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# On a multiple Hilbert-type integral inequality involving the upper limit functions

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**Abstract**

By applying the weight functions, the idea of introducing parameters and the technique of real analysis, a new multiple Hilbert-type integral inequality involving the upper limit functions is given. The constant factor related to the gamma function is proved to be the best possible in a condition. A corollary about the case of the nonhomogeneous kernel and some particular inequalities are obtained.

**MSC:** 26D15

**Keywords:** Weight function; Hilbert-type integral inequality; Upper limit function; Parameter; Gamma function

## 1 Introduction

Assuming that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, we have the following Hilbert's inequality with the best possible constant factor $\pi$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \tag{1}$$

If $0 < \int_0^{\infty} f^2(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^2(y)\,dy < \infty$, then we still have the following integral analogue of (1), named Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^{\infty} f^2(x)\,dx \int_0^{\infty} g^2(y)\,dy\right)^{1/2}, \tag{2}$$

where the constant factor $\pi$ is the best possible. Inequalities (1) and (2) play an important role in analysis and its applications (cf. [2–13]).

The following half-discrete Hilbert-type inequality was provided: If $K(x)$ $(x > 0)$ is a decreasing function, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^{\infty} K(x)x^{s-1}\,dx < \infty$, $f(x) \geq 0, 0 < \int_0^{\infty} f^p(x)\,dx < \infty$, then (cf. [1], Theorem 351)

$$\sum_{n=1}^{\infty} n^{p-2}\left(\int_0^{\infty} K(nx)f(x)\,dx\right)^p < \phi^p\left(\frac{1}{q}\right)\int_0^{\infty} f^p(x)\,dx. \tag{3}$$

In recent years, some new extensions of (3) were provided by [14–19].

In 2006, by using Euler–Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel $\frac{1}{(m+n)^{\lambda}}$ $(0 < \lambda \leq 4)$. In 2019, following the result of [20], Adiyasuren et al. [21] considered an extension of (1) involving the partial sums. In 2016–2017, by applying the weight functions, Hong [22, 23] obtained some equivalent statements of the extensions of (1) and (2) with a few parameters. A few similar works were provided by [24–38].

In this paper, following the idea of [21], by using the weight functions, the way of introducing parameters and the technique of real analysis, a new multiple Hilbert-type integral inequality with the kernel $\frac{1}{(x_1+\cdots+x_n)^{\lambda}}$ $(\lambda > 0)$ involving the upper limit functions is given. The constant factor related to the gamma function is proved to be the best possible in a condition. A corollary about the case of the nonhomogeneous kernel and some particular inequalities are obtained.

## 2 Some lemmas

In what follows, we assume that $n \in \mathrm{N}\backslash\{1\} := \{2,3,\ldots\}, p_i, r_i > 1$ $(i = 1,\ldots,n), \sum_{i=1}^{n} \frac{1}{p_i} = 1, \lambda > 0$, $c_\lambda := (1 - \sum_{j=1}^{n} \frac{1}{r_j})\lambda, f_i(x)$ $(i = 1,\ldots,n)$ are nonnegative measurable functions in $R_+ = (0,\infty)$ such that $f_i(x) = o(e^{tx})$ $(t > 0; x \to \infty)$, and for any $A = (0,a)$ $(a > 0), f_i \in L^1(A)$, the upper limit functions are defined by $F_i(x) := \int_0^x f_i(t)\,dt$ $(x \geq 0)$, satisfying

$$0 < \int_0^\infty x_i^{-p_i\frac{\lambda}{r_i}-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i < \infty \quad (i = 1,\ldots,n).$$

By the definition of the gamma function, for $x_i > 0$ $(i = 1,\ldots,n)$, the following expression holds:

$$\frac{1}{(\sum_{i=1}^{n} x_i)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t\sum_{i=1}^{n} x_i}\,dt. \tag{4}$$

**Lemma 1** *For $t > 0$, we have the following expressions*:

$$\int_0^\infty e^{-tx} f_i(x)\,dx = t\int_0^\infty e^{-tx} F_i(x)\,dx \quad (i =, 1,\ldots,n), \tag{5}$$

*Proof* In view of $F_i(0) = 0$,we find

$$\begin{aligned}\int_0^\infty e^{-tx} f_i(x)\,dx &= \int_0^\infty e^{-tx}\,dF_i(x)\\ &= e^{-tx}F_i(x)|_0^\infty - \int_0^\infty F_i(x)\,de^{-tx}\\ &= \lim_{x\to\infty}\frac{F_i(x)}{e^{tx}} + t\int_0^\infty e^{-tx}F_i(x)\,dx.\end{aligned}$$

If $F_i(\infty)$ = constant, then $\lim_{x\to\infty}\frac{F_i(x)}{e^{tx}} = 0$ and (5) follows; if $F_i(\infty) = \infty$, since $f_i(x) = o(e^{tx})$ $(x \to \infty)$, we find

$$\int_0^\infty e^{-tx} f_i(x)\,dx = \lim_{x\to\infty}\frac{F_i'(x)}{(e^{tx})_x'} + t\int_0^\infty e^{-tx}F_i(x)\,dx$$

$$= \lim_{x\to\infty} \frac{f_i(x)}{te^{tx}} + t\int_0^\infty e^{-tx}F_i(x)\,dx = 0 + t\int_0^\infty e^{-tx}F_i(x)\,dx,$$

and then (5) follows, too.

The lemma is proved. □

**Lemma 2** *For $x_i > 0$ $(i = 1,\ldots,n)$, the following expression holds*:

$$A := \prod_{i=1}^n \left[ x_i^{(\frac{\lambda}{r_i}-1)(1-p_i)} \prod_{j=1(j\neq i)}^n x_j^{\frac{\lambda}{r_j}-1} \right]^{\frac{1}{p_i}} = 1. \tag{6}$$

*Proof* We have

$$\begin{aligned} A &= \prod_{i=1}^n \left[ x_i^{(\frac{\lambda}{r_i}-1)(1-p_i)+1-\frac{\lambda}{r_i}} \prod_{j=1}^n x_j^{\frac{\lambda}{r_j}-1} \right]^{\frac{1}{p_i}} = \prod_{i=1}^n [x_i^{(\frac{\lambda}{r_i}-1)(-p_i)}]^{\frac{1}{p_i}} \left( \prod_{j=1}^n x_j^{\frac{\lambda}{r_j}-1} \right)^{\frac{1}{p_i}} \\ &= \prod_{i=1}^n x_i^{1-\frac{\lambda}{r_i}} \left( \prod_{j=1}^n x_j^{\frac{\lambda}{r_j}-1} \right)^{\sum_{i=1}^n \frac{1}{p_i}} = \prod_{i=1}^n x_i^{1-\frac{\lambda}{r_i}} \left( \prod_{j=1}^n x_j^{\frac{\lambda}{r_j}-1} \right) = 1, \end{aligned}$$

and then (6) follows.

The lemma is proved. □

**Lemma 3** *For $n \in \mathrm{N}\backslash\{1\}$, defining the following weight functions*:

$$\omega_\lambda^{(i)}(x_i) := x_i^{\frac{\lambda}{r_i}+c_\lambda} \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{j=1(j\neq i)}^n x_j^{\frac{\lambda}{r_j}-1}\,dx_1\cdots dx_{i-1}\,dx_{i+1}\cdots dx_n, \tag{7}$$

*we have*

$$\omega_\lambda^{(i)}(x_i) = k_\lambda^{(i)} := \frac{\Gamma(\lambda(1-\frac{1}{r_i}))}{\Gamma(\sum_{j=1(j\neq i)}^n \frac{\lambda}{r_j})} \cdot \frac{\prod_{j=1}^n \Gamma(\frac{\lambda}{r_j})}{\Gamma(\lambda)} \in \mathrm{R}_+ \quad (i = 1,\ldots,n). \tag{8}$$

*In particular, for $\sum_{i=1}^n \frac{1}{r_i} = 1$, we have*

$$k_\lambda^{(i)} = k_\lambda := \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{r_j}\right) \quad (i = 1,\ldots,n). \tag{9}$$

*Proof* For $j \neq i$, setting $u_j = \frac{x_j}{x_i}$ in (7), we have

$$\begin{aligned} \omega_\lambda^{(i)}(x_i) = {} & \int_0^\infty \cdots \int_0^\infty \frac{1}{(u_1 + \cdots u_{i-1} + 1 + u_{i+1} + \cdots + u_n)^\lambda} \\ & \times \prod_{j=1(j\neq i)}^n u_j^{\frac{\lambda}{r_j}-1}\,du_1\cdots du_{i-1}\,du_{i+1}\cdots du_n. \end{aligned}$$

Then by Lemma 9.15 and (9.1.19) (cf. [2], p. 341–342), we obtain (8).

The lemma is proved. □

**Lemma 4** *We have the following inequality*:

$$H_\lambda := \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n F_i(x_i)\,dx_1 \cdots dx_n$$
$$< \prod_{i=1}^n \left[ k_\lambda^{(i)} \int_0^\infty x_i^{p_i(1-\frac{\lambda}{r_i})-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i \right]^{\frac{1}{p_i}}. \tag{10}$$

*Proof* By (6) and Hölder's integral inequality (cf. [39]), we obtain

$$H_\lambda = \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n \left[ x_i^{(\frac{\lambda}{r_i}-1)(1-p_i)} \prod_{j=1 (j\neq i)}^n x_j^{\frac{\lambda}{r_j}-1} \right]^{\frac{1}{p_i}} F_i(x_i)\,dx_1 \cdots dx_n$$
$$\leq \prod_{i=1}^n \left\{ \int_0^\infty \left[ \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} x_i^{\frac{\lambda}{r_i}+c_\lambda} \prod_{j=1 (j\neq i)}^n x_j^{\frac{\lambda}{r_j}-1}\,dx_1 \cdots dx_{i-1}\,dx_{i+1} \cdots dx_n \right] \right.$$
$$\left. \times x_i^{p_i(1-\frac{\lambda}{r_i})-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i \right\}^{\frac{1}{p_i}}$$
$$= \prod_{i=1}^n \left[ \int_0^\infty \omega_\lambda^{(i)}(x_i) x_i^{p_i(1-\frac{\lambda}{r_i})-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i \right]^{\frac{1}{p_i}}. \tag{11}$$

If (11) takes the form of an equality, then there exist constants $C_i, C_k$ $(i \neq k)$ such that they are not all zero and

$$C_i x_i^{\frac{\lambda}{r_i}+c_\lambda} \prod_{j=1 (j\neq i)}^n x_j^{\frac{\lambda}{r_j}-1} x_i^{p_i(1-\frac{\lambda}{r_i})-c_\lambda-1} F_i^{p_i}(x_i)$$
$$= C_k x_k^{\frac{\lambda}{r_i}+c_\lambda} \prod_{j=1 (j\neq k)}^n x_j^{\frac{\lambda}{r_j}-1} x_k^{p_k(1-\frac{\lambda}{r_k})-c_\lambda-1} F_k^{p_k}(x_k) \quad \text{a.e. in } \mathrm{R}_+.$$

namely, $C_i x_i^{p_i(1-\frac{\lambda}{r_i})} F_i^{p_i}(x_i) = C_k x_k^{p_k(1-\frac{\lambda}{r_k})} F_k^{p_k}(x_k) = C$ a.e. in $\mathrm{R}_+$. Assuming that $C_i \neq 0$, we have

$$x_i^{p_i(1-\frac{\lambda}{r_i})-c_\lambda-1} F_i^{p_i}(x_i) = \frac{C}{C_i} x_i^{-c_\lambda-1} \quad \text{a.e. in } \mathrm{R}_+,$$

which contradicts the fact that $0 < \int_0^\infty x_i^{p_i(1-\frac{\lambda}{r_i})-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i < \infty$, in view of $\int_0^\infty x_i^{-c_\lambda-1}\,dx_i = \infty$. Then by (8) and (11), we have (10).

The lemma is proved. □

*Remark* 1 Replacing $\lambda$ (resp. $\frac{\lambda}{r_i}$) by $\lambda + n$ (resp. $\frac{\lambda}{r_i} + 1$) in (10), we have

$$H_{\lambda+n} = \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^{\lambda+n}} \prod_{i=1}^n F_i(x_i)\,dx_1 \cdots dx_n$$
$$< \prod_{i=1}^n \left( \tilde{k}_{\lambda+n}^{(i)} \int_0^\infty x_i^{-p_i\frac{\lambda}{r_i}-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i \right)^{\frac{1}{p_i}}, \tag{12}$$

where we denote

$$\tilde{k}_{\lambda+n}^{(i)} := \frac{\Gamma(\lambda(1-\frac{1}{r_i})+n-1)}{\Gamma(\sum_{j=1(j\neq i)}^{n}(\frac{\lambda}{r_j}+1))} \cdot \frac{\prod_{j=1}^{n}\Gamma(\frac{\lambda}{r_j}+1)}{\Gamma(\lambda+n)} \in \mathrm{R}_+ \quad (i=1,\ldots,n).$$

## 3 Main results and a corollary

**Theorem 1** *We have the following inequality*:

$$\begin{aligned} I &:= \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n \\ &< \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \prod_{i=1}^n \left(\tilde{k}_{\lambda+n}^{(i)} \int_0^\infty x_i^{-p_i\frac{\lambda}{r_i}-c_\lambda-1} F_i^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}}. \end{aligned} \tag{13}$$

*In particular, for* $\sum_{i=1}^n \frac{1}{r_i}=1$, *we have*

$$0 < \int_0^\infty x_i^{-p_i\frac{\lambda}{r_i}-1} F_i^{p_i}(x_i)\,dx_i < \infty \quad (i=1,\ldots,n),$$

*and the following inequality*:

$$\begin{aligned} I &= \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n \\ &< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{r_i}\Gamma\left(\frac{\lambda}{r_i}\right)\left(\int_0^\infty x_i^{-p_i\frac{\lambda}{r_i}-1} F_i^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}}. \end{aligned} \tag{14}$$

*Proof* By (4) and (5), we have

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n f_i(x_i) \int_0^\infty t^{\lambda-1} e^{-t(x_1+\cdots+x_n)}\,dt\,dx_1\cdots dx_n \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \prod_{i=1}^n \int_0^\infty e^{-tx_i} f_i(x_i)\,dx_i\,dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} \prod_{i=1}^n \int_0^\infty e^{-tx_i} F_i(x_i)\,dx_i\,dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n F_i(x_i) \int_0^\infty t^{\lambda+n-1} e^{-t(x_1+\cdots+x_n)}\,dt\,dx_1\cdots dx_n \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} H_{\lambda+n}. \end{aligned}$$

Then by (12), we have (13).

The theorem is proved. □

**Theorem 2** *The constant factor* $\frac{1}{\Gamma(\lambda)}\prod_{i=1}^n \frac{\lambda}{r_i}\Gamma(\frac{\lambda}{r_i})$ *in* (14) *is the best possible.*

*Proof* For any $0 < \varepsilon < \lambda \min_{1 \le i \le n}\{\frac{p_i}{r_i}\}$, we set

$$\tilde{f}_i(x_i) := \begin{cases} 0, & 0 < x_i \le 1, \\ x_i^{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i} - 1}, & x_i > 1, \end{cases} \quad (i = 1, \ldots, n).$$

We obtain that $\tilde{f}_i(x_i) = o(e^{tx_i})$ $(t > 0; x_i \to \infty)$, and $\tilde{F}_i(x_i) \equiv 0$ $(0 < x_i \le 1)$,

$$\tilde{F}_i(x_i) = \int_0^{x_i} \tilde{f}_i(t)\,dt = \int_1^{x_i} t^{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i} - 1}\,dt = \frac{x_i^{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}} - 1}{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}} < \frac{x_i^{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}}}{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}} \quad (x_i > 1; i = 1, \ldots, n).$$

If there exists a positive constant $M (M \le \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{r_i} \Gamma(\frac{\lambda}{r_i}))$ such that (14) is valid when replacing $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{r_i} \Gamma(\frac{\lambda}{r_i})$ by $M$, then in particular, by substitution of $f_i(x_i) = \tilde{f}_i(x_i)$ and $F_i(x_i) = \tilde{F}_i(x_i)$, we have

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n \tilde{f}_i(x_i)\,dx_1 \cdots dx_n \\ &< M \prod_{i=1}^n \left( \int_0^\infty x_i^{-p_i \frac{\lambda}{r_i} - 1} \tilde{F}_i^{p_i}(x_i)\,dx_i \right)^{\frac{1}{p_i}} \\ &= M \prod_{i=1}^n \frac{1}{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}} \left( \int_1^\infty x_i^{-\varepsilon - 1}\,dx_i \right)^{\frac{1}{p_i}} = \frac{M}{\varepsilon} \prod_{i=1}^n \frac{1}{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}}.. \end{aligned}$$

In view of Lemma 9.1.4 (9.1.5) in [2], we find

$$I_\varepsilon := \varepsilon \tilde{I} = \varepsilon \int_1^\infty \cdots \int_1^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n x_i^{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i} - 1}\,dx_1 \cdots dx_n = k_\lambda + o(1) \quad (\varepsilon \to 0^+).$$

Hence, we have

$$\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right) + o(1) = k_\lambda + o(1) = \varepsilon \tilde{I} < M \prod_{i=1}^n \frac{1}{\frac{\lambda}{r_i} - \frac{\varepsilon}{p_i}}.$$

For $\varepsilon \to 0^+$, we find

$$\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{r_i} \Gamma\left(\frac{\lambda}{r_i}\right) \le M,$$

which yields that the constant factor $M = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{r_i} \Gamma(\frac{\lambda}{r_i})$ in (14) is the best possible.

The theorem is proved. □

Setting $x = \frac{1}{x_1}, f(x) = x^{\lambda - 2} f_1(\frac{1}{x})$ in $I$ of (14), we have

$$I = \int_0^\infty \cdots \int_0^\infty \frac{f(x)}{(1 + \sum_{i=2}^n x x_i)^\lambda} \prod_{i=2}^n f_i(x_i)\,dx\,dx_2 \cdots dx_n.$$

For $f_1(t) = t^{\lambda-2}f(\frac{1}{t})$, we find

$$F_1(x_1) = \int_0^{x_1} f_1(t)\,dt = \int_0^{x_1} t^{\lambda-2}f\left(\frac{1}{t}\right)dt.$$

Then, replacing back $x$ (resp. $f(x)$) by $x_1$ (resp. $f_1(x_1)$), we have

**Corollary 1** *If* $\tilde{F}_1(x_1) = \int_0^{x_1} t^{\lambda-2}f_1(\frac{1}{t})\,dt$,

$$\tilde{F}_i(x_i) := \int_0^{x_i} f_i(t)\,dt \quad (i = 2,\dots,n),$$

*then we have the following inequality with the nonhomogeneous kernel*:

$$\begin{aligned}&\int_0^\infty \cdots \int_0^\infty \frac{1}{(1+\sum_{i=2}^n x_1x_i)^\lambda}\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n\\ &\quad< \frac{1}{\Gamma(\lambda)}\prod_{i=1}^n \frac{\lambda}{r_i}\Gamma\left(\frac{\lambda}{r_i}\right)\left(\int_0^\infty x_i^{-p_i\frac{\lambda}{r_i}-1}\tilde{F}_i^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}},\end{aligned} \tag{15}$$

*where the constant factor* $\frac{1}{\Gamma(\lambda)}\prod_{i=1}^n \frac{\lambda}{r_i}\Gamma(\frac{\lambda}{r_i})$ *in* (15) *is the best possible.*

*Remark* 2 (i) For $n = 2$, (14) reduces to (cf. [40])

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f_1(x_1)f_2(x_2)}{(x_1+x_2)^\lambda}\,dx_1\,dx_2\\ &\quad< \frac{\lambda^2}{r_1r_2}B\left(\frac{\lambda}{r_1},\frac{\lambda}{r_2}\right)\left(\int_0^\infty x_1^{-p_1\frac{\lambda}{r_1}-1}F_1^{p_1}(x_1)\,dx_1\right)^{\frac{1}{p_1}}\left(\int_0^\infty x_2^{-p_2\frac{\lambda}{r_2}-1}F_2^{p_2}(x_2)\,dx_2\right)^{\frac{1}{p_2}},\end{aligned} \tag{16}$$

and (15) reduces to the following new inequality:

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f_1(x_1)f_2(x_2)}{(1+x_1x_2)^\lambda}\,dx_1\,dx_2\\ &\quad< \frac{\lambda^2}{r_1r_2}B\left(\frac{\lambda}{r_1},\frac{\lambda}{r_2}\right)\left(\int_0^\infty x_1^{-p_1\frac{\lambda}{r_1}-1}\tilde{F}_1^{p_1}(x_1)\,dx_1\right)^{\frac{1}{p_1}}\left(\int_0^\infty x_2^{-p_2\frac{\lambda}{r_2}-1}\tilde{F}_2^{p_2}(x_2)\,dx_2\right)^{\frac{1}{p_2}}.\end{aligned} \tag{17}$$

(ii) For $r_i = p_i$ $(i = 1,\dots,n)$, (14) reduces to

$$\begin{aligned}&\int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda}\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n\\ &\quad< \frac{1}{\Gamma(\lambda)}\prod_{i=1}^n \frac{\lambda}{p_i}\Gamma\left(\frac{\lambda}{p_i}\right)\left(\int_0^\infty x_i^{-\lambda-1}F_i^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}},\end{aligned} \tag{18}$$

and (15) reduces to

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(1+\sum_{i=2}^n x_1x_i)^\lambda}\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{p_i} \Gamma\left(\frac{\lambda}{p_i}\right) \left( \int_0^\infty x_i^{-\lambda-1} \tilde{F}_i^{p_i}(x_i)\, dx_i \right)^{\frac{1}{p_i}}. \tag{19}$$

The constant factors in the above inequalities are the best possible.

## 4 Conclusions

In this paper, following the idea of [21], by the use of the weight functions, the way of introducing parameters and the technique of real analysis, a new multiple Hilbert-type integral inequality with the kernel $\frac{1}{(x_1+\cdots+x_n)^\lambda}$ $(\lambda > 0)$ involving the upper limit functions is given in Theorem 1. In a condition, the best possible constant factor related to the gamma function and a few parameters is proved in Theorem 2. A corollary about the case of nonhomogeneous kernel and some particular inequalities are obtained in Corollary 1 and Remark 2. The lemmas and theorems provide an extensive account of this type of inequalities.

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**Availability of data and materials**
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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. JZ participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

## Publisher's Note



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