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Convergence of λ-Bernstein operators based on (p,q)-integers

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Abstract

In the present paper, we construct a new class of positive linear λ-Bernstein operators based on (p,q)-integers. We obtain a Korovkin type approximation theorem, study the rate of convergence of these operators by using the conception of K-functional and moduli of continuity, and also give a convergence theorem for the Lipschitz continuous functions.

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1 Introduction

In 2018, Cai et al. [1] introduced the following new family of Bernstein operators with parameter $\lambda \in [-1,1]$:

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \widetilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{1}$$

where $\widetilde{b}_{n,k}(\lambda;x)$ $(k = 0,1,\ldots,n)$ are the λ-Bernstein basis functions defined as

$$\begin{cases} \widetilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \widetilde{b}_{n,k}(\lambda;x) = b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right), \\ \quad (k = 1,2,\ldots,n-1), \\ \widetilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \end{cases}$$

and $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ $(k = 0,1,\ldots,n)$ are the classical Bernstein basis functions. They call these operators (1) λ-Bernstein operators, they investigated some approximation theorems and also gave some numerical examples. In the same year, Acu et al. [2] studied some approximation properties of Kantorovich type of operators (1). Later, Özger [3] gave some statistical approximation results of (1). In 2019, Cai et al. [4] proposed new λ-Bernstein operators based on q-integers and established a statistical approximation theorem. Some other papers also mention λ-Bernstein operators, see [5, 6].

Recently, Mursaleen et al. [7] defined the following (p, q)-analogue of Bernstein operators:

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}(x;p,q) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \quad x\in[0,1], \tag{2}$$

where $b_{n,k}(x;p,q)$ $(k=0,1,\dots,n)$ are (p, q)-Bernstein basis functions defined as follows:

$$b_{n,k}(x;p,q)=\frac{1}{p^{\frac{n(n-1)}{2}}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1}\left(p^s-q^s x\right), \quad x\in[0,1]. \tag{3}$$

There are many papers about the research and application of (p, q) operators, we mention some of them [8–22].

Motivated by the above work, in this paper, we introduce positive linear λ-Bernstein operators based on (p, q)-integers as follows:

$$B^{\lambda}_{n,p,q}(f;x)=\sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \quad x\in[0,1], \tag{4}$$

where $b^{\lambda}_{n,k}(x;p,q)$ $(k=0,1,\dots,n)$ are called (p, q)-analogue of λ-Bernstein basis functions and defined as

$$\begin{cases} b^{\lambda}_{n,0}(x;p,q)=b_{n,0}(x;p,q)-\frac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,1}(x;p,q), \\ b^{\lambda}_{n,k}(x;p,q)=b_{n,k}(x;p,q)+\lambda\big(\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]^2_{p,q}-1} b_{n+1,k}(x;p,q) \\ \qquad -\frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]^2_{p,q}-1} b_{n+1,k+1}(x;p,q)\big), \quad (k=1,2,\dots,n-1), \\ b^{\lambda}_{n,n}(x;p,q)=b_{n,n}(x;p,q)-\frac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,n}(x;p,q), \end{cases}$$

$b_{n,k}(x;p,q)$ $(k=0,1,\dots,n)$ are defined in (3), $\lambda\in[-1,1]$, $n\ge 2$, $x\in[0,1]$, and $0<q<p\le 1$. Obviously, when $p\to 1^-$, $B^{\lambda}_{n,p,q}(f;x)$ reduces to q-analogue of λ-Bernstein operators in [4]; when $p,q\to 1^-$, $B^{\lambda}_{n,p,q}(f;x)$ reduces to (1).

Here we mention certain notations on (p, q)-calculus, details can be found in [23–27].

For any fixed real numbers $p>0$ and $q>0$, the (p, q)-integers $[n]_{p,q}$ are defined as follows:

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+pq^{n-2}+q^{n-1}=\begin{cases} \frac{p^n-q^n}{p-q}, & p\neq q\neq 1; \\ \frac{1-q^n}{1-q}, & p=1; \\ n, & p=q=1. \end{cases}$$

(p, q)-factorial and (p, q)-binomial coefficients are defined as follows:

$$[n]_{p,q}!=\begin{cases} [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}, & n=1,2,\dots; \\ 1, & n=0, \end{cases} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The aims of the present paper are to construct a new class of λ-Bernstein operators based on (p, q)-integers and give the rate of convergence of these operators.

The rest of this paper is mainly organized as follows. In Sect. 2, we compute some moments and central moments of $B^{\lambda}_{n,p,q}(f;x)$ which are needed to prove our main results. In Sect. 3, we obtain a Korovkin approximation theorem and rate of convergence of operators $B^{\lambda}_{n,p,q}(f;x)$.

2 Some lemmas

In order to obtain the main results, we need the following lemmas.

Lemma 2.1 (see [7]) *For $x \in [0,1]$, $0 < q < p \leq 1$, we have*

$$B_{n,p,q}(1;x) = 1; \qquad B_{n,p,q}(t;x) = x; \qquad B_{n,p,q}\left(t^2;x\right) = \frac{p^{n-1}}{[n]_{p,q}}x + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2.$$

Lemma 2.2 *Let $\lambda \in [-1,1]$, $x \in [0,1]$, and $0 < q < p \leq 1$, for the operators $B^{\lambda}_{n,p,q}(f;x)$, we have*

$$B^{\lambda}_{n,p,q}(1;x) = 1. \tag{5}$$

Proof Actually, by (4) and the fact that $q[n-1]_{p,q} = [n]_{p,q} - p^{n-1}$, we have

$$\begin{aligned}
&B^{\lambda}_{n,p,q}(1;x) \\
&\quad= \sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q) \\
&\quad= \sum_{k=0}^{n} b_{n,k}(x;p,q) - \frac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,1}(x;p,q) \\
&\qquad+ \lambda \frac{p^{1-n}[n]_{p,q} - 2[1]_{p,q} + 1}{p^{2-2n}[n]^2_{p,q} - 1} b_{n+1,1}(x;p,q) \\
&\qquad- \lambda \frac{p^{1-n}[n]_{p,q} - 2qp^{-1}[1]_{p,q} - 1}{p^{2-2n}[n]^2_{p,q} - 1} b_{n+1,2}(x;p,q) \\
&\qquad+ \lambda \frac{p^{1-n}[n]_{p,q} - 2p^{-1}[2]_{p,q} + 1}{p^{2-2n}[n]^2_{p,q} - 1} b_{n+1,2}(x;p,q) \\
&\qquad- \lambda \frac{p^{1-n}[n]_{p,q} - 2qp^{-2}[2]_{p,q} - 1}{p^{2-2n}[n]^2_{p,q} - 1} b_{n+1,3}(x;p,q) \\
&\qquad+ \lambda \frac{p^{1-n}[n]_{p,q} - 2p^{-2}[3]_{p,q} + 1}{p^{2-2n}[n]^2_{p,q} - 1} b_{n+1,3}(x;p,q) \\
&\qquad- \cdots \\
&\qquad- \lambda \frac{p^{1-n}[n]_{p,q} - 2qp^{1-n}[n-1]_{p,q} - 1}{p^{2-2n}[n]^2_{p,q} - 1} b_{n+1,n}(x;p,q) - \frac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,n}(x;p,q) \\
&\quad= \sum_{k=0}^{n} b_{n,k}(x;p,q).
\end{aligned}$$

Then we can obtain (5) since $B_{n,p,q}(1;x) = \sum_{k=0}^{n} b_{n,k}(x;p,q) = 1$. □

Lemma 2.3 *Let $\lambda \in [-1,1]$, $x \in [0,1]$, and $0 < q < p \leq 1$, for the operators $B^{\lambda}_{n,p,q}(f;x)$, we have*

$$\begin{aligned}B^{\lambda}_{n,p,q}(t;x) &= x + \frac{2\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{p^n(p^{2-2n}[n]^2_q - 1)}\left(1-\frac{q}{p}\right) + \frac{\lambda p^n(1-x^{n+1})}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} \\ &\quad + \frac{\lambda[n+1]_{p,q}x(1-x^n)}{[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}\left(\frac{1}{p}-\frac{1}{q}-\frac{2}{p(p^{1-n}[n]_{p,q}+1)}\right) \\ &\quad - \frac{\lambda\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}. \end{aligned} \tag{6}$$

Proof From (4), we have

$$\begin{aligned}&B^{\lambda}_{n,p,q}(t;x) \\ &\quad = \sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q)\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \\ &\quad = \sum_{k=1}^{n-1}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\Bigg[b_{n,k}(x;p,q) + \lambda\bigg(\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]^2_{p,q}-1}b_{n+1,k}(x;p,q) \\ &\quad\quad - \frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]^2_{p,q}-1}b_{n+1,k+1}(x;p,q)\bigg)\Bigg] + b_{n,n}(x;p,q) \\ &\quad\quad - \frac{\lambda}{p^{1-n}[n]_{p,q}+1}b_{n+1,n}(x;p,q) \\ &\quad = \sum_{k=0}^{n}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}b_{n,k}(x;p,q) \\ &\quad\quad + \lambda\sum_{k=0}^{n}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]^2_{p,q}-1}b_{n+1,k}(x;p,q) \\ &\quad\quad - \lambda\sum_{k=1}^{n-1}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]^2_{p,q}-1}b_{n+1,k+1}(x;p,q) \\ &\quad = \triangle_{1,1} + \triangle_{1,2} + \triangle_{1,3}, \end{aligned} \tag{7}$$

where

$$\begin{aligned}\triangle_{1,1} &= \sum_{k=0}^{n}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}b_{n,k}(x;p,q); \\ \triangle_{1,2} &= \lambda\sum_{k=0}^{n}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]^2_{p,q}-1}b_{n+1,k}(x;p,q); \\ \triangle_{1,3} &= -\lambda\sum_{k=1}^{n-1}\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]^2_{p,q}-1}b_{n+1,k+1}(x;p,q).\end{aligned}$$

By Lemma 2.1, we get

$$\triangle_{1,1} = B_{n,p,q}(t;x) = x. \tag{8}$$

Next, since $[k]_{p,q}^2 = q[k]_{p,q}[k-1]_{p,q} + p^{k-1}[k]_{p,q}$, we have

$$
\begin{aligned}
&\triangle_{1,2} \\
&= \lambda \sum_{k=0}^{n} \frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \left(\frac{1}{p^{1-n}[n]_{p,q} - 1} - \frac{2p^{1-k}[k]_{p,q}}{p^{2-2n}[n]_{p,q}^2 - 1} \right) b_{n+1,k}(x;p,q) \\
&= \frac{\lambda[n+1]_{p,q}x}{p[n]_{p,q}(p^{1-n}[n]_{p,q} - 1)} \sum_{k=0}^{n-1} b_{n,k}(x;p,q) - \frac{2\lambda[n+1]_{p,q}x}{p[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=0}^{n-1} b_{n,k}(x;p,q) \\
&\quad - \frac{2q\lambda[n+1]_{p,q}x^2}{p^{n+1}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=0}^{n-2} b_{n-1,k}(x;p,q) \\
&= \frac{\lambda[n+1]_{p,q}x}{p[n]_{p,q}(p^{1-n}[n]_{p,q} - 1)} (1 - x^n) - \frac{2\lambda[n+1]_{p,q}x}{p[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} (1 - x^n) \\
&\quad - \frac{2q\lambda[n+1]_{p,q}x^2}{p^{n+1}(p^{2-2n}[n]_{p,q}^2 - 1)} (1 - x^{n-1}).
\end{aligned}
\tag{9}
$$

Finally, using the following two identities:

$$
[k]_{p,q} = \frac{[k+1]_{p,q}}{q} - \frac{p^k}{q}, \qquad [k]_{p,q}^2 = \frac{[k+1]_{p,q}[k]_{p,q}}{q} - \frac{p^k[k]_{p,q}}{q},
$$

and some computations, we have

$$
\begin{aligned}
&\triangle_{1,3} \\
&= -\lambda \sum_{k=1}^{n-1} \frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \left(\frac{1}{p^{1-n}[n]_{p,q} + 1} - \frac{2qp^{-k}[k]_{p,q}}{p^{2-2n}[n]_{p,q}^2 - 1} \right) b_{n+1,k+1}(x;p,q) \\
&= \frac{-\lambda}{q(p^{1-n}[n]_{p,q} + 1)} \sum_{k=1}^{n-1} \frac{[k+1]_{p,q} - p^k}{p^{k-n}[n]_{p,q}} b_{n+1,k+1}(x;p,q) \\
&\quad + \frac{2\lambda}{[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=1}^{n-1} \frac{[k+1]_{p,q}[k]_{p,q} - p^k[k]_{p,q}}{p^{2k-n}} b_{n+1,k+1}(x;p,q) \\
&= \frac{-\lambda[n+1]_{p,q}x}{q[n]_{p,q}(p^{1-n}[n]_{p,q} + 1)} \sum_{k=1}^{n-1} b_{n,k}(x;p,q) + \frac{p^n\lambda}{q[n]_{p,q}(p^{1-n}[n]_{p,q} + 1)} \sum_{k=1}^{n-1} b_{n+1,k+1}(x;p,q) \\
&\quad + \frac{2\lambda[n+1]_{p,q}x^2}{p^n(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=0}^{n-2} b_{n-1,k}(x;p,q) + \frac{-2\lambda[n+1]_{p,q}x}{q[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=1}^{n-1} b_{n,k}(x;p,q) \\
&\quad + \frac{2p^n\lambda}{q[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=1}^{n-1} b_{n+1,k+1}(x;p,q) \\
&= \frac{-\lambda[n+1]_{p,q}x}{q[n]_{p,q}(p^{1-n}[n]_{p,q} - 1)} \left(1 - \frac{\prod_{s=0}^{n-1}(p^s - q^s x)}{p^{\frac{n(n-1)}{2}}} - x^n \right) + \frac{2\lambda[n+1]_{p,q}x^2}{p^n(p^{2-2n}[n]_{p,q}^2 - 1)} (1 - x^{n-1}) \\
&\quad + \frac{p^n\lambda}{q[n]_{p,q}(p^{1-n}[n]_{p,q} - 1)} \left(1 - \frac{\prod_{s=0}^{n}(p^s - q^s x)}{p^{\frac{n(n+1)}{2}}} - \frac{[n+1]_{p,q}x\prod_{s=0}^{n-1}(p^s - q^s x)}{p^{\frac{n(n+1)}{2}}} - x^{n+1} \right)
\end{aligned}
$$

$$= \frac{-\lambda[n+1]_{p,q}x}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}\left(1-x^n\right) + \frac{2\lambda[n+1]_{p,q}x^2}{p^n(p^{2-2n}[n]_{p,q}^2-1)}\left(1-x^{n-1}\right) + \frac{p^n\lambda}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}\left(1-\frac{\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n+1)}{2}}}-x^{n+1}\right). \tag{10}$$

Combining (7)–(10), we can obtain (6), Lemma 2.2 is proved. □

Lemma 2.4 *Let $\lambda \in [-1,1]$, $x \in [0,1]$, $n > 1$, and $0 < q < p \le 1$, for the operators $B^{\lambda}_{n,p,q}(f;x)$, we have*

$$\begin{aligned}
&B^{\lambda}_{n,p,q}\left(t^2;x\right)\\
&= x^2 + \frac{p^{n-1}x(1-x)}{[n]_{p,q}} + \frac{\lambda(q^2-p^2)[n+1]_{p,q}x^2(1-x^{n-1})}{p^2q[n]_{p,q}(p^{1-n}[n]_{p,q}+1)}\\
&\quad - \frac{p^{2n}\lambda(1-x^{n+1})}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}-1)} + \frac{2\lambda(p^2q-p^3-pq^2-q^3)[n+1]_{p,q}x^2(1-x^{n-1})}{p^3q[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)}\\
&\quad + \frac{p^{n-2}(p^2+q^2)\lambda[n+1]_{p,q}x(1-x^n)}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}+1)} + \frac{p\lambda\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{(n-1)(n-2)}{2}}q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}-1)}\\
&\quad + \frac{2q(p^2-q^2)\lambda[n+1]_{p,q}[n-1]_{p,q}x^3(1-x^{n-2})}{p^{n+2}[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)}\\
&\quad + \frac{2p^{n-1}(2q-p)\lambda[n+1]_{p,q}x}{q^2[n]_{p,q}^2(p^{2-2n}[n]_{p,q}^2-1)}\left(\frac{\prod_{s=0}^{n-1}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}}-\left(1-x^n\right)\right).
\end{aligned} \tag{11}$$

Proof From (4), we have

$$\begin{aligned}
&B^{\lambda}_{n,p,q}\left(t^2;x\right)\\
&= \sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q)\frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2}\\
&= \sum_{k=1}^{n-1}\frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2}\Bigg[b_{n,k}(x;p,q) + \lambda\Bigg(\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}b_{n+1,k}(x;p,q)\\
&\quad - \frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]_{p,q}^2-1}b_{n+1,k+1}(x;p,q)\Bigg)\Bigg] + b_{n,n}(x;p,q)\\
&\quad - \frac{\lambda}{p^{1-n}[n]_{p,q}+1}b_{n+1,n}(x;p,q)\\
&= \sum_{k=0}^{n}\frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2}b_{n,k}(x;p,q)\\
&\quad + \lambda\sum_{k=0}^{n}\frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2}\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}b_{n+1,k}(x;p,q)\\
&\quad - \lambda\sum_{k=1}^{n-1}\frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2}\frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]_{p,q}^2-1}b_{n+1,k+1}(x;p,q)\\
&= \triangle_{2,1} + \triangle_{2,2} + \triangle_{2,3},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\triangle_{2,1} &= \sum_{k=0}^{n} \frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2} b_{n,k}(x;p,q); \\
\triangle_{2,2} &= \lambda \sum_{k=0}^{n} \frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2} \frac{p^{1-n}[n]_{p,q} - 2p^{1-k}[k]_{p,q} + 1}{p^{2-2n}[n]_{p,q}^2 - 1} b_{n+1,k}(x;p,q); \\
\triangle_{2,3} &= -\lambda \sum_{k=1}^{n-1} \frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2} \frac{p^{1-n}[n]_{p,q} - 2qp^{-k}[k]_{p,q} - 1}{p^{2-2n}[n]_{p,q}^2 - 1} b_{n+1,k+1}(x;p,q).
\end{aligned}$$

By Lemma 2.1, we have

$$\triangle_{2,1} = B_{n,p,q}\left(t^2;x\right) = x^2 + \frac{p^{n-1}x(1-x)}{[n]_{p,q}}. \tag{13}$$

Indeed, by using

$$\begin{aligned}
&[k]_{p,q}^2 = q[k]_{p,q}[k-1]_{p,q} + p^{k-1}[k]_{p,q}; \\
&[k]_{p,q}^3 = q^3[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q} + qp^{k-2}(q+2p)[k]_{p,q}[k-1]_{p,q} + p^{2k-2}[k]_{p,q},
\end{aligned}$$

$\sum_{k=0}^{n} b_{n,k}(x;p,q) = 1$, and some computations, we obtain

$$\begin{aligned}
&\triangle_{2,2} \\
&\quad = \lambda \sum_{k=0}^{n} \frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2} \left(\frac{1}{p^{1-n}[n]_{p,q} - 1} - \frac{2p^{1-k}[k]_{p,q}}{p^{2-2n}[n]_{p,q}^2 - 1} \right) b_{n+1,k}(x;p,q) \\
&\quad = \frac{\lambda}{p^{1-n}[n]_{p,q} - 1} \sum_{k=0}^{n} \frac{q[k]_{p,q}[k-1]_{p,q} + p^{k-1}[k]_{p,q}}{p^{2k-2n}[n]_{p,q}^2} b_{n+1,k}(x;p,q) \\
&\qquad + \frac{-2p\lambda}{[n]_{p,q}^2} \sum_{k=0}^{n} \frac{q^3[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q} + qp^{k-2}(q+2p)[k]_{p,q}[k-1]_{p,q} + p^{2k-2}[k]_{p,q}}{(p^{2-2n}[n]_{p,q}^2 - 1)p^{3k-2n}} \\
&\quad = \frac{q\lambda[n+1]_{p,q}x^2}{p^2[n]_{p,q}(p^{1-n}[n]_{p,q} - 1)} \sum_{k=0}^{n-2} b_{n-1,k}(x;p,q) + \frac{p^{n-2}\lambda[n+1]_{p,q}x}{[n]_{p,q}^2(p^{1-n}[n]_{p,q} - 1)} \sum_{k=0}^{n-1} b_{n,k}(x;p,q) \\
&\qquad - \frac{2q^3\lambda[n+1]_{p,q}[n-1]_{p,q}x^3}{p^{n+2}[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=0}^{n-3} b_{n-2,k}(x;p,q) \\
&\qquad - \frac{2q(q+2p)\lambda[n+1]_{p,q}x^2}{p^3[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=0}^{n-2} b_{n-1,k}(x;p,q) \\
&\qquad - \frac{2p^{n-2}\lambda[n+1]_{p,q}x}{[n]_{p,q}^2(p^{2-2n}[n]_{p,q}^2 - 1)} \sum_{k=0}^{n-1} b_{n,k}(x;p,q) \\
&\quad = \frac{q\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{p^2[n]_{p,q}(p^{1-n}[n]_{p,q} + 1)} - \frac{2[2]_{p,q}q\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{p^3[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)} \\
&\qquad + \frac{p^{n-2}\lambda[n+1]_{p,q}x(1-x^n)}{[n]_{p,q}^2(p^{1-n}[n]_{p,q} + 1)} - \frac{2q^3\lambda[n+1]_{p,q}[n-1]_{p,q}x^3(1-x^{n-2})}{p^{n+2}[n]_{p,q}(p^{2-2n}[n]_{p,q}^2 - 1)}.
\end{aligned} \tag{14}$$

Finally, using the following identities:

$$
\begin{aligned}
[k]_{p,q}^2 &= \frac{1}{q}[k+1]_{p,q}[k]_{p,q} - \frac{p^k}{q^2}[k+1]_{p,q} + \frac{p^{2k}}{q^2}; \\
[k]_{p,q}^3 &= [k+1]_{p,q}[k]_{p,q}[k-1]_{p,q} + \frac{p^{k-1}}{q}\left(1-\frac{p}{q}\right)[k+1]_{p,q}[k]_{p,q} \\
&\quad - \frac{p^{2k-1}}{q^2}\left(2-\frac{p}{q}\right)[k+1]_{p,q} - \frac{p^{3k}}{q^3},
\end{aligned}
$$

$\sum_{k=0}^{n} b_{n,k}(x;p,q) = 1$, and some computations, we have

$$
\begin{aligned}
\triangle_{2,3} &= -\lambda \sum_{k=1}^{n-1} \frac{[k]_{p,q}^2}{p^{2k-2n}[n]_{p,q}^2}\left(\frac{1}{p^{1-n}[n]_{p,q}+1} - \frac{2qp^{-k}[k]_{p,q}}{p^{2-2n}[n]_{p,q}^2-1}\right) b_{n+1,k+1}(x;p,q) \\
&= -\frac{\lambda[n+1]_{p,q}x^2}{q[n]_{p,q}(p^{1-n}[n]_{p,q}+1)} \sum_{k=0}^{n-2} b_{n-1,k}(x;p,q) \\
&\quad + \frac{p^n\lambda[n+1]_{p,q}x}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}+1)} \sum_{k=1}^{n-1} b_{n,k}(x;p,q) \\
&\quad - \frac{p^{2n}\lambda}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}+1)} \sum_{k=1}^{n-1} b_{n+1,k+1}(x;p,q) \\
&\quad + \frac{2\lambda q[n+1]_{p,q}[n-1]_{p,q}x^3}{p^n[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)} \sum_{k=0}^{n-3} b_{n-2,k}(x;p,q) \\
&\quad + \frac{2(q-p)\lambda[n+1]_{p,q}x^2}{pq[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)} \sum_{k=0}^{n-2} b_{n-1,k}(x;p,q) \\
&\quad - \frac{2p^{n-1}(2q-p)\lambda[n+1]_{p,q}x}{q^2[n]_{p,q}^2(p^{2-2n}[n]_{p,q}^2-1)} \sum_{k=1}^{n-1} b_{n,k}(x;p,q) \\
&\quad - \frac{2\lambda p^{2n}}{q^2[n]_{p,q}^2(p^{2-2n}[n]_{p,q}^2-1)} \sum_{k=1}^{n-1} b_{n+1,k+1}(x;p,q) \\
&= -\frac{\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{q[n]_{p,q}(p^{1-n}[n]_{p,q}+1)} + \frac{p^n\lambda[n+1]_{p,q}x(1-x^n)}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}+1)} \\
&\quad - \frac{p^{2n}\lambda}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}-1)}\left(1 - \frac{\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n+1)}{2}}} - x^{n+1}\right) \\
&\quad + \frac{2\lambda q[n+1]_{p,q}[n-1]_{p,q}x^3(1-x^{n-2})}{p^n[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)} + \frac{2(q-p)\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{pq[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)} \\
&\quad - \frac{2p^{n-1}(2q-p)\lambda[n+1]_{p,q}x}{q^2[n]_{p,q}^2(p^{2-2n}[n]_{p,q}^2-1)}\left(1 - \frac{\prod_{s=0}^{n-1}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}} - x^n\right).
\end{aligned}
\tag{15}
$$

Combining (12)–(15) and some computations, we can get (11), Lemma 2.4 is proved. □

Corollary 2.5 *Let* $\lambda \in [-1,1]$, $x \in [0,1]$, $n > 1$, *and* $0 < q < p \le 1$, *we have*

$$\begin{aligned} &B^{\lambda}_{n,p,q}(t-x;x) \\ &\quad= \frac{2\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{p^n(p^{2-2n}[n]_q^2-1)}\left(1-\frac{q}{p}\right) + \frac{\lambda p^n(1-x^{n+1})}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} \\ &\qquad+ \frac{\lambda[n+1]_{p,q}x(1-x^n)}{[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}\left(\frac{1}{p}-\frac{1}{q}-\frac{2}{p(p^{1-n}[n]_{p,q}+1)}\right) \\ &\qquad- \frac{\lambda\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} := \gamma^{\lambda}_{n,p,q}(x) \qquad (16) \\ &\quad\le \frac{2}{p^{1-n}[n]_{p,q}-1} + \frac{4}{p^{2-2n}[n]_{p,q}^2-1} + \frac{2}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} \\ &\qquad+ \frac{2}{p[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)} \\ &\quad:= \alpha(n;p,q); \qquad (17) \\ &B^{\lambda}_{n,p,q}\bigl((t-x)^2;x\bigr) \\ &\quad\le \frac{1}{4[n]_{p,q}} + \frac{5}{p^2(p^{1-n}[n]_{p,q}-1)} + \frac{6}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} + \frac{6}{p^{2-2n}[n]_{p,q}^2-1} \\ &\qquad+ \frac{3}{q^2[n]_{p,q}^2(p^{1-n}[n]_{p,q}-1)} + \frac{8}{q[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)} + \frac{2}{q^2[n]_{p,q}^2(p^{2-2n}[n]_{p,q}^2-1)} \\ &\quad:= \beta(n;p,q). \qquad (18) \end{aligned}$$

Proof By Lemmas 2.2–2.3, we have

$$\begin{aligned} &B^{\lambda}_{n,p,q}(t-x;x) \\ &\quad= \frac{2\lambda[n+1]_{p,q}x^2(1-x^{n-1})}{p^n(p^{2-2n}[n]_{p,q}^2-1)}\left(1-\frac{q}{p}\right) + \frac{\lambda p^n(1-x^{n+1})}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} \\ &\qquad+ \frac{\lambda[n+1]_{p,q}x(1-x^n)}{[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}\left(\frac{1}{p}-\frac{1}{q}-\frac{2}{p(p^{1-n}[n]_{p,q}+1)}\right) - \frac{\lambda\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}q[n]_{p,q}(p^{1-n}-1)} \\ &\quad\le \begin{cases} \frac{2[n+1]_{p,q}x^2(1-x^{n-1})}{p^n(p^{2-2n}[n]_{p,q}^2-1)}(1-\frac{q}{p}) + \frac{p^n(1-x^{n+1})}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}, & \lambda \in [0,1], \\ \frac{[n+1]_{p,q}x(1-x^n)}{[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}(\frac{1}{q}+\frac{2}{p(p^{1-n}[n]_{p,q}+1)}) + \frac{\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}, & \lambda \in [-1,0]. \end{cases} \end{aligned}$$

Since

$$\prod_{s=0}^{n}(p^s-q^sx) = p^{\frac{n(n+1)}{2}}(1-x)\left(1-\frac{q}{p}x\right)\cdots\left(1-\frac{q^n}{p^n}x\right) \le p^{\frac{n(n+1)}{2}},$$

we have

$$\frac{\prod_{s=0}^{n}(p^s-q^sx)}{p^{\frac{n(n-1)}{2}}q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} \le \frac{p^{\frac{n(n+1)}{2}}}{p^{\frac{n(n-1)}{2}}q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} = \frac{p^n}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}.$$

We also have

$$\begin{aligned}\frac{2[n+1]_{p,q}}{p^n(p^{2-2n}[n]_{p,q}^2-1)} &= \frac{2q[n]_{p,q}+2p^n}{p^n(p^{2-2n}[n]_{p,q}^2-1)}\\ &\leq \frac{2(p[n]_{p,q}-p^n)+2p^n}{(p^{1-n}[n]_{p,q}+1)(p[n]_{p,q}-p^n)} + \frac{2}{p^{2-2n}[n]_{p,q}^2-1}\\ &= \frac{2}{p^{1-n}[n]_{p,q}+1} + \frac{4}{p^{2-2n}[n]_{p,q}^2-1}\end{aligned}$$

and

$$\frac{[n+1]_{p,q}}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} = \frac{1}{p^{1-n}[n]_{p,q}-1} + \frac{p^n}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}.$$

Then we obtain

$$\begin{aligned}&B_{n,p,q}^{\lambda}(t-x;x)\\ &\quad\leq \begin{cases}\frac{2}{p^{1-n}[n]_{p,q}-1} + \frac{4}{p^{2-2n}[n]_{p,q}^2-1} + \frac{1}{q[n]_{p,q}(p^{1-n}-1)}, & \lambda\in[0,1],\\ \frac{1}{p^{1-n}-1} + \frac{2}{p^{2-2n}[n]_{p,q}^2-1} + \frac{2}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)} + \frac{2}{p[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)}, & \lambda\in[-1,0]\end{cases}\\ &\quad\leq \frac{2}{p^{1-n}[n]_{p,q}-1} + \frac{4}{p^{2-2n}[n]_{p,q}^2-1} + \frac{2}{q[n]_{p,q}(p^{1-n}[n]_{p,q}-1)}\\ &\qquad + \frac{2}{p[n]_{p,q}(p^{2-2n}[n]_{p,q}^2-1)}.\end{aligned}$$

Therefore, (16) and (17) are obtained. Similarly, by using Lemmas 2.2–2.4 and some computations, we can get (18). □

Lemma 2.6 *Let* $\lambda\in[-1,1]$, $x\in[0,1]$, $n>1$, *and* $0<q<p\leq 1$, *then* $B_{n,p,q}^{\lambda}(f;x)$ *are positive linear operators.*

Proof Apparently, $B_{n,p,q}^{\lambda}(f;x)$ are linear operators, we only need to prove the positive property. Actually, we have

$$\begin{aligned}b_{n,0}^{\lambda}(x;p,q) &= b_{n,0}(x;p,q) - \frac{\lambda}{p^{1-n}+1}b_{n+1,1}(x;p,q)\\ &= b_{n,0}(x;p,q)\left(1-\frac{\lambda[n+1]_{p,q}x}{(p^{1-n}[n]_{p,q}+1)p^n}\right),\end{aligned}$$

since $b_{n,0}(x;p,q)\geq 0$ and $1-\frac{\lambda[n+1]_{p,q}x}{p[n]_{p,q}+p^n}\geq 0$ with the fact that $[n+1]_{p,q}=p[n]_{p,q}+q^n<p[n]_{p,q}+p^n$, then we obtain $b_{n,0}^{\lambda}(x;p,q)\geq 0$. Similarly, we can prove $b_{n,n}^{\lambda}(x;p,q)\geq 0$. Finally, we will prove $b_{n,k}^{\lambda}(x;p,q)\geq 0$ $(1\leq k\leq n-1)$. Indeed,

$$\begin{aligned}&b_{n,k}^{\lambda}(x;p,q)\\ &\quad= b_{n,k}(x;p,q)\left[1+\lambda\left(\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}\,\frac{[n+1]_{p,q}(p^{n-k}-q^{n-k}x)}{p^n[n+1-k]_{p,q}}\right.\right.\end{aligned}$$

$$-\frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]_{p,q}^2-1}\frac{p^k[n+1]_{p,q}x}{p^n[k+1]_{p,q}}\biggr)\biggr],$$

we need to prove

$$\begin{aligned}&\biggl|\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}\frac{[n+1]_{p,q}(p^{n-k}-q^{n-k}x)}{p^n[n+1-k]_{p,q}}\\&\quad-\frac{p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]_{p,q}^2-1}\frac{p^k[n+1]_{p,q}x}{p^n[k+1]_{p,q}}\biggr|\le 1.\end{aligned}\tag{19}$$

We can prove the above inequality in three cases:

$$\begin{aligned}&\text{Case 1}: \quad \text{for } p^{k-1}\le[k]_{p,q}\le\min\biggl\{\frac{p^k}{2q}\bigl(p^{1-n}[n]_{p,q}-1\bigr),\frac{1}{2}\bigl(p^{k-n}[n]_{p,q}+p^{k-1}\bigr)\biggr\};\\&\text{Case 2}: \quad \text{for } \max\biggl\{\frac{p^k}{2q}\bigl(p^{1-n}[n]_{p,q}-1\bigr),\frac{1}{2}\bigl(p^{k-n}[n]_{p,q}+p^{k-1}\bigr)\biggr\}\\&\qquad\qquad\le[k]_{p,q}\le p^{1-k-n}[n-1]_{p,q};\\&\text{Case 3}: \quad \text{for } \min\biggl\{\frac{p^k}{2q}\bigl(p^{1-n}[n]_{p,q}-1\bigr),\frac{1}{2}\bigl(p^{k-n}[n]_{p,q}+p^{k-1}\bigr)\biggr\}\\&\qquad\qquad\le[k]_{p,q}\le\max\biggl\{\frac{p^k}{2q}\bigl(p^{1-n}[n]_{p,q}-1\bigr),\frac{1}{2}\bigl(p^{k-n}[n]_{p,q}+p^{k-1}\bigr)\biggr\}.\end{aligned}$$

We mainly prove case 1. In fact, under the condition of case 1, we have

$$0\le p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1\le p^{1-n}[n]_{p,q}-1,$$

then we get

$$\begin{aligned}0&\le\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}\frac{[n+1]_{p,q}}{p^n[n+1-k]_{p,q}}\\&\le\frac{[n+1]_{p,q}}{(p^{1-n}[n]_{p,q}+1)p^n}\frac{1}{[n+1-k]_{p,q}}\le\frac{1}{p^{n-k}}\end{aligned}$$

with the help of $[n+1-k]_{p,q}\ge p^{n-k}$ and $[n+1]_{p,q}\le p[n]_{p,q}+p^n$. That is to say,

$$0\le\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}\frac{[n+1]_{p,q}p^{n-k}}{p^n[n+1-k]_{p,q}}\le 1.$$

Also we have

$$0\le\frac{p^{1-n}[n]_{p,q}-2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1}\frac{[n+1]_{p,q}q^{n-k}x}{p^n[n+1-k]_{p,q}}\le 1.$$

On the other hand, since

$$0\le p^{1-n}[n]_{p,q}-2qp^{-k}[k]_{p,q}-1\le p^{1-n}[n]_{p,q}-1,$$

we obtain

$$0 \le \frac{p^{1-n}[n]_{p,q} - 2qp^{-k} - 1}{p^{2-2n}[n]_{p,q}^2 - 1} \frac{p^k[n+1]_{p,q}x}{p^n[k+1]_{p,q}} \le \frac{[n+1]_{p,q}}{(p^{1-n}[n]_{p,q}+1)p^n} \frac{p^k}{[k+1]_{p,q}} \le 1$$

by the fact that $x \in [0,1]$, $[n+1]_{p,q} \le p[n]_{p,q} + p^n$, and $[k+1]_{p,q} \ge p^k$. Hence, (19) is proved in case 1. Case 2 and case 3 are similar. Lemma 2.6 is proved. □

Lemma 2.7 (see [28]) *Let the sequences $q := \{q_n\} = \{1-\alpha_n\}$, $p := \{p_n\} = \{1-\beta_n\}$ such that $0 \le \beta_n < \alpha_n < 1$, $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$. The following statements are true*:

(A) *If $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ and $e^{n\beta_n}/n \to 0$, then $[n]_{p_n,q_n} \to \infty$.*

(B) *If $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ and $e^{n\beta_n}(\alpha_n - \beta_n) \to 0$, then $[n]_{p_n,q_n} \to \infty$.*

(C) *If $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$, and $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n-\beta_n)\} \to 0$, then $[n]_{p_n,q_n} \to \infty$.*

3 Main results

In the sequel, let the sequences $q := \{q_n\}$ and $p := \{p_n\}$ satisfy the conditions of Lemma 2.7 and $p^n \to a \in (0,b]$, b is a finite positive real number.

Now we give a Korovkin type approximation theorem for $B_{n,p,q}^{\lambda}(f;x)$.

Theorem 3.1 *For $f \in C_{[0,1]}$, $\lambda \in [-1,1]$, $x \in [0,1]$, and $n > 1$, $B_{n,p,q}^{\lambda}(f;x)$ converge uniformly to f on $[0,1]$.*

Proof The well-known Korovkin theorem (see [29], pp. 8–9) implies that positive linear operators $B_{n,p,q}^{\lambda}(f;x)$ converge to $f(x)$ uniformly on $[0,1]$ as $n \to \infty$ for any $f \in C_{[0,1]}$ if and only if

$$B_{n,p,q}^{\lambda}(t^i;x) \to x^i, \quad i = 0,1,2. \tag{20}$$

Obviously, (20) can be easily obtained by Lemmas 2.2–2.4 as $n \to \infty$. Therefore, Theorem 3.1 is proved. □

Next, in order to get the rate of convergence of $B_{n,p,q}^{\lambda}(f;x)$, we give the following definitions about Peetre's K-functional and modulus of smoothness. Let $f \in C_{[0,1]}$, endowed with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2} \{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $C^2 = \{g \in C_{[0,1]} : g', g'' \in C_{[0,1]}\}$. It is known that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}),$$

where C is a positive constant, and

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \sup_{x,x+h,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness. We also denote the usual of modulus of continuity by

$$\omega(f;\delta) = \sup_{0<h\leq\delta}\sup_{x,x+h\in[0,1]}\bigl|f(x+h)-f(x)\bigr|.$$

Theorem 3.2 *For $f\in C_{[0,1]}$, $\lambda\in[-1,1]$, $x\in[0,1]$, and $n>1$, we have*

$$\bigl|B^{\lambda}_{n,p,q}(f;x)-f(x)\bigr|\leq 2\omega\bigl(f;\sqrt{\beta(n;p,q)}\bigr),$$

where $\beta(n;p,q)$ is defined in (18).

Proof Since $|f(t)-f(x)|\leq\omega(f;|t-x|)\leq(1+\frac{|t-x|}{\delta})\omega(f;\delta)$, applying $B^{\lambda}_{n,p,q}(f;x)$ to both ends and using Lemma 2.2, we have

$$\bigl|B^{\lambda}_{n,p,q}(f;x)-f(x)\bigr|\leq B_{n,p,q}\bigl(\bigl|f(t)-f(x)\bigr|;x\bigr)\leq\left(1+\frac{1}{\delta}B^{\lambda}_{n,p,q}\bigl(|t-x|;x\bigr)\right)\omega(f;\delta).$$

By the Cauchy–Schwarz inequality, we obtain

$$\bigl|B^{\lambda}_{n,p,q}(f;x)-f(x)\bigr|\leq\left(1+\frac{1}{\delta}\sqrt{B^{\lambda}_{n,p,q}\bigl((t-x)^2;x\bigr)}\right)\omega(f;\delta).$$

Then we have

$$\bigl|B^{\lambda}_{n,p,q}(f;x)-f(x)\bigr|\leq 2\omega\Bigl(f;\sqrt{B^{\lambda}_{n,p,q}\bigl((t-x)^2;x\bigr)}\Bigr)\leq 2\omega\bigl(f;\sqrt{\beta(n;p,q)}\bigr)$$

by taking $\delta=\sqrt{B^{\lambda}_{n,p,q}((t-x)^2;x)}$, $\beta(n;p,q)$ is defined in (18). Theorem 3.2 is proved. □

Theorem 3.3 *For $f\in C_{[0,1]}$, $\lambda\in[-1,1]$, $x\in[0,1]$, and $n>1$, we have*

$$\bigl|B^{\lambda}_{n,p,q}(f;x)-f(x)\bigr|\leq C\omega_2\bigl(f;\sqrt{\alpha(n;p,q)^2+\beta(n;p,q)}/2\bigr)+\omega\bigl(f;\alpha(n;p,q)\bigr),$$

where C is a positive constant, $\alpha(n;p,q)$ and $\beta(n;p,q)$ are defined in (17) *and* (18).

Proof Let us define the auxiliary operators $\overline{B}^{\lambda}_{n,p,q}(f;x)$,

$$\overline{B}^{\lambda}_{n,p,q}(f;x)=B^{\lambda}_{n,p,q}(f;x)-f\bigl(x+\gamma^{\lambda}_{n,p,q}(x)\bigr)+f(x),\tag{21}$$

where $\gamma^{\lambda}_{n,p,q}(x)$ is defined in (16). Obviously, operators $\overline{B}^{\lambda}_{n,p,q}(f;x)$ preserve not only constant functions but also linear functions, say,

$$\overline{B}^{\lambda}_{n,p,q}(t-x;x)=0.\tag{22}$$

Letting $g\in C^2$, $x,t\in[0,1]$, then by Taylor's expansion

$$g(t)=g(x)+g'(t-x)+\int_x^t(t-u)g''(u)\,du\tag{23}$$

and (22), applying $\overline{B}^{\lambda}_{n,p,q}(g;x)$ to (23), we have

$$\overline{B}^{\lambda}_{n,p,q}(g;x)-g(x)=\overline{B}^{\lambda}_{n,p,q}\left(\int_x^t (t-u)g''(u)\,du;x\right).$$

Therefore, using (21), we get

$$\begin{aligned}
&\left|\overline{B}^{\lambda}_{n,p,q}(g;x)\right| \\
&\quad\le \left|B^{\lambda}_{n,p,q}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| + \left|\int_x^{x+\gamma^{\lambda}_{n,p,q}(x)}\left(x+\gamma^{\lambda}_{n,p,q}(x)-u\right)g''(u)\,du\right| \\
&\quad\le \left|B^{\lambda}_{n,p,q}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| + \int_x^{x+\gamma^{\lambda}_{n,p,q}(x)}\left|x+\gamma^{\lambda}_{n,p,q}(x)-u\right|\left|g''(u)\right|du \\
&\quad\le \left(B^{\lambda}_{n,p,q}\left((t-x)^2;x\right)+\left(\gamma^{\lambda}_{n,p,q}(x)\right)^2\right)\left|g''\right| \\
&\quad\le \left(\alpha(n;p,q)^2+\beta(n;p,q)\right)\left|g''\right|,
\end{aligned} \tag{24}$$

where $\alpha(n;p,q)$ and $\beta(n;p,q)$ are defined in (17) and (18). We also have the following fact by Lemma 2.2 and (21):

$$\left|\overline{B}^{\lambda}_{n,p,q}(f;x)\right|\le\left|B^{\lambda}_{n,p,q}(f;x)\right|+2\|f\|\le 3\|f\|. \tag{25}$$

Combining (21), (24), and (25), we have

$$\begin{aligned}
&\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right| \\
&\quad\le \left|\overline{B}^{\lambda}_{n,p,q}(f-g;x)-(f-g)(x)\right| + \left|\overline{B}^{\lambda}_{n,p,q}(g;x)-g(x)\right| + \left|f\left(x+\gamma^{\lambda}_{n,p,q}(x)\right)-f(x)\right| \\
&\quad\le 4\|f-g\| + \left(\alpha(n;p,q)^2+\beta(n;p,q)\right)\left|g''\right| + \omega\left(f;\alpha(n;p,q)\right).
\end{aligned}$$

Taking the infimum on the right-hand side over all $g\in C^2$ and using the relationship between K-functional and the second order modulus of smoothness, we can obtain

$$\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right|\le C\omega_2\left(f;\sqrt{\alpha(n;p,q)^2+\beta(n;p,q)}/2\right)+\omega\left(f;\alpha(n;p,q)\right).$$

This completes the proof of Theorem 3.3. □

Theorem 3.4 *For* $f'\in C_{[0,1]}$, $\lambda\in[-1,1]$, $x\in[0,1]$, *and* $n>1$, *we have*

$$\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right|\le\alpha(n;p,q)\left|f'(x)\right|+2\sqrt{\beta(n;p,q)}\,\omega\left(f';\sqrt{\beta(n;p,q)}\right),$$

where $\alpha(n;p,q)$, $\beta(n;p,q)$ *are defined in* (17) *and* (18), *respectively.*

Proof Applying $B^{\lambda}_{n,p,q}(f;x)$ to both sides of $f(t)=f(x)+f'(x)(t-x)+f(t)-f(x)-f'(x)(t-x)$, we have

$$\begin{aligned}
&\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right| \\
&\quad\le \left|f'(x)\right|\left|B^{\lambda}_{n,p,q}(t-x;x)\right| + B^{\lambda}_{n,p,q}\left(\left|f(t)-f(x)-f'(x)(t-x)\right|;x\right)
\end{aligned}$$

$$\begin{aligned}
&\leq |f'(x)|\left|B^{\lambda}_{n,p,q}(t-x;x)\right| + B^{\lambda}_{n,p,q}\left(|t-x|\left(1+\frac{|t-x|}{\delta}\right)\omega(f';\delta);x\right)\\
&\leq |f'(x)|\left|B^{\lambda}_{n,p,q}(t-x;x)\right| + \sqrt{B^{\lambda}_{n,p,q}\left((t-x)^2;x\right)}\left(1+\frac{1}{\delta}\sqrt{B^{\lambda}_{n,p,q}\left((t-x)^2;x\right)}\right)\\
&\quad\times\omega(f';\delta)
\end{aligned}$$

with the help of mean value theorem and the Cauchy–Schwarz inequality. Taking $\delta = \sqrt{B^{\lambda}_{n,p,q}((t-x)^2;x)}$ and by Corollary 2.5, we can get the desired result. Theorem 3.4 is proved. □

For $x, y \in [0,1]$, a function belongs to Lipschitz class $\mathrm{Lip}_M(\xi)$ if

$$\left|f(y)-f(x)\right| \leq M|y-x|^{\xi}, \tag{26}$$

where $M > 0$ and $\xi \in (0,1]$. We end up giving the rate of convergence of $B^{\alpha}_{n,p,q}(f;x)$ on $f \in \mathrm{Lip}_M(\xi)$.

Theorem 3.5 *Let $f \in \mathrm{Lip}_M(\xi)$, $\xi \in (0,1]$, then for $\lambda \in [-1,1]$, $x \in [0,1]$ and $n > 1$, we have*

$$\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right| \leq M\left(\sqrt{\beta(n;p,q)}\right)^{\xi},$$

where $\beta(n;p,q)$ is defined in (18).

Proof Since $f \in \mathrm{Lip}_M(\xi)$, we have

$$\begin{aligned}
&\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right|\\
&\quad\leq B^{\lambda}_{n,p,q}\left(\left|f(t)-f(x)\right|;x\right)\\
&\quad\leq M\sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q)\left|\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}-x\right|^{\xi}\\
&\quad\leq M\sum_{k=0}^{n}\left(b^{\lambda}_{n,k}(x;p,q)\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}-x\right)^2\right)^{\frac{\xi}{2}}\left(b^{\lambda}_{n,k}(x;p,q)\right)^{\frac{2-\xi}{2}}.
\end{aligned}$$

Applying Hölder's inequality for sums, we have

$$\begin{aligned}
&\left|B^{\lambda}_{n,p,q}(f;x)-f(x)\right|\\
&\quad\leq M\left(\sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q)\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}-x\right)^2\right)^{\frac{\xi}{2}}\left(\sum_{k=0}^{n} b^{\lambda}_{n,k}(x;p,q)\right)^{\frac{2-\xi}{2}}\\
&\quad= M\left(B^{\lambda}_{n,p,q}\left((t-x)^2;x\right)\right)^{\frac{\xi}{2}}\\
&\quad\leq M\left(\beta(n;p,q)\right)^{\frac{\xi}{2}},
\end{aligned}$$

where $\beta(n;p,q)$ is defined in (18). Theorem 3.5 is proved. □

Remark 3.6 Apparently, if the sequences $q := \{q_n\}$ and $p := \{p_n\}$ satisfy the conditions of Lemma 2.7 and $p^n \to a \in (0, b]$, b is finite, then $\alpha(n; p, q), \beta(n; p, q) \to 0$ as $n \to \infty$. Therefore, by Theorems 3.2–3.5, the convergence rate of $B^{\lambda}_{n,p,q}(f; x)$ to f is obtained.

4 Conclusion

In this paper, λ-Bernstein operators based on (p, q)-integers are constructed, a Korovkin type approximation theorem is established, and also the rate of convergence of $B^{\lambda}_{n,p,q}(f; x)$ to $f(x)$ is obtained.

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Authors' contributions
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