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A double inequality for $\tanh x$

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Abstract

In this paper, we prove that, for $x > 0$,

$$\sqrt{1-\exp\Bigl(-\frac{x^2}{\sqrt{x^2+1}}\Bigr)} < \tanh x < \sqrt[3]{1-\exp\Bigl(-\frac{x^3}{\sqrt{x^3+1}}\Bigr)}.$$

This solves an open problem proposed by Ivády.

MSC: 26D07

Keywords: Inequality; Exponential function; Hyperbolic function

1 Introduction

Ivády (see [1, Problem 51]) proposed the following problem: Show that, for $x > 0$,

$$\sqrt{1-\exp\biggl(-\frac{x^2}{\sqrt{x^2+1}}\biggr)} < \tanh x < \sqrt[3]{1-\exp\biggl(-\frac{x^3}{\sqrt{x^3+1}}\biggr)} \tag{1.1}$$

holds. Subsequently, a solution was presented by the proposer (see [2]).

In [2], the proof of the left-hand side of (1.1) is correct, but the proof of the right-hand side of (1.1) is not correct.

Using the inverse function of $\tanh x$, the second inequality in (1.1) has the equivalent form

$$\frac{1}{2}\ln\left(\frac{1+\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}{1-\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}\right) > x \quad \text{for } x > 0. \tag{1.2}$$

According to the mean-value theorem, Ivády [2, Eq. (8)] got on $[0, x]$

$$\begin{aligned}&\frac{1}{2x}\ln\left(\frac{1+\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}{1-\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}\right)\\ &\quad=\frac{\frac{1}{2}\ln(\frac{1+\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}{1-\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}})}{x}=\frac{\eta^2(\eta^3+2)}{2(\eta^3+1)^{3/2}(1-\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}}))^{2/3}(1-(1-\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}}))^{2/3})}\end{aligned} \tag{1.3}$$

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for some $0 < \eta < x$, and then proved that, for all $\eta > 0$,

$$\frac{\eta^2(\eta^3+2)}{2(\eta^3+1)^{3/2}(1-\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}}))^{2/3}(1-(1-\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}}))^{2/3})} > 1.$$

We note that (1.3) may be corrected as

$$\begin{aligned}
&\frac{1}{2x}\ln\left(\frac{1+\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}{1-\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}\right)\\
&\quad=\frac{\eta^2(\eta^3+2)\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}})}{2(\eta^3+1)^{3/2}(1-\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}}))^{2/3}(1-(1-\exp(-\frac{\eta^3}{\sqrt{\eta^3+1}}))^{2/3})}
\end{aligned}$$

for some $0 < \eta < x$.

In this paper, we provide a proof of the right-hand side of (1.1).

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2 Lemma

Lemma 2.1 *Let*

$$\begin{aligned}
G(x) &= \frac{x^2(x^3+2)}{2(x^3+1)^{3/2}}\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)-\left(1-\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3}\\
&\quad+\left(1-\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{4/3}.
\end{aligned}\tag{2.1}$$

Then, for $x > 0$,

$$G(x) > 0. \tag{2.2}$$

Proof We split the proof into three cases.

Case 1. $0 < x < 0.5$.

We first prove the following inequalities:

$$\frac{x^3+2}{2(x^3+1)^{3/2}}\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right) > 1-2x^3, \tag{2.3}$$

$$\left(1-\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3} < x^2-\frac{2}{3}x^5+\frac{7}{12}x^8, \tag{2.4}$$

and

$$\left(1-\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{4/3} > x^4-\frac{4}{3}x^7 \tag{2.5}$$

for $0 < x < 0.5$.

The inequality (2.3) can be converted to

$$\frac{y}{\sqrt{y+1}} + \ln\left(\frac{2(y+1)^{3/2}(1-2y)}{y+2}\right) < 0 \quad \text{for } 0 < y < 0.125.$$

We consider the function $f_1(y)$ defined, for $0 < y < 0.125$, by

$$f_1(y) = \frac{y}{\sqrt{y+1}} + \ln 2 + \frac{3}{2}\ln(y+1) + \ln(1-2y) - \ln(y+2).$$

Differentiation yields

$$-2(y+1)f_1'(y) = \frac{6y^2+19y+4}{(1-2y)(y+2)} - \frac{y+2}{(y+1)^{1/2}}.$$

By direct computation, we get, for $0 < y < 0.125$,

$$\left(\frac{6y^2+19y+4}{(1-2y)(y+2)}\right)^2 - \frac{(y+2)^2}{y+1} = \frac{y(4y^4(2-y)+199y^3+597y^2+601y+200)}{(1-2y)^2(y+2)^2(y+1)} > 0.$$

We then obtain $f_1'(y) < 0$ for $0 < y < 0.125$. Hence, $f_1(y)$ is strictly decreasing for $0 < y < 0.125$, and we have

$$f_1(y) = \frac{y}{\sqrt{y+1}} + \ln\left(\frac{2(y+1)^{3/2}(1-2y)}{y+2}\right) < f_1(0) = 0.$$

The inequality (2.4) can be written for $0 < x < 0.5$ as

$$\frac{x^3}{\sqrt{x^3+1}} + \ln\left(1 - x^3\left(1 - \frac{2}{3}x^3 + \frac{7}{12}x^6\right)^{3/2}\right) < 0. \tag{2.6}$$

In order to prove (2.6), it suffices to show that

$$f_2(y) < 0 \quad \text{for } 0 < y < 0.125,$$

where

$$f_2(y) = \frac{y}{\sqrt{y+1}} + \ln\left(1 - y\left(1 - \frac{2}{3}y + \frac{7}{12}y^2\right)^{3/2}\right).$$

Differentiation yields

$$-f_2'(y) = \frac{4(3-5y+7y^2)\sqrt{36-24y+21y^2}}{72-(12y-8y^2+7y^3)\sqrt{36-24y+21y^2}} - \frac{y+2}{2(y+1)^{3/2}}.$$

We now prove $f_2'(y) < 0$ for $0 < y < 0.125$. It suffices to show that, for $0 < y < 0.125$,

$$\frac{4(3-5y+7y^2)\sqrt{36-24y+21y^2}}{72-(12y-8y^2+7y^3)\sqrt{36-24y+21y^2}} > \frac{y+2}{2(y+1)^{3/2}}.$$

It is not difficult to prove that

$$\frac{4(3-5y+7y^2)\sqrt{36-24y+21y^2}}{72-(12y-8y^2+7y^3)\sqrt{36-24y+21y^2}} > 1-y+\frac{9}{8}y^2$$

and

$$\frac{y+2}{2(y+1)^{3/2}} < 1-y+\frac{9}{8}y^2$$

for $0 < y < 0.125$ (we here omit the proofs). Hence, $f_2'(y) < 0$ holds for $0 < y < 0.125$. So, $f_2(y)$ is strictly decreasing for $0 < y < 0.125$, and we have

$$f_2(y) = \frac{y}{\sqrt{y+1}} + \ln\left(1-y\left(1-\frac{2}{3}y+\frac{7}{12}y^2\right)^{3/2}\right) < f_2(0) = 0$$

for $0 < y < 0.125$.

The inequality (2.5) can be written for $0 < x < 0.5$ as

$$\frac{x^3}{\sqrt{x^3+1}} + \ln\left(1-x^3\left(1-\frac{4}{3}x^3\right)^{3/4}\right) > 0. \tag{2.7}$$

In order to prove (2.7), it suffices to show that

$$f_3(y) > 0 \quad \text{for } 0 < y < 0.125,$$

where

$$f_3(y) = \frac{y}{\sqrt{y+1}} + \ln\left(1-y\left(1-\frac{4}{3}y\right)^{3/4}\right).$$

Differentiation yields

$$f_3'(y) = \frac{y+2}{2(y+1)^{3/2}} - \frac{3-7y}{3(1-\frac{4}{3}y)^{1/4}(1-y(1-\frac{4}{3}y)^{3/4})}.$$

We now prove $f_3'(y) > 0$ for $0 < y < 0.125$. It suffices to show that, for $0 < y < 0.125$,

$$1-y\left(1-\frac{4}{3}y\right)^{3/4} > \frac{2(3-7y)(y+1)^{3/2}}{3(y+2)(1-\frac{4}{3}y)^{1/4}}.$$

It is not difficult to prove that

$$1-y\left(1-\frac{4}{3}y\right)^{3/4} > 1-y \quad \text{and} \quad \frac{2(3-7y)(y+1)^{3/2}}{3(y+2)(1-\frac{4}{3}y)^{1/4}} < 1-y$$

for $0 < y < 0.125$ (we here omit the proofs). Hence, $f_3'(y) > 0$ holds for $0 < y < 0.125$. So, $f_3(y)$ is strictly increasing for $0 < y < 0.125$, and we have

$$f_3(y) = \frac{y}{\sqrt{y+1}} + \ln\left(1-y\left(1-\frac{4}{3}y\right)^{3/4}\right) > f_3(0) = 0$$

for $0 < y < 0.125$.

We then obtain by (2.3), (2.4) and (2.5), for $0 < x < 0.5$,

$$\begin{aligned} G(x) &> x^2\left(1-2x^3\right) - \left(x^2 - \frac{2}{3}x^5 + \frac{7}{12}x^8\right) + x^4 - \frac{4}{3}x^7 \\ &= \frac{1}{12}x^4\left(12 - 16x - 16x^3 - 7x^4\right) > 0. \end{aligned}$$

Case 2. $0.5 \le x \le 3$.

Let

$$G(x) = G_1(x) + G_2(x),$$

where

$$G_1(x) = \frac{x^2(x^3+2)}{2(x^3+1)^{3/2}} \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right) - \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3} \tag{2.8}$$

and

$$G_2(x) = \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{4/3}. \tag{2.9}$$

It is not difficult to prove that

$$\left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{1/3} < x, \quad x > 0 \tag{2.10}$$

(we here omit the proof). Differentiating $G_1(x)$ and using (2.10), we obtain, for $x > 0$,

$$\begin{aligned} &-\frac{(x^3+1)^{3/2}}{x} \exp\left(\frac{x^3}{\sqrt{x^3+1}}\right) G_1'(x) \\ &\quad = \frac{-(x^6+8)\sqrt{x^3+1} + 3x^3(x^3+2)^2}{4(x^3+1)^{3/2}} + \frac{x(x^3+2)}{(1-\exp(-\frac{x^3}{\sqrt{x^3+1}}))^{1/3}} \\ &\quad > \frac{-(x^6+8)\sqrt{x^3+1} + 3x^3(x^3+2)^2}{4(x^3+1)^{3/2}} + x^3 + 2 \\ &\quad = \frac{3x^3((x^3+4)\sqrt{x^3+1} + x^6 + 4x^3 + 4)}{4(x^3+1)^{3/2}} > 0. \end{aligned}$$

Therefore, the function $G_1(x)$ is strictly decreasing for $x > 0$.

Differentiation yields

$$G_2'(x) = \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{1/3} \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right) \frac{2x^2(x^3+2)}{(x^3+1)^{3/2}} > 0.$$

Therefore, the function $G_2(x)$ is strictly increasing for $x > 0$.

Let $0.5 \le r \le x \le s \le 3$. Since $G_1(x)$ is decreasing and $G_2(x)$ is increasing, we obtain

$$G(x) \ge G_1(s) + G_2(r) =: \sigma(r,s).$$

We divide the interval $[0.5, 3]$ into 250 subintervals:

$$[0.5,3]=\bigcup_{k=0}^{249}\left[0.5+\frac{k}{100},0.5+\frac{k+1}{100}\right]\quad\text{for } k=0,1,2,\ldots,249.$$

By direct computation we get

$$\sigma\left(0.5+\frac{k}{100},0.5+\frac{k+1}{100}\right)>0\quad\text{for } k=0,1,2,\ldots,249.$$

Hence,

$$G(x)>0\quad\text{for } x\in\left[0.5+\frac{k}{100},0.5+\frac{k+1}{100}\right]\text{ and } k=0,1,2,\ldots,249.$$

This implies that $G(x)$ is positive for $0.5\le x\le 3$.

Case 3. $x>3$.

We first prove that, for $x>3$,

$$\frac{3x^3(x^3+2)-(x^3+1)^{3/2}}{4\sqrt{x}}\left(1-\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{1/3}>x^2\left(x^3+2\right),\tag{2.11}$$

which can be written for $x>3$ as

$$\frac{x^3}{\sqrt{x^3+1}}+\ln\left(1-\left(\frac{4x^{5/2}(x^3+2)}{3x^3(x^3+2)-(x^3+1)^{3/2}}\right)^3\right)>0,\tag{2.12}$$

which can be converted to

$$\frac{y}{\sqrt{y+1}}+\ln\left(1-\left(\frac{4y^{5/6}(y+2)}{3y(y+2)-(y+1)^{3/2}}\right)^3\right)>0\quad\text{for } y>27.\tag{2.13}$$

It is not difficult to prove that

$$\frac{y}{\sqrt{y+1}}>\sqrt{y}-\frac{1}{2\sqrt{y}}$$

and

$$1-\left(\frac{4y^{5/6}(y+2)}{3y(y+2)-(y+1)^{3/2}}\right)^3>1-\frac{64}{27\sqrt{y}}-\frac{64}{27y}-\frac{128}{81y^{3/2}}$$

for $y>27$ (we here omit the proofs).

In order to prove (2.13), it suffices to show that

$$\sqrt{y}-\frac{1}{2\sqrt{y}}+\ln\left(1-\frac{64}{27\sqrt{y}}-\frac{64}{27y}-\frac{128}{81y^{3/2}}\right)>0\quad\text{for } y>27.\tag{2.14}$$

We consider the function $f_4(z)$ defined, for $z>3\sqrt{3}$, by

$$f_4(z)=z-\frac{1}{2z}+\ln\left(1-\frac{64}{27z}-\frac{64}{27z^2}-\frac{128}{81z^3}\right).$$

Differentiation yields, for $z > 3\sqrt{3}$,

$$f_4'(z) = \frac{162z^5 - 384z^4 + 81z^3 + 320z^2 + 576z - 128}{2z^2(81z^3 - 192z^2 - 192z - 128)} > 0.$$

Hence, $f_4(z)$ is strictly increasing for $z > 3\sqrt{3}$, and we have

$$f_4(z) = z - \frac{1}{2z} + \ln\left(1 - \frac{64}{27z} - \frac{64}{27z^2} - \frac{128}{81z^3}\right) > f_4(3\sqrt{3}) = 4.289\ldots > 0$$

for $z > 3\sqrt{3}$. Hence, (2.14) holds for $y > 27$.

We now prove $G(x) > 0$ for $x > 3$. It is easy to see that

$$\frac{x^2(x^3+2)}{(x^3+1)^{3/2}} > \sqrt{x} \quad \text{for } x > 3.$$

In order to prove $G(x) > 0$ for $x > 3$, it is enough to prove the following inequality:

$$H(x) > 0 \quad \text{for } x > 3,$$

where

$$H(x) = \frac{\sqrt{x}}{2}\exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right) - \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3} + \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{4/3}.$$

Differentiating $H(x)$ and using (2.11) yield, for $x > 3$,

$$\begin{aligned} &-\left(x^3+1\right)^{3/2}\exp\left(\frac{x^3}{\sqrt{x^3+1}}\right)H'(x) \\ &\quad = \frac{3x^3(x^3+2) - (x^3+1)^{3/2}}{4\sqrt{x}}\left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{1/3} \\ &\qquad + x^2\left(x^3+2\right) - 2x^2\left(x^3+2\right)\left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3} \\ &\quad > 2x^2\left(x^3+2\right)\left\{1 - \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3}\right\} > 0. \end{aligned}$$

Therefore, the function $H(x)$ is strictly decreasing for $x \geq 3$, and we have

$$H(x) > \lim_{t\to\infty} H(t) = 0 \quad \text{for } x \geq 3.$$

Hence, we have $G(x) > 0$ for all $x > 0$. The proof of Lemma 2.1 is complete. □

3 Proof of the right-hand side of (1.1)

It is sufficient to prove the following inequality:

$$F(x) = \frac{1}{2}\ln\left(\frac{1 + \sqrt[3]{1 - \exp(-\frac{x^3}{\sqrt{x^3+1}})}}{1 - \sqrt[3]{1 - \exp(-\frac{x^3}{\sqrt{x^3+1}})}}\right) - x > 0 \quad \text{for } x > 0. \tag{3.1}$$

Differentiating $F(x)$ and using (2.2), we obtain, for $x > 0$,

$$\left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3}\left[1 - \left(1 - \exp\left(-\frac{x^3}{\sqrt{x^3+1}}\right)\right)^{2/3}\right]F'(x) = G(x) > 0,$$

where $G(x)$ is given in (2.1). Therefore, $F(x)$ is strictly increasing for $x > 0$, and we have

$$F(x) = \frac{1}{2}\ln\left(\frac{1+\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}{1-\sqrt[3]{1-\exp(-\frac{x^3}{\sqrt{x^3+1}})}}\right) - x > F(0) = 0$$

for $x > 0$. The proof is complete.

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Authors' contributions
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