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Some dynamic Hilbert-type inequalities for two variables on time scales

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Abstract

In this paper, we discuss some new Hilbert-type dynamic inequalities on time scales in two separate variables. We also deduce special cases, like some integral and their respective discrete inequalities.

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1 Introduction

It is evident that Hilbert-type inequalities play a major role in mathematics, for complex pattern analysis, numerical analysis, qualitative theory of differential equations and their implementations. Hilbert's discrete inequality and its integral formula [1, Theorem 316] have been generalized in many ways (for example, see [2–6]). Lately, Pachpatte in [6], obtained the following inequality: if $A_q = \sum_{s=1}^{q} a_s \geq 0$ and $B_n = \sum_{t=1}^{n} b_t \geq 0$, for $q = 1, 2, \ldots, p$ and $n = 1, 2, \ldots, r$, where p and r are the natural numbers, then

$$\sum_{q=1}^{p}\sum_{n=1}^{r}\frac{A_q B_n}{q+n} \leq C(p,r)\left(\sum_{q=1}^{p}(p-q+1)(a_q)^2\right)^{\frac{1}{2}} \times \left(\sum_{n=1}^{r}(r-n+1)(b_n)^2\right)^{\frac{1}{2}}, \tag{1}$$

where

$$C(p,r) = \frac{1}{2}\sqrt{pr}.$$

The integral analogue of (1) is given by

$$\int_0^x \int_0^y \frac{F(s)G(t)}{s+t}\,ds\,dt \leq D(x,y)\left(\int_0^x (x-s)f^2(s)\,ds\right)^{\frac{1}{2}} \times \left(\int_0^y (y-t)g^2(t)\,dt\right)^{\frac{1}{2}}, \tag{2}$$

where $F(s) = \int_0^s f(\tau)\,d\tau \geq 0$, $G(t) = \int_0^t g(\nu)\,d\nu \geq 0$, and

$$D(x,y) = \frac{1}{2}\sqrt{xy}.$$

In the past few years, several researchers have suggested the study of dynamic time scale inequalities. In [7] the authors deduced some generalizations of the inequalities (1) and (2) on time scales. Namely, they proved that if $A(x_1) = \int_w^{x_1} a(\tau_1)\Delta\tau_1$, $B(y_1) = \int_w^{y_1} b(\tau_1)\Delta\tau_1$, and $p_1 > 1$, $q_1 > 1$ with $p_1^{-1} + q_1^{-1} = 1$, then

$$\begin{aligned}
&\int_w^{x_2}\int_w^{y_2} \frac{A(x_1)B(y_1)}{q_1(x_1-w)^{p_1-1} + p_1(y_1-w)^{q_1-1}}\Delta y_1 \Delta x_1 \\
&\quad \leq M(p_1,q_1)\biggl(\int_w^{x_2}\bigl(\sigma(x_2)-x_1\bigr)\bigl(a(x_1)\bigr)^{p_1}\Delta x_1\biggr)^{p_1^{-1}} \\
&\quad\quad \times (\int_w^{y_2}\bigl(\sigma(y_2)-y_1\bigr)\bigl(b(y_1)^{q_1}\Delta y_1\bigr)^{q_1^{-1}},
\end{aligned} \tag{3}$$

where

$$M(p_1,q_1) = (p_1q_1)^{-1}(x_2-w)^{p_1^{-1}(p_1-1)}(y_2-w)^{q_1^{-1}(q_1-1)}. \tag{4}$$

In order to develop dynamic time scale inequalities, we moved the reader to the articles [8–19].

Motivated by the above results, our major aim in this paper is to establish some dynamic Hilbert-type inequalities in two separate variables on time scales. These inequalities can be considered as extensions and generalizations of some Hilbert-type inequalities proved in [7] for the two-dimensional on time scales.

The paper is governed as follows: In Sect. 2, we remember some basic notions, definitions and results on time scales calculus which will be required in proving our main outcomes. In Sect. 3, we will exemplify the major results.

2 Preliminaries and basic lemmas

In this section, we will present some fundamental concepts and effects on time scales which will be beneficial for deducing our main results. The following definitions and theorems are referred from [20, 21].

A time scale $\mathbb{T}$ is defined as an arbitrary nonempty closed subset of the real numbers. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for any $t^* \in \mathbb{T}$ by

$$\sigma\bigl(t^*\bigr) = \inf\bigl\{s^* \in \mathbb{T} : s^* > t^*\bigr\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ for any $t^* \in \mathbb{T}$ by

$$\rho\bigl(t^*\bigr) = \sup\bigl\{s^* \in \mathbb{T} : s^* < t^*\bigr\}.$$

From the above two definitions, it can be stated that a point $t^* \in \mathbb{T}$ with $\inf\mathbb{T} < t^* < \sup\mathbb{T}$ is called right-scattered if $\sigma(t^*) > t^*$, right-dense if $\sigma(t^*) = t^*$, left-scattered if $\rho(t^*) < t^*$, and left-dense if $\rho(t^*) = t^*$. If $\mathbb{T}$ has a left-scattered maximum t_m^*, then $\mathbb{T}^k = \mathbb{T} - \{t_m^*\}$, otherwise

$\mathbb{T}^k = \mathbb{T}$. Moreover, the forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ for any $t^* \in \mathbb{T}$ is defined by $\mu(t^*) = \sigma(t^*) - t^*$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative of f at $t^* \in \mathbb{T}^k$ is defined as $f^\Delta(t^*)$ if for each $\varepsilon > 0$ there exists a neighborhood U^* of t^* such that

$$\left|\left[f\left(\sigma\left(t^*\right)\right) - f\left(s^*\right)\right] - f^\Delta\left(t^*\right)\left[\sigma\left(t^*\right) - s^*\right]\right| \leq \varepsilon\left|\sigma\left(t^*\right) - s^*\right|, \quad \text{for all } s^* \in U^*.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by $\mathrm{C}_{rd}(\mathbb{T},\mathbb{R})$. We will frequently use the following useful relations between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$, as well as difference calculus on $\mathbb{Z}$. Note that

(i) if $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} &\sigma\left(t^*\right) = t^*, \qquad \mu\left(t^*\right) = 0, \qquad f^\Delta\left(t^*\right) = f'\left(t^*\right), \quad \text{and} \\ &\int_a^b f^\Delta\left(t^*\right)\Delta t^* = \int_a^b f\left(t^*\right)dt^*; \end{aligned} \tag{5}$$

(ii) if $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma\left(t^*\right) = t^* + 1, \qquad \mu\left(t^*\right) = 1, \qquad f^\Delta\left(t^*\right) = \Delta f\left(t^*\right), \quad \text{and} \\ &\int_a^b f^\Delta\left(t^*\right)\Delta t^* = \sum_{t^*=a}^{b-1} f\left(t^*\right). \end{aligned} \tag{6}$$

Also, we must know some essentials about partial derivatives on time scales. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be any two time scales. Let σ_1, Δ_1 and σ_2, Δ_2 denote the forward jump operator and the delta differentiation operator on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Assume that $u < w$ are points in $\mathbb{T}_1$, $e < f$ are points in $\mathbb{T}_2$, $[u,w)$ is a semiclosed bounded interval in $\mathbb{T}_1$, and $[e,f)$ is a semiclosed bounded interval in $\mathbb{T}_2$. Let us consider a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ given by

$$\mathrm{R} = [u,w)_{\mathbb{T}_1} \times [e,f)_{\mathbb{T}_2} = \left\{\left(t_1^*, t_2^*\right) : t_1^* \in [u,v)_{\mathbb{T}_1}, t_2^* \in [e,f)_{\mathbb{T}_2}\right\}.$$

Suppose $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is a real-valued function. At $(t_1^*, t_2^*) \in \mathbb{T}_1 \times \mathbb{T}_2$, we say that f has a Δ_1 partial derivative with respect to t_1^* if for each $\varepsilon > 0$ there exists a neighborhood $U_{t_1^*}$ of t_1^* such that

$$\left|\left[f\left(\sigma_1\left(t_1^*\right), t_2^*\right) - f\left(s^*, t_2^*\right)\right] - f^{\Delta_1}\left(t_1^*, t_2^*\right)\left[\sigma_1\left(t_1^*\right) - s^*\right]\right| \leq \varepsilon\left|\sigma_1\left(t_1^*\right) - s^*\right|,$$

for all $s^* \in U_{t_1^*}$. At $(t_1^*, t_2^*) \in \mathbb{T}_1 \times \mathbb{T}_2$, we say that f has a Δ_2 partial derivative with respect to t_2^* if for each $\varepsilon > 0$ there exists a neighborhood $U_{t_2^*}$ of t_2^* such that

$$\left|\left[f\left(t_1^*, \sigma_2\left(t_2^*\right)\right) - f\left(t_1^*, l^*\right)\right] - f^{\Delta_2}\left(t_1^*, t_2^*\right)\left[\sigma_2\left(t_2^*\right) - l^*\right]\right| \leq \varepsilon\left|\sigma_2\left(t_2^*\right) - l^*\right|,$$

for all $l^* \in U_{t_2^*}$.

A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is said to be rd-continuous in t_2^* if for every $\beta_1^* \in \mathbb{T}_1$, the function $f(\beta_1^*, t_2^*)$ is rd-continuous on $\mathbb{T}_2$ and is rd-continuous in t_1^* if for every $\beta_2^* \in \mathbb{T}_2$, the function $f(t_1^*, \beta_2^*)$ is rd-continuous on $\mathbb{T}_1$. Let CC_{rd} denote the set of functions $f(t_1^*, t_2^*)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the properties:

(A1) f is rd-continuous in t_1^*;

(A2) f is rd-continuous in t_2^*;

(A3) if $(x_1^*, x_2^*) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_1^* right-dense or maximal and x_2^* right-dense or maximal, then f is continuous at (x_1^*, x_2^*);

(A4) if x_1^* and x_1^* are both left-dense, then the limit of $f(t_1^*, t_2^*)$ exists as (t_1^*, t_2^*) approaches (x_1^*, x_2^*) along any path in the region

$$\mathrm{R}_{LL}\left(x_1^*, x_2^*\right) = \left\{\left(t_1^*, t_2^*\right) : t_1^* \in [u, x_1^*) \cap \mathbb{T}_1, t_2^* \in [e, x_2^*) \cap \mathbb{T}_2\right\}.$$

Let CC_{rd}^1 be the set of all functions in CC_{rd} for which both the Δ_1 partial derivative and the Δ_2 partial derivative exist in CC_{rd}.

In the following, we present Fubini theorem on time scales which plays a key role in proving the main results of this paper.

Theorem 1 (Fubini's theorem [22]) *Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. Suppose that $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is a Δ-integrable function with respect to both time scales. Define*

$$\phi\left(t^*\right) = \int_{\mathbb{T}_1} f\left(x^*, t^*\right)\Delta x^*, \quad \textit{for a.e. } t^* \in \mathbb{T}_1,$$

and

$$\psi\left(x^*\right) = \int_{\mathbb{T}_2} f\left(x^*, t^*\right)\Delta t^*, \quad \textit{for a.e. } x^* \in \mathbb{T}_2.$$

Then ϕ and ψ are Δ-integrable on $\mathbb{T}_1$, $\mathbb{T}_2$, respectively, and

$$\int_{\mathbb{T}_1} \Delta x^* \int_{\mathbb{T}_2} f\left(x^*, t^*\right)\Delta t^* = \int_{\mathbb{T}_2} \Delta t^* \int_{\mathbb{T}_1} f\left(x^*, t^*\right)\Delta x^*. \tag{7}$$

Next, we present Hölder's and Jensen's inequalities in two dimensions on time scales.

Theorem 2 (Hölder's inequality [23, Theorem 2.3.10]) *Let $u, v \in \mathbb{T}$ with $u < v$. If $f, g \in \mathrm{CC}_{rd}^1([u, v]_{\mathbb{T}} \times [u, v]_{\mathbb{T}}, \mathbb{R})$ are integrable functions and $p^{-1} + q^{-1} = 1$ with $p > 1$, then*

$$\begin{aligned}\int_u^v \int_u^v \left|f\left(r^*, t^*\right)g\left(r^*, t^*\right)\right| \Delta r^* \Delta t^* &\leq \left(\int_u^v \int_u^v \left|f\left(r^*, t^*\right)\right|^p \Delta r^* \Delta t^*\right)^{p^{-1}} \\ &\quad \times \left(\int_u^v \int_u^v \left|g\left(r^*, t^*\right)\right|^q \Delta r^* \Delta t^*\right)^{q^{-1}}.\end{aligned} \tag{8}$$

Theorem 3 (Jensen's inequality [24, Theorem 3.1]) *Let $r^*, t^* \in \mathrm{R}$ and $-\infty \leq m^* < n^* \leq \infty$. If $f \in \mathrm{CC}_{rd}^1(\mathbb{R}, (m^*, n^*))$ and $\Phi : (m^*, n^*) \to \mathbb{R}$ is convex, then*

$$\Phi\left(\frac{\int_u^v \int_w^s f(r^*, t^*)\Delta_1 r^* \Delta_2 t^*}{\int_u^v \int_w^s \Delta_1 r^* \Delta_2 t^*}\right) \leq \frac{\int_u^v \int_w^s \Phi(f(r^*, t^*))\Delta_1 r^* \Delta_2 t^*}{\int_u^v \int_w^s \Delta_1 r^* \Delta_2 t^*}. \tag{9}$$

Lemma 1 (Young's inequality [25]) *If $l, u \in \mathbb{R}_+$ and $p^{-1} + q^{-1} = 1$ with $p > 1$, then*

$$lu \le p^{-1} l^p + q^{-1} u^q, \tag{10}$$

we get equality iff $l^p = u^q$.

3 Main results

In this section, we state and prove our main results. In particular, we establish the two-dimensional versions of the inequalities given in [7]. Throughout this section, we will assume that the following hypotheses hold:

(H1) $\mathbb{T}_1$ and $\mathbb{T}_2$ are any two time scales with (i) $t_0, s_1, k_1, x, z \in \mathbb{T}_1$; (ii) $t_0, t_1, r_1, y, w \in \mathbb{T}_2$.

(H2) p_1, q_1 are any two real numbers such that $p_1 > 1$, $q_1 > 1$with $1/p_1 + 1/q_1 = 1$.

(H3) For $t_0 \in \mathbb{T}_1, \mathbb{T}_2$ we denote the subintervals of $\mathbb{T}_1$, $\mathbb{T}_2$ by $I_x = [t_0, x)_{\mathbb{T}_1}$, $I_z = [t_0, z)_{\mathbb{T}_1}$, $I_y = [t_0, y)_{\mathbb{T}_2}$ and $I_w = [t_0, w)_{\mathbb{T}_2}$, where $x, z \in \Omega_1 = [t_0, \infty) \cap \mathbb{T}_1$ and $y, w \in \Omega_2 = [t_0, \infty) \cap \mathbb{T}_2$.

(H4) There exist two functions Φ and Ψ which are real-valued, nonnegative, convex, and submultiplicative, defined on $[0, \infty)$. A function f^* is submultiplicative if $f^*(x_1 y_1) \le f^*(x_1) f^*(y_1)$ for $x_1, y_1 \ge 0$.

Theorem 4 *Let (H1), (H2) be satisfied and $f^*(s_1, t_1) \in \mathrm{CC}^1_{rd}(I_x \times I_y, \mathbb{R}^+)$, $g^*(k_1, r_1) \in \mathrm{CC}^1_{rd}(I_z \times I_w, \mathbb{R}^+)$. Suppose that $F^*(s_1, t_1)$ and $G^*(k_1, r_1)$ are defined as*

$$F^*(s_1, t_1) = \int_{t_0}^{s_1} \int_{t_0}^{t_1} f^*(\xi, \eta) \Delta\xi \, \Delta\eta, G^*(k_1, r_1) = \int_{t_0}^{k_1} \int_{t_0}^{r_1} g^*(\xi, \eta) \Delta\xi \, \Delta\eta. \tag{11}$$

Then for $(s_1, t_1) \in I_x \times I_y$ and $(k_1, r_1) \in I_z \times I_w$, one gets

$$\begin{aligned} &\int_{t_0}^{x} \int_{t_0}^{y} \left(\int_{t_0}^{z} \int_{t_0}^{w} \frac{F^*(s_1, t_1) G^*(k_1, r_1)}{q_1 [(s_1 - t_0)(t_1 - t_0)]^{p_1 - 1} + p_1 [(k_1 - t_0)(r_1 - t_0)]^{q_1 - 1}} \Delta k_1 \Delta r_1 \right) \Delta s_1 \Delta t_1 \\ &\quad \le C(p_1, q_1) \left(\int_{t_0}^{x} \int_{t_0}^{y} \big(\sigma(x) - s_1\big)\big(\sigma(y) - t_1\big) \big[f^*(s_1, t_1)\big]^{p_1} \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}} \\ &\qquad \times \left(\int_{t_0}^{z} \int_{t_0}^{w} \big(\sigma(w) - k_1\big)\big(\sigma(z) - r_1\big) \big[g^*(k_1, r_1)\big]^{q_1} \Delta k_1 \Delta r_1 \right)^{\frac{1}{q_1}}, \end{aligned} \tag{12}$$

where

$$C(p_1, q_1) = \frac{1}{p_1 q_1} \big[(x - t_0)(y - t_0)\big]^{\frac{p_1 - 1}{p_1}} \big[(w - t_0)(z - t_0)\big]^{\frac{q_1 - 1}{q_1}}. \tag{13}$$

Proof By assumption and applying Hölder's inequality (8) with respect to p_1, $p_1/(p_1 - 1)$ and q_1, $q_1/(q_1 - 1)$, respectively, we find that

$$F^*(s_1, t_1) \le \big[(s_1 - t_0)(t_1 - t_0)\big]^{\frac{p_1 - 1}{p_1}} \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \big[f^*(\xi, \eta)\big]^{p_1} \Delta\xi \, \Delta\eta \right)^{\frac{1}{p_1}} \tag{14}$$

and

$$G^*(k_1, r_1) \le \big[(k_1 - t_0)(r_1 - t_0)\big]^{\frac{q_1 - 1}{q_1}} \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \big[g^*(\xi, \eta)\big]^{q_1} \Delta\xi \, \Delta\eta \right)^{\frac{1}{q_1}}. \tag{15}$$

By multiplying (14) and (15), we get

$$F^*(s_1,t_1)G^*(k_1,r_1) \le \big[(s_1-t_0)(t_1-t_0)\big]^{\frac{p_1-1}{p_1}}\big[(k_1-t_0)(r_1-t_0)\big]^{\frac{q_1-1}{q_1}} \times \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big[f^*(\xi,\eta)\big]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}} \times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big[g^*(\xi,\eta)\big]^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}. \tag{16}$$

Applying Young's inequality on the term $[(s_1-t_0)(t_1-t_0)]^{(p_1-1)/p_1} \times [(k_1-t_0)(r_1-t_0)]^{(q_1-1)/q_1}$ with $u=[(s_1-t_0)(t_1-t_0)]^{(p_1-1)/p_1}$ and $l=[(k_1-t_0)(r_1-t_0)]^{(q_1-1)/q_1}$, we observe that

$$\begin{aligned} F^*(s_1,t_1)G^*(k_1,r_1) &\le \left(\frac{[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}}{p_1}+\frac{[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}{q_1}\right) \\ &\quad\times \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big[f^*(\xi,\eta)\big]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}} \\ &\quad\times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big[g^*(\xi,\eta)\big]^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}} \\ &= \left(\frac{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}{p_1q_1}\right) \\ &\quad\times \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big[f^*(\xi,\eta)\big]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}} \\ &\quad\times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big[g^*(\xi,\eta)\big]^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}. \end{aligned} \tag{17}$$

Dividing both sides of (17) by $q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}$, we obtain

$$\begin{aligned} &\frac{F^*(s_1,t_1)G^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}} \\ &\quad\le \frac{1}{p_1q_1}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big[f^*(\xi,\eta)\big]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}} \\ &\qquad\times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big[g^*(\xi,\eta)\big]^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}. \end{aligned} \tag{18}$$

Integrating both sides of (18) first with respect to r_1 and k_1 and then with respect to s_1 and t_1, respectively, and applying Hölder's inequality (8) with indices p_1, $p_1/(p_1-1)$ and q_1, $q_1/(q_1-1)$, we see that

$$\begin{aligned} &\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{F^*(s_1,t_1)G^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1 \\ &\quad\le \frac{1}{p_1q_1}\big[(x-t_0)(y-t_0)\big]^{\frac{p_1-1}{p_1}}\big[(z-t_0)(w-t_0)\big]^{\frac{q_1-1}{q_1}} \\ &\qquad\times \left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\big[f^*(\xi,\eta)\big]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}} \end{aligned}$$

$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[g^*(\xi,\eta)\right]^{q_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}$$
$$= C(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left[f^*(\xi,\eta)\right]^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[g^*(\xi,\eta)\right]^{q_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}. \tag{19}$$

Applying Fubini's theorem on (19), we conclude that

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{F^*(s_1,t_1)G^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\le C(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}(x-s_1)(y-t_1)\left[f^*(s_1,t_1)\right]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}(z-k_1)(w-r_1)\left[g^*(k_1,r_1)\right]^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}},$$

and then, by using the fact that $\sigma(n)\ge n$, one gets

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{F^*(s_1,t_1)G^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\le C(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\sigma(x)-s_1\right)\left(\sigma(y)-t_1\right)\left[f^*(s_1,t_1)\right]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\sigma(z)-r_1\right)\left(\sigma(w)-k_1\right)\left[g^*(k_1,r_1)\right]^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}},$$

which proves (12). This completes the proof. □

Using the relations (5) and taking $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$, $t_0=0$ in Theorem 4 leads to the following result.

Corollary 1 *Assume that $f^*(s_1,t_1)$ and $g^*(k_1,r_1)$ are real-valued continuous functions and define*

$$F^*(s_1,t_1)=\int_0^{s_1}\int_0^{t_1}f^*(\xi,\eta)\,d\xi\,d\eta,\qquad G^*(k_1,r_1)=\int_0^{k_1}\int_0^{r_1}g^*(\xi,\eta)\,d\xi\,d\eta.$$

Then for $(s_1,t_1)\in I_x\times I_y$ and $(k_1,r_1)\in I_z\times I_w$, we have

$$\int_0^{x}\int_0^{y}\left(\int_0^{z}\int_0^{w}\frac{F^*(s_1,t_1)G^*(k_1,r_1)}{q_1(s_1t_1)^{p_1-1}+p_1(k_1r_1)^{q_1-1}}\,dk_1\,dr_1\right)ds_1\,dt_1$$
$$\le C^*(p_1,q_1)\left(\int_0^{x}\int_0^{y}(x-s_1)(y-t_1)\left[f^*(s_1,t_1)\right]^{p_1}ds_1\,dt_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_0^{z}\int_0^{w}(z-k_1)(w-r_1)\left[g^*(k_1,r_1)\right]^{q_1}dk_1\,dr_1\right)^{\frac{1}{q_1}},$$

where

$$C^{*}(p_1,q_1)=\frac{1}{p_1q_1}(xy)^{\frac{p_1-1}{p_1}}(zw)^{\frac{q_1-1}{q_1}}.$$

By using the relations (5) and letting $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, $t_0=0$ in Theorem 4, we get the following result.

Corollary 2 *Assume that* $\{a_{m_1,n_1}\}_{0\le m_1,n_1\le N}$ *and* $\{b_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ *are two nonnegative sequences of real numbers and define*

$$A_{m_1,n_1}=\sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1}a_{\xi,\eta},\qquad B_{k_1,r_1}=\sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1}b_{\xi,\eta}.$$

Then

$$\begin{aligned}&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1}\frac{A_{s_1,t_1}B_{k_1,r_1}}{q_1(s_1t_1)^{p_1-1}+p_1(k_1r_1)^{q_1-1}}\right)\\&\quad\le C^{**}(p_1,q_1)\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}(m_1-s_1+1)(n_1-t_1+1)(a_{s_1,t_1})^{p_1}\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1}(z_1-k_1+1)(w_1-r_1+1)(b_{k_1,r_1})^{q_1}\right)^{\frac{1}{q_1}},\end{aligned}$$

where

$$C^{**}(p_1,q_1)=\frac{1}{p_1q_1}(m_1n_1)^{\frac{p_1-1}{p_1}}(z_1w_1)^{\frac{q_1-1}{q_1}}.$$

In the following theorems, we give a further generalization of (12) obtained in Theorem 4.

Theorem 5 *Let* $(H1)$, $(H2)$, *and* $(H4)$ *be satisfied,* $f^{*}(s_1,t_1)\in \mathrm{CC}_{rd}^{1}(I_x\times I_y,\mathbb{R}^{+})$, $g^{*}(k_1,r_1)\in \mathrm{CC}_{rd}^{1}(I_z\times I_w,\mathbb{R}^{+})$ *and* $p^{*}(\xi,\eta)$, $q^{*}(\xi,\eta)$ *be two positive functions. Suppose that* $F^{*}(s_1,t_1)$ *and* $G^{*}(k_1,r_1)$*are as defined in Theorem 4 and let*

$$P^{*}(s_1,t_1)=\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^{*}(\xi,\eta)\Delta\xi\Delta\eta,\qquad Q^{*}(k_1,r_1)=\int_{t_0}^{k_1}\int_{t_0}^{r_1}q^{*}(\xi,\eta)\Delta\xi\Delta\eta.\tag{20}$$

Then for $(s_1,t_1)\in I_x\times I_y$ *and* $(k_1,r_1)\in I_z\times I_w$, *we have*

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^{*}(s_1,t_1))\Psi(G^{*}(k_1,r_1))}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\quad\le D(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\left(p^{*}(s_1,t_1)\Phi\left[\frac{f^{*}(s_1,t_1)}{p^{*}(s_1,t_1)}\right]\right)^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\big(\sigma(z)-k_1\big)\big(\sigma(w)-r_1\big)\left(q^{*}(k_1,r_1)\Psi\left[\frac{g^{*}(k_1,r_1)}{q^{*}(k_1,r_1)}\right]\right)^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}},\end{aligned}\tag{21}$$

where

$$D(p_1,q_1) = \frac{1}{p_1 q_1}\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}}\Delta s_1\Delta t_1\right)^{\frac{p_1-1}{p_1}} \times \left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\right)^{\frac{q_1}{q_1-1}}\Delta k_1\Delta r_1\right)^{\frac{q_1-1}{q_1}}. \tag{22}$$

Proof By assumption and using the Jensen's inequality (9), it is clear that

$$\begin{aligned}\Phi\bigl(F^*(s_1,t_1)\bigr) &= \Phi\left(\frac{P^*(s_1,t_1)\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}]\Delta\xi\Delta\eta}{\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)\Delta\xi\Delta\eta}\right)\\ &\le \Phi\bigl(P^*(s_1,t_1)\bigr)\Phi\left(\frac{\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}]\Delta\xi\Delta\eta}{\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)\Delta\xi\Delta\eta}\right)\\ &\le \frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\Delta\xi\Delta\eta. \end{aligned}\tag{23}$$

Applying Hölder's inequality (8) with indices p_1 and $p_1/(p_1-1)$ on the right-hand side of (23), we have

$$\Phi\bigl(F^*(s_1,t_1)\bigr) \le \bigl[(s_1-t_0)(t_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)} \times \left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}. \tag{24}$$

Analogously,

$$\Psi\bigl(G^*(k_1,r_1)\bigr) \le \bigl[(k_1-t_0)(r_1-t_0)\bigr]^{\frac{q_1-1}{q_1}}\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)} \times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1}\Delta\xi\Delta\eta\right)^{\frac{1}{q_1}}. \tag{25}$$

Thus, from (24) and (25), it can be concluded that

$$\begin{aligned}&\Phi\bigl(F^*(s_1,t_1)\bigr)\Psi\bigl(G^*(k_1,r_1)\bigr)\\ &\le \bigl[(s_1-t_0)(t_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\bigl[(k_1-t_0)(r_1-t_0)\bigr]^{\frac{q_1-1}{q_1}}\\ &\quad\times\left(\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}\right)\\ &\quad\times\left(\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1}\Delta\xi\Delta\eta\right)^{\frac{1}{q_1}}\right).\end{aligned}\tag{26}$$

Applying Young's inequality on the term $[(s_1-t_0)(t_1-t_0)]^{(p_1-1)/p_1}[(k_1-t_0)(r_1-t_0)]^{(q_1-1)/q_1}$, we get

$$\begin{aligned}
&\Phi\bigl(F^*(s_1,t_1)\bigr)\Psi\bigl(G^*(k_1,r_1)\bigr)\\
&\quad\le\left(\frac{[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}}{p_1}+\frac{[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}{q_1}\right)\\
&\qquad\times\left(\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right)\\
&\qquad\times\left(\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}\right).
\end{aligned}\tag{27}$$

From (27), we observe that

$$\begin{aligned}
&\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}\\
&\quad\le\frac{1}{p_1q_1}\left(\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\right)\\
&\qquad\times\left(\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}\right).
\end{aligned}\tag{28}$$

Integrating both sides of (28) first with respect to r_1 and k_1 and then with respect to s_1 and t_1, respectively, we get

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\le\frac{1}{p_1q_1}\left(\int_{t_0}^{x}\int_{t_0}^{y}\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\right.\\
&\qquad\times\left.\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}\Delta s_1\Delta t_1\right)\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\right.\\
&\qquad\times\left.\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}\Delta k_1\Delta r_1\right).
\end{aligned}\tag{29}$$

Using Hölder's inequality (8) again with respect to p_1, $p_1/(p_1-1)$ and q_1, $q_1/(q_1-1)$, respectively, on (29), we may write

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\le\frac{1}{p_1q_1}\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}}\Delta s_1\Delta t_1\right)^{\frac{p_1-1}{p_1}}\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\right)^{\frac{q_1}{q_1-1}}\Delta k_1\Delta r_1\right)^{\frac{q_1-1}{q_1}}
\end{aligned}$$

$$\times \left(\int_{t_0}^{x} \int_{t_0}^{y} \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1} \Delta\xi\,\Delta\eta \right) \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_{t_0}^{z} \int_{t_0}^{w} \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1} \Delta\xi\,\Delta\eta \right) \Delta k_1 \Delta r_1 \right)^{\frac{1}{q_1}}$$

$$= D(p_1,q_1)\left(\int_{t_0}^{x} \int_{t_0}^{y} \left(\int_{t_0}^{s_1} \int_{t_0}^{t_1} \left(p^*(\xi,\eta)\Phi\left[\frac{f^*(\xi,\eta)}{p^*(\xi,\eta)}\right]\right)^{p_1} \Delta\xi\,\Delta\eta \right) \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_{t_0}^{z} \int_{t_0}^{w} \left(\int_{t_0}^{k_1} \int_{t_0}^{r_1} \left(q^*(\xi,\eta)\Psi\left[\frac{g^*(\xi,\eta)}{q^*(\xi,\eta)}\right]\right)^{q_1} \Delta\xi\,\Delta\eta \right) \Delta k_1 \Delta r_1 \right)^{\frac{1}{q_1}}.$$

Applying Fubini's theorem and using the fact that $\sigma(n) \geq n$, we get

$$\int_{t_0}^{x} \int_{t_0}^{y} \left(\int_{t_0}^{z} \int_{t_0}^{w} \frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)} + p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}} \Delta k_1 \Delta r_1 \right) \Delta s_1 \Delta t_1$$

$$\leq D(p_1,q_1)\left(\int_{t_0}^{x} \int_{t_0}^{y} (x-s_1)(y-t_1)\left(p^*(s_1,t_1)\Phi\left[\frac{f^*(s_1,t_1)}{p^*(s_1,t_1)}\right]\right)^{p_1} \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_{t_0}^{z} \int_{t_0}^{w} (z-k_1)(w-r_1)\left(q^*(k_1,r_1)\Psi\left[\frac{g^*(k_1,r_1)}{q^*(k_1,r_1)}\right]\right)^{q_1} \Delta k_1 \Delta r_1 \right)^{\frac{1}{q_1}}$$

$$\leq D(p_1,q_1)\left(\int_{t_0}^{x} \int_{t_0}^{y} \big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\left(p^*(s_1,t_1)\Phi\left[\frac{f^*(s_1,t_1)}{p^*(s_1,t_1)}\right]\right)^{p_1} \Delta s_1 \Delta t_1 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_{t_0}^{z} \int_{t_0}^{w} \big(\sigma(z)-k_1\big)\big(\sigma(w)-r_1\big)\left(q^*(k_1,r_1)\Psi\left[\frac{g^*(k_1,r_1)}{q^*(k_1,r_1)}\right]\right)^{q_1} \Delta k_1 \Delta r_1 \right)^{\frac{1}{q_1}},$$

which is (21). This completes the proof. □

By using the relations (5) and taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $t_0 = 0$ in Theorem 5, we get the following result.

Corollary 3 *Assume that $f^*(s_1,t_1)$, $g^*(k_1,r_1)$ are real-valued continuous functions, $p^*(s_1,t_1)$, $q^*(k_1,r_1)$ are two positive functions, and define*

$$F^*(s_1,t_1) = \int_0^{s_1}\int_0^{t_1} f^*(\xi,\eta)\,d\xi\,d\eta, \qquad G^*(k_1,r_1) = \int_0^{k_1}\int_0^{r_1} g^*(\xi,\eta)\,d\xi\,d\eta,$$

$$P^*(s_1,t_1) = \int_0^{s_1}\int_0^{t_1} p^*(\xi,\eta)\,d\xi\,d\eta, \qquad Q^*(k_1,r_1) = \int_0^{k_1}\int_0^{r_1} q^*(\xi,\eta)\,d\xi\,d\eta.$$

Then for $(s_1,t_1) \in I_x \times I_y$ and $(k_1,r_1) \in I_z \times I_w$, we have

$$\int_0^{x}\int_0^{y}\left(\int_0^{z}\int_0^{w} \frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1(s_1t_1)^{p_1-1} + p_1(k_1r_1)^{q_1-1}}\,dk_1\,dr_1 \right) ds_1\,dt_1$$

$$\leq D^*(p_1,q_1)\left(\int_0^{x}\int_0^{y} (x-s_1)(y-t_1)\left(p^*(s_1,t_1)\Phi\left[\frac{f^*(s_1,t_1)}{p^*(s_1,t_1)}\right]\right)^{p_1} ds_1\,dt_1 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_0^{z}\int_0^{w} (z-k_1)(w-r_1)\left(q^*(k_1,r_1)\Psi\left[\frac{g^*(k_1,r_1)}{q^*(k_1,r_1)}\right]\right)^{q_1} dk_1\,dr_1 \right)^{\frac{1}{q_1}},$$

where

$$D^*(p_1,q_1) = \frac{1}{p_1q_1}\left(\int_0^x\int_0^y\left(\frac{\Phi(P^*(s_1,t_1))}{P^*(s_1,t_1)}\right)^{\frac{p_1}{p_1-1}} ds_1\,dt_1\right)^{\frac{p_1-1}{p_1}} \times\left(\int_0^z\int_0^w\left(\frac{\Psi(Q^*(k_1,r_1))}{Q^*(k_1,r_1)}\right)^{\frac{q_1}{q_1-1}} dk_1\,dr_1\right)^{\frac{q_1-1}{q_1}}.$$

By using the relations (5) and taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $t_0 = 0$ in Theorem 5, we get the following result.

Corollary 4 *Assume that* $\{a_{m_1,n_1}\}_{0\le m_1,n_1\le N}$, $\{b_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ *are two nonnegative sequences of real numbers,* $\{p_{m_1,n_1}\}_{0\le m_1,n_1\le N}$, $\{q_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ *are positive sequences, and define*

$$A_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} a_{\xi,\eta}, \qquad B_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} b_{\xi,\eta},$$
$$P_{m_1,n_1} = \sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} p_{\xi,\eta}, \qquad Q_{k_1,r_1} = \sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} q_{\xi,\eta}.$$

Then

$$\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1}\frac{\Phi(A_{s_1,t_1})\Psi(B_{k_1,r_1})}{q_1(s_1t_1)^{p_1-1}+p_1(k_1r_1)^{q_1-1}}\right)$$
$$\le D^{**}(p_1,q_1)\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}(m_1-s_1+1)(n_1-t_1+1)\left(p_{s_1,t_1}\Phi\left[\frac{a_{s_1,t_1}}{p_{s_1,t_1}}\right]\right)^{p_1}\right)^{\frac{1}{p_1}}$$
$$\times\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1}(z_1-k_1+1)(w_1-r_1+1)\left(q_{k_1,r_1}\Psi\left[\frac{b_{k_1,r_1}}{q_{k_1,r_1}}\right]\right)^{q_1}\right)^{\frac{1}{q_1}},$$

where

$$D^{**}(p_1,q_1) = \frac{1}{p_1q_1}\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\frac{\Phi(P_{s_1,t_1})}{P_{s_1,t_1}}\right)^{\frac{p_1}{p_1-1}}\right)^{\frac{p_1-1}{p_1}}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1}\left(\frac{\Psi(Q_{k_1,r_1})}{Q_{k_1,r_1}}\right)^{\frac{q_1}{q_1-1}}\right)^{\frac{q_1-1}{q_1}}.$$

Remark 1 By applying (10) on (12) in Theorem 4 and (21) in Theorem 5, respectively, we get the following inequalities:

$$\int_{t_0}^x\int_{t_0}^y\left(\int_{t_0}^z\int_{t_0}^w\frac{F^*(s_1,t_1)G^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\le C(p_1,q_1)\left\{\frac{1}{p_1}\left(\int_{t_0}^x\int_{t_0}^y\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\big[f^*(s_1,t_1)\big]^{p_1}\Delta s_1\Delta t_1\right)\right.$$
$$\left.+\frac{1}{q_1}\left(\int_{t_0}^z\int_{t_0}^w\big(\sigma(z)-k_1\big)\big(\sigma(w)-r_1\big)\big[g^*(k_1,r_1)\big]^{q_1}\Delta k_1\Delta r_1\right)\right\}, \tag{30}$$

where $C(p_1,q_1)$ is defined in (13), and

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1[(s_1-t_0)(t_1-t_0)]^{(p_1-1)}+p_1[(k_1-t_0)(r_1-t_0)]^{(q_1-1)}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\leq D(p_1,q_1)\left\{\frac{1}{p_1}\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\left(p^*(s_1,t_1)\Phi\left[\frac{f^*(s_1,t_1)}{p^*(s_1,t_1)}\right]\right)^{p_1}\Delta s_1\Delta t_1\right)\right.$$
$$+\frac{1}{q_1}\left(\int_{t_0}^{z}\int_{t_0}^{w}\big(\sigma(z)-k_1\big)\big(\sigma(w)-r_1\big)\right.$$
$$\left.\left.\times\left(q^*(k_1,r_1)\Psi\left[\frac{g^*(k_1,r_1)}{q^*(k_1,r_1)}\right]\right)^{q_1}\Delta k_1\Delta r_1\right)\right\}, \tag{31}$$

where $D(p_1,q_1)$ is defined in (22).

The following theorems present slight variants of (21) in Theorem 5.

Theorem 6 *Let (H1), (H2), and (H4) be satisfied and* $f^*(s_1,t_1)\in \mathrm{CC}^1_{rd}(I_x\times I_y,\mathbb{R}^+)$, $g^*(k_1,r_1)\in \mathrm{CC}^1_{rd}(I_z\times I_w,\mathbb{R}^+)$. *Define*

$$F^*(s_1,t_1)=\frac{1}{(s_1-t_0)(t_1-t_0)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}f^*(\xi,\eta)\Delta\xi\Delta\eta,$$
$$G^*(k_1,r_1)=\frac{1}{(k_1-t_0)(r_1-t_0)}\int_{t_0}^{k_1}\int_{t_0}^{r_1}g^*(\xi,\eta)\Delta\xi\Delta\eta. \tag{32}$$

Then for $(s_1,t_1)\in I_x\times I_y$ *and* $(k_1,r_1)\in I_z\times I_w$, *one gets*

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{q_1[(s-t_0)(t-t_0)]^{p_1-1}+p_1[(k-t_0)(r-t_0)]^{q_1-1}}\right.$$
$$\left.\times\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\leq K(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\big(\sigma(x)-s_1\big)\big(\sigma(y)-t_1\big)\big[\Phi\big(f^*(s_1,t_1)\big)\big]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\big(\sigma(w)-k_1\big)\big(\sigma(z)-r_1\big)\big[\Psi\big(g^*(k_1,r_1)\big)\big]^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}, \tag{33}$$

where

$$K(p_1,q_1)=\frac{1}{p_1q_1}\big[(s_1-t_0)(t_1-t_0)\big]^{\frac{p_1-1}{p_1}}\big[(k_1-t_0)(r_1-t_0)\big]^{\frac{q_1-1}{q_1}}. \tag{34}$$

Proof By assumption and using the Jensen's inequality (9), we obtain

$$\Phi\big(F^*(s_1,t_1)\big)=\Phi\left(\frac{1}{(s_1-t_0)(t_1-t_0)}\int_{t}^{s_1}\int_{t_0}^{t_1}f^*(\xi,\eta)\Delta\xi\Delta\eta\right)$$
$$\leq\frac{1}{(s_1-t_0)(t_1-t_0)}\int_{t}^{s_1}\int_{t_0}^{t_1}\Phi\big(f^*(\xi,\eta)\big)\Delta\xi\Delta\eta. \tag{35}$$

Similarly,

$$\begin{aligned}\Psi\big(G^{*}(k_1,r_1)\big) &= \Psi\bigg(\frac{1}{(k_1-t_0)(r_1-t_0)}\int_{t_0}^{k_1}\int_{t_0}^{r_1} g^{*}(\xi,\eta)\Delta\xi\,\Delta\eta\bigg)\\ &\le \frac{1}{(k_1-t_0)(r_1-t_0)}\int_{t_0}^{k_1}\int_{t_0}^{r_1}\Psi\big(g^{*}(\xi,\eta)\big)\Delta\xi\,\Delta\eta. \qquad (36)\end{aligned}$$

By multiplying (35) and (36), we get

$$\begin{aligned}&\Phi\big(F^{*}(s_1,t_1)\big)\Psi\big(G^{*}(k_1,r_1)\big)\\ &\quad\le \frac{1}{(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}\\ &\qquad\times\bigg(\int_{t}^{s_1}\int_{t_0}^{t_1}\Phi\big(f^{*}(\xi,\eta)\big)\Delta\xi\,\Delta\eta\bigg)\\ &\qquad\times\bigg(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\Psi\big(g^{*}(\xi,\eta)\big)\Delta\xi\,\Delta\eta\bigg).\end{aligned}$$

This implies that

$$\begin{aligned}&\Phi\big(F^{*}(s_1,t_1)\big)\Psi\big(G^{*}(k_1,r_1)\big)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le\bigg(\int_{t}^{s_1}\int_{t_0}^{t_1}\Phi\big(f^{*}(\xi,\eta)\big)\Delta\xi\,\Delta\eta\bigg)\bigg(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\Psi\big(g^{*}(\xi,\eta)\big)\Delta\xi\,\Delta\eta\bigg). \qquad (37)\end{aligned}$$

By Hölder's inequality (8), we find

$$\begin{aligned}&\Phi\big(F^{*}(s_1,t_1)\big)\Psi\big(G^{*}(k_1,r_1)\big)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le\big[(s_1-t_0)(t_1-t_0)\big]^{\frac{p_1-1}{p_1}}\big[(k_1-t_0)(r_1-t_0)\big]^{\frac{q_1-1}{q_1}}\\ &\qquad\times\bigg(\int_{t}^{s_1}\int_{t_0}^{t_1}\big[\Phi\big(f^{*}(\xi,\eta)\big)\big]^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}\bigg(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big[\Psi\big(g^{*}(\xi,\eta)\big)\big]^{q_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{q_1}}.\end{aligned}$$

Applying Young's inequality on the term $[(s_1-t_0)(t_1-t_0)]^{(p_1-1)/p_1}[(k_1-t_0)(r_1-t_0)]^{(q_1-1)/q_1}$, we get

$$\begin{aligned}&\Phi\big(F^{*}(s_1,t_1)\big)\Psi\big(G^{*}(k_1,r_1)\big)(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)\\ &\quad\le\bigg(\frac{[(s_1-t_0)(t_1-t_0)]^{p_1-1}}{p_1}+\frac{[(k_1-t_0)(r_1-t_0)]^{q_1-1}}{q_1}\bigg)\\ &\qquad\times\bigg(\int_{t}^{s_1}\int_{t_0}^{t_1}\big[\Phi\big(f^{*}(\xi,\eta)\big)\big]^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}\bigg(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\big[\Psi\big(g^{*}(\xi,\eta)\big)\big]^{q_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{q_1}}.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{\Phi(F^{*}(s_1,t_1))\Psi(G^{*}(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\\ &\quad\le\frac{1}{p_1q_1}\bigg(\int_{t}^{s_1}\int_{t_0}^{t_1}\big[\Phi\big(f^{*}(\xi,\eta)\big)\big]^{p_1}\Delta\xi\,\Delta\eta\bigg)^{\frac{1}{p_1}}\end{aligned}$$

$$\times \left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\Delta\eta\right)^{\frac{1}{q_1}}. \tag{38}$$

Integrating both sides of (28) first with respect to r_1 and k_1 and then with respect to s_1 and t_1, respectively, and applying Hölder's inequality (8) with indices p_1, $p_1/(p_1-1)$ and q_1, $q_1/(q_1-1)$, we get

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\right.\\
&\qquad\left.\times\,\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\le\frac{1}{p_1q_1}\left[(x-t_0)(y-t_0)\right]^{\frac{p_1-1}{p_1}}\left[(z-t_0)(w-t_0)\right]^{\frac{q_1-1}{q_1}}\\
&\qquad\times\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t}^{s_1}\int_{t_0}^{t_1}\left[\Phi\left(f^*(\xi,\eta)\right)\right]^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}\\
&\quad=K(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t}^{s_1}\int_{t_0}^{t_1}\left[\Phi\left(f^*(\xi,\eta)\right)\right]^{p_1}\Delta\xi\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}.
\end{aligned}\tag{39}$$

Applying Fubini's theorem and using the fact that $\sigma(n)\ge n$, we get

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\right.\\
&\qquad\left.\times\,\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\le K(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\sigma(x)-s_1\right)\left(\sigma(y)-t_1\right)\left[\Phi\left(f^*(s_1,t_1)\right)\right]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\sigma(z)-k_1\right)\left(\sigma(w)-r_1\right)\left[\Psi\left(g^*(k_1,r_1)\right)\right]^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}},
\end{aligned}$$

which is (33). This completes the proof. □

By using the relations (5) and taking $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$, $t_0=0$ in Theorem 6, we get the following result.

Corollary 5 *Assume that $f^*(s_1,t_1)$, $g^*(k_1,r_1)$ are real-valued continuous functions and define*

$$F^*(s_1,t_1)=\frac{1}{s_1t_1}\int_0^{s_1}\int_0^{t_1}f^*(\xi,\eta)\,d\xi\,d\eta,\qquad G^*(k_1,r_1)=\frac{1}{k_1r_1}\int_0^{k_1}\int_0^{r_1}g^*(\xi,\eta)\,d\xi\,d\eta.$$

Then for $(s_1,t_1) \in I_x \times I_y$ and $(k_1,r_1) \in I_z \times I_w$, we have

$$\begin{aligned}
&\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{(s_1t_1)(k_1r_1)\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))}{q_1(s_1t_1)^{p_1-1} + p_1(kr)^{q-1}}\, dk_1\, dr_1 \right) ds_1\, dt_1 \\
&\quad \le K^*(p_1,q_1) \left(\int_0^x \int_0^y (x-s_1)(y-t_1)\bigl[\Phi\bigl(f^*(s_1,t_1)\bigr)\bigr]^{p_1}\, ds_1\, dt_1 \right)^{\frac{1}{p_1}} \\
&\qquad \times \left(\int_0^z \int_0^w (z-k_1)(w-r_1)\bigl[\Psi\bigl(g^*(k_1,r_1)\bigr)\bigr]^{q_1}\, dk_1\, dr_1 \right)^{\frac{1}{q_1}},
\end{aligned}$$

where

$$K^*(p_1,q_1) = \frac{1}{p_1q_1}(xy)^{\frac{p_1-1}{p_1}}(zw)^{\frac{q_1-1}{q_1}}.$$

By using the relations (5) and taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $t_0 = 0$ in Theorem 6, we get the following result.

Corollary 6 *Assume that $\{a_{m_1,n_1}\}_{0\le m_1,n_1\le N}$, $\{b_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ are two nonnegative sequences of real numbers and define*

$$A_{m_1,n_1} = \frac{1}{m_1n_1}\sum_{\xi=1}^{m_1}\sum_{\eta=1}^{n_1} a_{\xi,\eta}, \qquad B_{k_1,r_1} = \frac{1}{k_1r_1}\sum_{\xi=1}^{k_1}\sum_{\eta=1}^{r_1} b_{\xi,\eta}.$$

Then

$$\begin{aligned}
&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1} \frac{(s_1t_1)(k_1r_1)\Phi(A_{s_1,t_1})\Psi(B_{k_1,r_1})}{q_1(s_1t_1)^{p_1-1} + p_1(k_1r_1)^{q_1-1}} \right) \\
&\quad \le K^{**}(p_1,q_1) \left\{ \sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1} (m_1-s_1+1)(n_1-t_1+1)\bigl(\Phi(a_{s_1,t_1})\bigr)^{p_1} \right\}^{\frac{1}{p_1}} \\
&\qquad \times \left\{ \sum_{k_1=1}^{z_1}\sum_{r_1=1}^{w_1} (z_1-k_1+1)(w_1-r_1+1)\bigl(\Psi(b_{k_1,r_1})\bigr)^{q_1} \right\}^{\frac{1}{q_1}},
\end{aligned}$$

where

$$K^{**}(p_1,q_1) = \frac{1}{p_1q_1}(m_1n_1)^{\frac{p_1-1}{p_1}}(z_1w_1)^{\frac{q_1-1}{q_1}}.$$

Theorem 7 *Let $(H1)$, $(H2)$, and $(H4)$ be satisfied, $f^*(s_1,t_1) \in \mathrm{CC}^1_{rd}(I_x \times I_y, \mathbb{R}^+)$, $g^*(k_1,r_1) \in \mathrm{CC}^1_{rd}(I_z \times I_w, \mathbb{R}^+)$, and $p^*(\xi,\eta)$, $q^*(\xi,\eta)$ be two positive functions. Suppose that P^* and Q^* are as defined in Theorem 5 and let*

$$\begin{aligned}
&F^*(s_1,t_1) = \frac{1}{P^*(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1} p^*(\xi,\eta)f^*(\xi,\eta)\Delta\xi\,\Delta\eta, \\
&G^*(k_1,r_1) = \frac{1}{Q^*(k_1,r_1)}\int_{t_0}^{k_1}\int_{t_0}^{r_1} q^*(\xi,\eta)g^*(\xi,\eta)\Delta\xi\,\Delta\eta.
\end{aligned} \tag{40}$$

Then for $(s_1,t_1)\in I_x\times I_y$ *and* $(k_1,r_1)\in I_z\times I_w$, *we have*

$$\begin{aligned}
&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))P^*(s_1,t_1)Q^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\
&\quad\le H(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\bigl[P^*(s_1,t_1)\Phi\bigl(f^*(s_1,t_1)\bigr)\bigr]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}\\
&\qquad\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(w)-r_1\bigr)\bigl[Q^*(k_1,r_1)\Psi\bigl(g^*(k_1,r_1)\bigr)\bigr]^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}},
\end{aligned}\tag{41}$$

where

$$H(p_1,q_1)=\frac{1}{p_1q_1}\bigl[(s_1-t_0)(t_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\bigl[(k_1-t_0)(r_1-t_0)\bigr]^{\frac{q_1-1}{q_1}}.\tag{42}$$

Proof By assumption and using the Jensen's inequality (9), it follows that

$$\begin{aligned}
\Phi\bigl(F^*(s_1,t_1)\bigr)&=\Phi\left(\frac{1}{P^*(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)f^*(\xi,\eta)\Delta\xi\Delta\eta\right)\\
&\le\frac{1}{P^*(s_1,t_1)}\int_{t_0}^{s_1}\int_{t_0}^{t_1}p^*(\xi,\eta)\Phi\bigl(f^*(\xi,\eta)\bigr)\Delta\xi\Delta\eta
\end{aligned}\tag{43}$$

and

$$\begin{aligned}
\Psi\bigl(G^*(k_1,r_1)\bigr)&=\Psi\left(\frac{1}{Q^*(k_1,r_1)}\int_{t_0}^{k_1}\int_{t_0}^{r_1}q^*(\xi,\eta)g^*(\xi,\eta)\Delta\xi\Delta\eta\right)\\
&\le\frac{1}{Q^*(k_1,r_1)}\int_{t_0}^{k_1}\int_{t_0}^{r_1}q^*(\xi,\eta)\Psi\bigl(g^*(\xi,\eta)\bigr)\Delta\xi\Delta\eta.
\end{aligned}\tag{44}$$

From (43) and (44) and using Hölder's inequality (8) with $p_1, p_1/(p_1-1)$ and $q_1, q_1/(q_1-1)$, respectively, we get

$$\begin{aligned}
\Phi\bigl(F^*(s_1,t_1)\bigr)&\le\frac{1}{P^*(s_1,t_1)}\bigl[(s_1-t_0)(t_1-t_0)\bigr]^{\frac{p_1-1}{p_1}}\\
&\quad\times\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\bigl[p^*(\xi,\eta)\Phi\bigl(f^*(\xi,\eta)\bigr)\bigr]^{p_1}\Delta\xi\Delta\eta\right)^{\frac{1}{p_1}}
\end{aligned}\tag{45}$$

and

$$\begin{aligned}
\Psi\bigl(G^*(k_1,r_1)\bigr)&\le\frac{1}{Q^*(k_1,r_1)}\bigl[(k_1-t_0)(r_1-t_0)\bigr]^{\frac{q_1-1}{q_1}}\\
&\quad\times\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\bigl[q^*(\xi,\eta)\Psi\bigl(g^*(\xi,\eta)\bigr)\bigr]^{q_1}\Delta\xi\Delta\eta\right)^{\frac{1}{q_1}}.
\end{aligned}\tag{46}$$

From (45) and (46) and using the elementary inequality (10), we get

$$\begin{aligned}
&\Phi\bigl(F^*(s_1,t_1)\bigr)\Psi\bigl(G^*(k_1,r_1)\bigr)P^*(s_1,t_1)Q^*(k_1,r_1)\\
&\quad\le\left(\frac{[(s_1-t_0)(t_1-t_0)]^{p_1-1}}{p_1}+\frac{[(k_1-t_0)(r_1-t_0)]^{q_1-1}}{q_1}\right)
\end{aligned}$$

$$\times\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left[p^*(\xi,\eta)\Phi\left(f^*(\xi,\eta)\right)\right]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[q^*(\xi,\eta)\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}. \qquad (47)$$

This implies that

$$\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))P^*(s_1,t_1)Q^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}$$
$$\le\frac{1}{p_1q_1}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left[p^*(\xi,\eta)\Phi\left(f^*(\xi,\eta)\right)\right]^{p_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[q^*(\xi,\eta)\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\,\Delta\eta\right)^{\frac{1}{q_1}}. \qquad (48)$$

Integrating both sides of (48) with respect to r_1 and k_1 and then with respect to s_1 and t_1, respectively, and applying Hölder's inequality (8) with indices p_1, $p_1/(p_1-1)$ and q_1, $q_1/(q_1-1)$, we see that

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))P^*(s_1,t_1)Q^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\le\frac{1}{p_1q_1}\left[(x-t_0)(y-t_0)\right]^{\frac{p_1-1}{p_1}}\left[(z-t_0)(w-t_0)\right]^{\frac{q_1-1}{q_1}}$$
$$\times\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left[p^*(\xi,\eta)\Phi\left(f^*(\xi,\eta)\right)\right]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[q^*(\xi,\eta)\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\,\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}$$
$$=H(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{s_1}\int_{t_0}^{t_1}\left[p^*(\xi,\eta)\Phi\left(f^*(\xi,\eta)\right)\right]^{p_1}\Delta\xi\,\Delta\eta\right)\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\int_{t_0}^{k_1}\int_{t_0}^{r_1}\left[q^*(\xi,\eta)\Psi\left(g^*(\xi,\eta)\right)\right]^{q_1}\Delta\xi\,\Delta\eta\right)\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}}. \qquad (49)$$

Applying Fubini's theorem and using the fact that $\sigma(n)\ge n$, we get

$$\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))P^*(s_1,t_1)Q^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1$$
$$\le H(p_1,q_1)\left(\int_{t_0}^{x}\int_{t_0}^{y}\left(\sigma(x)-s_1\right)\left(\sigma(y)-t_1\right)\left[p^*(s_1,t_1)\Phi\left(f^*(s_1,t_1)\right)\right]^{p_1}\Delta s_1\Delta t_1\right)^{\frac{1}{p_1}}$$
$$\times\left(\int_{t_0}^{z}\int_{t_0}^{w}\left(\sigma(z)-k_1\right)\left(\sigma(w)-r_1\right)\left[q^*(k_1,r_1)\Psi\left(g^*(k_1,r_1)\right)\right]^{q_1}\Delta k_1\Delta r_1\right)^{\frac{1}{q_1}},$$

which is (41). This completes the proof. □

By using the relations (5) and taking $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$, $t_0=0$ in Theorem 7, we get the following result.

Corollary 7 *Assume that $f^*(s_1,t_1)$, $g^*(k_1,r_1)$ are real-valued continuous functions, $p^*(s_1,t_1)$, $q^*(k_1,r_1)$ are two positive functions, and define*

$$F^*(s_1,t_1) = \frac{1}{P^*(s_1,t_1)} \int_0^{s_1} \int_0^{t_1} p^*(\xi,\eta) f^*(\xi,\eta)\, d\xi\, d\eta,$$

$$P^*(s_1,t_1) = \int_0^{s_1} \int_0^{t_1} p^*(\xi,\eta)\, d\xi\, d\eta,$$

$$G^*(k_1,r_1) = \frac{1}{Q^*(k_1,r_1)} \int_0^{k_1} \int_0^{r_1} q^*(\xi,\eta) g^*(\xi,\eta)\, d\xi\, d\eta,$$

$$Q^*(k_1,r_1) = \int_0^{k_1} \int_0^{r_1} q^*(\xi,\eta)\, d\xi\, d\eta.$$

Then for $(s_1,t_1) \in I_x \times I_y$ and $(k_1,r_1) \in I_z \times I_w$, we get

$$\begin{aligned} &\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))P^*(s_1,t_1)Q^*(k_1,r_1)}{q_1(s_1t_1)^{p_1-1} + p_1(k_1r_1)^{q_1-1}}\, dk_1\, dr_1 \right) ds_1\, dt_1 \\ &\quad \le H^*(p_1,q_1) \left(\int_0^x \int_0^y (x-s_1)(y-t_1) \left[p^*(s_1,t_1)\Phi\big(f^*(s_1,t_1)\big) \right]^{p_1} ds_1\, dt_1 \right)^{\frac{1}{p_1}} \\ &\quad\quad \times \left(\int_0^z \int_0^w (z-k_1)(w-r_1) \left[q^*(k_1,r_1)\Psi\big(g^*(k_1,r_1)\big) \right]^{q_1} dk_1\, dr_1 \right)^{\frac{1}{q_1}}, \end{aligned}$$

where

$$H^*(p_1,q_1) = \frac{1}{p_1q_1}(xy)^{\frac{p_1-1}{p_1}}(zw)^{\frac{q_1-1}{q_1}}.$$

By using the relations (5) and taking $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $t_0 = 0$ in Theorem 7, we get the following result.

Corollary 8 *Assume that $\{a_{m_1,n_1}\}_{0\le m_1,n_1\le N}$, $\{b_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ be two nonnegative sequences of real numbers and $\{p_{m_1,n_1}\}_{0\le m_1,n_1\le N}$, $\{q_{k_1,r_1}\}_{0\le k_1,r_1\le N}$ be positive sequences and define*

$$A_{m_1,n_1} = \frac{1}{P_{m_1,n_1}} \sum_{\xi=1}^{m_1} \sum_{\eta=1}^{n_1} a_{\xi,\eta}, \qquad P_{m_1,n_1} = \sum_{\xi=1}^{m_1} \sum_{\eta=1}^{n_1} p_{\xi,\eta},$$

$$B_{k_1,r_1} = \frac{1}{Q_{k_1,r_1}} \sum_{\xi=1}^{k_1} \sum_{\eta=1}^{r_1} b_{\xi,\eta}, \qquad Q_{k_1,r_1} = \sum_{\xi=1}^{k_1} \sum_{\eta=1}^{r_1} q_{\xi,\eta}.$$

Then

$$\begin{aligned} &\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} \left(\sum_{k_1=1}^{z_1} \sum_{r_1=1}^{w_1} \frac{\Phi(A_{s_1,t_1})\Psi(B_{k_1,r_1})P_{s_1,t_1}Q_{k_1,r_1}}{q_1(s_1t_1)^{p_1-1} + p_1(k_1r_1)^{q_1-1}} \right) \\ &\quad \le H^{**}(p_1,q_1) \left(\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} (m_1-s_1+1)(n_1-t_1+1) \left[p_{s_1,t_1}\Phi(a_{s_1,t_1}) \right]^{p_1} \right)^{\frac{1}{p_1}} \\ &\quad\quad \times \left(\sum_{k_1=1}^{z_1} \sum_{r_1=1}^{w_1} (z_1-k_1+1)(w_1-r_1+1) \left[q_{k_1,r_1}\Psi(b_{k_1,r_1}) \right]^{q_1} \right)^{\frac{1}{q_1}}, \end{aligned}$$

where

$$H^{**}(p_1,q_1)=\frac{1}{p_1q_1}(m_1n_1)^{\frac{p_1-1}{p_1}}(z_1w_1)^{\frac{q_1-1}{q_1}}.$$

Remark 2 By applying (10) on (33) in Theorem 6 and (41) in Theorem 7, respectively, we get the following inequalities:

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))(s_1-t_0)(t_1-t_0)(k_1-t_0)(r_1-t_0)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\right.\\&\qquad\left.\times\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\qquad\le K(p_1,q_1)\left\{\frac{1}{p_1}\left(\int_{t_0}^{x}\int_{t_0}^{y}\bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\bigl[\Phi\bigl(f^*(s_1,t_1)\bigr)\bigr]^{p_1}\Delta s_1\Delta t_1\right)\right.\\&\qquad\left.+\frac{1}{q_1}\left(\int_{t_0}^{z}\int_{t_0}^{w}\bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(w)-r_1\bigr)\bigl[\Psi\bigl(g^*(k_1,r_1)\bigr)\bigr]^{q_1}\Delta k_1\Delta r_1\right)\right\},\end{aligned}$$

where $K(p_1,q_1)$ is defined in (34), and

$$\begin{aligned}&\int_{t_0}^{x}\int_{t_0}^{y}\left(\int_{t_0}^{z}\int_{t_0}^{w}\frac{\Phi(F^*(s_1,t_1))\Psi(G^*(k_1,r_1))P^*(s_1,t_1)Q^*(k_1,r_1)}{q_1[(s_1-t_0)(t_1-t_0)]^{p_1-1}+p_1[(k_1-t_0)(r_1-t_0)]^{q_1-1}}\Delta k_1\Delta r_1\right)\Delta s_1\Delta t_1\\&\qquad\le H(p_1,q_1)\left\{\frac{1}{p_1}\left(\int_{t_0}^{x}\int_{t_0}^{y}\bigl(\sigma(x)-s_1\bigr)\bigl(\sigma(y)-t_1\bigr)\bigl[p^*(s_1,t_1)\Phi\bigl(f^*(s_1,t_1)\bigr)\bigr]^{p_1}\Delta s_1\Delta t_1\right)\right.\\&\qquad\left.+\frac{1}{q_1}\left(\int_{t_0}^{z}\int_{t_0}^{w}\bigl(\sigma(z)-k_1\bigr)\bigl(\sigma(w)-r_1\bigr)\bigl[q^*(k_1,r_1)\Psi\bigl(g^*(k_1,r_1)\bigr)\bigr]^{q_1}\Delta k_1\Delta r_1\right)\right\},\end{aligned}$$

where $H(p_1,q_1)$ is defined in (42).

Remark 3 Clearly, for the one-dimensional case, Theorems 4, 5, 6, and 7, coincide with Corollary 3.3, Theorems 3.2, 3.3, and 3.4, respectively, of [7].

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