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A spectral element Crank–Nicolson model to the 2D unsteady conduction–convection problems about vorticity and stream functions

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Abstract

In this study, a time semi-discretized Crank–Nicolson (CN) scheme of the two-dimensional (2D) unsteady conduction–convection problems for vorticity and stream functions is first built together with showing the existence and stability along with error estimates to the semi-discretized CN solutions. Afterwards, a fully discretized spectral element CN (SECN) model of the 2D unsteady conduction–convection problems as regards the vorticity and stream functions is set up together with showing the proof of the existence and stability along with error estimates of the SECN solution. Lastly, a set of numerical experiments are offered for checking the correctness of the theoretical conclusions.

MSC: 65N12; 65M15; 65N35

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded interconnected region. The 2D unsteady conduction–convection problems are stated as follows (see [1, 2]).

Problem 1 Seek u , v , Q , and p obeying

$$\left\{ \begin{array}{ll} \partial_t u - \mu \Delta u + u \partial_x u + v \partial_y u + \partial_x p = 0, & (t, x, y) \in (0, T) \times \Omega, \\ \partial_t v - \mu \Delta v + u \partial_x v + v \partial_y v + \partial_y p = 0, & (t, x, y) \in (0, T) \times \Omega, \\ \partial_x u + \partial_y v = 0, & (t, x, y) \in (0, T) \times \Omega, \\ \partial_t Q - \gamma_0 \Delta Q + u \partial_x Q + v \partial_y Q = 0, & (t, x, y) \in (0, T) \times \Omega, \\ u(t, x, y) = \varphi_u(t, x, y), \quad v(t, x, y) = \varphi_v(t, x, y), & (t, x, y) \in (0, T) \times \Omega, \\ Q(t, x, y) = Q_0(t, x, y), & (t, x, y) \in (0, T) \times \partial \Omega, \\ u(0, x, y) = u^0(x, y), \quad v(0, x, y) = v^0(x, y), & (x, y) \in \Omega, \\ Q(0, x, y) = Q^0(x, y), & (x, y) \in \Omega, \end{array} \right. \quad (1)$$

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where $\partial_z = \partial/\partial z$ ($z = t, x, y$), $(u, v)^T$ stands for the velocity vector of flow, p stands for the pressure, Q stands for the temperature or heat energy, T stands for the total time, $\mu = \sqrt{Pr/Re}$, Re stands for the Reynolds, Pr stands for Prandtl's number, $\gamma_0 = 1/\sqrt{RePr}$, $(\varphi_u(x, y, t), \varphi_v(x, y, t))^T$ and $Q_0(t, x, y)$ stand, respectively, for the known boundary values to the flow velocity and temperature, and $(u^0(x, y), v^0(x, y))^T$, and $Q^0(x, y)$ stand, respectively, for the known initial values to the flow velocity and the temperature.

For convenience of the theoretic argumentation, we will presume that $Q_0(t, x, y) = \varphi_u(t, x, y) = \varphi_v(t, x, y) = 0$ in the following.

The 2D unsteady conduction–convection problems possess very momentous physical background and can be applied for simulating the real-world natural phenomena (see [1–4]). But, due to the nonlinearity for Problem 1, most of all when the computational region for Problem 1 is of an irregular geometrical shape, one cannot usually find any genuine solution so one has to find numerical ones.

It is universally acknowledged that the spectral and finite element (FE) together with finite difference (FD) along with finite volume element (FVE) methods are four welcome numerical means (see [5–10]). Nevertheless, the spectral method possesses the highest precision among four numerical ones because the unknowns to the spectral method are approximated with the smooth functions, including trigonometric functions or the Chebyshev, Jacobi, and Legendre polynomials, but the unknowns to the FE and FVE methods are usually approximated by the classic polynomials, while the derivatives to the FD method are approached with difference quotients. Specially, the spectral element (SE) method possesses a similar principle to the FVE and FE ones so as to be adapt to the calculated regions of the non-regular shapes. Hence, it is more popular than the FE and FVE FD methods and has proverbially been applied for solving the various PDEs such as the hyperbolic and parabolic along with hydromechanics equations (see [11–15]).

Though the reduced-order extrapolating (SECN) method of the 2D unsteady conduction–convection problems to the vorticity and stream functions has been developed in [16], the SECN method has not been minutely developed. Specially, there have been no theoretic proofs as regards the existence along with stability as well as error estimates to the SECN solutions. Therefore, in Sect. 2, we firstly intend to set up a semi-discretized CN scheme as a function of time with second-order temporal precision to the 2D unsteady conduction–convection problems to the vorticity and stream functions, as well as a proof of the error estimates to the semi-discretized CN solutions. Afterwards, in Sect. 3, we intend to build the fully discretized SECN model of the 2D unsteady conduction–convection problems to the vorticity and stream functions, as well as prove the existence along with stability together with error estimates to the SECN solutions. In the end, in Sects. 4 and 5, we intend to pose a set of numeric experiments to verify the validity to the obtained theoretic consequences and give the primary conclusions and discussion, respectively.

What is noteworthy is that the SECN model of the 2D unsteady conduction–convection problems to the vorticity and stream functions is not only spilt into three sets of relatively linearly independent equations, but also that it possesses the second-order precision as a function of time. Specially, it is able to avoid the restriction for Babuška–Brezzi's stability conditions to spectral subspaces so as to be able to easily seek the SECN solutions, which is different from the previous other SE methods as stated above. As a consequence, the SECN model is fully distinguished from the spectral ones (see [8–21]) and is a development or a supplement to the previous ones.

2 The generalized solution and semi-discrete solution as a function of time

Thanks to the connectivity and boundedness of Ω and $\partial_x u + \partial_y v = 0$, there is only a stream function θ fulfilling $u = \partial_y \theta$ and $v = -\partial_x \theta$. In addition, there is a vorticity function ϖ meeting $\varpi = \partial v / \partial x - \partial u / \partial y = -\Delta \theta$.

Thereupon, Problem 1 may be turned into the next systems of equations:

$$\begin{cases} -\Delta \theta = \varpi, & (t, x, y) \in (0, T) \times \Omega, \\ \theta = 0, & (t, x, y) \in (0, T) \times \Omega, \end{cases} \quad (2)$$

$$\begin{cases} \partial_t \varpi - \mu \Delta \varpi + \partial_y \theta \partial_x \varpi - \partial_x \theta \partial_y \varpi = \partial_x Q, & (t, x, y) \in (0, T) \times \Omega, \\ \varpi = 0, & (t, x, y) \in (0, T) \times \Omega, \\ \varpi(0, x, y) = \varpi^0 = \partial v^0 / \partial x - \partial u^0 / \partial y, & (x, y) \in \Omega, \end{cases} \quad (3)$$

$$\begin{cases} \partial_t Q - \gamma_0 \Delta Q + \partial_y \theta \partial_x Q - \partial_x \theta \partial_y Q = 0, & (t, x, y) \in (0, T) \times \Omega, \\ Q = 0, & (t, x, y) \in (0, T) \times \Omega, \\ Q(0, x, y) = Q^0, & (x, y) \in \Omega. \end{cases} \quad (4)$$

The Sobolev spaces along with norms adopted in the following are normative (see [1, 22]). Set $V = H_0^1(\Omega)$. Using the Green formula, we may gain the next weak format.

Problem 2 Find $(\varpi, \theta, Q) \in H^1(0, T; V) \times H^1(0, T; V) \times H^1(0, T; V)$ that satisfies

$$a(\theta, \varphi) = (\varpi, \varphi), \quad \forall \varphi \in V; \quad (5)$$

$$(\partial_t \varpi, \chi) + \mu a(\varpi, \chi) + a_1(\theta, \varpi, \chi) = (\partial_x Q, \chi), \quad \forall \chi \in V, \quad (6)$$

$$(\partial_t Q, \varphi) + \gamma_0 a(Q, \varphi) + a_1(\theta, Q, \varphi) = 0, \quad \forall \varphi \in V, \quad (7)$$

$$\varpi(0, x, y) = \varpi^0, \quad Q(0, x, y) = Q^0, \quad (x, y) \in \Omega, \quad (8)$$

where $(\varpi, \varphi) = \int_{\Omega} \varpi \varphi \, dx \, dy$, $a(\theta, \chi) = \int_{\Omega} (\partial_x \theta \partial_x \chi + \partial_y \theta \partial_y \chi) \, dx \, dy$, and $a_1(\theta, \varpi, \varphi) = \int_{\Omega} (\partial_x \theta \partial_y \varpi - \partial_y \theta \partial_x \varpi) \varphi \, dx \, dy$.

Then $a_1(\theta, \varpi, \varphi)$ possesses the next properties (see [1, 16, 23, 24]):

$$a_1(\theta, \varpi, \varphi) = -a_1(\theta, \varphi, \varpi), \quad a_1(\theta, \varpi, \varpi) = 0, \quad \forall \theta, \varphi, \varpi \in H_0^1(\Omega); \quad (9)$$

$$|a_1(\theta, \varpi, \varphi)| \leq \tilde{C} \|\nabla \theta\|_0 \|\nabla \varpi\|_0 \|\varphi\|_0, \quad \forall \theta, \varphi, \varpi \in H_0^1(\Omega), \quad (10)$$

where $\tilde{C} > 0$ stands for the constant that is independent of θ , ϖ , and φ .

Theorem 1 If $(u^0, v^0, Q^0) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, then Problem 2 possesses only a solution $(\varpi, \theta, Q) \in H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \times H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \times H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$ that meets

$$\|\nabla \theta\|_0 + \|\varpi\|_0 + \|\nabla \varpi\|_{L^2(L^2)} + \|Q\|_0 + \|\nabla Q\|_{L^2(L^2)} \leq \sigma(g_1, g_2, u^0, v^0, \mu), \quad (11)$$

where $\|\cdot\|_{H^m(H^k)}$ stands for the norms of space $H^m(0, T; H^k(\Omega))$ and $\sigma(g_1, g_2, u^0, v^0, \mu) \geq 0$ stands for a constant that is dependent on g_1, g_2, u^0, v^0 , and μ .

Proof When $(u^0, v^0, Q^0) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$, from the approaches in Refs. [16, 23], or [24] we know that the direct variational format for Problem 1 has a unique a set of solution and from the regularity for PDEs (see [16, 23, 24]) we have $(u, v, Q) \in H^1(0, T; H_0^1(\Omega) \cap L^2(\Omega)) \times H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \times H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$. Therefore, Problem 2 has at least a solution $\varpi \in H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, $\theta \in H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, and $Q \in H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega))$. Thus, we only prove the uniqueness of solution to Problem 2. Namely, we only need to prove that, when $\varpi^0 = Q^0 = 0$, Problem 2 has only a zero solution.

Selecting $\varphi = \theta$ in (5), by the Hölder and Poincaré inequalities (see [1]) we obtain

$$\|\nabla\theta\|_0^2 = (\varpi, \theta) \leq \|\varpi\|_0 \|\theta\|_0 \leq C_0 \|\varpi\|_0 \|\nabla\theta\|_0, \quad (12)$$

where $C_0 > 0$ stands for the coefficient in the Poincaré inequality: $\|\theta\|_1 \leq C_0 \|\nabla\theta\|_0$. From (12) we get

$$\|\nabla\theta\|_0 \leq C_0 \|\varpi\|_0. \quad (13)$$

Selecting $\chi = \varpi$ in (6), by the Hölder and Poincaré inequalities (see [1]) and (9) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d\|\varpi\|_0^2}{dt} + \mu \|\nabla\varpi\|_0^2 &= (\partial_x Q, \varpi) \leq \|\partial_x Q\|_0 \|\varpi\|_0 \\ &\leq C_0 \|\nabla Q\|_0 \|\nabla\varpi\|_0 \leq \frac{C_0^2}{2\mu} \|\nabla Q\|_0^2 + \frac{\mu}{2} \|\nabla\varpi\|_0^2. \end{aligned} \quad (14)$$

Thus, we can get

$$\frac{d\|\varpi\|_0^2}{dt} + \mu \|\nabla\varpi\|_0^2 \leq \frac{C_0^2}{\mu} \|\nabla Q\|_0^2. \quad (15)$$

Integrating (15) on $[0, t]$ ($0 \leq t \leq T$) yields

$$\|\varpi\|_0^2 + \mu \|\nabla\varpi\|_{L^2(L^2)}^2 \leq \frac{C_0^2}{\mu} \|\nabla Q\|_{L^2(L^2)}^2 + \|\varpi^0\|_0^2. \quad (16)$$

Selecting $\varphi = Q$ in (7), by (9) we obtain

$$\frac{1}{2} \frac{d\|Q\|_0^2}{dt} + \gamma_0 \|\nabla Q\|_0^2 = 0. \quad (17)$$

Integrating (17) on $[0, t]$ ($0 \leq t \leq T$), we get

$$\|Q\|_0^2 + 2\gamma_0 \|\nabla Q\|_{L^2(L^2)}^2 = \|Q^0\|_0^2. \quad (18)$$

When $\varpi^0 = Q^0 = 0$, from (13), (16), and (18) we immediately obtain $\varpi = \theta = Q = 0$. And from (13), (16), and (18) we also acquire (11). This fulfils the proof of Theorem 1. \square

Let $M > 0$ stand for an integer, let $\Delta t = TM^{-1}$ stand for the temporal step, let $\varpi^n(x, y)$, $\theta^n(x, y)$, and $Q^n(x, y)$ stand, respectively, for the approximations of $\varpi(t, x, y)$, $\theta(x, y, t)$, and $Q(t, x, y)$ at $t_n = n\Delta t$, as well as let $\bar{\varphi} = (\varphi^n + \varphi^{n-1})/2$. If $\partial_t \varpi$ and $\partial_t Q$ are, respectively,

approximated with $(\varpi^n - \varpi^{n-1})/\Delta t$ and $(Q^n - Q^{n-1})/\Delta t$, then the semi-discretized CN scheme as a function of time with the second-order temporal accuracy is built in the following.

Problem 3 Seek $(\varpi^n, \theta^n, Q^n) \in V \times V \times V$ ($1 \leq n \leq M$) that satisfy

$$a(\theta^{n-1}, \varphi) = (\varpi^{n-1}, \varphi), \quad \forall \varphi \in V, 1 \leq n \leq M+1; \quad (19)$$

$$(\varpi^n, \chi) + \mu \Delta t a(\bar{\varpi}^n, \chi) + \Delta t a_1(\theta^{n-1}, \bar{\varpi}^n, \chi) = (\varpi^{n-1}, \chi) + \Delta t (\partial_x \bar{Q}^n, \chi), \quad (20)$$

$$(Q^n, \varphi) + \gamma_0 \Delta t a(\bar{Q}^n, \varphi) + \Delta t a_1(\theta^{n-1}, \bar{Q}^n, \varphi) = (Q^{n-1}, \varphi), \quad (21)$$

Problem 3 has the next consequence.

Theorem 2 Under the hypotheses in Theorem 1, Problem 3 possesses only a series of solutions $\{\varpi^n, \theta^n, Q^n\}_{n=1}^M \subset V \times V \times V$ meeting

$$\|\nabla \theta^n\|_0^2 + \|\varpi^n\|_0^2 + \|Q^n\|_0^2 + \Delta t \sum_{i=1}^n (\|\nabla \bar{\varpi}^i\|_0^2 + \|\nabla \bar{Q}^i\|_0^2) \leq \delta_0(u^0, v^0, Q^0, \mu, \gamma_0), \quad (22)$$

where $\delta_0(u^0, v^0, Q^0, \mu, \gamma_0)$ is a non-negative constant relying on u^0, v^0, Q^0, μ and γ_0 . When the solution of Eq. (6) meets $\varpi \in [H^3(0, T; L^2(\Omega)) \cap H^2(0, T; H_0^1(\Omega))]$ and $Q \in [H^3(0, T; L^2(\Omega)) \cap H^2(0, T; H_0^1(\Omega))]$, we obtain the following estimate errors:

$$\begin{aligned} & \|\nabla(\theta^n - \theta(t_n))\|_0 + \|\varpi^n - \varpi(t_n)\|_0 + \Delta t \|\nabla(\varpi^n - \varpi(t_n))\|_0 \\ & + \|Q^n - Q(t_n)\|_0 + \Delta t \|\nabla(Q^n - Q(t_n))\|_0 \leq C \Delta t^2, \quad 1 \leq n \leq M, \end{aligned} \quad (23)$$

where $C > 0$ is the generic constant that is independent of Δt .

Proof (1) The existence along with uniqueness to solution of Problem 3

Firstly, it is easily seen that the bilinear functional $a(\theta, \varphi)$ to the left hand side in (19) is bounded and coercive in $V \times V$ for given $\varpi^{n-1} \in V$ ($1 \leq n \leq M+1$). Thus, from Lax–Milgram’s theorem (see [1]) we conclude that Eq. (19) possesses only a series of solutions $\{\theta^n\}_{n=0}^M \subset H_0^1(\Omega)$.

Next, for the obtained θ^{n-1} , let $\tilde{A}(Q, \varphi) = (Q, \varphi) + \frac{\gamma_0 \Delta t}{2} a(Q, \varphi) + \frac{\Delta t}{2} a_1(\theta^{n-1}, Q, \varphi)$ and $\hat{F}(\varphi) = (Q^{n-1}, \varphi) - \frac{\gamma_0 \Delta t}{2} a(Q^{n-1}, \varphi) - \frac{\Delta t}{2} a_1(\theta^{n-1}, Q^{n-1}, \varphi)$. Then $\tilde{A}(Q, \varphi)$ is the bilinear functional and $\hat{F}(\varphi)$ is the linear functional. By (9) and the Hölder inequality we have

$$\begin{aligned} \hat{A}(Q, \varphi) &= (Q, \varphi) + \frac{\gamma_0 \Delta t}{2} a(Q, \varphi) + \frac{\Delta t}{2} a_1(\theta^{n-1}, Q, \varphi) \\ &\leq \|Q\|_0 \|\varphi\|_0 + \frac{\gamma_0 \Delta t}{2} \|\nabla \varphi\|_0 \|\nabla Q\|_0 + \frac{\Delta t}{2} \|\nabla \theta^{n-1}\|_0 \|\nabla \varphi\|_0 \|\nabla Q\|_0 \\ &\leq \left(1 + \frac{\gamma_0 \Delta t}{2} + \frac{\Delta t}{2} \|\nabla \theta^{n-1}\|_0\right) \|Q\|_1 \|\varphi\|_1, \quad \forall Q, \varphi \in V; \end{aligned}$$

$$\begin{aligned}\hat{A}(\varphi, \varphi) &= (\varphi, \varphi) + \frac{\gamma_0 \Delta t}{2} a(\varphi, \varphi) + \frac{\Delta t}{2} a_1(\theta^{n-1}, \varphi, \varphi) \\ &= \|\varphi\|_0^2 + \frac{\gamma_0 \Delta t}{2} \|\nabla \varphi\|_0^2 \geq \hat{\alpha} \|\varphi\|_1^2, \quad \forall \varphi \in V,\end{aligned}$$

where $\hat{\alpha} = \min\{1, \gamma_0 \Delta t/2\}$. Therefore, the bilinear functional $\hat{A}(\cdot, \cdot)$ is bounded and coercive on $V \times V$ for obtained $\theta^{n-1} \in V$. Moreover, by (9) and the Hölder inequality we have

$$\begin{aligned}\hat{F}(\varphi) &= (Q^{n-1}, \varphi) - \frac{\gamma_0 \Delta t}{2} a(Q^{n-1}, \varphi) - \frac{\Delta t}{2} a_1(\theta^{n-1}, Q^{n-1}, \varphi) \\ &\leq \|Q^{n-1}\|_0 \|\varphi\|_0 + \frac{\gamma_0 \Delta t}{2} \|\nabla \varphi\|_0 \|\nabla Q^{n-1}\|_0 + \frac{\Delta t}{2} \|\nabla Q^{n-1}\|_0 \|\nabla \theta^{n-1}\|_0 \|\nabla \varphi\|_0 \\ &\leq \left(\|Q^{n-1}\|_0 + \frac{\gamma_0 \Delta t}{2} \|\nabla Q^{n-1}\|_0 + \frac{\Delta t}{2} \|\nabla \theta^{n-1}\|_0 \|\nabla Q^{n-1}\|_0 \right) \|\varphi\|_1, \quad \forall \varphi \in V.\end{aligned}$$

Therefore, the linear function $\hat{F}(\varphi)$ is bounded in V for the known θ^{n-1} and Q^{n-1} . Consequently, by the Lax–Milgram theorem (see [1]) we may assert that Eq. (21) possesses only a series of solutions $\{Q^n\}_{n=1}^M \subset V$ for the known θ^{n-1} and Q^{n-1} .

Further, let $A(\varpi, \chi) = (\varpi, \chi) + \frac{\mu \Delta t}{2} a(\varpi, \chi) + \frac{\Delta t}{2} a_1(\theta^{n-1}, \varpi, \chi)$ and let $F(\chi) = (\varpi^{n-1}, \chi) - \frac{\mu \Delta t}{2} a(\varpi^{n-1}, \chi) - \frac{\Delta t}{2} a_1(\theta^{n-1}, \varpi^{n-1}, \chi) + \Delta t (\partial_x \bar{Q}^n, \chi)$. Then, by (9) and the Hölder inequality, we obtain

$$\begin{aligned}A(\varpi, \chi) &= (\varpi, \chi) + \frac{\mu \Delta t}{2} a(\varpi, \chi) + \frac{\Delta t}{2} a_1(\theta^{n-1}, \varpi, \chi) \\ &\leq \|\varpi\|_0 \|\chi\|_0 + \frac{\mu \Delta t}{2} \|\nabla \varpi\|_0 \|\nabla \chi\|_0 + \frac{\Delta t}{2} \|\nabla \theta^{n-1}\|_0 \|\nabla \varpi\|_0 \|\nabla \chi\|_0 \\ &\leq \left(1 + \frac{\mu \Delta t}{2} + \frac{\Delta t}{2} \|\nabla \theta^{n-1}\|_0 \right) \|\varpi\|_1 \|\chi\|_1, \quad \forall \varpi, \chi \in V; \\ A(\varpi, \varpi) &= (\varpi, \varpi) + \frac{\mu \Delta t}{2} a(\varpi, \varpi) + \frac{\Delta t}{2} a_1(\theta^{n-1}, \varpi, \varpi) \\ &= \|\varpi\|_0^2 + \frac{\mu \Delta t}{2} \|\nabla \varpi\|_0^2 \geq \alpha \|\varpi\|_1^2, \quad \forall \varpi \in V,\end{aligned}$$

where $\alpha = \min\{1, \mu \Delta t/2\}$. Therefore, the bilinear functional $A(\cdot, \cdot)$ is bounded and coercive on $V \times V$ for obtained $\theta^{n-1} \in V$. The linear functional $F(\cdot)$ is obviously bounded for the obtained ϖ^{n-1} , θ^{n-1} , $\bar{Q}^n \in V$, thereupon, from Lax–Milgram's theorem (see [1]) we conclude that Eq. (20) possesses only a series of solutions $\{\varpi^n\}_{n=1}^M \subset H_0^1(\Omega)$.

(2) The stability to solution of Problem 3

First, selecting $\varphi = \theta^{n-1}$ in (19), by the Hölder and Poincaré inequalities we have

$$\begin{aligned}\|\nabla \theta^{n-1}\|_0^2 &= a(\theta^{n-1}, \theta^{n-1}) = (\theta^{n-1}, \varpi^{n-1}) \\ &\leq \|\theta^{n-1}\|_0 \|\varpi^{n-1}\|_0 \leq C_0 \|\nabla \theta^{n-1}\|_0 \|\varpi^{n-1}\|_0.\end{aligned}\tag{24}$$

Thus, we get

$$\|\nabla \theta^{n-1}\|_0 \leq C_0 \|\varpi^{n-1}\|_0, \quad 1 \leq n \leq M+1.\tag{25}$$

Next, selecting $\chi = \varpi^n + \varpi^{n-1}$ in (20), by (9) along with the Hölder, Cauchy–Schwarz and the Poincaré inequalities we get

$$\begin{aligned} & \|\varpi^n\|_0^2 - \|\varpi^{n-1}\|_0^2 + 2\mu\Delta t \|\nabla \bar{\varpi}^n\|_0^2 \\ &= (\varpi^n - \varpi^{n-1}, \varpi^n + \varpi^{n-1}) + \frac{\mu\Delta t}{2} a(\varpi^n + \varpi^{n-1}, \varpi^n + \varpi^{n-1}) \\ &= 2\Delta t (\partial_x \bar{Q}^n, \bar{\varpi}^n) \leq 2C_0\Delta t \|\nabla \bar{Q}^n\|_0 \|\nabla \bar{\varpi}^n\|_0 \\ &\leq C_0^2\Delta t\mu^{-1} \|\nabla \bar{Q}^n\|_0^2 + \mu\Delta t \|\nabla \bar{\varpi}^n\|_0^2, \quad n = 1, 2, \dots, M. \end{aligned} \quad (26)$$

Hence, we obtain

$$\|\varpi^n\|_0^2 - \|\varpi^{n-1}\|_0^2 + \mu\Delta t \|\nabla \bar{\varpi}^n\|_0^2 \leq C_0^2\Delta t\mu^{-1} \|\nabla \bar{Q}^n\|_0^2, \quad n = 1, 2, \dots, M. \quad (27)$$

Summing (27) from 1 to n , we obtain

$$\|\varpi^n\|_0^2 + \mu\Delta t \sum_{i=1}^n \|\nabla \bar{\varpi}^i\|_0^2 \leq \|\varpi^0\|_0^2 + C_0^2\Delta t\mu^{-1} \sum_{i=1}^n \|\nabla \bar{Q}^i\|_0^2, \quad 1 \leq n \leq M. \quad (28)$$

Then selecting $\varphi = Q^n + Q^{-1}$ in (21), by (9) we get

$$\|Q^n\|_0^2 - \|Q^{n-1}\|_0^2 + 2\gamma_0\Delta t \|\nabla \bar{Q}^n\|_0^2 = 0, \quad n = 1, 2, \dots, M. \quad (29)$$

Summing (29) from 1 to n , we obtain

$$\|Q^n\|_0^2 + 2\gamma_0\Delta t \sum_{i=1}^n \|\nabla \bar{Q}^i\|_0^2 = \|Q^0\|_0^2, \quad n = 1, 2, \dots, M. \quad (30)$$

By (25), (27), and (30) we obtain (22).

(3) *The convergence of solution for Problem 3*

Let $E_n = \theta^n - \theta(t_n)$, $e_n = \varpi^n - \varpi(t_n)$, and $r_n = Q^n - Q(t_n)$.

First, selecting $\varphi = E_{n-1}$ after (19) subtracting (5), by means of the Hölder and Poincaré inequalities we obtain

$$\begin{aligned} \|\nabla E_{n-1}\|_0^2 &= a(E_{n-1}, E_{n-1}) = (E_{n-1}, e_{n-1}) \\ &\leq \|E_{n-1}\|_0 \|e_{n-1}\|_0 \leq C_0 \|\nabla E_{n-1}\|_0 \|e_{n-1}\|_0. \end{aligned} \quad (31)$$

Therefore, we obtain

$$\|\nabla E_{n-1}\|_0 \leq C_0 \|e_{n-1}\|_0, \quad 1 \leq n \leq M+1. \quad (32)$$

Next, selecting $\chi = e_n + e_{n-1}$ after (20) subtracts (6) at $t = t_{n-\frac{1}{2}}$, by (9), (10), the Hölder, Poincaré and the Cauchy–Schwarz inequalities along with (32) we obtain

$$\begin{aligned} & \|e_n\|_0^2 - \|e_{n-1}\|_0^2 + \frac{\mu\Delta t}{2} \|\nabla(e_{n-1} + e_n)\|_0^2 \\ &= (e_n - e_{n-1}, e_n + e_{n-1}) + \frac{\mu\Delta t}{2} a(e_{n-1} + e_n, e_{n-1} + e_n) \end{aligned}$$

$$\begin{aligned}
&= -\frac{\Delta t}{2} a_1(E_{n-1}, \varpi(t_n) + \varpi(t_{n-1}), e_n + e_{n-1}) \\
&\quad + \frac{\Delta t^3}{24} (\partial_{ttt} \varpi(\xi_{1n}), e_n + e_{n-1}) + \frac{\mu \Delta t^3}{16} a(\partial_{tt} \varpi(\xi_{2n}), e_n + e_{n-1}) \\
&\quad + \frac{\Delta t^3}{4} a_1(\theta(t_{n-\frac{1}{2}}), \partial_{tt} \varpi(\xi_{3n}), e_{n-1} + e_n) + \Delta t (\partial_x \bar{r}_n, e_{n-1} + e_n) \\
&\quad - \frac{\Delta t^3}{4} (\partial_{xttt} Q(\xi_{4n}), e_{n-1} + e_n) \\
&\leq \frac{\tilde{C} \Delta t}{2} \|\nabla E_{n-1}\|_0 \|\nabla(\varpi(t_{n-1}) + \varpi(t_n))\|_0 \|\nabla(e_{n-1} + e_n)\|_0 \\
&\quad + \frac{\Delta t^3}{24} \|\partial_{ttt} \varpi(\xi_{1n})\|_0 \|e_n + e_{n-1}\|_0 + \frac{\mu \Delta t^3}{16} \|\nabla \partial_{tt} \varpi(\xi_{2n})\|_0 \|\nabla(e_n + e_{n-1})\|_0 \\
&\quad + \frac{\tilde{C} \Delta t^3}{4} \|\nabla \theta(t_{n-\frac{1}{2}})\|_0 \|\nabla \partial_{tt} \varpi(\xi_{3n})\|_0 \|\nabla(e_{n-1} + e_n)\|_0 \\
&\quad + C_0 \Delta t \|\nabla \bar{r}_n\|_0 \|\nabla(e_n + e_{n-1})\|_0 + \frac{\Delta t^3}{4} \|\partial_x Q(\xi_{4n})\|_0 \|\nabla(e_{n-1} + e_n)\|_0 \\
&\leq C \Delta t \|e_{n-1}\|_0^2 + C \Delta t^5 + C \Delta t \|\nabla \bar{r}_n\|_0^2 + \frac{\mu \Delta t}{4} \|\nabla(e_{n-1} + e_n)\|_0^2, \tag{33}
\end{aligned}$$

where $t_{n-1} \leq \xi_{in} \leq t_n$ ($i = 1, 2, 3, 4$) and $n = 1, 2, \dots, M$. From (33) we get

$$\begin{aligned}
&\|e_n\|_0^2 - \|e_{n-1}\|_0^2 + \frac{\mu \Delta t}{4} \|\nabla(e_{n-1} + e_n)\|_0^2 \\
&\leq C \Delta t \|e_{n-1}\|_0^2 + C \Delta t^5 + C \Delta t \|\nabla \bar{r}_n\|_0^2, \quad n = 1, 2, \dots, M. \tag{34}
\end{aligned}$$

Noting that $e_0 = 0$ and summing (34) from 1 to n , we obtain

$$\begin{aligned}
&\|e_n\|_0^2 + \frac{\mu \Delta t}{4} \sum_{i=1}^n \|\nabla(e_i + e_{i-1})\|_0^2 \\
&\leq C n \Delta t^5 + C \Delta t \sum_{i=0}^{n-1} \|e_i\|_0^2 + C \Delta t \sum_{i=1}^n \|\nabla(r_i + r_{i-1})\|_0^2, \quad n = 1, 2, \dots, M. \tag{35}
\end{aligned}$$

And then, after (21) subtracting (7), selecting $\varphi = r_n + r_{n-1}$ and $t = t_{n-\frac{1}{2}}$, by (9), (10), the Hölder, Poincaré and Cauchy–Schwarz inequalities along with (32) we have

$$\begin{aligned}
&\|r_n\|_0^2 - \|r_{n-1}\|_0^2 + \frac{\gamma_0 \Delta t}{2} \|\nabla(r_n + r_{n-1})\|_0^2 \\
&= (r_n - r_{n-1}, r_n + r_{n-1}) + \frac{\gamma_0 \Delta t}{2} a(r_n + r_{n-1}, r_n + r_{n-1}) \\
&= -\frac{\Delta t}{2} a_1(E_{n-1}, Q(t_n) + Q(t_{n-1}), r_n + r_{n-1}) \\
&\quad + \frac{\Delta t^3}{24} (\partial_{ttt} Q(\xi_{1n}), r_n + r_{n-1}) + \frac{\gamma_0 \Delta t^3}{16} a(\partial_{tt} Q(\xi_{2n}), r_n + r_{n-1}) \\
&\quad + \frac{\Delta t^3}{4} a_1(\theta(t_{n-\frac{1}{2}}), \partial_{tt} Q(\xi_{3n}), r_n + r_{n-1}) \\
&\leq \frac{\tilde{C} \Delta t}{2} \|\nabla E_{n-1}\|_0 \|\nabla(Q(t_n) + Q(t_{n-1}))\|_0 \|\nabla(r_n + r_{n-1})\|_0
\end{aligned}$$

$$\begin{aligned}
& + \frac{\Delta t^3}{24} \|\partial_{ttt} Q(\zeta_{1n})\|_0 \|r_n + r_{n-1}\|_0 + \frac{\gamma_0 \Delta t^3}{16} \|\nabla \partial_{tt} Q(\zeta_{2n})\|_0 \|\nabla(r_n + r_{n-1})\|_0 \\
& + \frac{\tilde{C} \Delta t^3}{4} \|\nabla \theta(t_{n-\frac{1}{2}})\|_0 \|\nabla \partial_{tt} Q(\zeta_{3n})\|_0 \|\nabla(r_n + r_{n-1})\|_0 \\
& \leq C \Delta t \|e_{n-1}\|_0^2 + C \Delta t^5 + \frac{\gamma_0 \Delta t}{4} \|\nabla(r_n + r_{n-1})\|_0^2, \quad n = 1, 2, \dots, M,
\end{aligned} \quad (36)$$

where $t_{n-1} \leq \zeta_{in} \leq t_n$ ($i = 1, 2, 3$). From (36) we obtain

$$\|r_n\|_0^2 - \|r_{n-1}\|_0^2 + \frac{\gamma_0 \Delta t}{4} \|\nabla(r_n + r_{n-1})\|_0^2 \leq C \Delta t \|e_{n-1}\|_0^2 + C \Delta t^5, \quad (37)$$

where $n = 1, 2, \dots, M$. Note that $r_0 = 0$. Summing (37) from 1 to n , we get

$$\|r_n\|_0^2 + \frac{\gamma_0 \Delta t}{4} \sum_{i=1}^n \|\nabla(r_i + r_{i-1})\|_0^2 \leq C \Delta t \sum_{i=1}^n \|e_{i-1}\|_0^2 + C n \Delta t^5, \quad (38)$$

where $n = 1, 2, \dots, M$. By (35) and (38) we obtain

$$\|e_n\|_0^2 + \Delta t \sum_{i=1}^n \|\nabla(e_i + e_{i-1})\|_0^2 \leq C \Delta t^4 + C \Delta t \sum_{i=0}^{n-1} \|e_i\|_0^2, \quad 1 \leq n \leq M. \quad (39)$$

Applying the Gronwall inequality (see [1]) to (39) yields

$$\|e_n\|_0^2 + \Delta t \sum_{i=1}^n \|\nabla(e_i + e_{i-1})\|_0^2 \leq C \Delta t^4 \exp(C n \Delta t) \leq C \Delta t^4, \quad n = 1, 2, \dots, M. \quad (40)$$

From (32), (40), and (37) we obtain

$$\|\nabla E_n\|_0 \leq C \Delta t^2, \quad n = 1, 2, \dots, M; \quad (41)$$

$$\|e_n\|_0 \leq C \Delta t^2, \quad n = 1, 2, \dots, M; \quad (42)$$

$$\|r_n\|_0^2 \leq C \Delta t^4, \quad n = 1, 2, \dots, M; \quad (43)$$

$$\begin{aligned}
\Delta t \|\nabla e_n\|_0 & \leq \frac{\sqrt{\Delta t}}{\sqrt{n}} \sum_{i=1}^n (\|\nabla e_i\|_0 - \|\nabla e_{i-1}\|_0) \leq \frac{\sqrt{\Delta t}}{\sqrt{n}} \sum_{i=1}^n \|\nabla e_{i-1} + \nabla e_i\|_0 \\
& \leq \left(\Delta t \sum_{i=1}^n \|\nabla e_{i-1} + \nabla e_i\|_0^2 \right)^{1/2} \leq C \Delta t^2, \quad n = 1, 2, \dots, M;
\end{aligned} \quad (44)$$

$$\begin{aligned}
\Delta t \|\nabla r_n\|_0 & \leq \frac{\sqrt{\Delta t}}{\sqrt{n}} \sum_{i=1}^n (\|\nabla r_i\|_0 - \|\nabla r_{i-1}\|_0) \leq \frac{\sqrt{\Delta t}}{\sqrt{n}} \sum_{i=1}^n \|\nabla(r_{i-1} + r_i)\|_0 \\
& \leq \left(\Delta t \sum_{i=1}^n \|\nabla r_i + \nabla r_{i-1}\|_0^2 \right)^{1/2} \leq C \Delta t^2, \quad n = 1, 2, \dots, M.
\end{aligned} \quad (45)$$

From (41)–(45) we acquire (23), which fulfils the demonstration to Theorem 2. \square

Remark 1 The inequalities (22) and (23) to Theorem 2 signify that the sequence to solutions for Problem 3 is stable and convergent, respectively.

3 The SECN method for 2D unsteady conduction–convection problems

Let \mathfrak{S}_N stand for the quasi-uniform quadrangle partition for $\tilde{\Omega}$. A spectral element space is defined as

$$V_N = \{w_N \in H_0^1(\Omega) \cap C^0(\tilde{\Omega}) : w_N|_{K_j} \in \mathcal{P}_1(K_j), K_j \in \mathfrak{S}_N, 1 \leq j \leq N\},$$

where N stands for the number of quadrangles and $\mathcal{P}_1(K_j)$ is defined by the following:

$$\mathcal{P}_1(K_j) = \text{span}\{N_{ij} : 1 \leq i \leq 4\}, \quad j = 1, 2, \dots, N,$$

where $N_{ij} = \hat{N}_i \circ F_j^{-1}(x, y)$, $\hat{N}_i(\xi, \eta) = \frac{1}{4}[1 + \cos \pi(\xi - \xi_i)][1 + \cos \pi(\eta - \eta_i)]$, $(x, y) = F_j(\xi, \eta) = (\sum_{i=1}^4 \hat{N}_i(\xi, \eta)x_{ij}, \sum_{i=1}^4 \hat{N}_i(\xi, \eta)y_{ij})$ stands for an invertible mapping from the reference quadrangle $\hat{K} = [-1, 1] \times [-1, 1]$ to $K_j \in \mathfrak{S}_N$, and (x_{ij}, y_{ij}) and (ξ_i, η_i) are, respectively, the vertices of K_j and \hat{K} .

Let $R_N : H_0^1(\Omega) \rightarrow V_N$ stand for the H^1 -orthogonal operator, i.e., $\forall \varphi \in H_0^1(\Omega)$ satisfies

$$\int_{\Omega} \nabla(R_N \varphi - \varphi) \cdot \nabla v_N \, dx \, dy = 0, \quad \forall v_N \in V_N.$$

Note that when \mathfrak{S}_N is the quasi-uniform quadrangle partition to Ω , the number of nodes equals the number of quadrangles (see [1]). Hence, R_N shows the next consequence (see [7]).

Theorem 3 $\forall \varphi \in H^q(\Omega)$ ($m \geq 2$) meets

$$\|\nabla R_N \varphi\|_0 \leq C_r \|\nabla \varphi\|_0, \quad \|\partial^k(R_N \varphi - \varphi)\|_0 \leq CN^{k-1-m}, \quad 0 \leq k \leq m \leq N+1,$$

where $C > 0$ stands for a generic constant as well as N also stands for the number of nodes in \mathfrak{S}_N .

With the spectral element space, the SECN model is built in the following.

Problem 4 Find $(\varpi_N^n, \theta_N^n) \in V_N \times V_N$ ($1 \leq n \leq M$) satisfying

$$a(\theta_N^{n-1}, \varphi_N) = (\varpi_N^{n-1}, \varphi_N), \quad \forall \varphi_N \in V_N, \quad n = 1, 2, \dots, M+1; \quad (46)$$

$$\begin{aligned} & (\varpi_N^n, \chi_N) + \mu \Delta t a(\bar{\varpi}_N^n, \chi_N) + \Delta t a_1(\theta_N^{n-1}, \bar{\varpi}_N^n, \chi_N) \\ & = (\varpi_N^{n-1}, \chi) + \Delta t (\partial_x \bar{Q}_N^n, \chi_N), \quad \forall \chi_N \in V_N, n = 1, 2, \dots, M; \end{aligned} \quad (47)$$

$$\begin{aligned} & (Q_N^n, \varphi) + \gamma_0 \Delta t a(\bar{Q}_N^n, \varphi_N) + \Delta t a_1(\theta_N^{n-1}, \bar{Q}_N^n, \varphi_N) = (Q_N^{n-1}, \varphi_N), \\ & \forall \varphi_N \in V_N, n = 1, 2, \dots, M, \end{aligned} \quad (48)$$

where $\varpi_N^0 = R_N \varpi^0$ and $Q_N^0 = R_N Q^0$.

Problem 4 possesses the next result as regards existence, convergence, and stability.

Theorem 4 Under the hypotheses in Theorem 2, Problem 4 uniquely has three sets of solutions $\{\varpi_N^n\}_{n=1}^M \subset V_N$, $\{\theta_N^n\}_{n=0}^M \subset V_N$, and $\{Q_N^n\}_{n=0}^M \subset V_N$ meeting

$$\|\nabla \theta_N^n\|_0 + \|\varpi_N^n\|_0 + \Delta t \|\nabla \varpi_N^n\|_0 + \|Q_N^n\|_0 + \Delta t \|\nabla Q_N^n\|_0 \leq \tilde{\sigma}(u^0, v^0, Q^0, \mu, \gamma_0), \quad (49)$$

where $\tilde{\sigma}(u^0, v^0, Q^0, \mu, \gamma_0)$ is a non-negative constant dependent on u^0, v^0, Q^0, μ , and γ_0 . When the solution of Eq. (6) meets $(\varpi, \theta, Q) \in [H^3(0, T; H^q(\Omega) \cap H_0^1(\Omega))] \times [H^3(0, T; H^q(\Omega) \cap H_0^1(\Omega))] \times [H^3(0, T; H^q(\Omega) \cap H_0^1(\Omega))]$ ($2 \leq q \leq N+1$), we obtain the following estimate errors:

$$\begin{aligned} & \|\nabla(\theta_N^n - \theta(t_n))\|_0 + \|\varpi_N^n - \varpi(t_n)\|_0 + \Delta t \|\nabla(\varpi_N^n - \varpi(t_n))\|_0 + \|Q_N^n - Q(t_n)\|_0 \\ & + \Delta t \|\nabla(Q_N^n - Q(t_n))\|_0 \leq C(\Delta t^2 + N^{-q}), \quad 1 \leq n \leq M; 2 \leq q \leq N+1. \end{aligned} \quad (50)$$

Proof (1) The existence as well as stability to solution of Problem 4

Using the same proving approach as the existence and stability of solutions to Problem 3 in Theorem 2, we can prove that for Problem 4 there uniquely exist three series of solutions $\{\varpi_N^n\}_{n=1}^M \subset V_N$, $\{\theta_N^n\}_{n=0}^M \subset V_N$, and $\{Q_N^n\}_{n=0}^M \subset V_N$ to meet the stability (49).

(2) The convergence to solution of Problem 4

Let $\rho_n = \theta^n - R_N \theta^n$, $\tilde{e}_n = \theta^n - \theta_N^n$, $\tilde{E}_n = R_N \theta^n - \theta_N^n$, $\hat{\rho}_n = \varpi^n - R_N \varpi^n$, $\hat{E}_n = R_N \varpi^n - \varpi_N^n$, $\hat{e}_n = \varpi^n - \varpi_N^n$, $\hat{R}_n = R_N Q^n - Q_N^n$, $\tilde{\rho}_n = Q^n - R_N Q^n$, and $\hat{r}_n = Q^n - Q_N^n$.

First, selecting $\varphi_N = \tilde{E}_{n-1}$ after (19) subtracting (46), by the Hölder, Poincaré and the Cauchy–Schwarz inequalities as well as Theorem 3 we get

$$\begin{aligned} \|\nabla \tilde{e}_{n-1}\|_0^2 &= a(\tilde{e}_{n-1}, \tilde{e}_{n-1}) = a(\tilde{e}_{n-1}, \tilde{E}_{n-1}) + a(\tilde{e}_{n-1}, \rho_{n-1}) \\ &= (\hat{e}_{n-1}, \tilde{E}_{n-1}) + a(\rho_{n-1}, \rho_{n-1}) \\ &= \|\nabla \rho_{n-1}\|_0^2 + (\hat{e}_{n-1}, \tilde{e}_{n-1}) - (\hat{e}_{n-1}, \rho_{n-1}) \\ &\leq \|\nabla \rho_{n-1}\|_0^2 + C_0 \|\nabla \rho_{n-1}\|_0 \|\hat{e}_{n-1}\|_0 + C_0 \|\nabla \tilde{e}_{n-1}\|_0 \|\hat{e}_{n-1}\|_0 \\ &\leq C(N^{-2q+2}) + \|\hat{e}_{n-1}\|_0^2 + \frac{1}{2} \|\nabla \tilde{e}_{n-1}\|_0^2. \end{aligned} \quad (51)$$

Therefore, we acquire

$$\|\nabla \tilde{e}_n\|_0 \leq C(N^{-q} + \|\hat{e}_n\|_0), \quad n = 0, 1, 2, \dots, M. \quad (52)$$

Next, after (20) subtracting (47), by (9), (10), the Hölder, Poincaré and the Cauchy–Schwarz inequalities as well as Theorem 3 together with (52) we have

$$\begin{aligned} & \|\hat{e}_n\|_0^2 - \|\hat{e}_{n-1}\|_0^2 + \frac{\mu \Delta t}{2} \|\nabla \hat{e}_{n-1} + \nabla \hat{e}_n\|_0^2 \\ &= (\hat{e}_n - \hat{e}_{n-1}, \hat{e}_{n-1} + \hat{e}_n) + \frac{\mu \Delta t}{2} a(\hat{e}_{n-1} + \hat{e}_n, \hat{e}_{n-1} + \hat{e}_n) \\ &= (\hat{e}_n - \hat{e}_{n-1}, \hat{\rho}_{n-1} + \hat{\rho}_n) + (\hat{e}_n - \hat{e}_{n-1}, \hat{E}_{n-1} + \hat{E}_n) \\ &\quad + \frac{\mu \Delta t}{2} a(\hat{\rho}_{n-1} + \hat{\rho}_n, \hat{\rho}_{n-1} + \hat{\rho}_n) + \frac{\mu \Delta t}{2} a(\hat{e}_{n-1} + \hat{e}_n, \hat{E}_{n-1} + \hat{E}_n) \\ &= (\hat{e}_n - \hat{e}_{n-1}, \hat{\rho}_{n-1} + \hat{\rho}_n) + \frac{\mu \Delta t}{2} a(\hat{\rho}_{n-1} + \hat{\rho}_n, \hat{\rho}_{n-1} + \hat{\rho}_n) \end{aligned}$$

$$\begin{aligned}
& -\frac{\Delta t}{2}a_1(\tilde{e}_{n-1}, \varpi^n + \varpi^{n-1}, \hat{E}_n + \hat{E}_{n-1}) - \frac{\Delta t}{2}a_1(\theta_N^{n-1}, \hat{e}_n + \hat{e}_{n-1}, \hat{E}_n + \hat{E}_{n-1}) \\
& + \Delta t(\partial_x(\hat{r}_n + \hat{r}_{n-1}), \hat{E}_n + \hat{E}_{n-1}) \\
& = (\hat{e}_n - \hat{e}_{n-1}, \hat{\rho}_n + \hat{\rho}_{n-1}) + \frac{\mu\Delta t}{2}\|\nabla(\hat{\rho}_n + \hat{\rho}_{n-1})\|_0^2 \\
& - \frac{\Delta t}{2}a_1(\tilde{e}_{n-1}, \varpi^n + \varpi^{n-1}, \hat{\rho}_n + \hat{\rho}_{n-1}) - \frac{\Delta t}{2}a_1(\tilde{e}_{n-1}, \varpi^n + \varpi^{n-1}, \hat{e}_n + \hat{e}_{n-1}) \\
& + \frac{\Delta t}{2}a_1(\theta_N^{n-1}, \hat{e}_n + \hat{e}_{n-1}, \hat{\rho}_n + \hat{\rho}_{n-1}) \\
& + \Delta t(\partial_x(\hat{r}_n + \hat{r}_{n-1}), \hat{e}_n - \hat{\rho}_n + \hat{e}_{n-1} - \hat{\rho}_{n-1}) \\
& \leq CN^{-1}(\|\hat{e}_{n-1}\|_0^2 + \|\hat{e}_n\|_0^2) + CN^{-1-2q} + C\Delta t\|\nabla\tilde{e}_{n-1}\|_0^2 \\
& + C\Delta tN^{-2q} + C\Delta t\|\nabla(\hat{r}_{n-1} + \hat{r}_n)\|_0^2 + \frac{\mu\Delta t}{4}\|\nabla(\hat{e}_{n-1} + \hat{e}_n)\|_0^2. \tag{53}
\end{aligned}$$

Therefore, when $\Delta t = O(N^{-1})$, by (52) and (53) we get

$$\begin{aligned}
& \|\hat{e}_n\|_0^2 - \|\hat{e}_{n-1}\|_0^2 + \frac{\mu\Delta t}{4}\|\nabla(\hat{e}_{n-1} + \hat{e}_n)\|_0^2 \\
& \leq C\Delta t(\|\hat{e}_n\|_0^2 + \|\hat{e}_{n-1}\|_0^2) + C\Delta t\|\nabla(\hat{r}_n + \hat{r}_{n-1})\|_0^2 + C\Delta tN^{-2q}. \tag{54}
\end{aligned}$$

Summing for (54) from 1 to n , by Theorem 3 we obtain

$$\begin{aligned}
& \|\hat{e}_n\|_0^2 + \frac{\mu\Delta t}{4}\sum_{i=1}^n\|\nabla(\hat{e}_i + \hat{e}_{i-1})\|_0^2 \\
& \leq C\Delta t\sum_{i=0}^n(\|\hat{e}_i\|_0^2 + \|\nabla(\hat{r}_i + \hat{r}_{i-1})\|_0^2) + Cn\Delta tN^{-2q} + \|\varpi^0 - R_N\varpi^0\|_0^2 \\
& \leq C\Delta t\sum_{i=0}^n(\|\hat{e}_i\|_0^2 + \|\nabla\hat{r}_i\|_0^2) + CN^{-2q}. \tag{55}
\end{aligned}$$

As Δt is small enough to meet $C\Delta t \leq 3/4$ in (55), we get

$$\begin{aligned}
& \|\hat{e}_n\|_0^2 + \mu\Delta t\sum_{i=1}^n\|\nabla(\hat{e}_i + \hat{e}_{i-1})\|_0^2 \\
& \leq C\Delta t\sum_{i=0}^{n-1}\|\hat{e}_i\|_0^2 + C\Delta t\sum_{i=0}^n\|\nabla(\hat{r}_i + \hat{r}_{i-1})\|_0^2 + CN^{-2q}. \tag{56}
\end{aligned}$$

After (21) subtracting (49), with (9), (10), using the Hölder, Poincaré and the Cauchy–Schwarz inequalities as well as Theorem 3 and (52) we have

$$\begin{aligned}
& \|\hat{r}_n\|_0^2 - \|\hat{r}_{n-1}\|_0^2 + \frac{\gamma_0\Delta t}{2}\|\nabla\hat{r}_{n-1} + \nabla\hat{r}_n\|_0^2 \\
& = (\hat{r}_n - \hat{r}_{n-1}, \hat{r}_{n-1} + \hat{r}_n) + \frac{\gamma_0\Delta t}{2}a(\hat{r}_{n-1} + \hat{r}_n, \hat{r}_{n-1} + \hat{r}_n) \\
& = (\hat{r}_n - \hat{r}_{n-1}, \tilde{\rho}_{n-1} + \tilde{\rho}_n) + (\hat{r}_n - \hat{r}_{n-1}, \hat{R}_{n-1} + \hat{R}_n)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\mu \Delta t}{2} a(\tilde{\rho}_{n-1} + \tilde{\rho}_n, \tilde{\rho}_{n-1} + \tilde{\rho}_n) + \frac{\gamma_0 \Delta t}{2} a(\hat{r}_{n-1} + \hat{r}_n, \hat{R}_{n-1} + \hat{R}_n) \\
& = (\hat{r}_n - \hat{r}_{n-1}, \tilde{\rho}_{n-1} + \tilde{\rho}_n) + \frac{\gamma_0 \Delta t}{2} a(\tilde{\rho}_{n-1} + \tilde{\rho}_n, \tilde{\rho}_{n-1} + \tilde{\rho}_n) \\
& \quad - \frac{\Delta t}{2} a_1(\tilde{e}_{n-1}, Q^{n-1} + Q^n, \hat{R}_{n-1} + \hat{R}_n) - \frac{\Delta t}{2} a_1(\theta_N^{n-1}, \hat{r}_{n-1} + \hat{r}_n, \hat{R}_{n-1} + \hat{R}_n) \\
& = (\hat{r}_n - \hat{r}_{n-1}, \tilde{\rho}_n + \tilde{\rho}_{n-1}) + \frac{\gamma_0 \Delta t}{2} \|\nabla(\tilde{\rho}_n + \tilde{\rho}_{n-1})\|_0^2 \\
& \quad - \frac{\Delta t}{2} a_1(\tilde{e}_{n-1}, Q^{n-1} + Q^n, \tilde{\rho}_{n-1} + \tilde{\rho}_n) - \frac{\Delta t}{2} a_1(\tilde{e}_{n-1}, Q^{n-1} + Q^n, \hat{r}_{n-1} + \hat{r}_n) \\
& \quad + \frac{\Delta t}{2} a_1(\theta_N^{n-1}, \hat{r}_{n-1} + \hat{r}_n, \tilde{\rho}_{n-1} + \tilde{\rho}_n) \\
& \leq CN^{-1} (\|\hat{r}_{n-1}\|_0^2 + \|\hat{r}_n\|_0^2) + CN^{-1-2q} + C\Delta t \|\nabla \tilde{e}_{n-1}\|_0^2 \\
& \quad + C\Delta t N^{-2q} + \frac{\mu \Delta t}{4} \|\nabla(\hat{r}_n + \hat{r}_{n-1})\|_0^2. \tag{57}
\end{aligned}$$

Therefore, when $\Delta t = O(N^{-1})$, by (57) we obtain

$$\begin{aligned}
& \|\hat{r}_n\|_0^2 - \|\hat{r}_{n-1}\|_0^2 + \frac{\gamma_0 \Delta t}{4} \|\nabla(\hat{r}_n + \hat{r}_{n-1})\|_0^2 \\
& \leq C\Delta t (\|\hat{r}_n\|_0^2 + \|\hat{r}_{n-1}\|_0^2) + C\Delta t \|\nabla \tilde{e}_{n-1}\|_0^2 + C\Delta t N^{-2q}. \tag{58}
\end{aligned}$$

Summing (58) from 1 to n , by Theorem 3 and (52) we obtain

$$\begin{aligned}
& \|\hat{r}_n\|_0^2 + \frac{\gamma_0 \Delta t}{4} \sum_{i=1}^n \|\nabla(\hat{r}_i + \hat{r}_{i-1})\|_0^2 \\
& \leq C\Delta t \sum_{i=0}^n \|\hat{r}_i\|_0^2 + C\Delta t \sum_{i=0}^{n-1} \|\hat{e}_i\|_0^2 + Cn\Delta t N^{-2q} + \|Q^0 - R_N Q^0\|_0^2 \\
& \leq C\Delta t \sum_{i=0}^n \|\hat{r}_i\|_0^2 + C\Delta t \sum_{i=0}^{n-1} \|\hat{e}_i\|_0^2 + \|\nabla \tilde{e}_{n-1}\|_0^2 + CN^{-2q}. \tag{59}
\end{aligned}$$

As Δt is small enough to meet $C\Delta t \leq 3/4$ in (59), one gets

$$\begin{aligned}
& \|\hat{r}_n\|_0^2 + \gamma_0 \Delta t \sum_{i=1}^n \|\nabla(\hat{r}_i + \hat{r}_{i-1})\|_0^2 \\
& \leq C\Delta t \sum_{i=0}^{n-1} \|\hat{r}_i\|_0^2 + C\Delta t \sum_{i=0}^{n-1} \|\hat{e}_i\|_0^2 + CN^{-2q}. \tag{60}
\end{aligned}$$

By applying the discrete Gronwall inequality (see [1]) to (60), one gets

$$\begin{aligned}
& \|\hat{r}_n\|_0^2 + \gamma_0 \Delta t \sum_{i=1}^n \|\nabla(\hat{r}_i + \hat{r}_{i-1})\|_0^2 \leq \left(C\Delta t \sum_{i=0}^{n-1} \|\hat{e}_i\|_0^2 + CN^{-2q} \right) \exp(Cn\Delta t) \\
& \leq C\Delta t \sum_{i=0}^{n-1} \|\hat{e}_i\|_0^2 + CN^{-2q}. \tag{61}
\end{aligned}$$

From (56) and (61) we get

$$\|\hat{e}_n\|_0^2 + \mu \Delta t \sum_{i=1}^n \|\nabla(\hat{e}_i + \hat{e}_{i-1})\|_0^2 \leq C \Delta t \sum_{i=0}^{n-1} \|\hat{e}_i\|_0^2 + CN^{-2q}. \quad (62)$$

By using the discrete Gronwall inequality (see [1]) to (62), one obtains

$$\begin{aligned} \|\hat{e}_n\|_0^2 + \mu \Delta t \sum_{i=1}^n \|\nabla(\hat{e}_i + \hat{e}_{i-1})\|_0^2 \\ \leq CN^{-2q} \exp(Cn\Delta t) \leq CN^{-2q}, \quad n = 1, 2, \dots, M. \end{aligned} \quad (63)$$

By (62), (52), and (63) we obtain

$$\|\nabla \tilde{e}_n\|_0 \leq CN^{-q}, \quad n = 0, 1, 2, \dots, M; \quad (64)$$

$$\|\hat{r}_n\|_0^2 + \gamma_0 \Delta t \sum_{i=1}^n \|\nabla(\hat{r}_i + \hat{r}_{i-1})\|_0^2 \leq CN^{-2q}. \quad (65)$$

From (63) we have

$$\begin{aligned} \|\hat{e}_n\|_0 + \sqrt{\mu} \Delta t (\|\nabla \hat{e}_n\|_0 - \|\nabla(\varpi^0 - \varpi_N^0)\|_0) \\ \leq \frac{\sqrt{\mu \Delta t}}{\sqrt{n}} \sum_{i=1}^n (\|\nabla \hat{e}_i\|_0 - \|\nabla \hat{e}_{i-1}\|_0) + \|\hat{e}_n\|_0 \\ \leq \frac{\sqrt{\mu \Delta t}}{\sqrt{n}} \sum_{i=1}^n \|\nabla \hat{e}_i + \nabla \hat{e}_{i-1}\|_0 + \|\hat{e}_n\|_0 \\ \leq \|\hat{e}_n\|_0 + \left(\mu \Delta t \sum_{i=1}^n \|\nabla(\hat{e}_{i-1} + \hat{e}_i)\|_0^2 \right)^{1/2} \leq CN^{-q}. \end{aligned} \quad (66)$$

By Theorem 3 and (66) we obtain

$$\|\hat{e}_n\|_0 + \Delta t \|\nabla \hat{e}_n\|_0 \leq CN^{-q} + C \Delta t \|\nabla(\varpi^0 - R_N \varpi^0)\|_0 \leq CN^{-q}, \quad (67)$$

where $n = 1, 2, \dots, M$. Using the same ways as deducing (67) for (65), we obtain

$$\|\hat{r}_n\|_0 + \Delta t \|\nabla \hat{r}_n\|_0 \leq CN^{-q}, \quad n = 1, 2, \dots, M. \quad (68)$$

From (67), (68), and (64) along with Theorem 2 we acquire (50), which fulfills the proof to Theorem 4. \square

Remark 2 The inequalities (49) and (50) in Theorem 4 signify that the sequence to the SECN solutions of Problem 4 is stable and convergent, respectively.

By $u = \partial\theta/\partial y$ and $v = -\partial\theta/\partial x$ one directly obtains the next consequence.

Theorem 5 *Under the hypotheses in Theorems 1 and 4, Problem 4 possesses only a set of SECN solutions (u_N^n, v_N^n, Q_N^n) ($1 \leq n \leq M$) satisfying the stability estimates*

$$\|u_N^n\|_{0,\infty} + \|v_N^n\|_0 \leq \tilde{\sigma}(g_1, g_2, u^0, v^0, \mu), \quad (69)$$

and the error estimates

$$\begin{aligned} & \|u(t_n) - u_N^n\|_0 + \|v(t_n) - v_N^n\|_0 + \|Q(t_n) - Q_N^n\|_0 + \Delta t \|\nabla(Q(t_n) - Q_N^n)\|_0 \\ & \leq \sigma(\Delta t^2 + N^{-q}), \quad 2 \leq q \leq N+1. \end{aligned} \quad (70)$$

Remark 3 Even if Ω is a bounded polygonal region, the error estimations of Theorem 5 reach optimal order due to $u, v, Q \in H^3(0, T; H_0^1(\Omega) \cap H^2(\Omega))$.

4 Numerical examples

Here, we provide a set of experiments to check the correctness of the theoretical consequences.

Let the computational region Ω be a channel with a total length of 20 and a width of 6 that has two identical rectangular protrusions with a length of 4 and a width of 2 at the top and at the bottom (see Fig. 1). When the quadrilateral elements in \mathfrak{S}_N are the squares about edge length $\Delta x = \Delta y = 0.01$, $N = 3 \times 136 \times 10^4$. In addition to the outflow velocity $u(t, x, y) = u(20 - 1/M, y, t)$ ($20 - 1/M \leq x \leq 20$, $2 \leq y \leq 8$, $0 \leq t \leq T$) on the right boundary as well as the inflow velocity $(u, v) = (0.1(y - 2)(8 - y) \sin 2\pi t, 0)$ ($x = 0$, $2 \leq y \leq 8$) on the left boundary, the other boundary and initial values are chosen as 0. The temporal step $\Delta t = 0.0001$. In the case, the theoretic errors reach $O(10^{-8})$.

Using the SECN model (Problem 4), we seek the SECN solutions at $t = 4$ and 8, painted in Figs. 2 to 5, respectively. The numerical test results are very ideal.

When $0 \leq t \leq 8$, the errors of velocity and energy solutions are approximately estimated by $\|u_N^{n-1} - u_N^n\|_0 + \|v_N^{n-1} - v_N^n\|_0$ and $\|Q_N^{n-1} - Q_N^n\|_0$ ($1 \leq n \leq 80,000$), painted in Figs. 6 and 7, respectively, which also accord with the theoretic consequences since two types of errors do not exceed $O(10^{-8})$. This signifies that the SECN method is reliable and valid for settling the 2D unsteady conduction–convection problems to the vorticity and stream functions.

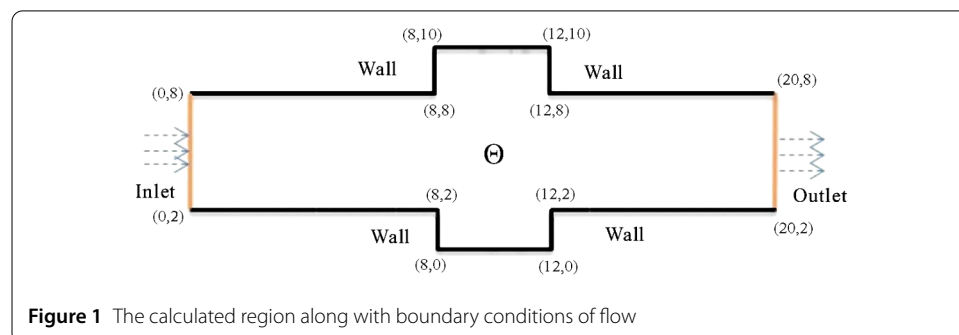


Figure 1 The calculated region along with boundary conditions of flow

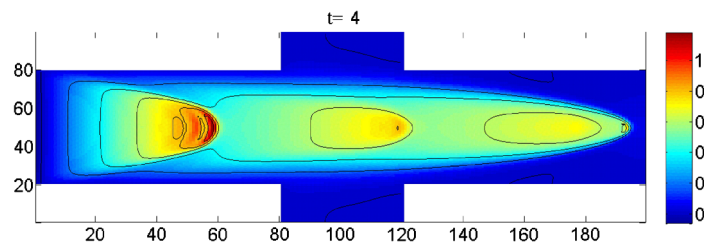


Figure 2 The SECN velocity solution at $t = 4$

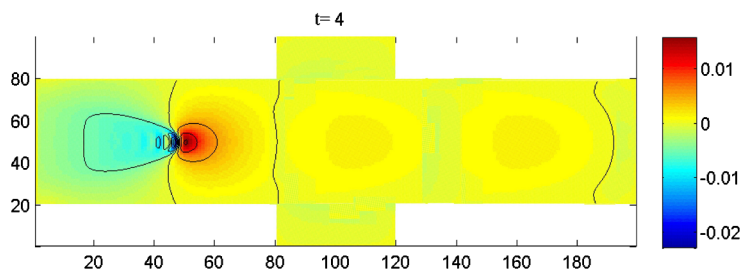


Figure 3 The SECN energy solution at $t = 4$

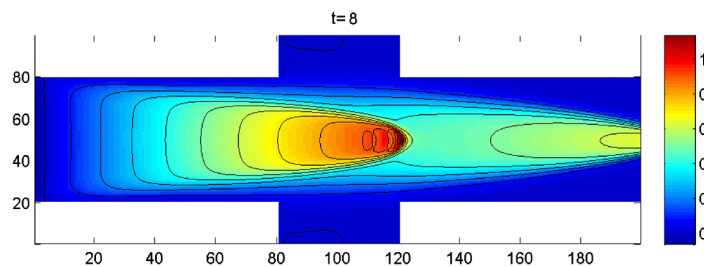


Figure 4 The SECN velocity solution at $t = 8$

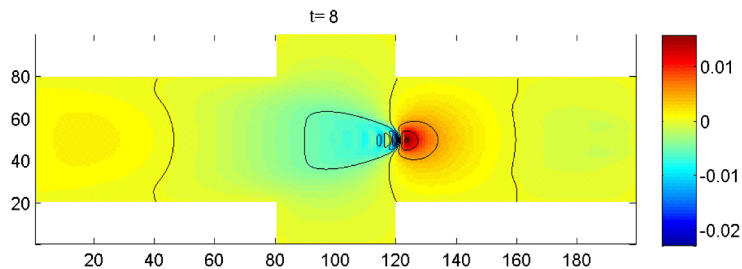


Figure 5 The SECN energy solution at $t = 8$

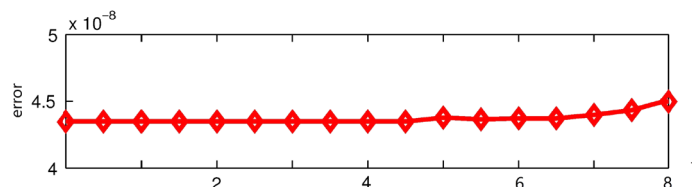


Figure 6 The errors of the SECN velocity solutions on $0 \leq t \leq 8$

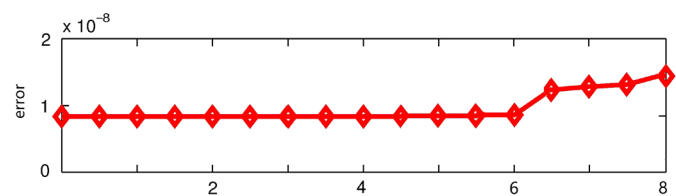


Figure 7 The errors of the SECN energy solutions on $0 \leq t \leq 8$

5 Conclusions and discussions

Hereto, we have built the time semi-discretized CN and fully discretized SECN models of the 2D unsteady conduction–convection problems to vorticity and stream functions and analyzed the existence, convergence, and stability to the time semi-discretized CN along with SECN solutions, respectively. We have also posed a set of numeric experiments to verify the reliability and validity to the SECN method and to verify that the numeric consequences accord with the theoretic ones.

Though we here only dealt with the 2D unsteady conduction–convection problems to the vorticity and stream functions, the SECN method may be popularized to the three-dimensional unsteady conduction–convection problems or more complicated flow dynamics problems, even to be used for the more complicated actual engineering computations. Thereupon, the SECN method shows an extensive prospect as regards applications.

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Availability of data and materials

The authors declare that all data and material in the article are available and veritable.

Competing interests

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Authors' contributions

All authors contributed equally and significantly in writing this article. All authors wrote, read, and approved the final manuscript.

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