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Positive solutions for a system of $2n$th-order boundary value problems involving semipositone nonlinearities

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Abstract
In this paper we use the fixed point index to study the existence of positive solutions for a system of $2n$th-order boundary value problems involving semipositone nonlinearities.

MSC: 34B18; 45J05; 47H11

Keywords: $2n$th-order boundary value problems; Fixed point index; Positive solution

1 Introduction

In this paper we investigate the existence of positive solutions for the following system of $2n$th-order boundary value problems involving semipositone nonlinearities:

$$\begin{cases} (-1)^n x^{(2n)} = f_1(t,x,x',\dots,(-1)^{n-2}x^{(2n-4)},(-1)^{n-2}x^{(2n-3)},(-1)^{n-1}x^{(2n-2)}, \\ \qquad y,y',\dots,(-1)^{n-2}y^{(2n-4)},(-1)^{n-2}y^{(2n-3)},(-1)^{n-1}y^{(2n-2)}), \\ (-1)^n y^{(2n)} = f_2(t,x,x',\dots,(-1)^{n-2}x^{(2n-4)},(-1)^{n-2}x^{(2n-3)},(-1)^{n-1}x^{(2n-2)}, \\ \qquad y,y',\dots,(-1)^{n-2}y^{(2n-4)},(-1)^{n-2}y^{(2n-3)},(-1)^{n-1}y^{(2n-2)}), \\ x^{(2i)}(0) = x^{(2i+1)}(1) = 0, \qquad y^{(2i)}(0) = y^{(2i+1)}(1) = 0, \quad i = 0,1,\dots,n-1, \end{cases} \tag{1.1}$$

where $n \in N$ with $n \geq 1$, and $f_j \in C([0,1] \times R_+^{4n-2}, R)$ ($R_+ := [0,\infty)$, $R := (-\infty,+\infty)$, $j = 1,2$) satisfy the semipositone condition:

(H0) there is a positive constant M such that

$$f_j(t,z_1,z_2,\dots,z_{4n-2}) \geq -M, \quad t \in [0,1], z_i \in R_+, i = 1,2,\dots,4n-2, j = 1,2.$$

In recent years, coupled systems of boundary value problems have been investigated by many authors since such systems appear naturally in many real-world situations. Some recent results on the topic can be found in a series of papers [1–26] and the references therein. In [1], Yang used nonnegative matrix theory to study the existence of positive solutions for the system of generalized Lidstone problems,

$$\begin{cases} (-1)^m u^{(2m)} = f_1(t, u, -u'', \dots, (-1)^{m-1} u^{(2m-2)}, v, -v'', \dots, (-1)^{n-1} v^{(2n-2)}), \\ (-1)^n v^{(2n)} = f_2(t, u, -u'', \dots, (-1)^{m-1} u^{(2m-2)}, v, -v'', \dots, (-1)^{n-1} v^{(2n-2)}), \\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = \alpha_1 u^{(2i)}(1) + \beta_1 u^{(2i+1)}(1) = 0, \quad i = 0, 1, \dots, m-1, \\ \alpha_0 v^{(2j)}(0) - \beta_0 v^{(2j+1)}(0) = \alpha_1 v^{(2j)}(1) + \beta_1 v^{(2j+1)}(1) = 0, \quad j = 0, 1, \dots, n-1, \end{cases} \tag{1.2}$$

where $f_1, f_2 \in C([0,1] \times R_+^{m+n}, R_+)$, and in [2] Xu and Yang used some concave functions to depict the coupling behaviors for the nonlinearities f_i ($i = 1, 2$), and they established the existence of positive solutions for (1.2). In [3], Wang and Yang used similar methods as in [1] to study the existence of positive solutions for the system of higher-order boundary value problems involving all derivatives of odd orders

$$\begin{cases} (-1)^m w^{(2m)} = f(t, w, w', -w''', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', -z''', \dots, (-1)^{n-1} z^{(2n-1)}), \\ (-1)^n z^{(2n)} = g(t, w, w', -w''', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', -z''', \dots, (-1)^{n-1} z^{(2n-1)}), \\ w^{(2i)}(0) = w^{(2i+1)}(1) = 0, \quad i = 0, 1, \dots, m-1, \\ z^{(2j)}(0) = z^{(2j+1)}(1) = 0, \quad j = 0, 1, \dots, n-1, \end{cases}$$

where $f, g \in C([0,1] \times R_+^{m+n+2}, R_+)$. Moreover, they used a condition of Berstein–Nagumo type to obtain a priori estimates for $w^{(2m-1)}$ and $z^{(2n-1)}$. For related papers, we refer the reader to [27–33]. In [27] the authors used topological degree theory to study the existence of nontrivial solutions for the higher-order nonlinear fractional boundary value problem involving Riemann–Liouville fractional derivatives:

$$\begin{cases} D_{0+}^{\alpha} u(t) = -f(t, u(t), D_{0+}^{\beta_1} u(t), D_{0+}^{\beta_2} u(t), \dots, D_{0+}^{\beta_{n-1}}(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = D_{0+}^{\beta} u(1) = 0, \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β}, $D_{0+}^{\beta_i}$ are the Riemann–Liouville fractional derivatives, and $f \in C([0,1] \times R^n, R)$.

Motivated by the above work, in this paper we investigate the positive solutions for the system of $2n$th-order boundary value problems (1.1) involving semipositone nonlinearities. We first use the method of order reduction to transform (1.1) into an equivalent system of integro-integral equations, and then we establish a system of nonnegative operator equations. Using the fixed point index and nonnegative matrix theory, we study the existence of positive fixed points for the operator equations, and obtain positive solutions for (1.1).

2 Preliminaries

Let $E = C[0,1]$, $\|z\| = \max_{t\in[0,1]} |z(t)|$, $P = \{t \in [0,1] : z(t) \geq 0, \forall t \in [0,1]\}$. Then $(E, \|\cdot\|)$ is a Banach space, and P a cone on E. Let

$$k_1(t,s) := \min\{t,s\}, \qquad k_i(t,s) := \int_0^1 k_{i-1}(t,\tau) k_1(\tau,s)\,d\tau, \quad t,s \in [0,1], i = 2, 3, \dots, n,$$

and

$$(B_i z)(t) := \int_0^1 k_i(t,s)z(s)\,ds, \qquad h_i(t,s) := \partial k_i(t,s)/\partial t, \quad i = 1,2,\dots,n-1.$$

Note

$$\big((B_i z)(t)\big)' := \int_0^1 h_i(t,s)z(s)\,ds, \quad i = 1,2,\dots,$$

and $B_i, B_i' : E \to E$ are completely continuous linear operators, B_i, B_i' are also positive operators, i.e., they will map P into P.

Lemma 2.1 ([28]) *Let $\kappa_\psi = 1 - 2/e$, and $\psi(t) = te^t$, $t \in [0,1]$. Then we have*

$$\kappa_\psi \psi(s) \le \int_0^1 k_1(t,s)\psi(t)\,dt \le \psi(s).$$

Lemma 2.2 ([28]) *Let $z \in P$. Then we have*

$$\int_0^1 \left[(B_{n-1}z)(t) + 2\sum_{i=0}^{n-2}\big((B_{n-1-i}z)(t)\big)'\right]\psi(t)\,dt = \int_0^1 z(t)\psi(t)\,dt.$$

Lemma 2.3 ([34]) *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If there exists a $\omega_0 \in P \setminus \{0\}$ such that*

$$\omega - A\omega \ne \lambda\omega_0, \quad \forall \lambda \ge 0, \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 2.4 ([34]) *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \ne 0, \quad \forall \lambda \in [0,1], \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

Now, we consider the following auxiliary problem associated with (1.1):

$$\begin{cases} (-1)^n x^{(2n)} = f(t, x, x', \dots, (-1)^{n-2}x^{(2n-4)}, (-1)^{n-2}x^{(2n-3)}, (-1)^{n-1}x^{(2n-2)}), \\ x^{(2i)}(0) = x^{(2i+1)}(1) = 0, \quad i = 0,1,\dots,n-1, \end{cases}$$

where $f \in C([0,1] \times R_+^{2n-1}, R)$ satisfies the condition:

(H0)$'$ there is a positive constant M such that

$$f(t, z_1, z_2, \dots, z_{2n-1}) \ge -M, \quad t \in [0,1], z_i \in R_+, i = 1,2,\dots,2n-1.$$

Let $(-1)^{n-1}x^{(2n-2)}(t) = z(t)$, $t \in [0,1]$. Then we have

$$\begin{cases} -z''(t) = f(t, (B_{n-1}z)(t), ((B_{n-1}z)(t))', \ldots, (B_1 z)(t), ((B_1 z)(t))', z(t)), \\ z(0) = z'(1) = 0, \end{cases} \tag{2.1}$$

which can be expressed in the integral form

$$z(t) = \int_0^1 k_1(t,s) f\bigl(s, (B_{n-1}z)(s), \bigl((B_{n-1}z)(s)\bigr)', \ldots, (B_1 z)(s), \bigl((B_1 z)(s)\bigr)', z(s)\bigr)\, ds. \tag{2.2}$$

For convenience, let

$$(A_i z)(t) = \bigl((B_i z)(t)\bigr)', \quad t \in [0,1], i = 1, 2, \ldots, n-1.$$

As a result, we can also write (2.2) in the form

$$z(t) = \int_0^1 k_1(t,s) f\bigl(s, (B_{n-1}z)(s), (A_{n-1}z)(s), \ldots, (B_1 z)(s), (A_1 z)(s), z(s)\bigr)\, ds.$$

Let $w(t) = M \int_0^1 k_1(t,s)\, ds = M(t - t^2/2)$. We need to consider the following problem:

$$\begin{cases} -z''(t) = \widetilde{f}(t, (B_{n-1}(z-w))(t), (A_{n-1}(z-w))(t), \ldots, \\ \qquad\qquad (B_1(z-w))(t), (A_1(z-w))(t), (z-w)(t)), \\ z(0) = z'(1) = 0, \end{cases} \tag{2.3}$$

where

$$\widetilde{f}(t, z_1, \ldots, z_{2n-1}) = \begin{cases} f(t, z_1, \ldots, z_{2n-1}) + M, & t \in [0,1], z_i \geq 0, i = 1, 2, \ldots, 2n-1, \\ f(t, 0, \ldots, 0) + M, & t \in [0,1], \text{for else cases.} \end{cases}$$

Note that (2.3) can be expressed in the integral form

$$\begin{aligned} z(t) = \int_0^1 & k_1(t,s) \widetilde{f}\bigl(s, \bigl(B_{n-1}(z-w)\bigr)(s), \bigl(A_{n-1}(z-w)\bigr)(s), \ldots, \\ & \bigl(B_1(z-w)\bigr)(s), \bigl(A_1(z-w)\bigr)(s), (z-w)(s)\bigr)\, ds. \end{aligned}$$

Using (H0)′, we see that $\widetilde{f} \in C([0,1] \times R_+^{2n-1}, R_+)$.

Lemma 2.5

(i) *If z^* is a positive solution of* (2.1), *then $z^* + w$ is a positive solution of* (2.3).

(ii) *If z^{**} is a positive solution of* (2.3), *and greater than w, then $z^{**} - w$ is a positive solution of* (2.1).

Proof Substituting $z^* + w$ into (2.3), we have

$$\begin{cases} -z^{*\prime\prime}(t) - w''(t) = \widetilde{f}(t, (B_{n-1}(z^* + w - w))(t), (A_{n-1}(z^* + w - w))(t), \ldots, \\ \qquad\qquad (B_1(z^* + w - w))(t), (A_1(z^* + w - w))(t), (z^* + w - w)(t)), \\ (z^* + w)(0) = (z^* + w)'(1) = 0. \end{cases} \tag{2.4}$$

Note that w satisfies the boundary value problem

$$\begin{cases} -z''(t) = M, \\ z(0) = z'(1) = 0. \end{cases}$$

By virtue of (2.4), we have

$$\begin{cases} -z^{*\prime\prime}(t) - w''(t) = f(t, (B_{n-1}z^*)(t), (A_{n-1}z^*)(t), \dots, (B_1z^*)(t), (A_1z^*)(t), z^*(t)) + M, \\ z^*(0) = z^{*\prime}(1) = 0, \end{cases}$$

which is (2.1).

On the other hand, we substitute $z^{**} - w$ into (2.1), and obtain

$$\begin{cases} -z^{**\prime\prime}(t) + w''(t) = f(t, (B_{n-1}(z^{**} - w))(t), (A_{n-1}(z^{**} - w))(t), \dots, \\ \qquad\qquad (B_1(z^{**} - w))(t), (A_1(z^{**} - w))(t), (z^{**} - w)(t)), \\ (z^{**} - w)(0) = (z^{**} - w)'(1) = 0. \end{cases}$$

Note that, from the definitions of w and $\widetilde{f}$, we have

$$\begin{cases} -z^{**\prime\prime}(t) = \widetilde{f}(t, (B_{n-1}(z^{**} - w))(t), (A_{n-1}(z^{**} - w))(t), \dots, \\ \qquad\qquad (B_1(z^{**} - w))(t), (A_1(z^{**} - w))(t), (z^{**} - w)(t)), \\ z^{**}(0) = z^{**\prime}(1) = 0, \end{cases}$$

which is (2.3). This completes the proof. □

From Lemma 2.5, if we wish to seek the positive solutions for (2.1), we only need to study the positive solutions for (2.3), which are greater than w. Consequently, we define an operator $T : P \to E$ as follows:

$$\begin{aligned} (Tz)(t) = \int_0^1 & k_1(t,s)\widetilde{f}\bigl(s, \bigl(B_{n-1}(z - w)\bigr)(s), \bigl(A_{n-1}(z - w)\bigr)(s), \dots, \\ & \bigl(B_1(z - w)\bigr)(s), \bigl(A_1(z - w)\bigr)(s), (z - w)(s)\bigr)\, ds. \end{aligned}$$

Then T is a completely continuous operator, and if there exists a $\bar{z} \in P$ with $\bar{z} \geq w$ such that $T\bar{z} = \bar{z}$, we see that $\bar{z} - w$ is a positive solution of (2.1).

Let

$$P_0 = \bigl\{z \in P : z(t) \geq t\|z\|, \forall t \in [0,1]\bigr\}.$$

Then P_0 is also a cone on E, and we have the following lemma.

Lemma 2.6 $T(P) \subset P_0$.

Note that, for $t, s \in [0,1]$, $tk_1(s,s) \leq k_1(t,s) \leq k_1(s,s)$ and $k_1(s,s) = s$, so we can easily obtain this lemma (the details are omitted).

From Lemmas 2.5 and 2.6, we have $\overline{z} \in P_0$ if $\overline{z}$ is a fixed point of T. Consequently, if $\|\overline{z}\| \geq M$ we have

$$\overline{z}(t) - w(t) \geq t\|\overline{z}\| - M\bigl(t - t^2/2\bigr) \geq t\|\overline{z}\| - tM \geq 0.$$

Hence, we only need to seek T's positive fixed point $\overline{z}$ with $\|\overline{z}\| \geq M$, and then $\overline{z} - w$ is a positive solution of (2.1).

3 Main results

In (1.1), let $(-1)^{n-1}x^{(2n-2)} = u$ and $(-1)^{n-1}y^{(2n-2)} = v$, then we obtain the following system of boundary value problems:

$$\begin{cases} -u''(t) = f_1(t, (B_{n-1}u)(t), (A_{n-1}u)(t), \dots, (B_1u)(t), (A_1u)(t), u(t), \\ \qquad (B_{n-1}v)(t), (A_{n-1}v)(t), \dots, (B_1v)(t), (A_1v)(t), v(t)), \\ -v''(t) = f_2(t, (B_{n-1}u)(t), (A_{n-1}u)(t), \dots, (B_1u)(t), (A_1u)(t), u(t), \\ \qquad (B_{n-1}v)(t), (A_{n-1}v)(t), \dots, (B_1v)(t), (A_1v)(t), v(t)), \\ u(0) = u'(1) = 0, \qquad v(0) = v'(1) = 0, \end{cases} \tag{3.1}$$

which has the integral form

$$\begin{cases} u(t) = \int_0^1 k_1(t,s) f_1(s, (B_{n-1}u)(s), (A_{n-1}u)(s), \dots, (B_1u)(s), (A_1u)(s), u(s), \\ \qquad (B_{n-1}v)(s), (A_{n-1}v)(s), \dots, (B_1v)(s), (A_1v)(s), v(s))\,ds, \\ v(t) = \int_0^1 k_1(t,s) f_2(s, (B_{n-1}u)(s), (A_{n-1}u)(s), \dots, (B_1u)(s), (A_1u)(s), u(s), \\ \qquad (B_{n-1}v)(s), (A_{n-1}v)(s), \dots, (B_1v)(s), (A_1v)(s), v(s))\,ds. \end{cases}$$

For $j = 1, 2$, let

$$\begin{aligned} &F_j(t, z_1, z_2, \dots, z_{4n-2}) \\ &\quad = \begin{cases} f_j(t, z_1, z_2, \dots, z_{4n-2}) + M, & t \in [0,1], z_i \geq 0, i = 1, 2, \dots, 4n-2, \\ f_j(t, 0, 0, \dots, 0) + M, & t \in [0,1], \text{for other cases.} \end{cases} \end{aligned}$$

Then we can define the operators T_j $(j = 1,2) : P^{4n-2} \to P$ and $T : P^2 \to P^2$ as follows:

$$\begin{aligned} T_j(u,v)(t) = \int_0^1 k_1(t,s) F_j\bigl(s, &\bigl(B_{n-1}(u-w)\bigr)(s), \bigl(A_{n-1}(u-w)\bigr)(s), \dots, \\ &\bigl(B_1(u-w)\bigr)(s), \bigl(A_1(u-w)\bigr)(s), (u-w)(s), \\ &\bigl(B_{n-1}(v-w)\bigr)(s), \bigl(A_{n-1}(v-w)\bigr)(s), \dots, \\ &\bigl(B_1(v-w)\bigr)(s), \bigl(A_1(v-w)\bigr)(s), (v-w)(s)\bigr)\,ds \end{aligned}$$

and

$$T(u,v)(t) = (T_1, T_2)(u,v)(t), \quad t \in [0,1].$$

Then, if we find the positive fixed point (u^*, v^*) of T with $u^*, v^* \geq w$, then $(u^* - w, v^* - w)$ is a positive solution for (3.1). Let

$$x(t) = \int_0^1 k_{n-1}(t,s)\big(u^*(s) - w(s)\big)\,ds, \qquad y(t) = \int_0^1 k_{n-1}(t,s)\big(v^*(s) - w(s)\big)\,ds, \tag{3.2}$$

and we will obtain the positive solution for (1.1) (note from the discussion in Sect. 2, we need the norms of u^*, v^* to be greater than M).

Now, we list our assumptions for F_j $(j = 1, 2)$:

(H1) There exist $a_{j1}, b_{j1}, c_{j1}, d_{j1}, l_j > 0$ $(j = 1, 2)$ such that

$$\kappa_\psi\big(b_{11} + a_{11}(n-1)\big) < 1, \qquad \kappa_\psi\big(d_{21} + c_{21}(n-1)\big) < 1,$$
$$\Delta_{11} = \det\begin{pmatrix} \kappa_\psi(d_{11} + c_{11}(n-1)) & \kappa_\psi(b_{11} + a_{11}(n-1)) - 1 \\ \kappa_\psi(d_{21} + c_{21}(n-1)) - 1 & \kappa_\psi(b_{21} + a_{21}(n-1)) \end{pmatrix} > 0,$$

and, for all $t \in [0,1]$, $z_i, \widetilde{z}_i \in R_+$, $i = 1, 2, \ldots, 2n-1$,

$$\begin{aligned} &F_1(t, z_1, z_2, \ldots, z_{2n-3}, z_{2n-2}, z_{2n-1}, \widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{2n-3}, \widetilde{z}_{2n-2}, \widetilde{z}_{2n-1}) \\ &\quad \geq a_{11}(z_1 + z_3 + \cdots + z_{2n-3}) + a_{11}\big(2z_2 + 4z_4 + \cdots + 2(n-1)z_{2n-2}\big) + b_{11}z_{2n-1} \\ &\qquad + c_{11}(\widetilde{z}_1 + \widetilde{z}_3 + \cdots + \widetilde{z}_{2n-3}) + c_{11}\big(2\widetilde{z}_2 + 4\widetilde{z}_4 + \cdots + 2(n-1)\widetilde{z}_{2n-2}\big) + d_{11}\widetilde{z}_{2n-1} - l_1, \\ &F_2(t, z_1, z_2, \ldots, z_{2n-3}, z_{2n-2}, z_{2n-1}, \widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{2n-3}, \widetilde{z}_{2n-2}, \widetilde{z}_{2n-1}) \\ &\quad \geq a_{21}(z_1 + z_3 + \cdots + z_{2n-3}) + a_{21}\big(2z_2 + 4z_4 + \cdots + 2(n-1)z_{2n-2}\big) + b_{21}z_{2n-1} \\ &\qquad + c_{21}(\widetilde{z}_1 + \widetilde{z}_3 + \cdots + \widetilde{z}_{2n-3}) + c_{21}\big(2\widetilde{z}_2 + 4\widetilde{z}_4 + \cdots + 2(n-1)\widetilde{z}_{2n-2}\big) + d_{21}\widetilde{z}_{2n-1} - l_2. \end{aligned}$$

(H2) There exist Q_j $(j = 1, 2) : [0,1] \to R$ such that

$$\int_0^1 k_1(s,s)Q_j(s)\,ds < M$$

and

$$F_j(t, z_1, z_2, \ldots, z_{2n-3}, z_{2n-2}, z_{2n-1}, \widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{2n-3}, \widetilde{z}_{2n-2}, \widetilde{z}_{2n-1}) \leq Q_j(t),$$

for all $t \in [0,1]$, $z_i, \widetilde{z}_i \in [0, M]$, $i = 1, 2, \ldots, 2n-1$, $j = 1, 2$.

(H3) There exist $\widetilde{a}_{j1}, \widetilde{b}_{j1}, \widetilde{c}_{j1}, \widetilde{d}_{j1}, \widetilde{l}_j > 0$ $(j = 1, 2)$ such that

$$\widetilde{b}_{11} + \widetilde{a}_{11}(n-1) < 1, \qquad \widetilde{d}_{21} + \widetilde{c}_{21}(n-1) < 1,$$
$$\Delta_{22} = \det\begin{pmatrix} 1 - [\widetilde{b}_{11} + \widetilde{a}_{11}(n-1)] & -[\widetilde{d}_{11} + \widetilde{c}_{11}(n-1)] \\ -[\widetilde{b}_{21} + \widetilde{a}_{21}(n-1)] & 1 - [\widetilde{d}_{21} + \widetilde{c}_{21}(n-1)] \end{pmatrix} > 0,$$

and, for all $t \in [0,1]$, $z_i, \widetilde{z}_i \in R_+$, $i = 1, 2, \ldots, 2n-1$,

$$\begin{aligned} &F_1(t, z_1, z_2, \ldots, z_{2n-3}, z_{2n-2}, z_{2n-1}, \widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{2n-3}, \widetilde{z}_{2n-2}, \widetilde{z}_{2n-1}) \\ &\quad \leq \widetilde{a}_{11}(z_1 + z_3 + \cdots + z_{2n-3}) + \widetilde{a}_{11}\big(2z_2 + 4z_4 + \cdots + 2(n-1)z_{2n-2}\big) + \widetilde{b}_{11}z_{2n-1} \end{aligned}$$

$$\begin{aligned}&+\widetilde{c}_{11}(\widetilde{z}_1+\widetilde{z}_3+\cdots+\widetilde{z}_{2n-3})+\widetilde{c}_{11}\big(2\widetilde{z}_2+4\widetilde{z}_4+\cdots+2(n-1)\widetilde{z}_{2n-2}\big)+\widetilde{d}_{11}\widetilde{z}_{2n-1}+\widetilde{l}_1,\\
F_2&(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})\\
&\le\widetilde{a}_{21}(z_1+z_3+\cdots+z_{2n-3})+\widetilde{a}_{21}\big(2z_2+4z_4+\cdots+2(n-1)z_{2n-2}\big)+\widetilde{b}_{21}z_{2n-1}\\
&\quad+\widetilde{c}_{21}(\widetilde{z}_1+\widetilde{z}_3+\cdots+\widetilde{z}_{2n-3})+\widetilde{c}_{21}\big(2\widetilde{z}_2+4\widetilde{z}_4+\cdots+2(n-1)\widetilde{z}_{2n-2}\big)+\widetilde{d}_{21}\widetilde{z}_{2n-1}+\widetilde{l}_2.\end{aligned}$$

(H4) There exist $\widetilde{Q}_j$ $(j=1,2):[0,1]\to R$ and $t_1,t_2\in(0,1]$ such that

$$\int_0^1 k_1(t_j,s)\widetilde{Q}_j(s)\,ds>M$$

and

$$F_j(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})\ge\widetilde{Q}_j(t),$$

for all $t\in[0,1]$, $z_i,\widetilde{z}_i\in[0,M]$, $i=1,2,\ldots,2n-1$, $j=1,2$.

Let $B_\rho=\{u\in P:\|u\|<\rho\}$ for $\rho>0$ in the sequel. Then we easily have $\partial B_\rho=\{u\in P:\|u\|=\rho\}$, $\overline{B}_\rho=\{u\in P:\|u\|\le\rho\}$.

Theorem 3.1 *Suppose that* (H0)–(H2) *hold. Then* (1.1) *has at least one positive solution.*

Proof We first prove that there exists $R_1>M$ such that

$$(u,v)\ne T(u,v)+\lambda(\phi_1,\phi_2),\quad \text{for } (u,v)\in\partial B_{R_1}\cap(P\times P),\lambda\ge0, \tag{3.3}$$

where ϕ_i $(i=1,2)$ are given elements in the cone P_0. We argue by contradiction. Suppose there exist $(u,v)\in\partial B_{R_1}\cap(P\times P)$ and $\lambda_0\ge0$ with

$$(u,v)=T(u,v)+\lambda_0(\phi_1,\phi_2). \tag{3.4}$$

This, together with Lemma 2.6, implies that $u,v\in P_0$. Moreover, from (H1) we have

$$\begin{aligned}&\begin{pmatrix}u(t)\\v(t)\end{pmatrix}\\
&=\begin{pmatrix}T_1(u,v)(t)+\lambda_0\phi_1(t)\\T_2(u,v)(t)+\lambda_0\phi_2(t)\end{pmatrix}\ge\begin{pmatrix}T_1(u,v)(t)\\T_2(u,v)(t)\end{pmatrix}\\
&\ge\begin{pmatrix}\begin{array}{l}\int_0^1 k_1(t,s)(a_{11}\sum_{i=1}^{n-1}[(B_i(u-w))(s)+2(n-i)(A_i(u-w))(s)]+b_{11}(u-w)(s))\,ds\\
\quad+\int_0^1 k_1(t,s)(c_{11}\sum_{i=1}^{n-1}[(B_i(v-w))(s)+2(n-i)(A_i(v-w))(s)]+d_{11}(v-w)(s))\,ds\\
\quad-l_1\int_0^1 k_1(t,s)\,ds\\
\int_0^1 k_1(t,s)(a_{21}\sum_{i=1}^{n-1}[(B_i(u-w))(s)+2(n-i)(A_i(u-w))(s)]+b_{21}(u-w)(s))\,ds\\
\quad+\int_0^1 k_1(t,s)(c_{21}\sum_{i=1}^{n-1}[(B_i(v-w))(s)+2(n-i)(A_i(v-w))(s)]+d_{21}(v-w)(s))\,ds\\
\quad-l_2\int_0^1 k_1(t,s)\,ds\end{array}\end{pmatrix}.\end{aligned}$$

Multiply by $\psi(t)$ on both sides, integrate over $[0,1]$, and use Lemma 2.1, and we have

$$\begin{aligned}
&\begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \\
&\quad\geq \begin{pmatrix} \kappa_\psi \int_0^1 \psi(t)(a_{11}\sum_{i=1}^{n-1}[(B_i(u-w))(t)+2(n-i)(A_i(u-w))(t)]+b_{11}(u-w)(t))\,dt \\ \quad + \kappa_\psi \int_0^1 \psi(t)(c_{11}\sum_{i=1}^{n-1}[(B_i(v-w))(t)+2(n-i)(A_i(v-w))(t)]+d_{11}(v-w)(t))\,dt \\ \quad - l_1\int_0^1 \psi(t)\,dt \\ \kappa_\psi \int_0^1 \psi(t)(a_{21}\sum_{i=1}^{n-1}[(B_i(u-w))(t)+2(n-i)(A_i(u-w))(t)]+b_{21}(u-w)(t))\,dt \\ \quad + \kappa_\psi \int_0^1 \psi(t)(c_{21}\sum_{i=1}^{n-1}[(B_i(v-w))(t)+2(n-i)(A_i(v-w))(t)]+d_{21}(v-w)(t))\,dt \\ \quad - l_2\int_0^1 \psi(t)\,dt \end{pmatrix} \\
&\quad= \begin{pmatrix} \kappa_\psi \int_0^1 \psi(t)(a_{11}\sum_{i=1}^{n-1}[(B_i(u-w))(t)+2(n-i)(B_i(u-w))'(t)]+b_{11}(u-w)(t))\,dt \\ \quad + \kappa_\psi \int_0^1 \psi(t)(c_{11}\sum_{i=1}^{n-1}[(B_i(v-w))(t)+2(n-i)(B_i(v-w))'(t)]+d_{11}(v-w)(t))\,dt - l_1 \\ \kappa_\psi \int_0^1 \psi(t)(a_{21}\sum_{i=1}^{n-1}[(B_i(u-w))(t)+2(n-i)(B_i(u-w))'(t)]+b_{21}(u-w)(t))\,dt \\ \quad + \kappa_\psi \int_0^1 \psi(t)(c_{21}\sum_{i=1}^{n-1}[(B_i(v-w))(t)+2(n-i)(B_i(v-w))'(t)]+d_{21}(v-w)(t))\,dt - l_2 \end{pmatrix}.
\end{aligned}$$

Using Lemma 2.2 we obtain

$$\begin{aligned}
&\begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \\
&\quad\geq \begin{pmatrix} \kappa_\psi(b_{11}+a_{11}(n-1))\int_0^1(u-w)(t)\psi(t)\,dt+\kappa_\psi(d_{11}+c_{11}(n-1))\int_0^1(v-w)(t)\psi(t)\,dt-l_1 \\ \kappa_\psi(b_{21}+a_{21}(n-1))\int_0^1(u-w)(t)\psi(t)\,dt+\kappa_\psi(d_{21}+c_{21}(n-1))\int_0^1(v-w)(t)\psi(t)\,dt-l_2 \end{pmatrix}.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{N}_1 &= \kappa_\psi\big[\big(b_{11}+a_{11}(n-1)\big)+\big(d_{11}+c_{11}(n-1)\big)\big]\int_0^1 w(t)\psi(t)\,dt+l_1 \\
&= \kappa_\psi\big[\big(b_{11}+a_{11}(n-1)\big)+\big(d_{11}+c_{11}(n-1)\big)\big](2e-5)M+l_1, \\
\mathcal{N}_2 &= \kappa_\psi\big[\big(b_{21}+a_{21}(n-1)\big)+\big(d_{21}+c_{21}(n-1)\big)\big]\int_0^1 w(t)\psi(t)\,dt+l_2 \\
&= \kappa_\psi\big[\big(b_{21}+a_{21}(n-1)\big)+\big(d_{21}+c_{21}(n-1)\big)\big](2e-5)M+l_2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\begin{pmatrix} [\kappa_\psi(b_{11}+a_{11}(n-1))-1]\int_0^1 u(t)\psi(t)\,dt+\kappa_\psi(d_{11}+c_{11}(n-1))\int_0^1 v(t)\psi(t)\,dt \\ \kappa_\psi(b_{21}+a_{21}(n-1))\int_0^1 u(t)\psi(t)\,dt+[\kappa_\psi(d_{21}+c_{21}(n-1))-1]\int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \\
&\quad\leq \begin{pmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{pmatrix}
\end{aligned}$$

and

$$\begin{pmatrix} \kappa_\psi(d_{11}+c_{11}(n-1)) & \kappa_\psi(b_{11}+a_{11}(n-1))-1 \\ \kappa_\psi(d_{21}+c_{21}(n-1))-1 & \kappa_\psi(b_{21}+a_{21}(n-1)) \end{pmatrix}\begin{pmatrix} \int_0^1 v(t)\psi(t)\,dt \\ \int_0^1 u(t)\psi(t)\,dt \end{pmatrix} \leq \begin{pmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{pmatrix}.$$

Solving this matrix inequality, we obtain

$$\begin{pmatrix} \int_0^1 v(t)\psi(t)\,dt \\ \int_0^1 u(t)\psi(t)\,dt \end{pmatrix} \le \frac{1}{\Delta_{11}} \begin{pmatrix} \kappa_\psi(b_{21}+a_{21}(n-1)) & 1-\kappa_\psi(b_{11}+a_{11}(n-1)) \\ 1-\kappa_\psi(d_{21}+c_{21}(n-1)) & \kappa_\psi(d_{11}+c_{11}(n-1)) \end{pmatrix} \begin{pmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{pmatrix}.$$

Consequently, there exist $\widetilde{\mathcal{N}}_1, \widetilde{\mathcal{N}}_2 > 0$ such that

$$\begin{pmatrix} \int_0^1 v(t)\psi(t)\,dt \\ \int_0^1 u(t)\psi(t)\,dt \end{pmatrix} \le \begin{pmatrix} \widetilde{\mathcal{N}}_1 \\ \widetilde{\mathcal{N}}_2 \end{pmatrix}.$$

Note that $u, v \in P_0$, and we have

$$\begin{pmatrix} \|v\| \\ \|u\| \end{pmatrix} \le \begin{pmatrix} \frac{\widetilde{\mathcal{N}}_1}{e-2} \\ \frac{\widetilde{\mathcal{N}}_2}{e-2} \end{pmatrix}.$$

Therefore, we can choose $R_1 > \max\{M, \frac{\widetilde{\mathcal{N}}_1}{e-2}, \frac{\widetilde{\mathcal{N}}_2}{e-2}\}$ such that (3.4) is false, and thus (3.3) holds. From Lemma 2.3 we have

$$i\bigl(T, B_{R_1} \cap (P \times P), P \times P\bigr) = 0. \tag{3.5}$$

Next we prove that

$$(u,v) \neq \lambda T(u,v), \quad \text{for } (u,v) \in \partial B_M \cap (P \times P), \forall \lambda \in [0,1]. \tag{3.6}$$

If not, there exist $(u,v) \in \partial B_M \cap (P \times P)$ and $\lambda_1 \in [0,1]$ such that

$$(u,v) = \lambda_1 T(u,v).$$

This, combining with (H2), implies that

$$\begin{pmatrix} M \\ M \end{pmatrix} = \begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix} \le \begin{pmatrix} \|T_1(u,v)\| \\ \|T_2(u,v)\| \end{pmatrix} \le \begin{pmatrix} \int_0^1 k_1(s,s)Q_1(s)\,ds \\ \int_0^1 k_1(s,s)Q_2(s)\,ds \end{pmatrix} < \begin{pmatrix} M \\ M \end{pmatrix}.$$

This is a contradiction, and thus (3.6) is true. From Lemma 2.4 we have

$$\bigl(T, B_M \cap (P \times P), P \times P\bigr) = 1. \tag{3.7}$$

From (3.5) and (3.7) we have

$$\begin{aligned} &i\bigl(T, (B_{R_1} \setminus \overline{B}_M) \cap (P \times P), P \times P\bigr) \\ &\quad = i\bigl(T, B_{R_1} \cap (P \times P), P \times P\bigr) - i\bigl(T, B_M \cap (P \times P), P \times P\bigr) = 0 - 1 = -1. \end{aligned}$$

Therefore the operator T has at least one fixed point (u^*, v^*) on $(B_{R_1} \setminus \overline{B}_M) \cap (P \times P)$ with $\|u^*\| \ge M$, $\|v^*\| \ge M$, and note from (3.2) we see that (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.2 *Suppose that* (H0), *and* (H3)–(H4) *hold*. *Then* (1.1) *has at least one positive solution*.

Proof We first prove that

$$(u,v) \neq T(u,v) + \lambda(\widetilde{\phi}_1,\widetilde{\phi}_2), \quad \text{for } (u,v) \in \partial B_M \cap (P \times P), \lambda \geq 0, \tag{3.8}$$

where $\widetilde{\phi}_i$ $(i = 1,2) \in P$ are fixed elements. If this claim is false, there exist $(u,v) \in \partial B_M \cap (P \times P)$ and $\lambda_2 \geq 0$ such that

$$(u,v) = T(u,v) + \lambda_2(\widetilde{\phi}_1,\widetilde{\phi}_2).$$

This, together with (H4), gives

$$\begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix} \geq \begin{pmatrix} u(t_1) \\ v(t_2) \end{pmatrix} \geq \begin{pmatrix} T_1(u,v)(t_1) \\ T_2(u,v)(t_2) \end{pmatrix} \geq \begin{pmatrix} \int_0^1 k_1(t_1,s)\widetilde{Q}_1(s)\,ds \\ \int_0^1 k_1(t_2,s)\widetilde{Q}_2(s)\,ds \end{pmatrix} > \begin{pmatrix} M \\ M \end{pmatrix}.$$

This is a contradiction, and thus (3.8) holds. From Lemma 2.3 we have

$$i\bigl(T, B_M \cap (P \times P), P \times P\bigr) = 0. \tag{3.9}$$

Next we show that there is a large number $R_2 > M$ such that

$$(u,v) \neq \lambda T(u,v), \quad \text{for } (u,v) \in \partial B_{R_2} \cap (P \times P), \forall \lambda \in [0,1]. \tag{3.10}$$

We argue by contradiction, so we assume there exist $(u,v) \in \partial B_{R_2} \cap (P \times P)$ and $\lambda_3 \in [0,1]$ such that

$$(u,v) = \lambda_3 T(u,v).$$

Lemma 2.6 implies that $u, v \in P_0$, and from (H3) we obtain

$$\begin{aligned} &\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \\ &\quad \leq \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} \\ &\quad \leq \begin{pmatrix} \begin{aligned} &\int_0^1 k_1(t,s)(\widetilde{a}_{11} \textstyle\sum_{i=1}^{n-1} [(B_i(u-w))(s) + 2(n-i)(A_i(u-w))(s)] + \widetilde{b}_{11}(u-w)(s))\,ds \\ &\quad + \int_0^1 k_1(t,s)(\widetilde{c}_{11} \textstyle\sum_{i=1}^{n-1} [(B_i(v-w))(s) + 2(n-i)(A_i(v-w))(s)] + \widetilde{d}_{11}(v-w)(s))\,ds \\ &\quad + \widetilde{l}_1 \int_0^1 k_1(t,s)\,ds \\ &\int_0^1 k_1(t,s)(\widetilde{a}_{21} \textstyle\sum_{i=1}^{n-1} [(B_i(u-w))(s) + 2(n-i)(A_i(u-w))(s)] + \widetilde{b}_{21}(u-w)(s))\,ds \\ &\quad + \int_0^1 k_1(t,s)(\widetilde{c}_{21} \textstyle\sum_{i=1}^{n-1} [(B_i(v-w))(s) + 2(n-i)(A_i(v-w))(s)] + \widetilde{d}_{21}(v-w)(s))\,ds \\ &\quad + \widetilde{l}_2 \int_0^1 k_1(t,s)\,ds \end{aligned} \end{pmatrix}. \end{aligned}$$

Multiply by $\psi(t)$ on both sides, integrate over $[0,1]$, and use Lemma 2.1, we have

$$\begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \leq \begin{pmatrix} \int_0^1 \psi(t)(\widetilde{a}_{11}\sum_{i=1}^{n-1}[(B_i(u-w))(t)+2(n-i)(A_i(u-w))(t)]+\widetilde{b}_{11}(u-w)(t))\,dt \\ \quad +\int_0^1 \psi(t)(\widetilde{c}_{11}\sum_{i=1}^{n-1}[(B_i(v-w))(t)+2(n-i)(A_i(v-w))(t)]+\widetilde{d}_{11}(v-w)(t))\,dt+\widetilde{l}_1 \\ \int_0^1 \psi(t)(\widetilde{a}_{21}\sum_{i=1}^{n-1}[(B_i(u-w))(t)+2(n-i)(A_i(u-w))(t)]+\widetilde{b}_{21}(u-w)(t))\,dt \\ \quad +\int_0^1 \psi(t)(\widetilde{c}_{21}\sum_{i=1}^{n-1}[(B_i(v-w))(t)+2(n-i)(A_i(v-w))(t)]+\widetilde{d}_{21}(v-w)(t))\,dt+\widetilde{l}_2 \end{pmatrix}.$$

This, combining with Lemma 2.2, implies that

$$\begin{aligned} \begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} &\leq \begin{pmatrix} \int_0^1 \psi(t)(\widetilde{a}_{11}\sum_{i=1}^{n-1}[(B_iu)(t)+2(n-i)(B_iu)'(t)]+\widetilde{b}_{11}u(t))\,dt \\ \quad +\int_0^1 \psi(t)(\widetilde{c}_{11}\sum_{i=1}^{n-1}[(B_iv)(t)+2(n-i)(B_iv)'(t)]+\widetilde{d}_{11}v(t))\,dt+\widetilde{l}_1 \\ \int_0^1 \psi(t)(\widetilde{a}_{21}\sum_{i=1}^{n-1}[(B_iu)(t)+2(n-i)(B_iu)'(t)]+\widetilde{b}_{21}u(t))\,dt \\ \quad +\int_0^1 \psi(t)(\widetilde{c}_{21}\sum_{i=1}^{n-1}[(B_iv)(t)+2(n-i)(B_iv)'(t)]+\widetilde{d}_{21}v(t))\,dt+\widetilde{l}_2 \end{pmatrix} \\ &= \begin{pmatrix} [\widetilde{b}_{11}+\widetilde{a}_{11}(n-1)]\int_0^1 u(t)\psi(t)\,dt+[\widetilde{d}_{11}+\widetilde{c}_{11}(n-1)]\int_0^1 v(t)\psi(t)\,dt+\widetilde{l}_1 \\ [\widetilde{b}_{21}+\widetilde{a}_{21}(n-1)]\int_0^1 u(t)\psi(t)\,dt+[\widetilde{d}_{21}+\widetilde{c}_{21}(n-1)]\int_0^1 v(t)\psi(t)\,dt+\widetilde{l}_2 \end{pmatrix}. \end{aligned}$$

Consequently, we have

$$\begin{pmatrix} 1-[\widetilde{b}_{11}+\widetilde{a}_{11}(n-1)] & -[\widetilde{d}_{11}+\widetilde{c}_{11}(n-1)] \\ -[\widetilde{b}_{21}+\widetilde{a}_{21}(n-1)] & 1-[\widetilde{d}_{21}+\widetilde{c}_{21}(n-1)] \end{pmatrix} \begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \leq \begin{pmatrix} \widetilde{l}_1 \\ \widetilde{l}_2 \end{pmatrix}.$$

Solving this matrix inequality, we obtain

$$\begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \leq \frac{1}{\Delta_{22}} \begin{pmatrix} 1-[\widetilde{d}_{21}+\widetilde{c}_{21}(n-1)] & \widetilde{d}_{11}+\widetilde{c}_{11}(n-1) \\ \widetilde{b}_{21}+\widetilde{a}_{21}(n-1) & 1-[\widetilde{b}_{11}+\widetilde{a}_{11}(n-1)] \end{pmatrix} \begin{pmatrix} \widetilde{l}_1 \\ \widetilde{l}_2 \end{pmatrix}.$$

Therefore, there exist $\widetilde{\mathcal{N}}_3, \widetilde{\mathcal{N}}_4 > 0$ such that

$$\begin{pmatrix} \int_0^1 u(t)\psi(t)\,dt \\ \int_0^1 v(t)\psi(t)\,dt \end{pmatrix} \leq \begin{pmatrix} \widetilde{\mathcal{N}}_3 \\ \widetilde{\mathcal{N}}_4 \end{pmatrix}.$$

Note that $u, v \in P_0$, and then we obtain

$$\begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix} \leq \begin{pmatrix} \frac{\widetilde{\mathcal{N}}_3}{e-2} \\ \frac{\widetilde{\mathcal{N}}_4}{e-2} \end{pmatrix}.$$

If we choose $R_2 > \max\{M, \frac{\widetilde{\mathcal{N}}_3}{e-2}, \frac{\widetilde{\mathcal{N}}_4}{e-2}\}$ then (3.10) holds. From Lemma 2.4 we have

$$i\big(T, B_{R_2} \cap (P \times P), P \times P\big) = 1. \tag{3.11}$$

From (3.9) and (3.11) we have

$$\begin{aligned} & i\big(T,(B_{R_2}\setminus\overline{B}_M)\cap(P\times P),P\times P\big) \\ &\quad = i\big(T,B_{R_2}\cap(P\times P),P\times P\big)-i\big(T,B_M\cap(P\times P),P\times P\big)=1-0=1. \end{aligned}$$

Therefore the operator T has at least one fixed point (u^*,v^*) on $(B_{R_2}\setminus\overline{B}_M)\cap(P\times P)$ with $\|u^*\|\geq M$, $\|v^*\|\geq M$, and note from (3.2) we see that (1.1) has at least one positive solution. This completes the proof. □

Example 3.3 Let

$$\begin{aligned} a_{11} &= \frac{e}{4(e-2)(n-1)}, & b_{11} &= \frac{e}{2(e-2)}, \\ c_{11} &= \frac{e}{(e-2)(n-1)}, & d_{11} &= \frac{e}{e-2}, \\ a_{21} &= \frac{e}{2(e-2)(n-1)}, & b_{21} &= \frac{e}{5(e-2)}, \\ c_{21} &= \frac{e}{3(e-2)(n-1)}, & d_{21} &= \frac{e}{2(e-2)}. \end{aligned}$$

Then

$$\begin{aligned} \kappa_\psi\big(b_{11}+a_{11}(n-1)\big) &= \frac{e-2}{e}\left(\frac{e}{2(e-2)}+\frac{e}{4(e-2)(n-1)}(n-1)\right)=\frac{3}{4}<1, \\ \kappa_\psi\big(d_{21}+c_{21}(n-1)\big) &= \frac{e-2}{e}\left(\frac{e}{2(e-2)}+\frac{e}{3(e-2)(n-1)}(n-1)\right)=\frac{5}{6}<1, \\ \Delta_{11} &= \begin{vmatrix} \frac{e-2}{e}(\frac{e}{e-2}+\frac{e}{(e-2)(n-1)}(n-1)) & -\frac{1}{4} \\ -\frac{1}{6} & \frac{e-2}{e}(\frac{e}{5(e-2)}+\frac{e}{2(e-2)(n-1)}(n-1)) \end{vmatrix} = \frac{163}{120}>0. \end{aligned}$$

Consider

$$\begin{aligned} & F_1(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1}) \\ &\quad = \frac{9}{5}M\left[\frac{eM}{e-2}\left(\frac{3}{2}+\frac{5}{4}(1+n)\right)\right]^{-\delta_1} \\ &\qquad \times\big[a_{11}(z_1+z_3+\cdots+z_{2n-3})+a_{11}\big(2z_2+4z_4+\cdots+2(n-1)z_{2n-2}\big) \\ &\qquad + b_{11}z_{2n-1}+c_{11}(\widetilde{z}_1+\widetilde{z}_3+\cdots+\widetilde{z}_{2n-3}) \\ &\qquad + c_{11}\big(2\widetilde{z}_2+4\widetilde{z}_4+\cdots+2(n-1)\widetilde{z}_{2n-2}\big)+d_{11}\widetilde{z}_{2n-1}\big]^{\delta_1}, \\ & F_2(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1}) \\ &\quad = \frac{19}{10}M\left[\frac{eM}{e-2}\left(\frac{7}{10}+\frac{5}{6}(1+n)\right)\right]^{-\delta_2} \\ &\qquad \times\big[a_{21}(z_1+z_3+\cdots+z_{2n-3})+a_{21}\big(2z_2+4z_4+\cdots+2(n-1)z_{2n-2}\big) \\ &\qquad + b_{21}z_{2n-1}+c_{21}(\widetilde{z}_1+\widetilde{z}_3+\cdots+\widetilde{z}_{2n-3}) \\ &\qquad + c_{21}\big(2\widetilde{z}_2+4\widetilde{z}_4+\cdots+2(n-1)\widetilde{z}_{2n-2}\big)+d_{21}\widetilde{z}_{2n-1}\big]^{\delta_2}, \end{aligned}$$

for all $t\in[0,1]$, $z_i,\widetilde{z}_i\in R_+$, $i=1,2,\ldots,2n-1$, and $\delta_1,\delta_2>1$.

For all $t \in [0,1]$, $z_i, \widetilde{z}_i \in [0,M]$, $i = 1,2,\ldots,2n-1$, $j = 1,2$, we have

$$F_1(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1}) \le \frac{9}{5}M,$$
$$F_2(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1}) \le \frac{19}{10}M.$$

Consequently, if let $Q_1(t) \equiv \frac{9}{5}M$, $Q_2(t) \equiv \frac{19}{10}M$ for $t \in [0,1]$, then (H2) holds.

On the other hand, for all $t \in [0,1]$ we note that

$$\begin{aligned}
&\liminf_{a_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{11}z_{2n-1}+c_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{11}\widetilde{z}_{2n-1}\to+\infty} \\
&\quad \frac{F_1(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})}{a_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{11}z_{2n-1}+c_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{11}\widetilde{z}_{2n-1}} \\
&= \liminf_{a_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{11}z_{2n-1}+c_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{11}\widetilde{z}_{2n-1}\to+\infty} \\
&\quad \frac{\frac{9}{5}M[\frac{eM}{e-2}(\frac{3}{2}+\frac{5}{4}(1+n))]^{-\delta_1}[a_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{11}z_{2n-1}+c_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{11}\widetilde{z}_{2n-1}]^{\delta_1}}{a_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{11}z_{2n-1}+c_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{11}\widetilde{z}_{2n-1}} \\
&= +\infty
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{a_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{21}z_{2n-1}+c_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{21}\widetilde{z}_{2n-1}\to+\infty} \\
&\quad \frac{F_2(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})}{a_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{21}z_{2n-1}+c_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{21}\widetilde{z}_{2n-1}} \\
&= \liminf_{a_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{21}z_{2n-1}+c_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{21}\widetilde{z}_{2n-1}\to+\infty} \\
&\quad \frac{\frac{19}{10}M[\frac{eM}{e-2}(\frac{7}{10}+\frac{5}{6}(1+n))]^{-\delta_2}[a_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{21}z_{2n-1}+c_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{21}\widetilde{z}_{2n-1}]^{\delta_2}}{a_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+b_{21}z_{2n-1}+c_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+d_{21}\widetilde{z}_{2n-1}} \\
&= +\infty.
\end{aligned}$$

Therefore, (H1) holds.

Example 3.4 Let $t_1 = \frac{1}{2}$, $t_2 = 1$, and note that $\int_0^1 k_1(t,s)\,ds = t - \frac{1}{2}t^2$ for $t \in [0,1]$, and if we consider the case $\widetilde{Q}_j \equiv$ constant, we have

$$\int_0^1 k_1(t_1,s)\widetilde{Q}_1(s)\,ds = \frac{3}{8}\widetilde{Q}_1, \qquad \int_0^1 k_1(t_2,s)\widetilde{Q}_2(s)\,ds = \frac{1}{2}\widetilde{Q}_2.$$

To obtain the first inequality in (H4), we can take $\widetilde{Q}_1 = 3M$, $\widetilde{Q}_2 = \frac{5}{2}M$.

Let

$$\begin{pmatrix} \widetilde{a}_{11} & \widetilde{b}_{11} & \widetilde{c}_{11} & \widetilde{d}_{11} \\ \widetilde{a}_{21} & \widetilde{b}_{21} & \widetilde{c}_{21} & \widetilde{d}_{21} \end{pmatrix} = \begin{pmatrix} \frac{1}{20(n-1)} & \frac{1}{10} & \frac{1}{80(n-1)} & \frac{1}{40} \\ \frac{1}{40(n-1)} & \frac{1}{20} & \frac{1}{100(n-1)} & \frac{1}{50} \end{pmatrix}.$$

Then we have

$$\widetilde{b}_{11}+\widetilde{a}_{11}(n-1)=\frac{1}{10}+\frac{1}{20}<1,\qquad \widetilde{d}_{21}+\widetilde{c}_{21}(n-1)=\frac{1}{50}+\frac{1}{100}<1,$$

$$\Delta_{22}=\begin{vmatrix} 0.85 & -[\frac{1}{40}+\frac{1}{80}] \\ -[\frac{1}{20}+\frac{1}{40}] & 0.97 \end{vmatrix}\approx 0.82>0.$$

Let

$$\begin{aligned}
&F_1(t,z_1,z_2,\dots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\dots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})\\
&\quad=3M\exp\bigl\{M\bigl(0.125+0.0625(1+n)\bigr)\bigr\}\\
&\qquad\times\exp\bigl\{-\widetilde{a}_{11}(z_1+z_3+\cdots+z_{2n-3})-\widetilde{a}_{11}\bigl(2z_2+4z_4+\cdots+2(n-1)z_{2n-2}\bigr)\\
&\qquad-\widetilde{b}_{11}z_{2n-1}-\widetilde{c}_{11}(\widetilde{z}_1+\widetilde{z}_3+\cdots+\widetilde{z}_{2n-3})\\
&\qquad-\widetilde{c}_{11}\bigl(2\widetilde{z}_2+4\widetilde{z}_4+\cdots+2(n-1)\widetilde{z}_{2n-2}\bigr)-\widetilde{d}_{11}\widetilde{z}_{2n-1}\bigr\},\\
&F_2(t,z_1,z_2,\dots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\dots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})\\
&\quad=2.5M\exp\bigl\{M\bigl(0.07+0.035(1+n)\bigr)\bigr\}\\
&\qquad\times\exp\bigl\{-\widetilde{a}_{21}(z_1+z_3+\cdots+z_{2n-3})-\widetilde{a}_{21}\bigl(2z_2+4z_4+\cdots+2(n-1)z_{2n-2}\bigr)\\
&\qquad-\widetilde{b}_{21}z_{2n-1}-\widetilde{c}_{21}(\widetilde{z}_1+\widetilde{z}_3+\cdots+\widetilde{z}_{2n-3})\\
&\qquad-\widetilde{c}_{21}\bigl(2\widetilde{z}_2+4\widetilde{z}_4+\cdots+2(n-1)\widetilde{z}_{2n-2}\bigr)-\widetilde{d}_{21}\widetilde{z}_{2n-1}\bigr\},
\end{aligned}$$

for all $t\in[0,1]$, $z_i,\widetilde{z}_i\in R_+$, $i=1,2,\dots,2n-1$.

For all $t\in[0,1]$, $z_i,\widetilde{z}_i\in[0,M]$, $i=1,2,\dots,2n-1$, $j=1,2$, we have

$$\begin{aligned}
&F_1(t,z_1,z_2,\dots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\dots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})\geq 3M,\\
&F_2(t,z_1,z_2,\dots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\dots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})\geq 2.5M,
\end{aligned}$$

and thus (H4) holds. On the other hand, for all $t\in[0,1]$ we also have

$$\begin{aligned}
&\limsup_{\widetilde{a}_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{11}z_{2n-1}+\widetilde{c}_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{11}\widetilde{z}_{2n-1}\to+\infty}\\
&\qquad\frac{F_1(t,z_1,z_2,\dots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\dots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})}{\widetilde{a}_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{11}z_{2n-1}+\widetilde{c}_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{11}\widetilde{z}_{2n-1}}\\
&\quad=\limsup_{\widetilde{a}_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{11}z_{2n-1}+\widetilde{c}_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{11}\widetilde{z}_{2n-1}\to+\infty}\\
&\qquad\Bigl(3M\exp\bigl\{M\bigl(0.125+0.0625(1+n)\bigr)\bigr\}\\
&\qquad\times\exp\Biggl\{-\Biggl[\widetilde{a}_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{11}z_{2n-1}+\widetilde{c}_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{11}\widetilde{z}_{2n-1}\Biggr]\Biggr\}\\
&\qquad\Bigg/\Biggl(\widetilde{a}_{11}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{11}z_{2n-1}+\widetilde{c}_{11}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{11}\widetilde{z}_{2n-1}\Biggr)\Biggr)\\
&\quad=0
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\widetilde{a}_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{21}z_{2n-1}+\widetilde{c}_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{21}\widetilde{z}_{2n-1}\to+\infty} \frac{F_2(t,z_1,z_2,\ldots,z_{2n-3},z_{2n-2},z_{2n-1},\widetilde{z}_1,\widetilde{z}_2,\ldots,\widetilde{z}_{2n-3},\widetilde{z}_{2n-2},\widetilde{z}_{2n-1})}{\widetilde{a}_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{21}z_{2n-1}+\widetilde{c}_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{21}\widetilde{z}_{2n-1}} \\
&= \limsup_{\widetilde{a}_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{21}z_{2n-1}+\widetilde{c}_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{21}\widetilde{z}_{2n-1}\to+\infty} \\
&\quad \Bigg(2.5M\exp\big\{M\big(0.07+0.035(1+n)\big)\big\} \\
&\quad \times \exp\Bigg\{-\Bigg[\widetilde{a}_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{21}z_{2n-1}+\widetilde{c}_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{21}\widetilde{z}_{2n-1}\Bigg]\Bigg\} \\
&\quad \Big/\Bigg(\widetilde{a}_{21}\sum_{i=1}^{n-1}(z_{2i-1}+2iz_{2i})+\widetilde{b}_{21}z_{2n-1}+\widetilde{c}_{21}\sum_{i=1}^{n-1}(\widetilde{z}_{2i-1}+2i\widetilde{z}_{2i})+\widetilde{d}_{21}\widetilde{z}_{2n-1}\Bigg)\Bigg) \\
&= 0.
\end{aligned}$$

Therefore, (H3) holds.

4 Conclusion

In this paper we use the fixed point index to study the existence of positive solutions for the system of $2n$th-order boundary value problems (1.1) involving semipositone nonlinearities. Our nonlinearities not only depend on all derivatives of unknown functions, but they also grow superlinearly and sublinearly at infinity.

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Authors' contributions

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