type.

MSC: 42B25; 42B20

Keywords: $A_{\vec{p}}^{\infty}(\varphi)$ weight; $\mathrm{BMO}_{\theta}(\varphi)$ function; Dini kernel; Commutator

1 Introduction

In the 1980s, the multilinear Calderón–Zygmund theory was first studied by Coifman and Meyer [1, 2]. The multilinear Calderón–Zygmund operators with standard kernels were then further investigated by many authors, such as [3–8]. Meanwhile, many authors weakened the standard kernel conditions to rough associated kernel conditions; see [9–13]. Particularly, in 1985, Yabuta [10] introduced the Calderón–Zygmund operators of type $\omega(t)$ (the definition given below) and obtained weighted norm inequalities of the Calderón–Zygmund operators of type $\omega(t)$ on L^p spaces, here weight functions belong to Muckenhoupt's class A_p. In 2014, Lu and Zhang [12] obtained the weighted boundedness of multilinear Calderón–Zygmund operators of type $\omega(t)$ and their commutators with BMO functions from weighted L^p spaces to weighted product of L^p spaces. In 2016, Zhang and Sun [13] further considered weighted norm inequalities of iterated commutators that multilinear Calderón–Zygmund operators of type $\omega(t)$ with BMO functions.

Throughout this paper, $\omega(t) : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$.

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

For $a > 0$, we say that $\omega \in \mathrm{Dini}(a)$, if

$$|\omega|_{\mathrm{Dini}(a)} := \int_0^1 \frac{\omega^a(t)}{t}\,dt < \infty.$$

It is worth mentioning that $\mathrm{Dini}(a_1) \subset \mathrm{Dini}(a_2)$ when $0 < a_1 < a_2$.

Definition 1.1 Let $K(x, y_1, \ldots, y_m)$ be a locally integrable function, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, it is said to belong a certain class of multilinear Calderón–Zygmund kernel of type $\omega(t)$, if there exist constant $A > 0$, $N > 0$ such that

$$\left|K(x, y_1, \ldots, y_m)\right| \le \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}(1 + \sum_{j=1}^m |x - y_j|)^N} \tag{1.1}$$

for all $(x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some $j = 1, 2, \ldots, m$ and

$$\begin{aligned} &\left|K(x, y_1, \ldots, y_m) - K\left(x', y_1, \ldots, y_m\right)\right| \\ &\quad \le \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}(1 + \sum_{j=1}^m |x - y_j|)^N}\,\omega\left(\frac{|x - x'|}{\sum_{j=1}^m |x - y_j|}\right) \end{aligned} \tag{1.2}$$

whenever $|x - x'| \le \frac{1}{2}\max_{1\le j\le m}|x - y_j|$, and

$$\begin{aligned} &\left|K(x, y_1, \ldots, y_j \ldots, y_m) - K\left(x, y_1, \ldots, y_j', \ldots, y_m\right)\right| \\ &\quad \le \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}(1 + \sum_{j=1}^m |x - y_j|)^N}\,\omega\left(\frac{|y_j - y_j'|}{\sum_{j=1}^m |x - y_j|}\right) \end{aligned} \tag{1.3}$$

whenever $|y_j - y_j'| \le \frac{1}{2}\max_{1\le j\le m}|x - y_j|$.

Let $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ (from the product of Schwarz spaces to the space of tempered distributions) be a multilinear operator with certain classes of multilinear Calderón–Zygmund kernels of type $\omega(t)$ if there exists a $K(x, y_1, \ldots, y_m)$ that satisfies (1.1)–(1.3), such that

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\,dy_1 \cdots dy_m \tag{1.4}$$

whenever $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and each $f_j \in C_c^\infty(\mathbb{R}^n), j = 1, \ldots, m$.

If T can be extended to a bounded multilinear operator:

$$L^{q_1}\left(\mathbb{R}^n\right) \times \cdots \times L^{q_m}\left(\mathbb{R}^n\right) \to L^q\left(\mathbb{R}^n\right) \tag{1.5}$$

for some $1 < q_1, \ldots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, or

$$L^{q_1}\left(\mathbb{R}^n\right) \times \cdots \times L^{q_m}\left(\mathbb{R}^n\right) \to L^{q,\infty}\left(\mathbb{R}^n\right) \tag{1.6}$$

for some $1 \le q_1, \ldots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, then T is said to belong to the class of multilinear Calderón–Zygmund operators of type $\omega(t)$.

Remark When $N = 0$ in Eqs. (1.1)–(1.3), such kernels have a standard kernel of type $\omega(t)$ as Lu and Zhang [12] and Zhang and Sun [13] considered.

The assumption of (1.6) is reasonable, one may refer to [12, Theorem 1.2].

Let T be a multilinear operator and $\vec{b} = (b_1, \dots, b_m)$ be a locally integrable vector function in $\mathrm{BMO}^m(\mathbb{R}^n)$, the multilinear commutators of T with $\vec{b}$ is defined by

$$T_{\Sigma_{\vec{b}}}(f_1,\dots,f_m) = \sum_{j=1}^{m} T_{\vec{b}}^{j}(\vec{f}),$$

where

$$T_{\vec{b}}^{j}(\vec{f}) \equiv b_j T(f_1,\dots,f_j,\dots,f_m) - T(f_1,\dots,b_j f_j,\dots,f_m).$$

In 2003, Pérez and Torres [14] first introduced multilinear commutators of multilinear Calderón–Zygmund operators and established their boundedness from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < q, q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, also, from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for $1 \le q, q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$. In 2009, Lerner et al. [8] obtained some weighted boundedness of multilinear commutators as follows:

$$\left\| T_{\Sigma_{\vec{b}}}(\vec{f}) \right\|_{L^q(\nu_{\vec{w}})} \le C \sum_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)},$$

and for the weak end-point also it was proved that

$$\nu_{\vec{w}}\left\{x \in \mathbb{R}^n : \left|T_{\Sigma_{\vec{b}}}(\vec{f})(x)\right| > t^m\right\} \le C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{t} \right) w_j(x)\, dx \right)^{\frac{1}{m}}.$$

To clarify the notation, if T is associated in the usual way with a kernel $K(x, y_1, \dots, y_m)$ satisfying (1.1)–(1.3), then at a formal level

$$\begin{aligned} &T_{\Pi_{\vec{b}}}(f_1,f_2,\dots,f_m)(x) \\ &\quad = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} \bigl(b_j(x) - b_j(y_j)\bigr) K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m. \end{aligned}$$

Lerner [8] obtained weighted norm inequalities of classical multilinear Calderón–Zygmund operators and their commutators with BMO functions through new maximal functions. In 2014, end-point estimates for iterated commutators of multilinear singular integrals were shown by Pérez et al. [15]. Lu and Zhang [12] studied multilinear Calderón–Zygmund operators with type $\omega(t)$ and multilinear commutators with BMO functions. Simultaneously, they established some weighted norm inequalities, such as strong type and weak end-point estimates. The corresponding result of iterated commutators by Zhang and Sun [13] was shown, where the weights belong to $A_{\vec{p}}$. In 2015, Pan and Tang [16] and Bui [17], respectively, established weighted norm inequalities for certain classes of multilinear Calderón–Zygmund operators and their commutators with $\mathrm{BMO}_\theta(\varphi)$. The difference is that Pan and Tang also considered weak end-point results. In 2019, Hu and

Zhou [18] obtained weighted norm inequalities of Calderón–Zygmund operators of type $\omega(t)$ and their commutators with $\mathrm{BMO}_\theta(\varphi)$ functions, here weights belong to $A_p(\varphi)$ functions.

Inspired by the work above, this paper's primary purpose is to obtain weighted norm inequalities for certain classes of multilinear operators of type $\omega(t)$ and their commutators, including the pointwise estimate, strong type, and weak end-point estimates.

2 Some preliminaries and notations

In this section, we first recall some notations. For a measure set E, we define $|E|$ as the Lebesgue measure of E and χ_E as the characteristic function of E. $Q(x,r)$ denotes the cube centered at x with the side length r and $\lambda Q = Q(x,\lambda r)$. $\vec{q} = (q_1,q_2,\dots,q_m)$ and $\vec{\theta} = (\theta_1,\theta_2,\dots,\theta_m)$. For a locally integrable function f, f_Q denotes the average $f_Q = (1/|Q|)\int_Q f(y)\,dy$. In this paper, let $\varphi_\theta(Q) = (1+r)^\theta$, where r is the side length of the cube Q.

2.1 The $A_{\vec{p}}^{\infty}(\varphi)$ weights

According to [16], we say that a weight w belongs to the class $A_p^\theta(\varphi)$ for $1<p<\infty$, if there exists a constant C such that, for all cubes Q,

$$\left(\frac{1}{\varphi_\theta(Q)|Q|}\int_Q w(y)\,dy\right)\left(\frac{1}{\varphi_\theta(Q)|Q|}\int_Q w(y)^{\frac{-1}{p-1}}\,dy\right)^{p-1} \le C.$$

In particular, when $p=1$,

$$\left(\frac{1}{\varphi_\theta(Q)|Q|}\int_Q w(y)\,dy\right) \le C\inf_{x\in Q} w(x).$$

Notice that $A_p^\infty(\varphi) = \bigcup_{\theta\ge 0} A_p^\theta(\varphi)$, $A_\infty^\infty(\varphi) = \bigcup_{p\ge 1} A_p^\infty(\varphi)$ and $A_p^0(\varphi)$ is equivalent to the Muckenhoupt class of weights A_p in [19] for all $1\le p<\infty$. However, in general, the class $A_p^\infty(\varphi)$ is strictly larger than the class A_p for all $1\le p<\infty$.

Next, we give some necessary properties of $A_{\vec{p}}^\theta(\varphi)$ functions.

Lemma 2.1 ([20]) *The following statements hold*:

(i) $A_p^\infty(\varphi)\subset A_q^\infty(\varphi)$ *for* $1\le p\le q<\infty$.

(ii) *If* $w\in A_p^\infty(\varphi)$, *with* $p>1$ *then there exists* $\epsilon>0$ *such that* $w\in A_{p-\epsilon}^\infty(\varphi)$. *Consequently,* $A_p^\infty(\varphi) = \bigcup_{q<p} A_q^\infty(\varphi)$.

(iii) *If* $w\in A_p^\infty(\varphi)$ *with* $p\ge 1$, *then exist positive numbers* δ, l *and* C *so that, for all cubes* Q,

$$\left(\frac{1}{|Q|}\int_Q w^{1+\delta}(x)\,dx\right)^{\frac{1}{1+\delta}} \le C\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\varphi^l(Q).$$

Lemma 2.2 ([21]) *The following statements hold*:

(i) $w\in A_p^\theta(\varphi)$ *if and only if* $w^{-\frac{1}{p-1}}\in A_{p'}^\theta(\varphi)$, *where* $\frac{1}{p}+\frac{1}{p'}=1$;

(ii) *if* $w_1, w_2\in A_p^\theta(\varphi)$, $p\ge 1$, *then* $w_1^\alpha w_2^{1-\alpha}\in A_p^\theta(\varphi)$ *for any* $0<\alpha<1$;

(iii) *if* $w\in A_p^\theta(\varphi)$, *for* $1\le p<\infty$, *then*

$$\frac{1}{\varphi_\theta(Q)|Q|}\int_Q |f(y)|\,dy \le C\left(\frac{1}{w(5Q)}\int_Q |f(y)|^p w(y)\,dy\right)^{\frac{1}{p}}.$$

In particular, let $f = \chi_E$ for any measurable set $E \subset Q$,

$$\frac{|E|}{\varphi_\theta(Q)|Q|} \le C\left(\frac{w(E)}{w(5Q)}\right)^{\frac{1}{p}}.$$

Let $\vec{p} = (p_1, \dots, p_m)$ and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \le p_1, \dots, p_m < \infty$. Given $\vec{w} = (w_1, \dots, w_m)$, each w_j being nonnegative measurable, we set

$$\nu_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}.$$

For $\theta \ge 0$, we say that $\vec{w}$ satisfies the $A_{\vec{p}}^{\theta}(\varphi)$ condition and denote $\vec{w} \in A_{\vec{p}}^{\theta}(\varphi)$, if

$$\sup_Q \left(\frac{1}{\varphi_\theta(Q)|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\right)^{1/p} \prod_{j=1}^{m}\left(\frac{1}{\varphi_\theta(Q)|Q|}\int_Q w_j(x)^{1-p_j'}\,dx\right)^{1/p_j'} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and the term $(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'})^{1/p_j'}$ coincides with $(\inf_{x\in Q} w_j)^{-1}$ when $p_j = 1$ $j = 1, 2, \dots, m$.

For $1 \le p_1, \dots, p_m < \infty$, set $A_{\vec{p}}^{\infty}(\varphi) = \bigcup_{\theta \ge 0} A_{\vec{p}}^{\theta}(\varphi)$. When $\theta = 0$, the class $A_{\vec{p}}^{0}(\varphi)$ coincides with the class of multiple weights $A_{\vec{p}}$ introduced by [15].

Lemma 2.3 ([17]) *Let $1 \le p_1, \dots, p_m < \infty$ and $\vec{w} = (w_1, \dots, w_m)$. Then the following statements are equivalent:*

(i) $\vec{w} \in A_{\vec{p}}^{\infty}(\varphi)$;

(ii) $w_j^{1-p_j'} \in A_{mp_j'}^{\infty}$, *$j = 1, \dots, m$, and* $\nu_{\vec{w}} \in A_{mp}^{\infty}(\varphi)$.

The class $A_{\vec{p}}^{\infty}(\varphi)$ is not increasing, which means that, for $\vec{p} = (p_1, \dots, p_m)$ and $\vec{q} = (q_1, \dots, q_m)$ with $p_j \le q_j, j = 1, \dots, m$, the following may not be true $A_{\vec{p}}^{\infty}(\varphi) \subset A_{\vec{q}}^{\infty}(\varphi)$.

Lemma 2.4 ([17]) *Let $1 \le p_1, \dots, p_m < \infty$ and $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{p}}^{\infty}(\varphi)$. Then*

(i) *for any $r \ge 1, \vec{w} \in A_{r\vec{p}}^{\infty}(\varphi)$;*

(ii) *if $1 < p_1, \dots, p_m < \infty$, then there exists $r > 1$ so that $\vec{w} \in A_{\vec{p}/r}^{\infty}(\varphi)$.*

2.2 $\mathrm{BMO}_\infty(\varphi)$ spaces

Now, recall the definition and properties of the BMO_∞ spaces introduced by [20].

A locally integrable function b is in $\mathrm{BMO}_\theta(\varphi)(\theta \ge 0)$ if

$$\|b\|_{\mathrm{BMO}_\theta(\varphi)} := \sup_Q \frac{1}{\varphi_\theta(Q)|Q|}\int_Q \big|b(y) - b_Q\big|\,dy < \infty.$$

When $\theta = 0, \mathrm{BMO}_0(\varphi) = \mathrm{BMO}(\mathbb{R}^n)$. Clearly $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_\theta(\varphi)$ and $\mathrm{BMO}_{\theta_1}(\varphi) \subset \mathrm{BMO}_{\theta_2}(\varphi)$ for $\theta_1 \le \theta_2$. We denote $\mathrm{BMO}_\infty(\varphi) = \bigcup_{\theta \ge 0}\mathrm{BMO}_\theta(\varphi)$.

Lemma 2.5 ([20]) *Let $\theta > 0, s \ge 1$. If $b \in \mathrm{BMO}_\theta(\varphi)$ then for all cubes $Q = Q(x, r)$*

(i) $(\frac{1}{|Q|}\int_Q |b(y) - b_Q|^s\,dy)^{\frac{1}{s}} \le \|b\|_{\mathrm{BMO}_\theta(\varphi)}\varphi_\theta(Q)$;

(ii) $(\frac{1}{|3^kQ|}\int_{3^kQ} |b(y) - b_Q|^s\,dy)^{\frac{1}{s}} \le k\|b\|_{\mathrm{BMO}_\theta(\varphi)}\varphi_\theta(3^kQ)$, *for all $k \in N$.*

2.3 The norm of Orlicz spaces

For $\Phi(t)=t(1+\log^+ t)$ and a cube Q in $\mathbb{R}^n$, we will consider the average $\|f\|_{\Phi,Q}$ of a function f given by the Luxemburg norm

$$\|f\|_{\Phi,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}.$$

The generalized Hölder inequality in Orlicz spaces together with the corresponding John–Nirenberg inequality in [18, Lemma 2.5] implies that

$$\frac{1}{|Q|}\int_Q\bigl|(b(y)-b_Q)f(y)\bigr|\,dy\le\|b\|_{\mathrm{BMO}_\theta(\varphi)}\|f\|_{L(\mathrm{Log}L),Q}\varphi_\theta(Q).$$

2.4 Maximal functions and Sharp maximal functions

Maximal functions and sharp maximal functions play an important role in the proof of the main theorem. Next, recall the relevant definition.

For $0<\eta<\infty$, the maximal operator $M_{\varphi,\eta}$ is defined by

$$M_{\varphi,\eta}f(x)=\sup_{x\in Q}\frac{1}{\varphi(Q)^\eta|Q|}\int_Q\bigl|f(y)\bigr|\,dy.$$

Definition 2.6 ([21]) Let $0<\eta<\infty$, then the dyadic maximal function $M^d_{\varphi,\eta}$ is defined by

$$M^d_{\varphi,\eta}f(x)=\sup_{x\in Q(dyadiccube)}\frac{1}{\varphi(Q)^\eta|Q|}\int_Q\bigl|f(y)\bigr|\,dy.$$

Let Q be a dyadic cube; f is a locally integral function, then the dyadic sharp maximal function $M^{\sharp,d}_{\varphi,\eta}$ is defined by

$$\begin{aligned}M^{\sharp,d}_{\varphi,\eta}f(x)&=\sup_{x\in Q,r<1}\frac{1}{|Q|}\int_{Q(x_0,r)}\bigl|f(y)-f_Q\bigr|\,dy+\sup_{x\in Q,r\ge 1}\frac{1}{\varphi(Q)^\eta|Q|}\int_{Q(x_0,r)}\bigl|f(y)\bigr|\,dy\\&\simeq\sup_{x\in Q,r<1}\inf_C\frac{1}{|Q|}\int_{Q(x_0,r)}\bigl|f(y)-C\bigr|\,dy+\sup_{x\in Q,r\ge 1}\frac{1}{\varphi(Q)^\eta|Q|}\int_{Q(x_0,r)}\bigl|f(y)\bigr|\,dy,\end{aligned}$$

where $f_Q=\frac{1}{|Q|}\int_Q f(y)\,dy$.

From the above definition, the variants of the dyadic maximal operator and the dyadic sharp maximal operator are as follows:

$$M^d_{\delta,\varphi,\eta}f(x)=\bigl[M^d_{\varphi,\eta}\bigl(|f|^\delta\bigr)\bigr]^{\frac{1}{\delta}},\qquad M^{\sharp,d}_{\delta,\varphi,\eta}f(x)=\bigl[M^{\sharp,d}_{\varphi,\eta}\bigl(|f|^\delta\bigr)\bigr]^{\frac{1}{\delta}}.$$

Lemma 2.7 ([21]) *Let $1<p<\infty$, $w\in A^\infty_p$, $0<\eta<\infty$ and $f\in L^p(w)$, then*

$$\|f\|_{L^p(w)}\le\bigl\|M^d_{\varphi,\eta}f\bigr\|_{L^p(w)}\le\bigl\|M^{\sharp,d}_{\varphi,\eta}f\bigr\|_{L^p(w)}.$$

Lemma 2.8 ([21]) *Let $1<p<\infty$, $\omega\in A^\infty_\infty$, $0<\eta<\infty$ and $\delta>0$ and let $\psi:(0,\infty)\mapsto(0,\infty)$ be doubling, that is, $\psi(2a)\le\psi(a)$ for $a>0$. Then there exists a constant C depending upon the A^∞_∞ condition of w and the doubling condition of ψ such that*

$$\sup_{\lambda>0}\psi(\lambda)w\bigl(\bigl\{y\in\mathbb{R}^n:M^d_{\delta,\varphi,\eta}f(y)>\lambda\bigr\}\bigr)\le C\sup_{\lambda>0}\psi(\lambda)w\bigl(\bigl\{y\in\mathbb{R}^n:M^{\sharp,d}_{\delta,\varphi,\eta}f(y)>\lambda\bigr\}\bigr),$$

$$M_{L(\log L),\varphi,\eta}f(x) = \sup_{x\in Q} \frac{1}{\varphi(Q)^{\eta}} \|f\|_{L(\log L),Q}.$$

Let $0 < \eta < \infty$, $\vec{f} = (f_1, f_2, \dots, f_m)$, *then the multilinear maximal operators* $\mathcal{M}_{\varphi,\eta}$ *and* $\mathcal{M}_{L(\mathrm{Log}L),\varphi,\eta}$ *are defined by*

$$\mathcal{M}_{\varphi,\eta}\vec{f}(x) = \sup_{x\in Q} \prod_{j=1}^{m} \frac{1}{\varphi(Q)^{\eta}} \|f_j\|_Q,$$

$$\mathcal{M}_{L(\mathrm{Log}L),\varphi,\eta}\vec{f}(x) = \sup_{x\in Q} \prod_{j=1}^{m} \frac{1}{\varphi(Q)^{\eta}} \|f_j\|_{L(\log L)}.$$

Lemma 2.9 ([16]) *Let* $1 < p_j < \infty$, $j = 1, 2, \dots, m$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$ *and* $\vec{w} \in A_{\vec{p}}^{\infty}$, *then there exists some* $\eta_0 > 0$ *depending on* p, m, p_j *such that*

$$\big\|\mathcal{M}_{\varphi,\eta_0}(\vec{f})\big\|_{L^p(\nu_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

3 Estimates for multilinear operators

Theorem 3.1 *Let* T *be a multilinear Calderón–Zygmund operator of type* $\omega(t)$ *as in Definition* 1.1, *assume that* $0 < \delta < \frac{1}{m}$, $0 < \eta$ *and* ω *is satisfying* $\omega \in \mathrm{Dini}(1)$. *Then there exists a constant* $C > 0$ *such that*

$$M^{\#,d}_{\delta,\varphi,\eta}\big(T(\vec{f})\big)(x) \le C\mathcal{M}_{\varphi,\eta}(\vec{f})(x)$$

for all $\vec{f}$ *in* $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ *with* $1 \le p_j < \infty$ *for* $j = 1, \dots, m$.

Proof If $\omega \in \mathrm{Dini}(1)$, then

$$\sum_{k=1}^{\infty} \omega\big(2^{-k}\big) \approx \int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

For a fixed point $x \in \mathbb{R}^n$ and let $x \in Q = Q(x_0, r)$, Q is a dyadic cube. To complete the proof, we consider the following two cases of the side length r: $r \le 1$ and $r > 1$.

Case 1. When $r \le 1$. Since $0 < \delta < \frac{1}{m} < 1$, $\eta > 0$ and $||a|^t - |b|^t| < |a - b|^t$ for $0 < t < 1$, for any number C we can estimate

$$\left(\frac{1}{|Q|}\int_Q \big|\big|T(\vec{f})(z)\big|^{\delta} - |C|^{\delta}\big|\,dz\right)^{\frac{1}{\delta}} \le \left(\frac{1}{|Q|}\int_Q \big|T(\vec{f})(z) - C\big|^{\delta}\,dz\right)^{\frac{1}{\delta}}.$$

Let $Q^* = 8\sqrt{n}Q$, we decompose $f_j = f_j^0 + f_j^{\infty}$ for each f_j, where $f_j^0 = f_j\chi_{Q^*}$. Then

$$\prod_{j=1}^{m} f_j(y_j) = \sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m) = \prod_{j=0}^{m} f_j^0(y_j) + \sum_{\alpha_1,\dots,\alpha_m\in\mathscr{L}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),$$

where $\mathscr{L} = \{(\alpha_1, \dots, \alpha_m) : \text{there is at least one } \alpha_j \neq 0\}$.

Let $C = \sum_{\alpha_1,\dots,\alpha_m \in \mathscr{L}} C_{\alpha_1,\dots,\alpha_m}$, from this condition, we can get the following a series of estimates:

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \bigl|T(\vec{f})(z) - C\bigr|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&\quad\le \left(\frac{1}{|Q|}\int_Q \bigl|T\bigl(f_1^0,\dots,f_m^0\bigr)(z)\bigr|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&\qquad + \sum_{\alpha_1,\dots,\alpha_m \in \mathscr{L}} \left(\frac{1}{|Q|}\int_Q \bigl|T\bigl(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\bigr)(z) - C_{\alpha_1,\dots,\alpha_m}\bigr|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&\quad := I + II.
\end{aligned}$$

Since $T: L^1 \times \cdots \times L^1 \to L^{\frac{1}{m},\infty}$ and using the Kolmogorov inequality with $p = \delta$ and $q = \frac{1}{m}$, we have

$$\begin{aligned}
I &\le C\bigl\| T\bigl(f_1^0,\dots,f_m^0\bigr)\bigr\|_{L^{\frac{1}{m},\infty}(Q,\frac{dx}{|Q|})} \\
&\le C\prod_{j=1}^m \frac{1}{|Q^*|}\int_{Q^*} \bigl|f_j(z)\bigr|\,dz \\
&\le C\prod_{j=1}^m \frac{1}{\varphi(Q^*)^{\eta}|Q^*|}\int_{Q^*} \bigl|f_j(z)\bigr|\,dz \\
&\le C\mathcal{M}_{\varphi,\eta}(\vec{f})(x).
\end{aligned}$$

To estimate II, we choose $C_{\alpha_1,\dots,\alpha_m} = T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)$, for any $z \in Q$, the following estimate holds:

$$\sum_{\alpha_1,\dots,\alpha_m \in \mathscr{L}} \bigl|T\bigl(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\bigr)(z) - T\bigl(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\bigr)(x)\bigr| \le C\mathcal{M}_{\varphi,\eta}(\vec{f})(x).$$

We consider first the case when $\alpha_1 = \cdots = \alpha_m = \infty$. For any $z \in Q$, we get

$$\begin{aligned}
&\bigl|T\bigl(f_1^{\infty},\dots,f_m^{\infty}\bigr)(z) - T\bigl(f_1^{\infty},\dots,f_m^{\infty}\bigr)(x)\bigr| \\
&\quad\le \int_{(\mathbb{R}^n\setminus Q^*)^m} \bigl|K(z,\vec{y}) - K(x,\vec{y})\bigr|\prod_{j=1}^m \bigl|f(y_j)\bigr|\,d\vec{y} \\
&\quad\le \sum_{k=1}^{\infty}\int_{(\Omega_k)^m} \bigl|K(z,\vec{y}) - K(x,\vec{y})\bigr|\prod_{j=1}^m \bigl|f(y_j)\bigr|\,d\vec{y},
\end{aligned}$$

where $\Omega_k = (2^{k+3}\sqrt{n}Q) \setminus (2^{k+2}\sqrt{n}Q)$ for $k = 1, 2, \dots$.

Note that, for $x, z \in Q$ and any $(y_1,\dots,y_m) \in (\Omega_k)^m$,

$$|z - y_j| \ge 2^k\sqrt{n}r \quad \text{and} \quad |z - x| \le \sqrt{n}r,$$

since ω is nondecreasing, and through the kernel condition (1.2), we have

$$\begin{aligned}\left|K(z,\vec{y})-K(x,\vec{y})\right| &\leq \frac{A}{(\sum_{j=1}^m |z-y_j|)^{mn}(1+\sum_{j=1}^m |z-y_j|)^N}\omega\left(\frac{|z-x|}{\sum_{j=1}^m |z-y_j|}\right)\\ &\leq \frac{C\omega(2^{-k})}{|2^k\sqrt{n}Q|^m(1+2^k\sqrt{n}r)^N}.\end{aligned}$$

Then, taking $N \geq m\eta$,

$$\begin{aligned}&\left|T\left(f_1^\infty,\dots,f_m^\infty\right)(z)-T\left(f_1^\infty,\dots,f_m^\infty\right)(x)\right|\\ &\quad\leq C\sum_{k=1}^\infty \omega\left(2^{-k}\right)\int_{(\Omega_k)^m}\frac{1}{|2^k\sqrt{n}Q|^m(1+2^k\sqrt{n}r)^N}\prod_{j=1}^m\left|f(y_j)\right|d\vec{y}\\ &\quad\leq C\sum_{k=1}^\infty \omega\left(2^{-k}\right)\prod_{j=1}^m\frac{1}{|2^k\sqrt{n}Q|(1+2^k\sqrt{n}r)^{\frac{N}{m}}}\int_{2^{k+3}\sqrt{n}Q}\left|f_j(y_j)\right|dy_i\\ &\quad\leq C|\omega|_{\mathrm{Dini}(1)}\mathcal{M}_{\varphi,\eta}(\overrightarrow{f})(x).\end{aligned}$$

We are now to consider $\alpha_{j_1}=\cdots=\alpha_{j_l}=0$ for $1\leq l<m$. Let $\mathscr{J}:=\{j_1,\dots,j_l\}$ then $\alpha_j=\infty$ for $j\notin\mathscr{J}$. Thus

$$\begin{aligned}&\left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(z)-T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right|\\ &\quad\leq \int_{(\mathbb{R}^n)^m}\left|K(z,\vec{y})-K(x,\vec{y})\right|\prod_{j=1}^m\left|f_j^{\alpha_j}(y_j)\right|d\vec{y}\\ &\quad\leq \int_{(Q^*)^l}\prod_{j\in\mathscr{J}}\left|f_j(y_j)\right|\int_{(\mathbb{R}^n\setminus Q^*)^{m-l}}\left|K(z,\vec{y})-K(x,\vec{y})\right|\prod_{j\notin\mathscr{J}}\left|f_j(y_j)\right|d\vec{y}\\ &\quad\leq \int_{(Q^*)^l}\prod_{j\in\mathscr{J}}\left|f_j(y_j)\right|\sum_{k=1}^\infty\int_{(\Omega_k)^{m-l}}\left|K(z,\vec{y})-K(x,\vec{y})\right|\prod_{j\notin\mathscr{J}}\left|f_j(y_j)\right|d\vec{y}.\end{aligned}$$

Similar to the above discussion, taking $N\geq m\eta$, we have

$$\begin{aligned}&\left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(z)-T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right|\\ &\quad\leq C\int_{(Q^*)^l}\prod_{j\in\mathscr{J}}\left|f_j(y_j)\right|\sum_{k=1}^\infty\omega\left(2^{-k}\right)\int_{(\Omega_k)^{m-l}}\frac{1}{|2^k\sqrt{n}Q|^m(1+2^k\sqrt{n}r)^N}\prod_{j\notin\mathscr{J}}\left|f_j(y_j)\right|d\vec{y}\\ &\quad\leq C\sum_{k=1}^\infty\omega\left(2^{-k}\right)\frac{1}{|2^k\sqrt{n}Q|^m(1+2^k\sqrt{n}r)^N}\left(\prod_{j\in\mathscr{J}}\int_{Q^*}\left|f_j(y_j)\right|dy_j\right)\\ &\qquad\times\left(\prod_{j\notin\mathscr{J}}\int_{2^{k+3}\sqrt{n}Q}\left|f_j(y_j)\right|dy_j\right)\\ &\quad\leq C\sum_{k=1}^\infty\omega\left(2^{-k}\right)\prod_{j=1}^m\frac{1}{|2^k\sqrt{n}Q|(1+2^k\sqrt{n}r)^{\frac{N}{m}}}\int_{2^{k+3}\sqrt{n}Q}\left|f_j(y_j)\right|dy_i\\ &\quad\leq C|\omega|_{\mathrm{Dini}(1)}\mathcal{M}_{\varphi,\eta}(\vec{y})(x).\end{aligned}$$

Case 2. When $r > 1$, since $0 < \delta < \frac{1}{m} < 1$, and $\eta > 0$, it follows that

$$\begin{aligned}
&\left(\frac{1}{\varphi(Q)^{\eta}|Q|}\int_Q \left|T(\vec{f})(x)\right|^{\delta}dx\right)^{\frac{1}{\delta}}\\
&\quad\le \frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q \left|T\left(f_1^0,\dots,f_m^0\right)(x)\right|^{\delta}dx\right)^{\frac{1}{\delta}}\\
&\qquad+\sum_{\alpha_1,\dots,\alpha_m\in\mathscr{L}}\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q \left|T\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(z)\right|^{\delta}dz\right)^{\frac{1}{\delta}}\\
&\quad:= I + II.
\end{aligned}$$

For I, by the Kolmogorov inequality and $T : L^1 \times \cdots \times L^1 \to L^{\frac{1}{m},\infty}$, we have

$$\begin{aligned}
I &\le C\frac{1}{\varphi(Q)^{m\eta}}\left\|T\left(f_1^0,\dots,f_m^0\right)\right\|_{L^{\frac{1}{m},\infty}(Q,\frac{dx}{|Q|})}\\
&\le C\frac{1}{\varphi(Q)^{m\eta}}\prod_{j=1}^{m}\frac{1}{|Q^*|}\int_{Q^*}\left|f_j(z)\right|dz\\
&\le C\mathcal{M}_{\varphi,\eta}(\vec{f})(x).
\end{aligned}$$

To estimate II, note that, for $z \in Q$ and any $(y_1,\dots,y_m) \in (\Omega_k)^m$, $|z - y_j| \ge 2^k\sqrt{n}r$. Consider now $\alpha_1 = \cdots = \alpha_m = \infty$, taking $N \ge m\eta + 1$, the following estimate holds:

$$\begin{aligned}
&\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q \left|T\left(f_1^{\infty},\dots,f_m^{\infty}\right)(z)\right|^{\delta}dz\right)^{\frac{1}{\delta}}\\
&\quad\le \frac{C}{|Q|}\int_Q \left|T\left(f_1^{\infty},\dots,f_m^{\infty}\right)(z)\right|dz\\
&\quad\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{(\Omega_k)^m}\left|K(z,\vec{y})\right|\prod_{j=1}^{m}\left|f(y_j)\right|d\vec{y}\,dz\\
&\quad\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{(\Omega_k)^m}\frac{\prod_{j=1}^{m}|f(y_j)|}{(\sum_{j=1}^{m}|z-y_j|)^{mn}(1+\sum_{j=1}^{m}|z-y_j|)^N}d\vec{y}\,dz\\
&\quad\le \sum_{k=1}^{m}\frac{C}{|2^{k+3}\sqrt{n}Q|^m(1+2^{k+3}\sqrt{n}r)^N}\int_{(2^{k+3}\sqrt{n}Q)^m}\prod_{j=1}^{m}\left|f(y_j)\right|dy_j\\
&\quad\le \sum_{k=1}^{m}\frac{C}{(1+2^{k+3}\sqrt{n}r)^N}\prod_{j=1}^{m}\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}\left|f(y_j)\right|dy_j\\
&\quad\le \sum_{k=1}^{m}\frac{C}{(1+2^{k+3}\sqrt{n}r)}\mathcal{M}_{\varphi,\eta}(\vec{f})(x)\\
&\quad\le C\mathcal{M}_{\varphi,\eta}(\vec{f})(x).
\end{aligned}$$

When $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for $1 \le l < m$. Let $\mathscr{J} := \{j_1, \dots, j_l\}$ then $\alpha_j = \infty$ for $j \notin \mathscr{J}$, taking $N \ge m\eta + 1$, then

$$\begin{aligned}
&\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q \left|T\left(f_1^{\alpha_1}, \dots, f_m^{\alpha_m}\right)(z)\right|^\delta dz\right)^{\frac{1}{\delta}} \\
&\quad \le C\int_{(Q^*)^l}\prod_{j\in\mathscr{J}}\left|f_j(y_j)\right|\int_{(R^n\setminus Q^*)^{m-l}}\frac{\prod_{j\notin\mathscr{J}}|f_j(y_j)|}{(\sum_{j=1}^m|z-y_j|)^{mn}(1+\sum_{j=1}^m|z-y_j|)^N}\,d\vec{y} \\
&\quad \le C\prod_{j\in\mathscr{J}}\int_{Q^*}\left|f_j(y_j)\right|\sum_{k=1}^\infty\int_{(\Omega_k)^{m-l}}\frac{\prod_{j\notin\mathscr{J}}|f_j(y_j)|}{|2^{k+3}\sqrt{n}Q|^m(1+2^{k+3}\sqrt{n}r)^N}\,d\vec{y} \\
&\quad \le C\sum_{k=1}^\infty\frac{1}{|2^{k+3}\sqrt{n}Q|^m(1+2^{k+3}\sqrt{n}r)^N}\left(\prod_{j\in\mathscr{J}}\int_{Q^*}\left|f_j(y_j)\right|dy_j\right)\left(\prod_{j\notin\mathscr{J}}\int_{2^{k+3}\sqrt{n}Q}\left|f_j(y_j)\right|dy_j\right) \\
&\quad \le C\mathcal{M}_{\varphi,\eta}(\vec{f})(x).
\end{aligned}$$

Pan and Tang in [16, Lemma 2.7] proved the result in our framework, which is similar to the classical Fefferman–Stein inequalities. Next, using Lemma 2.7 of our paper, we obtain the result as follows. □

Corollary 3.2 *Let T be a multilinear operator satisfying (1.1)–(1.5), and suppose that ω is satisfying $\omega \in \mathrm{Dini}(1)$, $w \in A_\infty^\infty$, $\eta > 0$ and $p > 0$. Then there exist constants $C > 0$, such that*

$$\left\|T(\vec{f})\right\|_{L^p(w)} \le C\left\|\mathcal{M}_{\varphi,\eta}(\vec{f})\right\|_{L^p(w)}$$

and

$$\left\|T(\vec{f})\right\|_{L^{p,\infty}(w)} \le C\left\|\mathcal{M}_{\varphi,\eta}(\vec{f})\right\|_{L^{p,\infty}(w)}.$$

Proof From Lemma 2.7 and Theorem 3.1, we get

$$\begin{aligned}
\left\|T(\vec{f})\right\|_{L^p(w)} &\le \left\|M_{\varphi,\eta}^d\left(T(\vec{f})\right)\right\|_{L^p(w)} \\
&\le C\left\|M_{\varphi,\eta}^{\sharp,d}\left(T(\vec{f})\right)\right\|_{L^p(w)} \\
&\le C\left\|\mathcal{M}_{\varphi,\eta}(\vec{f})\right\|_{L^p(w)}.
\end{aligned}$$ □

Similarly, with the help of Lemma 2.8, the weak-type estimate is obtained.

Theorem 3.3 *Let T be a multilinear operator satisfying (1.1)–(1.5), $\vec{w} \in A_{\vec{p}}^\infty(\varphi)$ and $1/p = 1/p_1 + \cdots + 1/p_m$. If ω is satisfying $\omega \in \mathrm{Dini}(1)$, then there exists a constant $C > 0$, such that:*

(i) *If $1 < p_j < \infty$, $j = 1, \dots, m$, then*

$$\left\|T(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \le C\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j)}.$$

(ii) *If* $1 \le p_j < \infty$, $j = 1, \dots, m$, *and at least one of* $p_j = 1$, *then*

$$\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

Proof The desired result directly is obtained from Theorem 3.1, Corollary 3.2, Lemma 2.4 and Lemma 2.9. The proof is completed. □

4 Estimates for multilinear commutators

To ensure the fluency of the demonstration in this section, we need first to explain the meaning of some notations. We write

$$C_j^m = \{\sigma : \sigma = \{\sigma(1), \sigma(2), \dots, \sigma(j)\}, 1 \le j \le m\},$$

We always take $\sigma(i) \le \sigma(j)$ if $i \le j$.

For any $\sigma' \in C_j^m$, we have $\sigma' = \{\sigma(1), \sigma(2), \dots, \sigma(m)\} \setminus \sigma$ and $\sigma' \in C_{m-j}^m$.

Let $\vec{b}$ be m-tuple functions and $\sigma \in C_j^m$, we have the j-tuple function $\vec{b} = (b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(j)})$. For all $b_{\sigma(j)} \in \mathrm{BMO}_\theta(\varphi)$, $1 \le j \le m$, we have $\vec{b} = (b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(m)}) \in \mathrm{BMO}_{\vec{\theta}}^m(\varphi)$. See [15, 16].

Corresponding to the classical form, can define the following form of the iterated commutators:

$$T_{\Pi_{\vec{b}_\sigma}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{i=1}^{m} \big(b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})\big) K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m.$$

Theorem 4.1 *Let* T *be a multilinear Calderón–Zygmund operator of type* $\omega(t)$ *as in Definition* 1.1, $T_{\Pi_{\vec{b}}}$ *be a multilinear commutator with* $\vec{b} \in \mathrm{BMO}_{\vec{\theta}}^m(\varphi)$. *We have* $0 < \delta < \epsilon < 1/m$ *and* $\eta > (\theta_1, \dots, \theta_m)/(1/\delta - 1/\epsilon)$, *assume that* ω *is satisfying*

$$\int_0^1 \frac{\omega(t)}{t} \left(1 + \log \frac{1}{t}\right)^m dt < \infty. \tag{4.1}$$

Then there exists a constant $C > 0$ *such that*

$$\begin{aligned} M_{\delta,\varphi,\eta}^{\#,d}\big(T_{\Pi_{\vec{b}}}(\vec{f})\big)(x) &\le C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)} \big(\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x) + M_{\epsilon,\varphi,\eta}^d\big(T(\vec{f})\big)(x)\big) \\ &\quad + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \prod_{i=1}^{j} \|b_{\sigma(i)}\|_{\mathrm{BMO}_{\sigma(i)}} M_{\epsilon,\varphi,\eta}\big(T_{\Pi_{\vec{b}_{\sigma'}}}(\vec{f})\big)(x), \end{aligned}$$

for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ *of bounded measurable functions with compact support.*

Proof For simplicity, we only prove the case $m = 2$ and $\theta_1 = \theta_2 = \theta$.

If ω is satisfying (4.1), then $\omega \in \mathrm{Dini}(1)$ and

$$\sum_{k=1}^{\infty} k^m \cdot \omega(2^{-k}) \approx \int_0^1 \frac{\omega(t)}{t} \left(1 + \log \frac{1}{t}\right)^m < \infty.$$

For $b_1, b_2 \in \mathrm{BMO}_\theta(\varphi)$, it suffices to prove that

$$\begin{aligned} M^{\#,d}_{\delta,\varphi,\eta}\big(T_{\Pi_{\vec{b}}}(f_1,f_2)\big)(x) \le{}& C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\big(\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x) \\ &+ CM^d_{\epsilon,\varphi,\eta}\big(T(f_1,f_2)\big)(x)\big) \\ &+ C\big(\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}M_{\epsilon,\varphi,\eta}\big(T^1_{b_1}\big)(f_1,f_2)(x) \\ &+ C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}M_{\epsilon,\varphi,\eta}\big(T^2_{b_2}\big)(f_1,f_2)(x)\big). \end{aligned}$$

For any constants λ_1, λ_2, it follows that

$$\begin{aligned} T_{\Pi_{\vec{b}}}(f_1,f_2)(x) ={}& \big(b_1(x)-\lambda_1\big)\big(b_2(x)-\lambda_2\big)T(f_1,f_2)(x) - \big(b_1(x)-\lambda_1\big)T\big(f_1,(b_2-\lambda_2)f_2\big)(x) \\ &- \big(b_2(x)-\lambda_2\big)T\big((b_1-\lambda_1)f_1,f_2\big)(x) + T\big((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2\big)(x) \\ ={}& -\big(b_1(x)-\lambda_1\big)\big(b_2(x)-\lambda_2\big)T(f_1,f_2)(x) + \big(b_1(x)-\lambda_1\big)T^2_{b_2-\lambda_2}(f_1,f_2)(x) \\ &+ \big(b_2(x)-\lambda_2\big)T^1_{b_1-\lambda_1}(f_1,f_2)(x) + T\big((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2\big)(x), \end{aligned}$$

where

$$T^1_{b_1-\lambda_1}(f_1,f_2)(x) = \big(b_1(x)-\lambda_1\big)T(f_1,f_2)(x) - T\big((b_1-\lambda_1)f_1,f_2\big)(x) \tag{4.2}$$

and

$$T^2_{b_2-\lambda_2}(f_1,f_2)(x) = \big(b_2(x)-\lambda_2\big)T(f_1,f_2)(x) - T\big(f_1,(b_2-\lambda_2)f_2\big)(x). \tag{4.3}$$

Now, we fix $x \in \mathbb{R}^n$, a dyadic cube $Q \ni x$ and a constant c, then, since $0 < \delta < \frac{1}{2}$, we only need to consider the two cases $r \le 1$ and $r > 1$.

Case 1: When $r \le 1$, the following estimate holds:

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \big||T_{\Pi_{\vec{b}}}(f_1,f_2)(z)|^\delta - |c|^\delta\big|\,dz\right)^{\frac{1}{\delta}} \\ &\quad\le \left(\frac{1}{|Q|}\int_Q \big|T_{\Pi_{\vec{b}}}(f_1,f_2)(z) - c\big|^\delta\,dz\right)^{\frac{1}{\delta}} \\ &\quad\le \left(\frac{C}{|Q|}\int_Q \big|\big(b_1(z)-\lambda_1\big)\big(b_2(z)-\lambda_2\big)T(f_1,f_2)(z)\big|^\delta\,dz\right)^{\frac{1}{\delta}} \\ &\qquad+ \left(\frac{C}{|Q|}\int_Q \big|\big(b_1(z)-\lambda_1\big)T^2_{b_2-\lambda_2}(f_1,f_2)(z)\big|^\delta\,dz\right)^{\frac{1}{\delta}} \\ &\qquad+ \left(\frac{C}{|Q|}\int_Q \big|\big(b_2(z)-\lambda_2\big)T^1_{b_1-\lambda_1}(f_1,f_2)(z)\big|^\delta\,dz\right)^{\frac{1}{\delta}} \\ &\qquad+ \left(\frac{C}{|Q|}\int_Q \big|T\big((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2\big)(z) - c\big|^\delta\,dz\right)^{\frac{1}{\delta}} \\ &\quad:= I + II + III + IV. \end{aligned}$$

Let $Q^* = 8\sqrt{n}Q$ and let $\lambda_j = (b_j)_{Q^*}$ be the average of b_j on Q^*, $j = 1, 2$. For any $1 < r_1, r_2, r_3 < \infty$ with $1/r_1 + 1/r_2 + 1/r_3 = 1$, choosing a δ to make $\delta r_i < 1$, $i = 1, 2$ and $r_3 < \epsilon/\delta$.

By Hölder's inequality, we have

$$
\begin{aligned}
I &\le C\left(\frac{1}{|Q|}\int_Q \left|b_1(z) - (b_1)_{Q^*}\right|^{r_1\delta}\,dz\right)^{\frac{1}{r_1\delta}}\left(\frac{1}{|Q|}\int_Q \left|b_2(z) - (b_2)_{Q^*}\right|^{r_2\delta}\,dz\right)^{\frac{1}{r_2\delta}} \\
&\quad\times\left(\frac{1}{|Q|}\int_Q \left|T(f_1,f_2)\right|^{r_3\delta}\,dz\right)^{\frac{1}{r_3\delta}} \\
&\le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}M^d_{\epsilon,\varphi,\eta}\bigl(T(f_1,f_2)\bigr)(x)).
\end{aligned}
$$

For II, let $1 < t_1, t_2 < \infty$ with $1 = 1/t_1 + 1/t_2$ and $t_2 < \epsilon/\delta$. By Hölder's inequality,

$$
\begin{aligned}
II &\le C\left(\frac{1}{|Q|}\int_Q \left|b_1(z) - (b_1)_{Q^*}\right|^{t_1\delta}\,dz\right)^{\frac{1}{t_1\delta}}\left(\frac{1}{|Q|}\int_Q \left|T^2_{b_2-\lambda_2}(f_1,f_2)(z)\right|^{t_2\delta}\,dz\right)^{\frac{1}{t_2\delta}} \\
&\le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}M_{t_2\delta,\varphi,\eta}\bigl(T^2_{b_2-\lambda_2}(f_1,f_2)\bigr)(x) \\
&\le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}M_{\epsilon,\varphi,\eta}\bigl(T^2_{b_2-\lambda_2}(f_1,f_2)\bigr)(x) \\
&= C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}M_{\epsilon,\varphi,\eta}\bigl(T^2_{b_2}(f_1,f_2)\bigr)(x).
\end{aligned}
$$

Similarly, we obtain

$$
III \le C\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}M_{\epsilon,\varphi,\eta}\bigl(T^1_{b_1}(f_1,f_2)\bigr)(x).
$$

Now for the last term IV. We split each f_j as $f_j = f_j^0 + f_j^\infty$ where $f_i^0 = f_j\chi_{Q^*}$ and $f_j^\infty = f_j - f_j^0$. Let $c = c_1 + c_2 + c_3$, where

$$
\begin{aligned}
c_1 &= T\bigl((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty\bigr)(x), \\
c_2 &= T\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0\bigr)(x), \\
c_3 &= T\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty\bigr)(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
IV &\le C\left(\frac{1}{|Q|}\int_Q \left|T\bigl((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0\bigr)(z)\right|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q \left|T\bigl((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty\bigr)(z) - c_1\right|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q \left|T\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0\bigr)(z) - c_2\right|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q \left|T\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty\bigr)(z) - c_3\right|^{\delta}\,dz\right)^{\frac{1}{\delta}} \\
&:= IV_1 + IV_2 + IV_3 + IV_4.
\end{aligned}
$$

For IV_1, choosing $1 < p < \frac{1}{2\delta}$ and applying Kolmogorov's inequality with $p = \delta < \frac{1}{2}$, $q = \frac{1}{2}$,

$$\begin{aligned}
IV_1 &= C\big\| T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda)f_2^0\big)\big\|_{L^\delta(Q,\frac{dx}{|Q|})} \\
&\le C\big\| T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda)f_2^0\big)\big\|_{L^{\frac{1}{2},\infty}(Q,\frac{dx}{|Q|})} \\
&\le C\frac{1}{|Q|}\int_Q\big|\big(b_1(z)-\lambda_1\big)f_1^0(z)\big|\,dz\frac{1}{|Q|}\int_Q\big|\big(b_2(z)-\lambda_2\big)f_2^0(z)\big|\,dz \\
&\le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).
\end{aligned}$$

Next to estimate IV_2. For any $z \in Q$, let $c_1 = T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x)$, we have

$$\begin{aligned}
IV_2 &= \left(\frac{1}{|Q|}\int_Q\big|T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\big)(z)-T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\big)(x)\big|^\delta\,dz\right)^{\frac{1}{\delta}} \\
&\le \int_{(\mathbb{R}^n)^2}\big|K(z,y_1,y_2)-K(x,y_1,y_2)\big|\big|\big(b_1(y_1)-\lambda_1\big)f_1^0(y_1)\big|\big|\big(b_2(y_2)-\lambda_2\big)f_2^\infty(y_2)\big|\,dy_1\,dy_2 \\
&\le \int_{Q^*}\big|\big(b_1(y_1)-\lambda_1\big)f_1(y_1)\big| \\
&\quad\times\left(\int_{\mathbb{R}^n\setminus Q^*}\big|K(z,y_1,y_2)-K(x,y_1,y_2)\big|\big|\big(b_2(y_2)-\lambda_2\big)f_2(y_2)\big|\,dy_2\right)dy_1.
\end{aligned}$$

For any $z \in Q, y_1 \in Q^*$ and $y_2 \in \Omega_k$,

$$\begin{aligned}
&\big|K(z,y_1,y_2)-K(x,y_1,y_2)\big| \\
&\quad\le \frac{A}{(|z-y_1|+|z-y_2|)^{2n}(1+|z-y_1|+|z-y_2|)^N}\,\omega\left(\frac{|z-x|}{|z-y_1|+|z-y_2|}\right) \\
&\quad\le C\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2(1+2^{k+3}\sqrt{n}r)^N},
\end{aligned}$$

then we have

$$\begin{aligned}
IV_2 &\le C\int_{Q^*}\big|\big(b_1(y_1)-\lambda_1\big)f_1(y_1)\big| \\
&\quad\times\left(\sum_{k=1}^\infty\int_{\Omega_k}\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2(1+2^{k+3}\sqrt{n}r)^N}\big|\big(b_2(y_2)-\lambda_2\big)f_2(y_2)\big|\,dy_2\right)dy_1 \\
&\le C\int_{Q^*}\big|\big(b_1(y_1)-\lambda_1\big)f_1(y_1)\big| \\
&\quad\times\left(\sum_{k=1}^\infty\omega\big(2^{-k}\big)\int_{2^{k+3}\sqrt{n}Q}\frac{|(b_2(y_2)-\lambda_2)f_2(y_2)|}{|2^{k+3}\sqrt{n}Q|^2(1+2^{k+3}\sqrt{n}r)^N}\,dy_2\right)dy_1 \\
&\le C\sum_{k=1}^\infty\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2(1+2^{k+3}\sqrt{n}r)^N} \\
&\quad\times\int_{(2^{k+3}\sqrt{n}Q)^2}\big|b_1(y_1)-\lambda_1\big|\big|f_1(y_1)\big|\big|b_2(y_2)-\lambda_2\big|\big|f_2(y_2)\big|\,dy_1\,dy_2.
\end{aligned}$$

Note that, for the constant $\lambda_j = (b_j)_{Q^*}$, the following holds:

$$\int_{2^{k+3}\sqrt{n}Q} \left|b(y) - b_{Q^*}\right| \left|f(y)\right| dy \le Ck\left|2^{k+3}\sqrt{n}Q\right| \varphi\left(2^{k+3}\sqrt{n}Q\right) \|b\|_{\mathrm{BMO}_\theta(\varphi)} \|f\|_{L\log L, 2^{k+3}\sqrt{n}Q}.$$

Taking $N \ge 2\eta$, then

$$\begin{aligned} IV_2 &\le C\sum_{k=1}^{\infty} k^2\omega\left(2^{-k}\right) \|b_1\|_{\mathrm{BMO}_\theta(\varphi)} \|b_2\|_{\mathrm{BMO}_\theta(\varphi)} \|f_1\|_{L\log L, 2^{k+3}\sqrt{n}Q} \|f_2\|_{L\log L, 2^{k+3}\sqrt{n}Q} \\ &\le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)} \|b_2\|_{\mathrm{BMO}_\theta(\varphi)} \mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x). \end{aligned}$$

Similarly to IV_2, we can estimate

$$IV_3 \le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)} \|b_2\|_{\mathrm{BMO}_\theta(\varphi)} \mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).$$

Now for the term IV_4. For any $z \in Q$ and $y_1, y_2 \in \Omega_k$,

$$\begin{aligned} &\left|K(z,y_1,y_2) - K(x,y_1,y_2)\right| \\ &\quad\le \frac{A}{(|z-y_1| + |z-y_2|)^{2n}(1 + |z-y_1| + |z-y_2|)^N} \omega\left(\frac{|z-x|}{|z-y_1| + |z-y_2|}\right) \\ &\quad\le C\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2(1 + 2^{k+3}\sqrt{n}r)^N}. \end{aligned}$$

Note $c_3 = T((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(x)$. Then

$$\begin{aligned} IV_4 &\le C\left(\frac{1}{|Q|}\int_Q \left|T\left((b_1-\lambda_1)f_1^\infty, (b_2-\lambda_2)f_2^\infty\right)(z)\right.\right. \\ &\qquad \left.\left. - T\left((b_1-\lambda_1)f_1^\infty, (b_2-\lambda_2)f_2^\infty\right)(x)\right|^\delta dz\right)^{\frac{1}{\delta}} \\ &\le C\int_{(\mathbb{R}^n\setminus Q^*)^2} \left|K(z,y_1,y_2) - K(x,y_1,y_2)\right| \left(\prod_{j=1}^{2}\left|\left(b_j(y_j) - \lambda_j\right)f_j(y_j)\right|\right) dy_1\,dy_2 \\ &\le C\sum_{k=1}^{\infty}\int_{(\Omega_k)^2} \left|K(z,y_1,y_2) - K(x,y_1,y_2)\right| \left(\prod_{j=1}^{2}\left|\left(b_j(y_j) - \lambda_j\right)f_j(y_j)\right|\right) dy_1\,dy_2 \\ &\le C\sum_{k=1}^{\infty}\int_{(2^{k+3}\sqrt{n}Q)^2} \frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2(1 + 2^{k+3}\sqrt{n}r)^N} \left(\prod_{j=1}^{2}\left|b_j(y_j) - \lambda_j\right|\left|f_j(y_j)\right|\right) dy_1\,dy_2 \\ &\le C\sum_{k=1}^{\infty} k^2\omega\left(2^{-k}\right) \|b_1\|_{\mathrm{BMO}_\theta(\varphi)} \|b_2\|_{\mathrm{BMO}_\theta(\varphi)} \|f_1\|_{L\log L, 2^{k+3}\sqrt{n}Q} \|f_2\|_{L\log L, 2^{k+3}\sqrt{n}Q} \\ &\le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)} \|b_2\|_{\mathrm{BMO}_\theta(\varphi)} \mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x). \end{aligned}$$

Case 2: When $r > 1$. Let $0 < \delta < \epsilon < 1$, the following holds:

$$\left(\frac{1}{\varphi(Q)^\eta|Q|}\int_Q \left|T_{\Pi_{\vec{b}}}(\vec{f})(z)\right|^\delta dy\right)^{\frac{1}{\delta}}$$

$$
\begin{aligned}
&\leq C\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q\left|(b_1-\lambda_1)(b_2-\lambda_2)T(f_1,f_2)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad+C\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q\left|(b_1-\lambda_1)T\bigl(f_1,(b_2-\lambda_2)f_2\bigr)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad+C\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q\left|(b_2-\lambda_2)T\bigl((b_1-\lambda_1)f_1,f_2\bigr)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad+C\frac{1}{\varphi(Q)^{\eta/\delta}}\left(\frac{1}{|Q|}\int_Q\left|T\bigl((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2\bigr)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&:=I+II+III+IV.
\end{aligned}
$$

Let $Q^*=8\sqrt{n}Q$ and let $\lambda_j=(b_j)_{Q^*}$ be the average of b_j on Q^*, $j=1,2$. For any $1<r_1,r_2,r_3<\infty$ with $1/r_1+1/r_2+1/r_3=1$, we choose a δ small enough to make $\delta r_i<1, i=1,2$ and $r_3<\epsilon/\delta$.

Using Hölder's inequality, choosing η so that $\eta(\frac{1}{\delta}-\frac{1}{\epsilon})>2\theta$, then

$$
\begin{aligned}
I&\leq C\frac{1}{\varphi(Q)^{\frac{\eta}{\delta}}}\left(\frac{1}{|Q|}\int_Q\left|b_1(z)-\lambda_1\right|^{r_1\delta}dz\right)^{\frac{1}{r_1\delta}}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q\left|b_2(z)-\lambda_2\right|^{r_2\delta}dz\right)^{\frac{1}{r_2\delta}}\left(\frac{1}{|Q|}\int_Q\left|T(f_1,f_2)(z)\right|^{r_3\delta}dz\right)^{\frac{1}{r_3\delta}}\\
&\leq C\frac{\varphi(Q)^{\eta/\epsilon}}{\varphi(Q)^{\eta/\delta}}\varphi_\theta^2\bigl(Q^*\bigr)\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\left(\frac{1}{\varphi(Q)^\eta|Q|}\int_Q\left|T(f_1,f_2)(z)\right|^\epsilon dz\right)^{\frac{1}{\epsilon}}\\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}M^d_{\epsilon,\varphi,\eta}\bigl(T(f_1,f_2)\bigr)(x).
\end{aligned}
$$

By the Hölder inequality, and Lemma 2.5, we have

$$
\begin{aligned}
II&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\left(\frac{1}{|Q|}\int_Q\left|T\bigl(f_1^0,(b_2-\lambda_2)f_2^0\bigr)(z)\right|^{p\delta}dz\right)^{\frac{1}{p\delta}}\\
&\quad+C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\left(\frac{1}{|Q|}\int_Q\left|T\bigl(f_1^0,(b_2-\lambda_2)f_2^\infty\bigr)(z)\right|^{p\delta}dz\right)^{\frac{1}{p\delta}}\\
&\quad+C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\left(\frac{1}{|Q|}\int_Q\left|T\bigl(f_1^\infty,(b_2-\lambda_2)f_2^0\bigr)(z)\right|^{p\delta}dz\right)^{\frac{1}{p\delta}}\\
&\quad+C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\left(\frac{1}{|Q|}\int_Q\left|T\bigl(f_1^\infty,(b_2-\lambda_2)f_2^\infty\bigr)(z)\right|^{p\delta}dz\right)^{\frac{1}{p\delta}}\\
&:=II_1+II_2+II_3+II_4.
\end{aligned}
$$

Now to estimate II_1, using the Kolmogorov inequality and the boundedness of operators, we have

$$
\begin{aligned}
II_1&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\left\|T\bigl(f_1^0,(b_2-\lambda_2)f_2^0\bigr)\right\|_{L^{\frac{1}{2},\infty}(Q,\frac{dx}{|Q|})}\\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\frac{1}{|Q|}\int_Q\left|f_1^0(z)\right|dz\int_Q\left|(b_2-\lambda_2)f_2^0(z)\right|dz
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\|f_2\|_{L(\mathrm{Log}L),Q}\varphi_\theta(Q)|f_1|_Q \\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta(1/\delta-2)-2\theta}}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\mathrm{Log}L),\varphi,\eta}(f_1,f_2)(x) \\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).
\end{aligned}
$$

The way of estimate II_2 is the same as II_3, we only prove II_2:

$$
\begin{aligned}
II_2 &\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\frac{1}{|Q|}\int_Q\bigl|T\bigl(f_1^0,(b_2-\lambda_2)f_2^\infty\bigr)(z)\bigr|\,dz \\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\frac{1}{|Q|}\int_Q\biggl(\int_{Q^*}\bigl|f_1(y_1)\bigr|\,dy_1\biggr) \\
&\quad\times\biggl(\int_{\mathbb{R}^n\setminus Q^*}\frac{|(b_2(y_2)-\lambda_2)f_2(y_2)\,dy_2|}{(|z-y_1|+|z-y_2|)^{2n}(1+|z-y_1|+|z-y_2|)^N}\biggr)\,dz \\
&\leq C\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\sum_{k=1}^\infty\frac{k(1+2^{k+3}\sqrt{n}r)^{\theta+2\eta}}{(1+2^{k+3}\sqrt{n}|Q|^{\frac{1}{n}})^N}\|b_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).
\end{aligned}
$$

Choosing η such that $\eta/\delta-1>0$, $N\geq\theta+2\eta+1$, we get

$$
II_2\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).
$$

Let $N\geq\theta+2\eta+1$ and $\eta/\delta-1>0$, then

$$
\begin{aligned}
II_4 &\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\frac{1}{|Q|}\int_Q\bigl|T\bigl(f_1^\infty,(b_2-\lambda_2)f_2^\infty\bigr)(z)\bigr|\,dz \\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\frac{1}{|Q|} \\
&\quad\times\biggl(\sum_{k=1}^\infty\int_{(\Omega_k)^2}\frac{|f_1(y_1)||(b_2(y_2)-\lambda_2)f_2(y_2)|\,dy_1\,dy_2}{(|z-y_1|+|z-y_2|)^{2n}(1+|z-y_1|+|z-y_2|)^N}\biggr)\,dz \\
&\leq C\frac{1}{\varphi(Q)^{\eta/\delta-\theta}}\sum_{k=1}^\infty\frac{k(1+2^{k+3}\sqrt{n}r)^{\theta+2\eta}}{(1+2^{k+3}\sqrt{n}|Q|^{\frac{1}{n}})^N}\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x) \\
&\leq C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).
\end{aligned}
$$

Now estimate IV. We first split any function

$$
\begin{aligned}
IV &\leq C\frac{1}{\varphi(Q)^{\eta/\delta}}\biggl(\frac{1}{|Q|}\int_Q\bigl|T\bigl((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0\bigr)(z)\bigr|^\delta\,dz\biggr)^{\frac{1}{\delta}} \\
&\quad+C\frac{1}{\varphi(Q)^{\eta/\delta}}\biggl(\frac{1}{|Q|}\int_Q\bigl|T\bigl((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\bigr)(z)\bigr|^\delta\,dz\biggr)^{\frac{1}{\delta}} \\
&\quad+C\frac{1}{\varphi(Q)^{\eta/\delta}}\biggl(\frac{1}{|Q|}\int_Q\bigl|T\bigl((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0\bigr)(z)\bigr|^\delta\,dz\biggr)^{\frac{1}{\delta}} \\
&\quad+C\frac{1}{\varphi(Q)^{\eta/\delta}}\biggl(\frac{1}{|Q|}\int_Q\bigl|T\bigl((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty\bigr)(z)\bigr|^\delta\,dz\biggr)^{\frac{1}{\delta}}
\end{aligned}
$$

$$:= IV_1 + IV_2 + IV_3 + IV_4.$$

Similar to the estimate of II_1, taking $\eta(\frac{1}{\delta} - 2) > 2\theta$, then

$$IV_1 \le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).$$

IV_2 and IV_3 are symmetric, here, only to estimate of IV_2, similar to II_2, taking $N \ge 2\theta + 2\eta + 1$, we get

$$IV_2 \le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).$$

Finally, similarly we estimate IV_4,

$$IV_4 \le C\|b_1\|_{\mathrm{BMO}_\theta(\varphi)}\|b_2\|_{\mathrm{BMO}_\theta(\varphi)}\mathcal{M}_{L(\log L),\varphi,\eta}(f_1,f_2)(x).$$

Thus, we completed the proof of Theorem 4.1. □

Theorem 4.2 *Let T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ as in Definition 1.1, $T_{\Pi_{\vec{b}}}$ be a multilinear commutator with $\vec{b} \in \mathrm{BMO}_{\vec{\theta}}^m(\varphi)$ and $\vec{w} \in A_{\vec{p}}^\infty(\varphi)$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p_j < \infty, j = 1, \ldots, m$. If ω satisfies*

$$\int_0^1 \frac{\omega(t)}{t}\left(1 + \log\frac{1}{t}\right)^m dt < \infty,$$

then there exists a constant $C > 0$ such that

$$\left\|T_{\Pi_{\vec{b}}}(\vec{f})\right\|_{L^p(v_{\vec{w}})} \le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)}\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Proof It is sufficient to prove that, $p > 0$, $w \in A_\infty^\infty$,

$$\int_{\mathbb{R}^n}\left|T_{\Pi_{\vec{b}}}(\vec{f})(x)\right|^p w(x) \le C\prod_{i=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)}\int_{\mathbb{R}^n}\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x)^p w(x)\,dx.$$

By the related Fefferman–Stein inequality (Lemma 2.7) and Theorem 4.1, we get

$$\begin{aligned}
\left\|T_{\Pi_{\vec{b}}}(\vec{f})\right\|_{L^p(w)} &\le \left\|\mathcal{M}_{\delta,\varphi,\eta}^d\left(T_{\Pi_{\vec{b}}}(\vec{f})\right)\right\|_{L^p(w)} \\
&\le \left\|\mathcal{M}_{\delta,\varphi,\eta}^{\sharp,d}\left(T_{\Pi_{\vec{b}}}(\vec{f})\right)\right\|_{L^p(w)} \\
&\le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j(\varphi)}}\left\|\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})\right\|_{L^p(w)} + \left\|M_{\epsilon,\varphi,\eta}^d\left(T(\vec{f})\right)\right\|_{L^p(w)} \\
&\quad + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{j=1}^m \|b_{\sigma(i)}\|_{\mathrm{BMO}_{\sigma(i)}}\left\|M_{\epsilon,\varphi,\eta}\left(T_{\Pi b_{\sigma'}^{-}}(\vec{f})\right)\right\|_{L^p(w)} \\
&\le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j(\varphi)}}\left\|\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})\right\|_{L^p(w)} + \left\|M_{\epsilon,\varphi,\eta}^{\sharp,d}\left(T(\vec{f})\right)\right\|_{L^p(w)}
\end{aligned}$$

$$+ C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i=1}^{m}\|b_{\sigma(i)}\|_{\mathrm{BMO}_{\sigma(i)}}\big\|M^{\sharp}_{\epsilon,\varphi,\eta}\big(T_{\Pi_{\vec{b}_{\sigma'}}}(\vec{f})\big)\big\|_{L^p(w)}.$$

Here ϵ,η are the same as in Theorem 4.1.

By Theorem 3.1, then

$$\begin{aligned}\big\|M^{\sharp,d}_{\epsilon,\varphi,\eta}\big(T(\vec{f})\big)\big\|_{L^p(w)} &\leq C\big\|\mathcal{M}_{\varphi,\eta}(\vec{f})\big\|_{L^p(w)}\\ &\leq C\big\|\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})\big\|_{L^p(w)}.\end{aligned}$$

For simplicity, we only prove the case $m = 2$ in the following:

$$\begin{aligned}\big\|M^{\sharp}_{\epsilon,\varphi,\eta}\big(T_{\Pi_{\vec{b}_{\sigma'}}}(\vec{f})\big)\big\|_{L^p(w)} \leq{}& C\big\|M^{\sharp}_{\epsilon,\varphi,\eta}\big(T^1_{b_1}(f_1,f_2)\big)\big\|_{L^p(w)}\\ &+ C\big\|M^{\sharp}_{\epsilon,\varphi,\eta}\big(T^1_{b_2}(f_1,f_2)\big)\big\|_{L^p(w)}.\end{aligned}$$

Similar to the estimate of $\|M^{\sharp,d}_{\epsilon,\varphi,\eta}(T(\vec{f}))\|_{L^p(w)}$, and Eqs. (4.2) and (4.3),

$$\big\|M^{\sharp}_{\epsilon,\varphi,\eta}\big(T^j_{b_j}(f_1,f_2)\big)\big\|_{L^p(w)} \leq C\|b_j\|_{\mathrm{BMO}_{\theta_j}}\big\|\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})\big\|_{L^p(w)}.$$

To sum up, we obtain

$$\int_{\mathbb{R}^n}\big|T_{\Pi_{\vec{b}}}(\vec{f})(x)\big|^p w(x) \leq C\prod_{i=1}^{m}\|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)}\int_{\mathbb{R}^n}\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x)^p w(x)\,dx. \tag{4.4}$$

By Lemma 2.3, we get

$$\int_{\mathbb{R}^n}\big|T_{\Pi_{\vec{b}}}(\vec{f})(x)\big|^p \nu_{\vec{w}} \leq C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}_{\theta_i}}\int_{\mathbb{R}^n}\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x)^p \nu_{\vec{w}}\,dx.$$

If $\mu > 1$, and since $\Phi(t) = t(1 + \log^+(t)) \leq t^{\mu}\,(t > 1)$, we easily get

$$\mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x) \leq \mathcal{M}_{\mu,\varphi,\eta}(\vec{f})(x).$$

By Lemma 2.9, then

$$\big\|\mathcal{M}_{\mu,\varphi,\eta}(\vec{f})\big\|_{L^p(\nu_{\vec{w}})} \leq C\prod_{j=1}^{m}\|f_j\|_{L^p_{(w_j)}}.$$

The inequality $\|\mathcal{M}_{\mu,\varphi,\eta}(\vec{f})\|_{L^p_{(\nu(\vec{w}))}} \leq C\prod_{j=1}^{m}\|f_j\|_{L^p_{(w_j)}}$ is equivalent to $\|\mathcal{M}_{\varphi,\eta}(\vec{f})\|_{L^{p/u}(\nu_{\vec{w}})} \leq C\prod_{j=1}^{m}\|f_j\|_{L^p_{(w_j)}}$, which was proved in [16]. For some $\mu > 1$, using Lemma 2.4, we get

$$\big\|T_{\Pi_{\vec{b}}}(\vec{f})\big\|_{L^p(\nu_{\vec{w}})} \leq C\prod_{j=1}^{m}\|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)}.$$

Thus, this proof is finished. By the proof of Theorem 4.2, the following results are obtained. □

Theorem 4.3 *Let $p > 0$ and w be a weight in $A_\infty^\infty(\varphi)$, and suppose that $T_{\Pi_{\vec{b}}}$ is a multilinear iterated commutator with $\vec{b} \in \mathrm{BMO}_{\vec{\theta}}^m(\varphi)$. Let $\eta > 0$ and ω is satisfying (4.1). Then there exist constants $C > 0$ depending on the $A_\infty^\infty(\varphi)$ constant of w such that*

$$\int_{\mathbb{R}^n} \left|T_{\Pi_{\vec{b}}}(\vec{f})(x)\right|^p w(x) \le C \prod_{i=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)} \int_{\mathbb{R}^n} \mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x)^p w(x)\,dx$$

and

$$\begin{aligned}&\sup_{t>0} \frac{1}{\Phi^{(m)}(\frac{1}{t})} w\big(\big\{y \in \mathbb{R}^n : \left|T_{\Pi_{\vec{b}}}(\vec{f})(y)\right| > t^m\big\}\big) \\ &\quad\le C \sup_{t>0} \frac{1}{\Phi^{(m)}(\frac{1}{t})} w\big(\big\{y \in \mathbb{R}^n : \mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(y) > t^m\big\}\big).\end{aligned}$$

Proof The first result is proved in Theorem 4.2, the proof of second result is similar to the first, also refer to the [8, Theorem 3.19], we need to use Lemma 2.8, Theorem 3.1 and Theorem 4.1. We omit the details here. □

Lemma 4.4 ([21]) *Let $w \in A_1^\theta(\varphi)$ and $\eta > 2\theta$. Then there exists a constant $C > 0$ such that*

$$\nu_{\vec{w}}\big\{x \in \mathbb{R}^n : \mathcal{M}_{L(\log L),\varphi,\eta}(\vec{f})(x) > t^m\big\} \le C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t}\right) w_j(x)\,dx\right)^{\frac{1}{m}},$$

where $\Phi(t) = t(1 + \log^+ t)$ and $\Phi^{(m)} = \underbrace{\Phi \circ \cdots \circ \Phi}$.

Theorem 4.5 *Let T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ as in Definition 1.1, $T_{\Pi_{\vec{b}}}$ be a multilinear commutator with $\vec{b} \in \mathrm{BMO}_{\vec{\theta}}^m(\varphi)$ and $\vec{w} \in A_{\vec{1}}^\infty(\varphi)$, assume that ω is satisfying (4.1). Then there exists a constant $C > 0$ such that*

$$\nu_{\vec{w}}\big\{x \in \mathbb{R}^{\cdot}\left|T_{\Pi_{\vec{b}}}(\vec{f})(x)\right| > t^m\big\} \le C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t}\right) w_j(x)\,dx\right)^{\frac{1}{m}}.$$

Proof Now by Theorem 4.1, Theorem 4.3 and Lemma 4.4, we can get the above result. Since this argument is the same as the proof of [16, Theorem 4.2], here, we omit the proof. □

Corollary 4.6 *Let T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ as in Definition 1.1, $T_{\Sigma_{\vec{b}}}$ be a multilinear commutator with $\vec{b} \in \mathrm{BMO}_{\vec{\theta}}^m(\varphi)$, assume that ω is satisfying (4.1), $0 < \delta < \epsilon < 1/m$ and $\eta \ge 2(\theta_1, \dots, \theta_m)/(1/\delta - 1/\epsilon)$. Then there exists a constant $C > 0$ such that*

$$M^{\sharp,d}_{\delta,\varphi,\eta,}\big(T_{\Sigma_{\vec{b}}}(\vec{f})\big)(x) \le C \sum_{j=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)} \left(\sum_{j=1}^m \mathcal{M}^i_{L(\log L),\varphi,\eta}(\vec{f})(x) + M^d_{\epsilon,\varphi,\eta}\big(T(\vec{f})\big)(x)\right).$$

For all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support.

Proof In fact, the multilinear commutator is a special case of iterate commutator, so we can directly obtain this result through Theorem 4.1. □

Corollary 4.7 *Let $T_{\Sigma_{\vec{b}}}$ be a multilinear commutator with $\vec{b} \in \mathrm{BMO}^m_{\vec{\theta}}(\varphi)$, T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ as in Definition 1.1 and $\vec{w} \in A^{\infty}_{\vec{p}}(\varphi)$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p_j < \infty$, $j = 1, \ldots, m$. If ω is satisfying*

$$\int_0^1 \frac{\omega(t)}{t}\left(1 + \log\frac{1}{t}\right)^m dt < \infty,$$

then there exists a constant $C > 0$ such that

$$\left\| T_{\Sigma\vec{b}}(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \le C\sum_{j=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Proof Obviously, the multilinear commutator is a special case of the iterate commutator, then, through Theorem 4.2 we can directly obtain this result. □

Corollary 4.8 *Let $p > 0$ and w be a weight in $A^{\infty}_{\infty}(\varphi)$, and suppose that $T_{\Sigma_{\vec{b}}}$ be a multilinear commutator with $\vec{b} \in \mathrm{BMO}^m_{\vec{\theta}}(\varphi)$, T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ as in Definition 1.1. Let $\eta > 0$ and ω is satisfying (4.1). Then there exist constants $C > 0$ depending on the $A^{\infty}_{\infty}(\varphi)$ constant of w such that*

$$\int_{\mathbb{R}^n} \left| T_{\Sigma_{\vec{b}}}(\vec{f})(x)\right|^p w(x) \le C\sum_{i=1}^m \|b_j\|_{\mathrm{BMO}_{\theta_j}(\varphi)} \int_{\mathbb{R}^n} \sum_{i=1}^m \mathcal{M}^i_{L(\log L),\varphi,\eta}(\vec{f})(x)^p w(x)\,dx$$

and

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\left(\left\{y \in \mathbb{R}^n : \left|T_{\Sigma_{\vec{b}}}(\vec{f})(y)\right| > t^m\right\}\right) \\
&\quad\le C\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\left(\left\{y \in \mathbb{R}^n : \sum_{i=1}^m \mathcal{M}^i_{L(\log L),\varphi,\eta}(\vec{f})(y) > t^m\right\}\right).
\end{aligned}$$

Proof Similar to the proof of [8, Theorem 3.19], we need to use Lemma 2.7, Lemma 2.8, Theorem 3.1, and Corollary 4.6. We omit the details here. □

Lemma 4.9 ([16]) *Let $w \in A^{\theta}_1(\varphi)$ and $\eta > 2\theta$. Then there exists a constant C such that*

$$\nu_{\vec{w}}\left\{x \in \mathbb{R}^n : \mathcal{M}^i_{L(\log L),\varphi,\eta}(\vec{f})(x) > t^m\right\} \le C\prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{t}\right) w_j(x)\,dx\right)^{\frac{1}{m}},$$

where $\Phi(t) = t(1 + \log^+ t)$.

Corollary 4.10 *Let $T_{\Sigma_{\vec{b}}}$ be a multilinear commutator with $\vec{b} \in \mathrm{BMO}^m_{\vec{\theta}}(\varphi)$, T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ as in Definition 1.1 and $\vec{w} \in A^{\infty}_{\vec{1}}(\varphi)$, assume that ω is satisfying (4.1). Then there exists a constant C such that*

$$\nu_{\vec{w}}\left\{x \in \mathbb{R}^n : \left|T_{\Sigma_{\vec{b}}}(\vec{f})(x)\right| > t^m\right\} \le C\prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{t}\right) w_j(x)\,dx\right)^{\frac{1}{m}}.$$

Proof Now by Corollary 4.8 and Lemma 4.9. We can get Corollary 4.10. Since this argument is the same as the proof of [16, Theorem 4.2], here, we omit the proof. □

Acknowledgements
The authors would like to express their thanks to the referees for valuable advice regarding a previous version of this paper.

Funding
The research was supported by Natural Science Foundation of China (Grant Nos. 12061069).

Availability of data and materials
This article is licensed under a Creative Commons Attribution 4.0 International License, which permits us, sharing, adaptation, distribution in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link tyo the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the articles Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the articles Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of licence, visit http://creativecommons.org/licenses/by/4.0/.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equality and significantly in writing this paper. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 August 2020 Accepted: 21 January 2021 Published online: 30 January 2021

References
1. Coifman, R.R., Meyer, Y.: On commutators of singular integrals and bilinear singular integrals. Trans. Am. Math. Soc. **212**, 315–331 (1975)
2. Coifman, R.R., Meyer, Y.: Commutateurs d'intégrales singulières et opérateurs multilinéaires. Ann. Inst. Fourier **28**(3), 177–202 (1978)
3. Grafakos, L., Torres, R.H.: Multilinear Calderón–Zygmund theory. Adv. Math. **165**(1), 124–164 (2002)
4. Kenig, C., Stein, E.M.: Multilinear estimates and fractional integration. Math. Res. Lett. **6**(1), 1–15 (1999)
5. Grafakos, L., Torres, R.H.: Maximal operator and weighted norm inequalities for multilinear singular integrals. Indiana Univ. Math. J. **51**(5), 1261–1276 (2000)
6. Hu, G., Meng, Y., Yang, D.: Weighted norm inequalities for multilinear Calderón–Zygmund operators on non-homogeneous metric measure spaces. Forum Math. **26**(5), 1289–1322 (2014)
7. Grafakos, L., He, D.: Multilinear Calderón–Zygmund operators on Hardy spaces. J. Math. Anal. Appl. **416**(2), 511–521 (2014)
8. Lerner, A.K., Ombrosi, S., Pérez, C., Torres, R.H., Trujillo-González, R.: New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory. Adv. Math. **220**(4), 1222–1264 (2009)
9. David, G., Journé, J.L.: A boundedness criterion for generalized Calderón–Zygmund operators. Ann. Math. **120**(2), 371–397 (1984)
10. Yabuta, K.: Generalizations of Calderón–Zygmund operators. Stud. Math. **1**, 17–31 (1985)
11. Hu, G.: Estimates for the maximal bilinear singular integral operators. Acta Math. Sin. **31**, 847–862 (2015)
12. Lu, G., Zhang, P.: Multilinear Calderón–Zygmund operators with kernels of Dini's type and applications. Nonlinear Anal. **107**, 92–117 (2014)
13. Zhang, P., Sun, J.: Commutators of multilinear Calderón–Zygmund operators with kernels of Dini's type and applications (2016) arXiv:1605.07449
14. Pérez, C., Torres, R.H.: Sharp maximal function estimates for multilinear singular integrals. Contemp. Math. **320**, 323–331 (2003)
15. Pérez, C., Pradolini, G., Torres, R.H., Trujillo-González, R.: End-point estimates for iterated commutators of multilinear singular integrals. Bull. Lond. Math. Soc. **46**, 26–42 (2014)
16. Pan, G., Tang, L.: New weighted norm inequalities for certain classes of multilinear operators and their iterated commutators. Potential Anal. **43**, 371–398 (2015)
17. Bui, T.A.: New class of multiple weights and new weighted inequalities for multilinear operators. Forum Math. **27**(2), 995–1023 (2015)
18. Hu, X., Zhou, J.: New weighted norm inequalities for Calderón–Zygmund operators with kernels of Dini's type and their commutators. Proc. Indian Acad. Sci. Math. Sci. **129**, 56–71 (2019)
19. García-Cuerva, J., Rubio de Francia, J.: Weighted Norm Inequalities and Related Topics. Amsterdam, New York (1985)
20. Bongioanni, B., Haboure, E., Salinas, O.: Classes of weights related to Schrodinger operators. J. Math. Anal. Appl. **373**, 563–579 (2011)
21. Tang, L.: Weighted norm inequalities for pseudo-differential operators with smooth symbols and their commutators. J. Funct. Anal. **262**, 1603–1629 (2012)