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Multipliers for Jacobi expansions in the Hardy spaces

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Abstract

The purpose of the paper is to study the coefficient multipliers of the Hardy spaces H^p associated with Jacobi expansions of exponential type. The main results are about the boundedness from H^p to ℓ^q of the multiplier operators in terms of Jacobi expansions of exponential type for (i) $p = 1, 2 \le q < \infty$; (ii) $\gamma(\alpha,\beta)^{-1} < p < 1 \le q < \infty$, under appropriate conditions, where $\gamma(\alpha,\beta) \in (1,\infty]$ is a number depending on the parameters of the Jacobi system.

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1 Introduction and main results

1.1 Jacobi expansions of exponential type

Assume that $\alpha,\beta > -1$. Let $R_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial on $[-1,1]$ of degree n normalized so that $R_n^{(\alpha,\beta)}(1) = 1$. It follows that the system $\{R_n^{(\alpha,\beta)}(\cos t)\}_{n=0}^{\infty}$ is orthogonal over $[0,\pi]$ with respect to the weight $\sin^{2\alpha+1}(t/2)\cos^{2\beta+1}(t/2)$. In particular,

$$R_n^{(-1/2,-1/2)}(\cos t) = \cos nt, \qquad R_n^{(1/2,1/2)}(\cos t) = \frac{\sin(n+1)t}{(n+1)\sin t}.$$

In analogy to the relation of $\cos nt$ and $\sin nt$, the system $\{R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t\}_{n=1}^{\infty}$ is introduced in [8], which is a conjugate one of $\{R_n^{(\alpha,\beta)}(\cos t)\}_{n=0}^{\infty}$ based on a pair of generalized Cauchy–Riemann equations. This allows us to define an exponential type system $\{E_n^{(\alpha,\beta)}(t)\}_{n=-\infty}^{\infty}$ as in [9], by $E_0^{(\alpha,\beta)} = 1/\sqrt{2}$, and for $n \ge 1$,

$$\begin{aligned} E_n^{(\alpha,\beta)}(t) &= \frac{1}{2}\left[R_n^{(\alpha,\beta)}(\cos t) + i\frac{\rho_n}{2\alpha+2}R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t\right], \\ E_{-n}^{(\alpha,\beta)}(t) &= \overline{E_n^{(\alpha,\beta)}(t)}, \end{aligned} \tag{1}$$

with $\rho_n = \sqrt{n(n+\alpha+\beta+1)}$, which is orthogonal over $[-\pi,\pi]$ with respect to the weight $w(t) = \phi_{\alpha,\beta}(t)^2$, where

$$\phi_{\alpha,\beta}(t) = \left|\sin(t/2)\right|^{\alpha+1/2}\left|\cos(t/2)\right|^{\beta+1/2}.$$

It is interesting that each $E_n^{(\alpha,\beta)}$ is an eigenfunction of the first order differential-difference operator T defined by

$$Tf(t) = \frac{d}{dt}f(t) + \eta_{\alpha,\beta}(t)\big[f(t) - f(-t)\big], \quad \eta_{\alpha,\beta}(t) = \frac{(\alpha+\beta+1)\cos t + \alpha - \beta}{2\sin t},$$

that is,

$$TE_{\pm n}^{(\alpha,\beta)}(t) = \pm i\rho_n E_{\pm n}^{(\alpha,\beta)}(t) \quad \text{for } n \ge 0.$$

This resembles the functions e^{int} satisfying $\frac{d}{dt}e^{\pm int} = \pm ine^{\pm int}$. The system $\{E_n^{(\alpha,\beta)}(t)\}_{n=-\infty}^{\infty}$ induces an orthonormal system $\{\mathcal{E}_n^{(\alpha,\beta)}(t)\}_{n=-\infty}^{\infty}$ over $[-\pi,\pi]$ with respect to the Lebesgue measure, that is,

$$\int_{-\pi}^{\pi} \mathcal{E}_m^{(\alpha,\beta)}(t)\overline{\mathcal{E}_n^{(\alpha,\beta)}(t)}\,dt = \delta_{m,n}, \quad m,n = 0,\pm1,\pm2,\ldots,$$

where, for $n = 0,\pm1,\pm2,\ldots$,

$$\mathcal{E}_n^{(\alpha,\beta)}(t) = \sqrt{\omega_n^{(\alpha,\beta)}}E_n^{(\alpha,\beta)}(t)\phi_{\alpha,\beta}(t),$$

and $1/\omega_n^{(\alpha,\beta)} = \int_{-\pi}^{\pi} |E_n^{(\alpha,\beta)}(t)|^2\phi_{\alpha,\beta}(t)^2\,dt$. From [9, Lemma 4] and [8, (2.2)] it follows that, for $n = 0,1,2,\ldots$,

$$\omega_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(n+\beta+1)\Gamma(n+1)},$$

and $\omega_{-n}^{(\alpha,\beta)} = \omega_n^{(\alpha,\beta)}$. It is easy to see that

$$\omega_n^{(\alpha,\beta)} = 2\Gamma(\alpha+1)^{-2}n^{2\alpha+1}\big(1 + O\big(n^{-1}\big)\big) \quad \text{for } n \ge 1. \tag{2}$$

Sometimes we write $E_n(t) = E_n^{(\alpha,\beta)}(t)$ and $\mathcal{E}_n(t) = \mathcal{E}_n^{(\alpha,\beta)}(t)$ for simplicity.

In what follows we assume that $\alpha,\beta \ge -1/2$, for which the functions $\mathcal{E}_n^{(\alpha,\beta)}(t)$ are continuous on $[-\pi,\pi]$. For $f \in L(-\pi,\pi)$, its Jacobi expansion of exponential type is defined by

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n(f)\mathcal{E}_n^{(\alpha,\beta)}(t), \quad c_n(f) = \int_{-\pi}^{\pi} f(t)\overline{\mathcal{E}_n^{(\alpha,\beta)}(t)}\,dt, \tag{3}$$

where $c_n(f)$ are called the Fourier–Jacobi coefficients of f.

1.2 The main results

The purpose of the paper is to study the coefficient multipliers of the real Hardy spaces $H^p(-\pi,\pi)$ associated with Jacobi expansions of exponential type. We recall that a function F analytic in the unit disk $\mathbb{D}$ is said to be in the Hardy space $H^p(\mathbb{D})$, $0 < p < \infty$, if $\|F\|_{H^p} := \sup_{0\le r<1} M_p(F;r) < \infty$, where

$$M_p(F;r) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|F\big(re^{i\theta}\big)\big|^p\,d\theta\right\}^{1/p}.$$

The real Hardy space $H^p(-\pi,\pi)$ consists of boundary values of real parts of functions F in $H^p(\mathbb{D})$, with real $F(0)$.

Since $H^1(-\pi,\pi) \subset L(-\pi,\pi)$, the Fourier–Jacobi coefficients of $f \in H^1(-\pi,\pi)$ may be defined as in (3); but if $f \in H^p(-\pi,\pi)$ for $0 < p < 1$, we need a substitute definition of its Fourier–Jacobi coefficients $c_n(f)$, which is based on the duality relation of the Hardy space $H^p(-\pi,\pi)$ and the Lipschitz space $\Lambda_{p^{-1}-1}(-\pi,\pi)$. For $m \geq 1$ and $m-1 < \delta \leq m$, $\Lambda_\delta(-\pi,\pi)$ is the set of $(m-1)$-times differentiable and 2π-period functions f satisfying

$$\|f\|_{\Lambda_\delta} := \sup_{x,h} \bigl|f^{(m-1)}(x+h) - f^{(m-1)}(x)\bigr| / |h|^{\delta+1-m} < \infty$$

for $\delta \neq m$, and

$$\|f\|_{\Lambda_\delta} := \sup_{x,h} \bigl|f^{(m-1)}(x+h) - 2f^{(m-1)}(x) + f^{(m-1)}(x-h)\bigr| / |h| < \infty$$

for $\delta = m$. Here we use a unified notation $\Lambda_\delta(-\pi,\pi)$ for all $\delta > 0$, without use of Zygmund's notation $\Lambda^*_\delta(-\pi,\pi)$ for $\delta = m$.

Lemma 1.1 ([1, Theorem 7.5]) *To each bounded linear functional $\mathcal{L}$ on $H^p(\mathbb{D})$, $0 < p < 1$, there is a function $g \in \Lambda_{p^{-1}-1}(-\pi,\pi)$ such that, for all $F(z) = \sum_{n=0}^{\infty} c_n z^n \in H^p(\mathbb{D})$,*

$$\mathcal{L}(F) = \lim_{r\to 1-} \int_{-\pi}^{\pi} F\bigl(re^{it}\bigr) g(t)\,dt. \tag{4}$$

Conversely, for any $g \in \Lambda_{p^{-1}-1}(-\pi,\pi)$, the above limit exists for all $F \in H^p(\mathbb{D})$ and defines a bounded linear functional satisfying

$$\bigl|\mathcal{L}(F)\bigr| \leq c\|g\|_{\Lambda_{p^{-1}-1}} \|F\|_{H^p},$$

where c is a constant independent of g and F.

A convenient notation is $\mathcal{L}_g = \mathcal{L}$ once $\mathcal{L}$ and g satisfy relation (4).

The linear functional $\mathcal{L}$ on the real Hardy space $H^p(-\pi,\pi)$, $0 < p < 1$, is identified with the associated one on $H^p(\mathbb{D})$, that means $\mathcal{L}(f) = \mathcal{L}(F)$, where $f(t) =$ the real part of $F(e^{it})$ for $F \in H^p(\mathbb{D})$ with real $F(0)$.

For $0 < p < 1$, the Fourier–Jacobi coefficients $c_n(f)$ of $f \in H^p(-\pi,\pi)$ are defined by

$$c_n(f) = \mathcal{L}_{\overline{\mathcal{E}_n^{(\alpha,\beta)}}}(f) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

It is easy to see that this definition is identical with the previous definition in (3) for "good" functions. However, it is not always meaningful in general for all $H^p(-\pi,\pi)$, $0 < p < 1$, since the functions $\mathcal{E}_n^{(\alpha,\beta)}(t)$ are not sufficiently smooth for most of α, β. Indeed we have the following.

Proposition 1.2 *Let $\alpha, \beta \geq -1/2$. The functions $\mathcal{E}_n(t) = \mathcal{E}_n^{(\alpha,\beta)}(t)$ are in $\Lambda_{p^{-1}-1}(-\pi,\pi)$ for $\gamma(\alpha,\beta)^{-1} \leq p < 1$, where*

$$\begin{aligned} \gamma(\alpha,\beta) &= +\infty \quad \textit{if both } \alpha + 1/2 \textit{ and } \beta + 1/2 \textit{ are nonnegative even integers} \\ &= \min\{\alpha,\beta\} + 3/2 \quad \textit{if neither } \alpha + 1/2 \textit{ nor } \beta + 1/2 \textit{ is an even integer} \end{aligned}$$

$$= \alpha + 3/2 \text{ or } \beta + 3/2 \quad \text{if } \alpha + 1/2 \text{ or } \beta + 1/2 \text{ is not an even integer and the other one is an even integer.}$$

Consequently, for $f \in H^p(-\pi,\pi)$, $\gamma(\alpha,\beta)^{-1} \le p < 1$, *its Fourier–Jacobi coefficients* $c_n(f) = \mathcal{L}_{\overline{\mathcal{E}_n^{(\alpha,\beta)}}}(f)$ *are well defined.*

It is obvious that $\gamma(\alpha,\beta) > 1$ for all $\alpha,\beta \ge -1/2$.

The main results in the present paper are about the boundedness of the multiplier operators associated with Jacobi expansions of exponential type from the Hardy spaces $H^p(-\pi,\pi)$ to the sequence spaces ℓ^q for some ranges of p and q. The details are stated in the following two theorems.

Theorem 1.3 *Let* $\alpha,\beta \ge -1/2$ *and* $2 \le q < \infty$. *If a bilateral sequence* $\{\lambda_n\}_{n=-\infty}^{\infty}$ *satisfies the condition*

$$\sum_{N\le|n|\le 2N} |\lambda_n|^q = O(1) \quad \text{for } N \ge 1, \tag{5}$$

then, for all $f \in H^1(-\pi,\pi)$ *having the Jacobi expansion* (3) *of exponential type,*

$$\sum_{n=-\infty}^{\infty} \bigl|\lambda_n c_n(f)\bigr|^q \le c\|f\|_{H^1}^q. \tag{6}$$

Theorem 1.4 *Let* $\alpha,\beta \ge -1/2$ *and*

$$\gamma(\alpha,\beta)^{-1} < p < 1 \le q < \infty.$$

If a bilateral sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ *satisfies the condition*

$$\sum_{N\le|n|\le 2N} |\lambda_n|^q = O\bigl(N^{q(1-1/p)}\bigr) \quad \text{for } N \ge 1, \tag{7}$$

then, for all $f \in H^p(-\pi,\pi)$, *the Fourier–Jacobi coefficients* $c_n(f) = \mathcal{L}_{\overline{\mathcal{E}_n^{(\alpha,\beta)}}}(f)$ *satisfy*

$$\sum_{n=-\infty}^{\infty} \bigl|\lambda_n c_n(f)\bigr|^q \le c\|f\|_{H^p}^q. \tag{8}$$

As a consequence of Theorems 1.3 and 1.4, a Paley-type inequality associated with Jacobi expansions (3) can be obtained. We first note that if $\{n_k\}$ is a Hadamard sequence satisfying $n_{k+1}/n_k \ge \rho > 1$ ($k = 1,2,\ldots$), then for any $N = 1,2,\ldots$, the number of elements in $\{n_k\}$ locating in $[N,2N]$ has a bound independent of N. Indeed, if $n_{k-1} < N \le n_k$, then for $j \ge k$ satisfying $n_j \le 2N$, we have $1 \le 2Nn_j^{-1} \le 2Nn_k^{-1}\rho^{k-j} \le 2\rho^{k-j}$, and hence the bound to be determined is $2/(1-\rho^{-1})$. Now if $\lambda_n = n_k^{1-1/p}$ for $|n| = n_k$, and 0 otherwise, then the sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ satisfies (5) or (7) for $p = 1$ or $0 < p < 1$ respectively, and hence applying Theorems 1.3 and 1.4 to $q = 2$, we have the following corollary.

Corollary 1.5 *Let $\alpha,\beta \geq -1/2$ and $\gamma(\alpha,\beta)^{-1} < p \leq 1$. If $\{n_k\}$ is a Hadamard sequence satisfying $n_{k+1}/n_k \geq \rho > 1$ $(k = 1,2,\ldots)$, then for $f \in H^p(-\pi,\pi)$, the Fourier–Jacobi coefficients $c_n(f)$ satisfy*

$$\sum_{k=1}^{\infty} n_k^{2(1-1/p)} \bigl(\bigl|c_{n_k}(f)\bigr|^2 + \bigl|c_{-n_k}(f)\bigr|^2\bigr) \leq c\|f\|_{H^p}^2. \tag{9}$$

The Paley-type inequality for usual Jacobi expansions is a special case of (9) with $p = 1$ and for even functions, which has been proved in [7] by the duality of H^1 and BMO.

Throughout the paper, c or c' denotes constants independent of variables, functions, n, k, etc., which may be different in different occurrences.

1.3 Backgrounds and remarks

1. The research on multipliers for power series and Fourier series has a long history and rich contents. One of the criteria of multipliers in the Hardy spaces, proved in Hardy and Littlewood [4, 5], is that, for $1 \leq p \leq 2 \leq q$ and $p^{-1} - q^{-1} = 1 - \delta^{-1}$, the sequence $\{\lambda_n\}$ is a multiplier of $H^p(\mathbb{D})$ into $H^q(\mathbb{D})$ if the function $h_\lambda(z) = \sum_{k=0}^{\infty} \lambda_k z^k$ satisfies

$$M_\delta\bigl(h_\lambda';r\bigr) \leq c(1-r)^{-1}. \tag{10}$$

It was pointed out in [13] (see [12] too) that (10) with $\delta = q$ is also necessary for the sequence $\{\lambda_n\}$ being a multiplier of $H^1(\mathbb{D})$ into $H^q(\mathbb{D})$ when $1 \leq q \leq \infty$, and hence, for $q \geq 2$, it provides a characterization of multipliers from $H^1(\mathbb{D})$ into $H^q(\mathbb{D})$. In particular, by Parseval's theorem, (10) for $\delta = 2$ is equivalent to

$$\sum_{n=N}^{2N} |\lambda_n|^2 = O(1) \quad \text{for } N \geq 1. \tag{11}$$

That means the sequence $\{\lambda_n\}$ is a multiplier of $H^1(\mathbb{D})$ into $H^2(\mathbb{D})$ if and only if (11) is satisfied.

Turning to the case for $1 \leq p \leq q < 2$, the situation is completely different from (10). A sequence $\{\lambda_n\}$ is constructed in [12] (see Theorem 3.1 there), which satisfies (10) but is not a multiplier of $H^p(\mathbb{D})$ into $H^q(\mathbb{D})$. However, it was shown in [2] (and first stated in [4, 5]) that a modification of (10) allows to extend the theorem of Hardy and Littlewood to smaller p. Indeed, if $0 < p < 1 \leq q \leq \infty$ and $(\nu+1)^{-1} \leq p < \nu^{-1}$ with $\nu = 1,2,\ldots$, then the sequence $\{\lambda_n\}$ is a multiplier of $H^p(\mathbb{D})$ into $H^q(\mathbb{D})$ if and only if the function $h_\lambda(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ satisfies $M_q(h_\lambda^{(\nu+1)};r) \leq c(1-r)^{\frac{1}{p}-\nu-2}$.

A different type of multipliers is the coefficient multipliers of the Hardy spaces $H^p(\mathbb{D})$ into the sequence spaces ℓ^q. For $0 < q < \infty$, $\ell^q = \{\{a_k\} : \|\{a_k\}\|_q = (\sum_{k=0}^{\infty} |a_k|^q)^{1/q} < \infty\}$; and ℓ^∞ is the set of bounded sequences. We use the same notations for a bilateral sequence. That a sequence $\{\lambda_n\}$ is such a multiplier means that $\{\lambda_n c_n\} \in \ell^q$ whenever $\sum_{n=0}^{\infty} c_n z^n \in H^p(\mathbb{D})$.

A basic criterion following from that on multipliers of $H^1(\mathbb{D})$ into $H^2(\mathbb{D})$ (alias ℓ^2) mentioned above is stated as follows: the sequence $\{\lambda_n\}$ is a multiplier of $H^1(\mathbb{D})$ into ℓ^q for $2 \leq q < \infty$ if and only if $\sum_{n=N}^{2N} |\lambda_n|^q = O(1)$ for $N \geq 1$. For details, see [2, pp. 72–73]. A general extension of this criterion to the case $0 < p < 1$ is given in [2]. It is proved in [2] that (see

Theorem 2 there), for $0 < p < 1$ and $p \le q < \infty$, the sequence $\{\lambda_n\}$ is a multiplier of $H^p(\mathbb{D})$ into ℓ^q if and only if $\sum_{n=N}^{2N} |\lambda_n|^q = O(N^{q(1-1/p)})$ for $N \ge 1$. We note that this condition is equivalent to what is given in [2] by the following proposition.

Proposition 1.6 ([10, Proposition 1.8]) *Let a, b be real and $a > 0$. Then, for a nonnegative sequence $\{\mu_n\}$, the following conditions are equivalent*:

(i) $\sum_{n=1}^{N} n^b \mu_n = O(N^a)$ *for* $N \ge 1$;

(ii) $\sum_{n=N}^{2N} \mu_n = O(N^{a-b})$ *for* $N \ge 1$;

(iii) $\sum_{n=2^k}^{2^{k+1}} \mu_n = O(2^{k(a-b)})$ *for* $k \ge 0$;

(iv) *For some* $\delta > 0$, $\sum_{n=N}^{\infty} n^{b-a-\delta} \mu_n = O(N^{-\delta})$ *for* $N \ge 1$.

Since an element in the real Hardy spaces $H^p(-\pi,\pi)$ has an expansion associated with $\{e^{in\theta}\}_{n=-\infty}^{\infty}$, or equivalently with $\{1, \cos n\theta, \sin n\theta, n = 1, 2, \dots\}$, the coefficient multiplier criteria given above can be restated as follows.

Theorem 1.7

(i) *If a bilateral sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ satisfies condition* (5), *then it is a multiplier of $H^1(-\pi,\pi)$ into ℓ^q for $2 \le q < \infty$, which means $\{\lambda_n c_n\} \in \ell^q$ whenever $\sum_{n=-\infty}^{\infty} c_n e^{int} \in H^1(-\pi,\pi)$.*

(ii) *Suppose $0 < p < 1$. If a bilateral sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ satisfies condition* (7), *then it is a multiplier of $H^p(-\pi,\pi)$ into ℓ^q for $p \le q < \infty$, which means $\{\lambda_n c_n\} \in \ell^q$ whenever $\sum_{n=-\infty}^{\infty} c_n e^{int} \in H^p(-\pi,\pi)$.*

2. If $f \in L(-\pi,\pi)$ is even or $f \in L(0,\pi)$, (3) is identical with the following usual Jacobi expansion:

$$f(t) \sim \sum_{n=0}^{\infty} a_n(f) p_n^{(\alpha,\beta)}(t), \qquad a_n(f) = \int_0^{\pi} f(t) p_n^{(\alpha,\beta)}(t)\,dt,$$

where

$$p_n^{(\alpha,\beta)}(t) = \sqrt{\omega_n^{(\alpha,\beta)}} R_n^{(\alpha,\beta)}(\cos t)\phi_{\alpha,\beta}(t), \qquad n = 0, 1, 2, \dots,$$

which form an orthonormal system over $[0,\pi]$ with respect to the Lebesgue measure.

3. In our previous work [10], the problem on coefficient multipliers of the Hardy spaces $H^p(\mathbb{R})$ associated with Hermite expansions was studied. A sufficient condition given in [10] for a sequence $\{\lambda_n\}_{n=0}^{\infty}$ to be a multiplier of $H^p(\mathbb{R})$ into the sequence space ℓ^q associated with Hermite expansions for (i) $p = 1$, $2 \le q < \infty$ and (ii) $0 < p < 1 \le q < \infty$ is

$$\sum_{k=n}^{2n} |\lambda_k|^q = O\bigl(n^{\frac{q}{2}(\frac{7}{6}-\frac{1}{p})}\bigr). \tag{12}$$

In comparison to (5) and (7), condition (12) might seem peculiar. For a Hadamard sequence $\{n_k\}$, a Paley-type inequality following (12) is of the form (see [10, Corollary 3.3])

$$\sum_{k=1}^{\infty} n_k^{\frac{7}{6}-\frac{1}{p}} \bigl|a_{n_k}(f)\bigr|^2 \le c\|f\|_{H^p(\mathbb{R})}^2, \quad f \in H^p(\mathbb{R}), \tag{13}$$

where $a_n(f)$, $n \geq 0$, are the coefficients of the Hermite expansion of f. It is noted that the Hardy inequality associated with Hermite expansions was proved in [11], that is,

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}} \left|a_n(f)\right| \leq c\|f\|_{H^1(\mathbb{R})}, \quad f \in H^1(\mathbb{R}). \tag{14}$$

The sharpness of (14) was verified by Kanjin in [6], who showed that there exists $f_0 \in L^1(\mathbb{R})$ such that

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}} \left|a_n(f_0)\right| = \infty.$$

4. The original proofs of the classical multiplier theorems depend on the complex variable structure of analytic functions, which is not workable for the Jacobi expansions of exponential type. We shall apply, in Sect. 3, the duality of H^1 and BMO in proving Theorem 1.3, and the duality of $H^p(-\pi,\pi)$ and $\Lambda_{p^{-1}-1}(-\pi,\pi)$ in the proof of Theorem 1.4. Applications of these principles base upon some evaluations of the exponential type Jacobi functions $\mathcal{E}_n^{(\alpha,\beta)}(t)$ which are given in Sect. 2.

2 Several lemmas

Lemma 2.1 *Let $\alpha, \beta > -1$ and let $0 < \epsilon_0 < \pi$ be fixed. Then, for $0 \leq |t| \leq \pi - \epsilon_0$ and for $n = 1, 2, \ldots$,*

$$\mathcal{E}_n^{(\alpha,\beta)}(t) = \frac{\sqrt{\omega_n^{(\alpha,\beta)}}}{2}\left(\frac{|t|}{2}\right)^{\alpha+1/2} e_{\alpha+1/2}(iNt) + O\left[\frac{1}{n}\left(\frac{n|t|}{1+n|t|}\right)^{\alpha+3/2}\right] \tag{15}$$

with $N = n + (\alpha + \beta + 1)/2$, where e_λ is the one-dimensional Dunkl kernel

$$e_\lambda(z) = j_{\lambda-1/2}(iz) + \frac{z}{2\lambda+1} j_{\lambda+1/2}(iz), \quad z \in \mathbb{C}, \tag{16}$$

and $j_\alpha(z)$ is the normalized Bessel function $j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) z^{-\alpha} J_\alpha(z)$.

Proof The proof is essentially based upon the well-known formula [14, (8.21.17)] of "Hilb's type", which is rewritten as

$$\phi_{\alpha,\beta}(t) R_n^{(\alpha,\beta)}(\cos t) = \left(\frac{t}{2}\right)^{\alpha+1/2} j_\alpha(Nt) + O\left[\left(\frac{t}{1+nt}\right)^{\alpha+3/2}\right]$$

for $0 \leq t \leq \pi - \epsilon_0$. Applying to $R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)$ and noting that $\rho_n = N + O(1)$, then equality (15) follows from (1)–(2) and these estimates. If $-\pi + \epsilon_0 \leq t \leq 0$, we use the relation $\mathcal{E}_n^{(\alpha,\beta)}(t) = \overline{\mathcal{E}_n^{(\alpha,\beta)}(-t)}$ to obtain (15) again. □

Corollary 2.2 *Let $\alpha, \beta > -1$ and let $0 < \epsilon_0 < \pi$ be fixed. Then, for $n^{-1} \leq |t| \leq \pi - \epsilon_0$ and for $n = 1, 2, \ldots$,*

$$\begin{aligned}\mathcal{E}_n^{(\alpha,\beta)}(t) = \frac{1}{\sqrt{2\pi}}\Bigg[&\left(1 + i\frac{(2\alpha+1)^2}{8Nt}\right) e^{i(\operatorname{sgn} t)(N|t| - \frac{\pi}{2}\alpha - \frac{\pi}{4})} \\ &+ i\frac{2\alpha+1}{4Nt} e^{-i(\operatorname{sgn} t)(N|t| - \frac{\pi}{2}\alpha - \frac{\pi}{4})}\Bigg] + O\left(\frac{1}{n^2t^2} + \frac{1}{n}\right).\end{aligned} \tag{17}$$

Proof We rewrite [3, 7–13(3)] as

$$j_\alpha(x) = \frac{c_{\alpha+1/2}}{|x|^{\alpha+1/2}}\left[\cos\left(|x| - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) + \frac{1-4\alpha^2}{8|x|}\sin\left(|x| - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{x^2}\right)\right]$$

for $x \to \infty$, where $c_\lambda = 2^\lambda \Gamma(\lambda + 1/2)/\sqrt{\pi}$. Applying to $j_{\lambda-1/2}$ and $j_{\lambda+1/2}$, then from (16) it follows that

$$e_\lambda(ix) = \frac{c_\lambda}{|x|^\lambda}\left[\left(1 + i\frac{\lambda^2}{2x}\right)e^{i(\operatorname{sgn} x)(|x|-\frac{\pi}{2}\lambda)} + i\frac{\lambda}{2x}e^{-i(\operatorname{sgn} x)(|x|-\frac{\pi}{2}\lambda)} + O\left(\frac{1}{x^2}\right)\right].$$

Putting $\lambda = \alpha + 1/2$, $x = Nt$, and then substituting into (15) proves (17) on account of (2). □

Lemma 2.3 *Let $\alpha, \beta > -1$ and let $0 < \epsilon_0 < \pi$ be fixed. Then, for $k = 0, 1, \ldots$ and $0 \le |t| \le \pi - \epsilon_0$,*

$$\left|\frac{d^k}{dt^k}E_n^{(\alpha,\beta)}(t)\right| \le cn^k\bigl(1 + n|t|\bigr)^{-\alpha-1/2}, \tag{18}$$

where c is a constant independent of n, t.

Proof The key step is the following equality:

$$\frac{d^k}{dt^k}\bigl[R_n^{(\alpha,\beta)}(\cos t)\bigr] = \sum_{j=0}^{k} U_j(n)R_{n-j}^{(\alpha+j,\beta+j)}(\cos t)\psi_j(t), \tag{19}$$

where $U_j(n)$ is a polynomial in n of degree $2j$ and $U_j(n) \asymp n^{2j}$ as $n \to \infty$, $\psi_j(t)$ is a homogeneous polynomial in $\cos t$ and $\sin t$ of degree j, and when $[(k+1)/2] \le j \le k$, $\psi_j(t) = (\sin t)^{2j-k} \times$ is a homogeneous polynomial in $\cos t$ and $\sin t$ of degree $k - j$.

The equality in (19) is a consequence of the formula (see [8, (2.9)])

$$\frac{d}{dx}R_n^{(\alpha,\beta)}(x) = \frac{\rho_n^2}{2\alpha + 2}R_{n-1}^{(\alpha+1,\beta+1)}(x)$$

after induction. And hence, by means of the estimate $|R_n^{(\alpha,\beta)}(\cos t)| \le c(1 + |nt|)^{-\alpha-1/2}$ (see [14, (7.32.5)]), $|\frac{d^k}{dt^k}[R_n^{(\alpha,\beta)}(\cos t)]|$ is dominated by a multiple of

$$\sum_{j=[(k+1)/2]}^{k} \frac{n^{2j}|t|^{2j-k}}{(1 + |nt|)^{j+\alpha+1/2}} + \sum_{j=0}^{[(k-1)/2]} \frac{n^{2j}}{(1 + |nt|)^{j+\alpha+1/2}},$$

and consequently by $cn^k(1 + n|t|)^{-\alpha-1/2}$, since $n^{2j}|t|^{2j-k} \le n^k(1 + |nt|)^{2j-k}$ in the first summation and $n^{2j} \le n^k$ in the second one. Furthermore, we also have

$$\begin{aligned}\frac{\rho_n}{2\alpha + 2}\bigl|\bigl[R_{n-1}^{(\alpha+1,\beta+1)}(\cos t)\sin t\bigr]_t^{(k)}\bigr| &= \frac{1}{\rho_n}\bigl|\bigl[R_n^{(\alpha,\beta)}(\cos t)\bigr]_t^{(k+1)}\bigr| \\ &\le cn^k\bigl(1 + n|t|\bigr)^{-\alpha-1/2},\end{aligned}$$

and then by (1) the desired estimate (18) follows immediately. □

Lemma 2.4 *Let $\alpha, \beta \geq -1/2$ and let $0 < \epsilon_0 < \pi$ be fixed. Then there is a constant c independent of n, t such that*

$$\left|\frac{d^k}{dt^k}\mathcal{E}_n^{(\alpha,\beta)}(t)\right| \leq cn^k\left(\frac{|nt|}{1+|nt|}\right)^{\alpha-k+1/2}$$

for (i) $0 \leq k < \alpha + 1/2$, $0 \leq |t| \leq \pi - \epsilon_0$, *and* (ii) $k \geq \alpha + 1/2$, $0 < |t| \leq \pi - \epsilon_0$. *Furthermore,* (iii) *if $\alpha + 1/2$ is an even integer, then $|\frac{d^k}{dt^k}\mathcal{E}_n^{(\alpha,\beta)}(t)| \leq cn^k$ for $k \geq \alpha + 1/2$ and $0 \leq |t| \leq \pi - \epsilon_0$.*

Proof By Leibnitz's rule,

$$\frac{d^k}{dt^k}\mathcal{E}_n^{(\alpha,\beta)}(t) = \sqrt{\omega_n^{(\alpha,\beta)}}\sum_{j=0}^{k}\binom{k}{j}E_n^{(k-j)}(t)\phi_{\alpha,\beta}^{(j)}(t). \tag{20}$$

It is obvious that $\phi_{\alpha,\beta}^{(j)}(t) = |\sin(t/2)|^{\alpha+1/2-j}h(t)$, where $h(t)$ is a continuous function for $0 \leq |t| \leq \pi - \epsilon_0$. We have $|\phi_{\alpha,\beta}^{(j)}(t)| \leq c|t|^{\alpha+1/2-j}$ for $t \neq 0$ and also for $t = 0$ when $j < \alpha + 1/2$, and $|\phi_{\alpha,\beta}^{(j)}(t)| \leq c$ for $j \geq \alpha + 1/2$ when $\alpha + 1/2$ is an even integer. Now by Lemma 2.3 we obtain

$$\left|\frac{d^k}{dt^k}\mathcal{E}_n^{(\alpha,\beta)}(t)\right| \leq cn^{\alpha+1/2}\sum_{j=0}^{k}\frac{n^{k-j}|t|^{\alpha-j+1/2}}{(1+n|t|)^{\alpha+1/2}} \leq cn^k\left(\frac{|nt|}{1+|nt|}\right)^{\alpha-k+1/2}$$

for $0 < |t| \leq \pi - \epsilon_0$ and also for $t = 0$ when $k < \alpha + 1/2$. If $\alpha + 1/2$ is an even integer and $k \geq \alpha + 1/2$, again by Lemma 2.3 the summation with $j < \alpha + 1/2$ is, as above, bounded by cn^k, and the remainder part is dominated by

$$cn^{\alpha+1/2}\sum_{j=\alpha+1/2}^{k}n^{k-j}\bigl(1+n|t|\bigr)^{-\alpha-1/2} \leq cn^k.$$

Combining all the evaluations finishes the proof of the lemma. □

Corollary 2.5 *Let $\alpha, \beta \geq -1/2$. Then, for $0 \leq k < \gamma(\alpha, \beta) - 1$,*

$$\left|\frac{d^k}{dt^k}\mathcal{E}_n^{(\alpha,\beta)}(t)\right| \leq cn^k.$$

The corollary is a consequence of Lemma 2.4 and the relation

$$\mathcal{E}_n^{(\alpha,\beta)}(t) = (-1)^n\mathcal{E}_n^{(\beta,\alpha)}(\pi - t). \tag{21}$$

Lemma 2.6 *Let $\alpha, \beta \geq -1/2$. Then if $1 \leq m < \gamma(\alpha, \beta) - 1$,*

$$\left|\mathcal{E}_n^{(m-1)}(s) - \mathcal{E}_n^{(m-1)}(t)\right| \leq cn^m|s-t|; \tag{22}$$

if $\gamma(\alpha, \beta)$ is finite and $\gamma(\alpha, \beta) - 1 \leq m < \gamma(\alpha, \beta)$, then

$$\left|\mathcal{E}_n^{(m-1)}(s) - \mathcal{E}_n^{(m-1)}(t)\right| \leq cn^{\gamma(\alpha,\beta)-1}|s-t|^{\gamma(\alpha,\beta)-m}. \tag{23}$$

Proof For $1 \le m < \gamma(\alpha,\beta)-1$, (22) is a consequence of the mean-value theorem and Corollary 2.5.

In what follows we assume that $\gamma(\alpha,\beta)$ is finite and $\gamma(\alpha,\beta)-1 \le m < \gamma(\alpha,\beta)$. If $|s-t| \ge (2n)^{-1}$, the estimate is obvious since $|\mathcal{E}_n^{(m-1)}(t)| \le cn^{m-1}$. Now we consider the case when $|s-t| \le (2n)^{-1}$ ($\le \pi/6$). Furthermore, we assume that $\gamma(\alpha,\beta) = \alpha + 3/2$ and restrict ourselves to the case for $-\pi/4 \le t \le 3\pi/4$, which implies $-5\pi/12 \le s \le 11\pi/12$. By (20), $|\mathcal{E}_n^{(m-1)}(s) - \mathcal{E}_n^{(m-1)}(t)|$ is dominated by a multiple of $n^{\alpha+1/2}$ times

$$\left|E_n(s)\phi_{\alpha,\beta}^{(m-1)}(s) - E_n(t)\phi_{\alpha,\beta}^{(m-1)}(t)\right| + \sum_{j=0}^{m-2}\left|\left(E_n^{(m-1-j)}(t)\phi_{\alpha,\beta}^{(j)}(t)\right)'_{t=\xi_1}\right||s-t| \tag{24}$$

with some ξ_1 lying between s and t. The second term above is bounded by, applying Lemma 2.3,

$$\begin{aligned}
&\sum_{j=0}^{m-2}\left|E_n^{(m-j)}(\xi_1)\phi_{\alpha,\beta}^{(j)}(\xi_1) + E_n^{(m-1-j)}(\xi_1)\phi_{\alpha,\beta}^{(j+1)}(\xi_1)\right||s-t| \\
&\quad \le c\sum_{j=0}^{m-2}\left[\frac{n^{m-j}|\xi_1|^{\alpha-j+1/2}}{(1+|n\xi_1|)^{\alpha+1/2}} + \frac{n^{m-j-1}|\xi_1|^{\alpha-j-1/2}}{(1+|n\xi_1|)^{\alpha+1/2}}\right]|s-t| \\
&\quad \le cn^{m-\alpha-1/2}|s-t|,
\end{aligned}$$

since $|n\xi_1|^{\alpha-j-1/2} \le (1+|n\xi_1|)^{\alpha-j-1/2}$ for $j \le m-2$. In the meantime, it is easy to see that the first term in (24) is bounded by

$$\left|E_n'(\xi_2)\right|\left|\phi_{\alpha,\beta}^{(m-1)}(t)\right||s-t| + \left|E_n(s)\right|\left|\phi_{\alpha,\beta}^{(m-1)}(s) - \phi_{\alpha,\beta}^{(m-1)}(t)\right|$$

for some ξ_2 lying between s and t. Since

$$\phi_{\alpha,\beta}^{(m-1)} \in \mathrm{Lip}(\alpha - m + 3/2), \quad 0 < \alpha - m + 3/2 \le 1,$$

and $(1+|n\xi_2|) \asymp (1+|nt|)$ for s, t under consideration, again applying Lemma 2.3 we obtain an upper bound of the first term in (24) as a multiple of

$$\frac{n|t|^{\alpha-m+3/2}}{(1+|n\xi_2|)^{\alpha+1/2}}|s-t| + |s-t|^{\alpha-m+3/2} \le cn^{m-\alpha-1/2}|s-t| + |s-t|^{\alpha-m+3/2}.$$

Substituting the two estimates into (24) yields

$$\left|\mathcal{E}_n^{(m-1)}(s) - \mathcal{E}_n^{(m-1)}(t)\right| \le cn^{\alpha+1/2}|s-t|^{\alpha-m+3/2}$$

for $|s-t| \le (2n)^{-1}$ and $\alpha + 1/2 \le m < \alpha + 3/2$.

If $3\pi/4 \le t \le 7\pi/4$ or $\gamma(\alpha,\beta) = \beta + 3/2$, the associated estimate in (23) is a consequence of the proved case and formula (21). □

Lemma 2.7 *Let $\alpha, \beta \ge -1/2$. There exists a constant c such that, for all interval I, $|I| \le \pi/4$,*

$$\left|\int_I \mathcal{E}_k(t)\mathcal{E}_j(t)\,dt\right| \le c\left(\frac{|j|}{|k|}|I| + \frac{1}{|k|}\right).$$

Proof If $|k|/2 \le |j| \le |k|$, then by Corollary 2.5, $|\int_I \mathcal{E}_k(t)\mathcal{E}_j(t)\,dt| \le c|I|$. In what follows, we assume that $|j| \le |k|/2$. Since $|I| \le \pi/4$, we may suppose $I \subseteq [-3\pi/4, 3\pi/4]$, and the case when $I \subseteq [\pi/4, 7\pi/4]$ has a similar result by appealing to (21). If I contains 0 as an interior point, then we divide it by 0 into two parts, and hence we may assume $I \subseteq [0, 3\pi/4]$. We also assume that $k, j > 0$ without loss of generality. At first we have

$$\left|\int_{I\cap\{t:t\le k^{-1}\}} \mathcal{E}_k(t)\mathcal{E}_j(t)\,dt\right| \le ck^{-1} \quad \text{by Corollary 2.5.}$$

It remains to show that, for $j \le k/2$,

$$\left|\int_{I\cap\{t:t\ge k^{-1}\}} \mathcal{E}_k(t)\mathcal{E}_j(t)\,dt\right| \le c\left(\frac{j}{k}|I| + \frac{1}{k}\right). \tag{25}$$

We apply (17) to the $\mathcal{E}_k(t)$ in (25). The contribution of the O-term, in conjunction with Corollary 2.5, to the integral is dominated by

$$c\int_{k^{-1}}^{3\pi/4} \left[(kt)^{-2} + k^{-1}\right] dt \le c'k^{-1}.$$

We need to evaluate the critical part of the integral in (25) according to (17), that is,

$$A_{j,k} := \int_{I\cap\{t:t\ge k^{-1}\}} e^{iKt}\mathcal{E}_j(t)\,dt,$$

where $K = k + (\alpha + \beta + 1)/2$. But the evaluation of the part of the integral associated with the terms in (17) with additional factor $(Kt)^{-1}$ is a little easier.

Taking integration by parts yields

$$A_{j,k} = \frac{i}{K}\int_{I\cap\{t:t\ge k^{-1}\}} e^{iKt}\mathcal{E}_j'(t)\,dt + O\bigl(k^{-1}\bigr). \tag{26}$$

By Lemma 2.4, $|\mathcal{E}_j'(t)| \le cj$ for $\alpha = -1/2$ and $|\mathcal{E}_j'(t)| \le cj(jt/(1 + jt))^{\alpha-1/2}$ for $\alpha > -1/2$. It follows from (26) that, for $\alpha = -1/2$, $|A_{j,k}| \le c(j|I| + 1)/k$, and for $\alpha > 1/2$,

$$|A_{j,k}| \le c\frac{j}{K}\int_I \left(\frac{jt}{1 + jt}\right)^{\alpha-1/2} dt + O\bigl(k^{-1}\bigr).$$

It is obvious that the integration over I is dominated by a multiple of

$$\int_0^{j^{-1}} (jt)^{\alpha-1/2}\,dt + \int_I dt \le c'\bigl(j^{-1} + |I|\bigr),$$

and immediately one has an upper bound for $|A_{j,k}|$ as in (25), as desired. □

3 Proofs of the main results

3.1 Proof of Theorem 1.3

We first note that the conclusion for $2 < q < \infty$ follows from that for $q = 2$. Indeed, if we put $\nu_n = |\lambda_n|^{q/2}$, then (5) implies $\sum_{N\le|n|\le 2N} |\nu_n|^2 = O(1)$, and since $|c_k(f)| \le c\|f\|_{H^1}$ by

Corollary 2.5 with $k = 0$, we have

$$\sum_{k=-\infty}^{\infty} \left|\lambda_k c_k(f)\right|^q \le c' \|f\|_{H^1}^{q-2} \sum_{k=-\infty}^{\infty} \left|\nu_k c_k(f)\right|^2 \le c\|f\|_{H^1}^q.$$

Now we turn to the proof of the theorem for $q = 2$. We fix a sequence $\{b_n\}_{n=-\infty}^{\infty} \in \ell^2$ and for $n = 1, 2, \ldots$, put

$$g_n(t) = \sum_{k=-n}^{n} \lambda_k b_k \overline{\mathcal{E}_k(t)}. \tag{27}$$

In terms of the duality of $H^1(-\pi,\pi)$ and BMO, one has $|\int_{-\pi}^{\pi} f(t) g_n(t)\,dt| \le c\|g_n\|_{BMO}\|f\|_{H^1}$, or equivalently,

$$\left|\sum_{k=-n}^{n} \lambda_k b_k c_k(f)\right| \le c\|g_n\|_{BMO}\|f\|_{H^1}, \tag{28}$$

where $\|g\|_{BMO} = \sup_I (1/|I|) \int_I |g(t) - g_I|\,dt$ for taking I to be all interval of the line and $g_I = (1/|I|) \int_I g(t)\,dt$ with $|I|$ being the length of I. We shall show that

$$\|g_n\|_{BMO} \le c' \left(\sum_{k=-n}^{n} |b_k|^2\right)^{1/2} \tag{29}$$

for a constant c' independent of n and $\{b_k\}_{k=-\infty}^{\infty} \in \ell^2$. Once (29) is true, then from (28) it follows that $(\sum_{k=-n}^{n} |\lambda_k c_k(f)|^2)^{1/2} \le c\|f\|_{H^1}$, which proves the theorem by letting $n \to \infty$.

As usual, in order to prove (29), it suffices to show that, for any interval I, there exists a constant γ_I satisfying

$$\frac{1}{|I|} \int_I \left|g_n(t) - \gamma_I\right|\,dt \le c' \left(\sum_{k=-n}^{n} |b_k|^2\right)^{1/2}. \tag{30}$$

For an interval I, if $2m\pi < |I| \le 2(m+1)\pi$ for some $m \ge 1$, then

$$\left(\frac{1}{|I|} \int_I \left|g_n(t)\right|\,dt\right)^2 \le \frac{1}{|I|} \int_I \left|g_n(t)\right|^2\,dt \le \frac{m+1}{2m\pi} \int_{-\pi}^{\pi} \left|g_n(t)\right|^2\,dt \le c \sum_{k=-n}^{n} |b_k|^2.$$

If $\pi/4 \le |I| \le 2\pi$, we have a similar estimate.

In what follows we assume that $2\pi/(m+1) < |I| \le 2\pi/m$ for some $m \ge 8$. For such an interval, if $m \ge n$, then choosing t_I to be one of the end points of I, we have

$$\left|g_n(t) - g_n(t_I)\right|^2 \le \sum_{k=-n}^{n} |b_k|^2 \sum_{k=-n}^{n} |\lambda_k|^2 \left|\mathcal{E}_k(t) - \mathcal{E}_k(t_I)\right|^2,$$

and by Lemma 2.6,

$$\left|g_n(t) - g_n(t_I)\right|^2 \le c \sum_{k=-n}^{n} |b_k|^2 \sum_{k=-n}^{n} |\lambda_k|^2 |k|^{2\delta} |t - t_I|^{2\delta},$$

where $\delta = \gamma(\alpha,\beta) - 1$ if $\gamma(\alpha,\beta)$ is finite and $1 < \gamma(\alpha,\beta) \le 2$, and $\delta = 1$ otherwise. From assumption (5), $\sum_{N\le|n|\le 2N} |\lambda_n|^2 = O(1)$, which implies $\sum_{k=-n}^{n} |\lambda_k|^2|k|^{2\delta} \le cn^{2\delta}$ by Proposition 1.6 ((ii) $\Rightarrow$ (i) with $b = a = 2\delta > 0$). Therefore

$$\bigl|g_n(t) - g_n(t_I)\bigr|^2 \le c \sum_{|k|\le n} |b_k|^2\bigl(n|I|\bigr)^{2\delta} \le c' \sum_{|k|\le n} |b_k|^2,$$

and (30) is true with $\gamma_I = g_n(t_I)$.

If $m < n$, we again choose t_I to be one of the end points of I to get

$$\bigl|g_n(t) - g_m(t_I)\bigr| \le \bigl|g_m(t) - g_m(t_I)\bigr| + \biggl|\sum_{m<|k|\le n} \lambda_k b_k \mathcal{E}_k(t)\biggr|.$$

Hence by what has been verified,

$$\frac{1}{|I|}\int_I \bigl|g_n(t) - g_m(t_I)\bigr|\,dt \le c'\Biggl(\sum_{k=-m}^{m} |b_k|^2\Biggr)^{1/2} + F_{m,n}, \tag{31}$$

where $F_{m,n} = |I|^{-1}\int_I |\sum_{m<|k|\le n} \lambda_k b_k \mathcal{E}_k(t)|\,dt$. But for $F_{m,n}$, we first note

$$\begin{aligned} F_{m,n}^2 &\le \frac{1}{|I|}\int_I \biggl|\sum_{m<|k|\le n} \lambda_k b_k \mathcal{E}_k(t)\biggr|^2 dt \\ &\le \sum_{m<|k|\le n}\sum_{m<|j|\le n} |\lambda_k b_k \overline{\lambda_j b_j}| \frac{1}{|I|}\biggl|\int_I \mathcal{E}_k(t)\overline{\mathcal{E}_j(t)}\,dt\biggr|. \end{aligned}$$

By symmetry, it suffices to evaluate the part $\sum_{m<|k|\le n}\sum_{m<|j|\le|k|}$, and for these j, k, $2\pi|I|^{-1} \le m+1 \le |j|$, and by Lemma 2.7,

$$|I|^{-1}\biggl|\int_I \mathcal{E}_k(t)\overline{\mathcal{E}_j(t)}\,dt\biggr| \le c\bigl(|j| + |I|^{-1}\bigr)/|k| \le 2c|j|/|k|.$$

Thus the evaluation of $F_{m,n}^2$ is reduced to showing the following inequality:

$$S_{m,n} := \sum_{m<|k|\le n}\sum_{m<|j|\le|k|} |\lambda_k b_k \overline{\lambda_j b_j}| \frac{|j|}{|k|} \le c \sum_{m<|k|\le n} |b_k|^2.$$

For the purpose, we rewrite $S_{m,n}$ as

$$\begin{aligned} S_{m,n} &\le \frac{1}{2}\sum_{m<|k|\le n}\sum_{m<|j|\le|k|} \bigl(|\lambda_j b_k|^2 + |\lambda_k b_j|^2\bigr)\frac{|j|}{|k|} \\ &= \frac{1}{2}\sum_{m<|k|\le n} \frac{|b_k|^2}{|k|}\sum_{m<|j|\le|k|} |\lambda_j|^2|j| + \frac{1}{2}\sum_{m<|j|\le n} |b_j|^2|j|\sum_{|j|\le|k|\le n} \frac{|\lambda_k|^2}{|k|}. \end{aligned} \tag{32}$$

Since assumption (5) ($q = 2$) implies

$$\sum_{|j|\le|k|} |\lambda_j|^2|j| \le c|k| \quad \text{and} \quad \sum_{|k|\ge|j|} \frac{|\lambda_k|^2}{|k|} \le c|j|^{-1},$$

by Proposition 1.6 ((ii) ⇒ (i) and (ii) ⇒ (iv) with $b = a = \delta = 1$). Incorporating these into (32) proves that $S_{m,n} \le c' \sum_{m<|k|\le n} |b_k|^2$; furthermore $F_{m,n} \le c(\sum_{m<|k|\le n} |b_k|^2)^{1/2}$. Inserting this into (31) proves (30) with $\gamma_I = g_m(t_I)$.

The proof of Theorem 1.3 is completed.

3.2 Proof of Theorem 1.4

We fix a sequence $\{b_n\}_{n=-\infty}^{\infty} \in \ell^{q'}$, $q^{-1} + q'^{-1} = 1$, and for $n = 1, 2, \dots$, define g_n as in (27). By Proposition 1.2 and Lemma 1.1,

$$\left|\sum_{k=-n}^{n} \lambda_k b_k c_k(f)\right| = \left|\sum_{k=-n}^{n} \lambda_k b_k \mathcal{L}_{\overline{\mathcal{E}}_k}(f)\right| = \left|\mathcal{L}_{g_n}(f)\right| \le c\|g_n\|_{\Lambda_{p^{-1}-1}} \|f\|_{H^p}.$$

In order to prove (6), it suffices to show that there is a constant c' independent of n and $\{b_k\} \in \ell^{q'}$ such that

$$\|g_n\|_{\Lambda_{p^{-1}-1}} \le c' \left\|\{b_k\}\right\|_{q'}. \tag{33}$$

Assume $m - 1 < \delta := p^{-1} - 1 < m \le \gamma(\alpha, \beta) - 1$. From (27) we have, for $h \ne 0$,

$$\left|g_n^{(m-1)}(t+h) - g_n^{(m-1)}(t)\right| \le \sum_{k=-n}^{n} |\lambda_k b_k| \left|\mathcal{E}_k^{(m-1)}(t+h) - \mathcal{E}_k^{(m-1)}(t)\right|. \tag{34}$$

If $n \le |h|^{-1}$, we apply Lemma 2.6 for $m \le \gamma(\alpha, \beta) - 1$ to get an upper bound of $|g_n^{(m-1)}(t + h) - g_n^{(m-1)}(t)|$ as a multiple of

$$\sum_{k=-n}^{n} |\lambda_k b_k| |k|^m |h| \le |h| \left\|\{b_k\}\right\|_{q'} \left(\sum_{k=-n}^{n} |\lambda_k|^q |k|^{mq}\right)^{1/q}. \tag{35}$$

Since $q(1 - p^{-1}) = a - b$, where $a = q(m + 1 - p^{-1}) > 0$, $b = mq$, condition (7) and Proposition 1.6 ((ii) ⇒ (i)) give

$$\sum_{k=-n}^{n} |\lambda_k|^q |k|^{mq} \le cn^{q(m+1-p^{-1})} \le c|h|^{q(p^{-1}-m-1)} \quad \text{for } n \le |h|^{-1}.$$

Substituting this into (35) yields

$$\left|g_n^{(m-1)}(t+h) - g_n^{(m-1)}(t)\right| \le c\left\|\{b_k\}\right\|_{q'} |h|^{p^{-1}-m}. \tag{36}$$

If $n > |h|^{-1}$, the summation of those terms in (34) for $|k| \le |h|^{-1}$ has the same bound $c\|\{b_k\}\|_{q'} |h|^{p^{-1}-m}$ as above and the summation of the terms for $|h|^{-1} < |k| \le n$, in virtue of Corollary 2.5, is dominated by

$$\begin{aligned} &\sum_{|h|^{-1}<|k|\le n} |\lambda_k b_k| \left(\left|\mathcal{E}_k^{(m-1)}(t+h)\right| + \left|\mathcal{E}_k^{(m-1)}(t)\right|\right) \\ &\quad \le \sum_{|h|^{-1}<|k|\le n} |\lambda_k b_k| |k|^{m-1} \le c\left\|\{b_k\}\right\|_{q'} \left(\sum_{|h|^{-1}<|k|\le n} |\lambda_k|^q |k|^{q(m-1)}\right)^{1/q}. \end{aligned} \tag{37}$$

Since

$$q(m-1) = q\bigl(p^{-1}-1\bigr) - q\bigl(p^{-1}-m\bigr) \quad \text{and} \quad q\bigl(p^{-1}-m\bigr) > 0,$$

condition (7) and Proposition 1.6 ((ii) $\Rightarrow$ (iv)) give

$$\sum_{|h|^{-1}<|k|\le n} |\lambda_k|^q |k|^{q(m-1)} \le c\bigl(|h|^{-1}\bigr)^{-q(p^{-1}-m)} = c|h|^{q(p^{-1}-m)}.$$

Substituting this into the previous evaluation yields an upper bound of the summation of the terms in (34) for $|h|^{-1} < |k| \le n$ as $c\|\{b_k\}\|_{q'}|h|^{p^{-1}-m}$. Thus (36) is proved to be true for all n and h, so that (33) is shown whenever

$$m-1 < \delta := p^{-1} - 1 < m \le \gamma(\alpha,\beta) - 1.$$

Next we consider the case when $\gamma(\alpha,\beta)$ is finite and

$$m-1 < \delta := p^{-1} - 1 < \gamma(\alpha,\beta) - 1 < m.$$

Similarly to (35), we apply (23) in (34) to obtain

$$\begin{aligned} &\bigl|g_n^{(m-1)}(t+h) - g_n^{(m-1)}(t)\bigr| \\ &\quad \le c|h|^{\gamma(\alpha,\beta)-m}\bigl\|\{b_k\}\bigr\|_{q'}\left(\sum_{k=-n}^{n} |\lambda_k|^q |k|^{q(\gamma(\alpha,\beta)-1)}\right)^{1/q}. \end{aligned} \tag{38}$$

Since $q(1-p^{-1}) = a - b$, where $a = q(\gamma(\alpha,\beta) - p^{-1}) > 0$, $b = q(\gamma(\alpha,\beta)-1)$, condition (7) and Proposition 1.6 ((ii) $\Rightarrow$ (i)) give

$$\sum_{k=-n}^{n} |\lambda_k|^q |k|^{q(\gamma(\alpha,\beta)-1)} \le cn^{q(\gamma(\alpha,\beta)-p^{-1})} \le c|h|^{q(p^{-1}-\gamma(\alpha,\beta))}$$

for $n \le |h|^{-1}$, and substituting this into (38) proves (36) again. If $n > |h|^{-1}$, we also break the summation in (34) into two parts according to $|k| \le |h|^{-1}$ and $|h|^{-1} < |k| \le n$, where the first part has the same bound $c\|\{b_k\}\|_{q'}|h|^{p^{-1}-m}$ as just proved and the second part is dealt with by the same way as in (37). That means (36) is true for all n and h, and hence (33) is proved for $m-1 < \delta := p^{-1} - 1 < \gamma(\alpha,\beta) - 1 < m$.

Finally, we prove (33) for

$$\delta := p^{-1} - 1 = m < \gamma(\alpha,\beta) - 1.$$

It is noted that the verification from (35) to (36) for $n \le |h|^{-1}$ does not work when $p^{-1} = m+1$. We shall need to evaluate the second order difference of $g_n^{(m-1)}$, which is also sufficient by our definition about Λ_δ for $\delta = m$. From (27) it follows, for $h \ne 0$, that $|g_n^{(m-1)}(t+h) - 2g_n^{(m-1)}(t) + g_n^{(m-1)}(t-h)|$ is bounded by

$$\sum_{k=-n}^{n} |\lambda_k b_k| \bigl|\mathcal{E}_k^{(m-1)}(t+h) - 2\mathcal{E}_k^{(m-1)}(t) + \mathcal{E}_k^{(m-1)}(t-h)\bigr|. \tag{39}$$

If $1 \le m < \gamma(\alpha,\beta) - 2$, this is dominated by $c\sum_{k=-n}^{n} |\lambda_k b_k||\mathcal{E}_k^{(m+1)}(\xi)||h|^2$; furthermore, by virtue of Corollary 2.5, by

$$c'|h|^2 \sum_{k=-n}^{n} |\lambda_k b_k||k|^{m+1} \le c'|h|^2 \big\|\{b_k\}\big\|_{q'} \left(\sum_{k=-n}^{n} |\lambda_k|^q |k|^{q(m+1)}\right)^{1/q}. \tag{40}$$

Since $q(1-p^{-1}) = -qm = a - b$, where $a = q > 0$, $b = q(m+1)$, condition (7) and Proposition 1.6 ((ii) $\Rightarrow$ (i)) give

$$\sum_{k=-n}^{n} |\lambda_k|^q |k|^{q(m+1)} \le cn^q \le c|h|^{-q} \quad \text{for } n \le |h|^{-1}.$$

Substituting this into (40) yields, for $n \le |h|^{-1}$,

$$\big|g_n^{(m-1)}(t+h) - 2g_n^{(m-1)}(t) + g_n^{(m-1)}(t-h)\big| \le c\big\|\{b_k\}\big\|_{q'}|h|. \tag{41}$$

If $\gamma(\alpha,\beta)$ is finite and $\gamma(\alpha,\beta) - 2 \le m < \gamma(\alpha,\beta) - 1$, we note that

$$\big|\mathcal{E}_k^{(m-1)}(t+h) - 2\mathcal{E}_k^{(m-1)}(t) + \mathcal{E}_k^{(m-1)}(t-h)\big| = \big|\mathcal{E}_k^{(m)}(\xi_1) - \mathcal{E}_k^{(m)}(\xi_2)\big||h|$$

by the mean-value theorem, where ξ_1 and ξ_2 lay between $t-h$ and $t+h$; furthermore, by (23) this is bounded by

$$cn^{\gamma(\alpha,\beta)-1}|h|^{\gamma(\alpha,\beta)-m-1}|h| = cn^{\gamma(\alpha,\beta)-1}|h|^{\gamma(\alpha,\beta)-m}.$$

Hence the expression in (39) is dominated by a multiple of

$$\sum_{k=-n}^{n} |\lambda_k b_k||k|^{\gamma(\alpha,\beta)-1}|h|^{\gamma(\alpha,\beta)-m},$$

which has the same bound as in (38), and also the bound $c\|\{b_k\}\|_{q'}|h|^{p^{-1}-m} = c\|\{b_k\}\|_{q'}|h|$ for $n \le |h|^{-1}$ as in (36). Thus (41) is shown to be true for $n \le |h|^{-1}$.

If $n > |h|^{-1}$, the summation of the terms for $|k| \le |h|^{-1}$ in (39) has the same bound as in (41), and the summation of those for $|h|^{-1} < |k| \le n$ is dealt with by the same way as in (37) to obtain its bound $c\|\{b_k\}\|_{q'}|h|^{p^{-1}-m} = c\|\{b_k\}\|_{q'}|h|$. Therefore (41) is verified for all n and h, and hence (33) is proved for $\delta := p^{-1} - 1 = m < \gamma(\alpha,\beta) - 1$.

The proof of Theorem 1.4 is completed.

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