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Complete monotonicity and inequalities related to generalized k -gamma and k -polygamma functions

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Abstract

In this paper, we prove new complete monotonicity properties of some functions related to generalized k -gamma and k -polygamma functions. Applications of the results yield various new inequalities. In the end, double inequalities are constructed involving the k -generalized digamma and polygamma functions.

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1 Introduction

The gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0.$$

The most important function related to the gamma function is the digamma or ψ function, which is defined as the logarithmic derivative of $\Gamma(x)$. The derivatives ψ' , ψ'' , $\dots$ are called polygamma functions.

In [7], the k -analogue of the gamma function is defined as

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt \quad (1.1)$$

$$= \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad (1.2)$$

where $k > 0$, $x > 0$, $\lim_{k \rightarrow 1} \Gamma_k(x) = \Gamma(x)$ and

$$(x)_{n,k} = x(x+k)(x+2k) \cdots (x+(n-1)k).$$

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The Weierstrass' product form of $\Gamma_k(x)$ is given as

$$\frac{1}{\Gamma_k(x)} = x k^{-\frac{x}{k}} e^{\frac{\gamma x}{k}} \prod_{n=1}^{\infty} \left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}}, \quad (1.3)$$

where

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.57721 \dots$$

is the Euler–Mascheroni constant, see [12]. Consequently, the k -analogue of the digamma function is defined as

$$\psi_k(x) = \frac{d}{dx} \log \Gamma_k(x) = \frac{\Gamma'_k(x)}{\Gamma_k(x)}, \quad (1.4)$$

and the corresponding k -polygamma functions are

$$\psi_k^{(m)}(x) = (-1)^{m+1} \int_0^{\infty} t^m \frac{e^{-(x+k)t}}{1 - e^{-kt}} dt \quad (1.5)$$

$$= (-1)^{m+1} m! \sum_{n=1}^{\infty} \frac{1}{[(n-1)k + x]^{m+1}}, \quad m = 1, 2, \dots \quad (1.6)$$

Recently, Nantomah, Prempeh, and Twum gave (p, k) -analogues of the gamma and the digamma functions in [15]. Furthermore, they established some inequalities related to these new functions. The reader may see [14, 15].

The above special functions are very important and have significant applications in mathematics and other disciplines such as physics and statistics. They are also connected to generalized harmonic numbers and many other special functions such as the Riemann zeta, Hurwitz zeta, and Clausen functions. These functions have been deeply and widely studied by many authors, and inequalities involving these functions, as well as their complete monotonicity and convexity, have been extensively researched. In particular, Qi and his coauthors made a great contribution. In [22], Qi et al. suggested very effective methods to handle the complete monotonicity of some functions involving gamma and polygamma functions. In [18], Qi and Guo gave the definition of logarithmically complete monotonicity and established some classes of logarithmically completely monotonic functions. In addition, they [16] also creatively defined the completely monotonic degree, and firstly established the completely monotonic degree of some functions involving gamma and polygamma functions. For more, we give a list of the literature on this topic. The reader may see [5, 6, 8, 9, 11, 13, 17, 19–21, 23, 27–31, 34–38] and the references therein.

Recently, new complete monotonicity properties of some functions involving the gamma and polygamma functions were presented by Sevli and Batir in [24]. As consequences, various new upper and lower bounds for the gamma function were established.

Motivated by their work, the first objective here is to generalize these results to the k -generalized gamma and polygamma functions. The second motivation comes from the article of Batir. In [4], some inequalities involving the polygamma functions were proved. Their conclusions can also be generalized to the k -generalized polygamma functions.

Our main results read as follows.

Theorem 1.1 For positive real numbers x and k , let

$$A_{k,a}(x) = k \log \Gamma_k(x+k) - x \log x + x - \frac{k}{2} \log x - \frac{k^3}{12} \psi'_k(x+a) - \frac{k}{2} \log \left(\frac{2\pi}{k} \right).$$

Then $A_{k,a}(x)$ is completely monotonic on $(0, \infty)$ if and only if $a \geq k/2$, and $-A_{k,b}(x)$ is completely monotonic on $(0, \infty)$ if and only if $b = 0$.

Corollary 1.2 Let x and k be positive real numbers. Then

$$\begin{aligned} \left(\frac{x}{e} \right)^x \sqrt[k]{\frac{2\pi x}{k}} \exp \left\{ \frac{k^3}{12} \psi'_k \left(x + \frac{k}{2} \right) \right\} &< (\Gamma_k(x+k))^k \\ &< \left(\frac{x}{e} \right)^x \sqrt[k]{\frac{2\pi x}{k}} \exp \left\{ \frac{k^3}{12} \psi'_k(x) \right\}. \end{aligned} \quad (1.7)$$

Theorem 1.3 For positive real numbers x and k , let

$$\begin{aligned} B_k(x) &= x \log x - x + \frac{k}{2} \log \left(\frac{2\pi}{k} \right) + \frac{k}{2} \log \left(x + \frac{k}{2} \right) - k \log \Gamma_k(x+k) \\ &\quad - \frac{k^2}{6(x + \frac{k}{2})} - \frac{k^3}{48(x + \frac{k}{2})^2}. \end{aligned}$$

Then $B_k(x)$ is completely monotonic on $(0, \infty)$.

Corollary 1.4 Let x and k be positive real numbers. Then

$$\begin{aligned} \sqrt[k]{2e^{\frac{5k}{12}}} \left(\frac{x}{e} \right)^x \sqrt[k]{\frac{x+k/2}{k}} \exp \left\{ -\frac{k^2}{6(x+k/2)} - \frac{k^3}{48(x+k/2)^2} \right\} \\ \leq (\Gamma_k(x+k))^k \\ < \sqrt[k]{2\pi} \left(\frac{x}{e} \right)^x \sqrt[k]{\frac{x+k/2}{k}} \exp \left\{ -\frac{k^2}{6(x+k/2)} - \frac{k^3}{48(x+k/2)^2} \right\}. \end{aligned} \quad (1.8)$$

Theorem 1.5 For positive real numbers x and k , let

$$F_k(x) = k \log \Gamma_k(x+k) - \left(x + \frac{k}{2} \right) \log \left(x + \frac{k}{2} \right) + x + \frac{k}{2} - \frac{k}{2} \log \frac{2\pi}{k} + \frac{k^2}{24(x+k/2)}.$$

Then $F_k(x)$ is completely monotonic on $(0, \infty)$.

Corollary 1.6 Let x and k be positive real numbers. Then

$$\begin{aligned} \sqrt[k]{\frac{2\pi}{k}} \left(\frac{x + \frac{k}{2}}{e} \right)^{x + \frac{k}{2}} \exp \left\{ -\frac{k^2}{24} \left(x + \frac{k}{2} \right) \right\} \\ < (\Gamma_k(x+k))^k \\ \leq \sqrt[k]{\frac{2}{k}} e^{\frac{7k}{12}} \left(\frac{x + \frac{k}{2}}{e} \right)^{x + \frac{k}{2}} \exp \left\{ -\frac{k^2}{24} \left(x + \frac{k}{2} \right) \right\}. \end{aligned} \quad (1.9)$$

Theorem 1.7 For positive real numbers x and k , the following inequalities hold:

$$\frac{k^2}{2} \psi'_k \left(x + \frac{k}{3} \right) < \log x - k \psi_k(x) < \frac{k^2}{2} \psi'_k \left(\frac{k}{\sqrt{\frac{2k}{x} - 2 \log(1 + \frac{k}{x})}} \right). \quad (1.10)$$

2 Lemmas

Lemma 2.1 ([25]) The gamma function $\Gamma(x)$ satisfies the following property:

$$\log \Gamma(x) = \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log(2\pi) + 2 \int_0^\infty \frac{\arctan \frac{t}{x}}{e^{2\pi t} - 1} dt, \quad x > 0.$$

Lemma 2.2 ([1–3]) For $x > 0$, the function $\psi(x)$ satisfies the following inequalities:

1. $\log(x) - \frac{1}{x} < \psi(x) < \log(x).$
2. $\frac{1}{x} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}.$
3. $-\frac{1}{x^2} - \frac{2}{x^3} < \psi''(x) < -\frac{1}{x^2} - \frac{1}{x^3}.$

Lemma 2.3 ([7, 32]) The k -generalized gamma function $\Gamma_k(x)$ satisfies the following properties:

1. $\Gamma_k(x+k) = x \Gamma_k(x).$
2. $\Gamma_k(k) = 1.$
3. $\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k}).$

Lemma 2.4 ([32]) The k -generalized function $\psi_k(x)$ satisfies the following properties:

1. $\psi_k(x) = \frac{\log k}{k} + \frac{1}{k} \psi(\frac{x}{k}).$
2. $\psi_k^{(m)}(x+k) - \psi_k^{(m)}(x) = \frac{(-1)^m m!}{x^{m+1}}.$
3. $\psi'_k(x) = \frac{1}{k^2} \psi'(\frac{x}{k}).$
4. $\psi''_k(x) = \frac{1}{k^3} \psi''(\frac{x}{k}).$

Lemma 2.5 (Complete monotonicity, [10]) A function $f(x)$ is said to be completely monotonic on an interval I if $f(x) : I \rightarrow \mathbf{R}$ has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n = 0, 1, 2, 3, \dots$

Lemma 2.6 (Hausdorff–Bernstein–Widder Theorem, [26]) A function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform. That is, there is a bounded and nondecreasing measure ν on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt} d\nu(t), \quad x > 0.$$

3 Proofs of main results

Proof of Theorem 1.1 Let $x > 0$ and $k > 0$. Differentiating $A_{k,a}(x)$ yields

$$A'_{k,a}(x) = k \psi_k(x+k) - \log x - \frac{k}{2x} - \frac{k^3}{12} \psi''_k(x+a).$$

In view of Lemma (2.4), the second order derivative of $A_{k,a}(x)$ is

$$\begin{aligned} A''_{k,a}(x) &= k\psi'_k(x+k) - \frac{1}{x} + \frac{k}{2x^2} - \frac{k^3}{12}\psi'''_k(x+a) \\ &= k\psi'_k(x) - \frac{1}{x} - \frac{k}{2x^2} - \frac{k^3}{12}\psi'''_k(x+a). \end{aligned}$$

Using (1.5) and the integral identity

$$\frac{1}{x^{n+1}} = \frac{1}{n!} \int_0^\infty t^n e^{-xt} dt, \quad (3.1)$$

we obtain

$$A''_{k,a}(x) = \frac{1}{12} \int_0^\infty \frac{\delta_{k,a}(t)}{e^{kt} - 1} e^{-xt} dt, \quad (3.2)$$

where

$$\delta_{k,a}(t) = 12kt - 12(e^{kt} - 1) + 6kt(e^{kt} - 1) - (kt)^3 e^{(k-a)t}, \quad a > 0. \quad (3.3)$$

Considering the power series expansions of exponential functions,

$$\delta_{k,a}(t) = \sum_{n=4}^{\infty} \frac{\zeta_n(a)}{n!} (kt)^n,$$

where

$$\zeta_n(a) = -12 + 6n - (n-2)(n-1)n \left(1 - \frac{a}{k}\right)^{n-3}. \quad (3.4)$$

If $a \geq k/2$, then

$$\begin{aligned} \zeta_n\left(\frac{k}{2}\right) &= (n-2) \left[6 - (n-1)n \left(1 - \frac{k/2}{k}\right)^{n-3} \right] \\ &= (n-2) [6 - n(n-1)2^{3-n}] > 0, \quad k \geq 5. \end{aligned} \quad (3.5)$$

Since the mapping $a \rightarrow \delta_{k,a}(t)$ is increasing for $a \geq k/2$, one has $\delta_{k,a}(t) > \delta_{k, \frac{k}{2}}(t) > 0$. This implies that $A''_{k,a}(x) > 0$ for $a \geq k/2$ by (3.2). That is, $A''_{k,a}(x)$ is strictly completely monotonic on $(0, \infty)$.

In view of Lemma 2.4, $A'_{k,a}(x)$ can be rewritten as

$$A'_{k,a}(x) = \log k + \psi\left(\frac{x}{k}\right) + \frac{k}{2x} - \log x - \frac{1}{12}\psi''\left(\frac{x}{k} + \frac{a}{k}\right).$$

In view of Lemma 2.2 and since $A'_{k,a}(x)$ is increasing on $(0, \infty)$, one gets that $\lim_{x \rightarrow \infty} A'_{k, \frac{k}{2}}(x) = 0$ and $A'_{k, \frac{k}{2}}(x) < 0$. For $\psi'''(x) > 0$, consequently $A'_{k,a}(x) < A'_{k, \frac{k}{2}}(x) < 0$ on $(0, \infty)$.

In view of Lemma 2.3 and 2.4, thus

$$A_{k,a}(x) = k \log x + (x-k) \log k + k \log \Gamma\left(\frac{x}{k}\right) - \frac{k}{12} \psi'_k\left(\frac{x+a}{k}\right) \\ - x \log x + x - \frac{k}{2} \log x - \frac{k}{2} \log\left(\frac{2\pi}{k}\right)$$

and $\lim_{x \rightarrow \infty} A_{k,a}(x) = 0$. For $A'_{k,a}(x) < 0$, so $A_{k,a}(x) > 0$ with $a \geq k/2$. Hence, $A_{k,a}(x)$ is strictly completely monotonic on $(0, \infty)$ with $a \geq k/2$.

If $0 < a < k/2$, in view of $\delta_{k,a}^{(n)}(t) = 0$ for $n \in \{0, 1, 2, 3\}$ and $\delta_{k,a}^{(4)}(0) = 24a - 12k$, then $\delta_{k,a}(t) < 0$ for $t > 0$ and very close to 0. Therefore, by the Hausdorff–Bernstein–Widder theorem, $A''_{k,a}(x)$ cannot be completely monotonic on $(0, \infty)$ and hence $A_{k,a}(x)$ is not either.

If $a = 0$, in view of (3.4), $\zeta_n(0) = -(n-3)(n-2)(n+2) < 0$ for $n \geq 5$, we have $\delta_{k,0}(t) < 0$. Hence $A''_{k,0}(x) < 0$, which indicates $-A''_{k,0}$ is strictly completely monotonic on $(0, \infty)$. The limits $\lim_{x \rightarrow \infty} A_{k,0}(x) = 0$ and $\lim_{x \rightarrow \infty} A'_{k,0}(x) = 0$, which indicates $-A_{k,0}(x) > 0$ and $-A'_{k,0}(x) < 0$. Hence, $-A_{k,0}(x)$ is strictly completely monotonic on $(0, \infty)$.

Now we assume that $-A_{k,b}(x)$ ($b > 0$) is completely monotonic on $(0, \infty)$. Then $A_{k,b}(x)$ is negative on $(0, \infty)$. But this contradicts with the former conclusions. The proof is finished. $\square$

Proof of Corollary 1.2 For $A_{k,0}(x) < 0$, since $A_{k,\frac{k}{2}}(x) > 0$ and $A_{k,a}(x)$ is decreasing on $(0, \infty)$, the double inequalities can be easily obtained. $\square$

Proof of Theorem 1.3 Letting $x > 0$ and $k > 0$, the first order derivative of $B_k(x)$ is

$$B'_k(x) = \log x - k\psi_k(x+k) + \frac{k}{2} \frac{1}{x+k/2} + \frac{k^2}{6(x+k/2)^2} + \frac{k^3}{24(x+k/2)^3}.$$

Considering Lemma 2.4, the second order derivative of $B_k(x)$ is

$$B''_k(x) = \frac{1}{x} - k\psi'_k(x+k) - \frac{k}{2} \frac{1}{(x+k/2)^2} - \frac{k^2}{3(x+k/2)^3} - \frac{k^3}{14(x+k/2)^4} \\ = -k\psi'_k(x) + \frac{1}{x} + \frac{k}{x^2} - \frac{k}{2} \frac{1}{(x+k/2)^2} - \frac{k^2}{3(x+k/2)^3} - \frac{k^3}{14(x+k/2)^4}.$$

In view of (1.5) and (3.1), $B''_k(x)$ can be rewritten as

$$B''_k(x) = \frac{1}{48} \int_0^\infty \frac{\eta_k(t)}{1-e^{-kt}} e^{-3t/2} e^{-xt} dt, \quad (3.6)$$

where

$$\eta_k(t) = 24kt(1-e^{kt}) + 8(kt)^2(1-e^{kt}) - (kt)^3(1-e^{kt}) + 48e^{3kt/2} \\ - 48e^{kt/2} - 48(kt)e^{kt/2}.$$

Using the power series expansions of exponential functions, we get

$$\eta_k(t) = \sum_{n=5}^{\infty} \frac{\theta(n)}{n!} \left(\frac{kt}{2}\right)^n,$$

where

$$\theta(n) = 48 \times 3^n - (n^3 + 5n^2 + 18n)2^n - 48 - 96n. \quad (3.7)$$

It is easy to prove $\theta(n) > 0$ for $n \geq 5$, which indicates $B''_{k,a}(x) > 0$ by (3.6). Namely, $B''_{k,a}(x)$ is completely monotonic on $(0, \infty)$.

In view of Lemmas 2.2 and 2.4, $B'_k(x)$ is rewritten as

$$B'_k(x) = -\frac{k}{x} - \log k - \psi\left(\frac{k}{x}\right) + \log x + \frac{k}{2} \frac{1}{x + \frac{k}{2}} + \frac{k^2}{6(x + \frac{k}{2})^2} + \frac{k^3}{24(x + \frac{k}{2})^3},$$

and $\lim_{x \rightarrow \infty} B'_k(x) = 0$. Hence, $B'_k(x) < 0$ on $(0, \infty)$.

In view of Lemmas 2.1–2.4, it is easy to get $\lim_{x \rightarrow \infty} B_k(x) = 0$. Since $B'_k(x) < 0$, so $B_k(x) > 0$. Hence, $B_k(x)$ is strictly completely monotonic on $(0, \infty)$. $\square$

Proof of Corollary 1.4 Since $\lim_{x \rightarrow \infty} B_k(x) = 0$, $\lim_{x \rightarrow 0} B_k(x) = (\frac{1}{2} \log \pi - \frac{5}{12})k$ and $B_k(x)$ is decreasing on $(0, \infty)$, so the corollary is trivial. $\square$

Proof of Theorem 1.5 Letting $x > 0$ and $k > 0$, then

$$F'_k(x) = k\psi_k(x+k) - \log(x+k/2) + \frac{k^2}{24(x+k/2)^2},$$

and, in view of (1.5) and (3.1),

$$\begin{aligned} F''_k(x) &= k\psi'_k(x+k) - \frac{1}{x+k/2} - \frac{k^2}{12(x+k/2)^3} \\ &= \frac{1}{24} \int_0^\infty \frac{\phi_k(t)}{1-e^{-kt}} e^{-3t/2} e^{-xt} dt, \end{aligned} \quad (3.8)$$

where

$$\phi_k(t) = 24kte^{kt/2} - 24(e^{kt} - 1) + (kt)^2(e^{kt} - 1).$$

Using the power series expansions of exponential functions, we obtain

$$\phi_k(t) = \sum_{n=5}^{\infty} \frac{\mu(n)}{n!} \left(\frac{kt}{2}\right)^n,$$

where

$$\mu(n) = (n^2 - n - 24)2^n + 48n. \quad (3.9)$$

It is easy to prove $\mu(n) > 0$ for $n \geq 5$, which indicates $F''_k(x) > 0$ by (3.8). Namely, $F''_k(x)$ is completely monotonic on $(0, \infty)$. Hence, $F'_k(x)$ is increasing on $(0, \infty)$.

In view of Lemmas 2.1–2.4, we can get $\lim_{x \rightarrow \infty} F'_k(x) = \lim_{x \rightarrow \infty} F_k(x) = 0$. Thus $F'_k(x) < 0$, and so $F_k(x) > 0$. Hence, $F_k(x)$ is strictly completely monotonic on $(0, \infty)$. $\square$

Proof of Corollary 1.6 Since $\lim_{x \rightarrow 0} F_k(x) = (\frac{7}{12} - \frac{1}{2} \log \pi)k$, $\lim_{x \rightarrow \infty} F_k(x) = 0$ and $F_k(x)$ is decreasing on $(0, \infty)$, the proof is easily completed. $\square$

Proof of Theorem 1.7 Based on $\Gamma_k(x+k) = x\Gamma_k(x)$ and (1.3), then

$$\begin{aligned}\log x &= \log \Gamma_k(x+k) - \log \Gamma_k(x) \\ &= -\gamma' + \log k + \sum_{n=1}^{\infty} \left[\frac{k}{nk} - \log \left(1 + \frac{k}{(n-1)k+x} \right) \right].\end{aligned}\quad (3.10)$$

Using Taylor's theorem,

$$\log \left(1 + \frac{k}{(n-1)k+x} \right) = \frac{k}{(n-1)k+x} - \frac{k^2}{2[(n-1)k+\xi(n)]^2}, \quad (3.11)$$

where

$$\xi(n) = k \left[\frac{2k}{(n-1)k+x} - 2 \log \left(1 + \frac{k}{(n-1)k+x} \right) \right]^{-\frac{1}{2}} - (n-1)k. \quad (3.12)$$

Substituting (3.11) into (3.10), then (3.10) is rewritten as

$$\begin{aligned}\log x &= -\gamma' + \log k + \sum_{n=1}^{\infty} \left[\frac{k}{nk} - \frac{k}{(n-1)k+x} + \frac{k^2}{2[(n-1)k+\xi(n)]^2} \right] \\ &= k\psi_k(x) + \sum_{n=1}^{\infty} \frac{k^2}{2[(n-1)k+\xi(n)]^2}.\end{aligned}$$

Hence,

$$\log x - k\psi_k(x) = \frac{k^2}{2} \sum_{n=1}^{\infty} \frac{1}{[(n-1)k+\xi(n)]^2}.$$

Let $u = \frac{k}{(n-1)k+x}$, then $\xi(n) = \vartheta_k(u) + x$, where $\vartheta_k(u) = \frac{k}{\sqrt{2u-2\log(1+u)}} - \frac{k}{u}$. Using a similar method as in [4], it is easy to prove that $\vartheta_k(u)$ is strictly increasing on $(0, \infty)$. Consequently,

$$\frac{k^2}{2} \sum_{n=1}^{\infty} \frac{1}{[(n-1)k+\xi(\infty)]^2} < \log x - k\psi_k(x) < \frac{k^2}{2} \sum_{n=1}^{\infty} \frac{1}{[(n-1)k+\xi(1)]^2}. \quad (3.13)$$

Letting $n = 1$ in (3.12), the value of $\xi(1)$ is equal to

$$\xi(1) = \frac{k}{\sqrt{\frac{2k}{x} - 2 \log(1 + \frac{k}{x})}}. \quad (3.14)$$

Using Taylor series expansion, the limit is

$$\xi(\infty) = \lim_{n \rightarrow \infty} \xi(n) = \lim_{u \rightarrow 0} \vartheta_k(u) + x = \frac{k}{3} + x. \quad (3.15)$$

Based on (1.6), (3.14) and (3.15), the proof is finished. $\square$

4 Remarks and applications

Remark 4.1 In view of Lemma 2.4, for positive real numbers x and k , the following equalities hold:

$$\begin{aligned}\frac{k^2}{2}\psi'_k\left(x + \frac{k}{3}\right) &= \frac{1}{2}\psi'\left(\frac{x}{k} + \frac{1}{3}\right), \\ \frac{k^2}{2}\psi'_k\left(\frac{k}{\sqrt{\frac{2k}{x} - 2\log(1 + \frac{k}{x})}}\right) &= \frac{1}{2}\psi'\left(\frac{1}{\sqrt{\frac{2k}{x} - 2\log(1 + \frac{k}{x})}}\right), \\ \log x - k\psi_k(x) &= \log x - \log k - \psi\left(\frac{x}{k}\right),\end{aligned}$$

using which (1.10) can be written as

$$\frac{1}{2}\psi'\left(\frac{x}{k} + \frac{1}{3}\right) < \log\left(\frac{x}{k}\right) - \psi\left(\frac{x}{k}\right) < \frac{1}{2}\psi'\left(\frac{1}{\sqrt{\frac{2k}{x} - 2\log(1 + \frac{k}{x})}}\right). \quad (4.1)$$

Letting $\frac{x}{k} = t$, (4.1) can be changed into

$$\frac{1}{2}\psi'\left(t + \frac{1}{3}\right) < \log t - \psi(t) < \frac{1}{2}\psi'\left(\frac{1}{\sqrt{\frac{2}{t} - 2\log(1 + \frac{1}{t})}}\right), \quad (4.2)$$

which is as in Theorem 2.1 of [4].

The following remark gives a much simpler upper bound, which clearly improves the upper bound of (1.10), and has the advantage of simplicity.

Remark 4.2 For positive real numbers x and $0 < k \leq 1$, the following inequalities hold:

$$\frac{k^2}{2}\psi'_k\left(x + \frac{k}{3}\right) < \log x - k\psi_k(x) < \frac{k^2}{2}\psi'_k\left(x + \frac{k}{3} - \frac{k}{12x+3}\right). \quad (4.3)$$

Proof Since $\psi'_k(x)$ is strictly decreasing on $(0, \infty)$, in the light of Theorem 1.7, for positive real numbers x and $0 < k \leq 1$, one only needs to show that

$$\frac{k}{\sqrt{\frac{2k}{x} - 2\log(1 + \frac{k}{x})}} > x + \frac{k}{3} - \frac{k}{12x+3}, \quad (4.4)$$

or equivalently, to prove

$$G_k(x) = \log\left(1 + \frac{k}{x}\right) - \frac{k}{x} + \frac{k^2}{2(x + \frac{k}{3} - \frac{k}{12x+3})} > 0. \quad (4.5)$$

It is easy to get

$$G'_k(x) = k^3 \times \frac{-9(3+4k) + 4(-63+16k^2)x + 576(-1+k)x^2}{x^3(k+x)(3+4k+12x)^3}. \quad (4.6)$$

If $0 < k \leq 1$, then

$$H_k(x) = -9(3+4k) + 4(-63+16k^2)x + 576(-1+k)x^2 < 0, \quad x \in (0, \infty), \quad (4.7)$$

which implies $G'_k(x) < 0$. Hence $G_k(x)$ is decreasing on $(0, \infty)$ and the limit is $\lim_{x \rightarrow \infty} G_k(x) = 0$, together these facts indicate $G_k(x) > 0$. $\square$

Remark 4.3 Finally, we give an application of inequality (1.10). Define the k -generalization of the Nielsen's β -function as [33]

$$\begin{aligned}\beta_k(x) &= \int_0^1 \frac{t^{x-1}}{1+t^k} dt \\ &= \int_0^\infty \frac{e^{-xt}}{1+e^{-kt}} dt \\ &= \sum_{n=0}^\infty \left(\frac{1}{2nk+x} - \frac{1}{2nk+k+x} \right) \\ &= \frac{1}{2} \left\{ \psi_k\left(\frac{x+k}{2}\right) - \psi_k\left(\frac{x}{2}\right) \right\}.\end{aligned}$$

By using (1.10), we easily obtain the following double inequalities of generalized Nielsen's β -function for $0 < k \leq 1$ and $x \in (0, \infty)$:

$$\begin{aligned}& \frac{1}{2k} \log\left(\frac{x+k}{x}\right) + \frac{k}{4} \psi'_k\left(\frac{x}{2} + \frac{k}{3}\right) - \frac{k}{4} \psi'_k\left(\frac{k}{\sqrt{\frac{4k}{x+k}} - 2\log\left(\frac{x+3k}{x+k}\right)}\right) \\ & < \beta_k(x) < \frac{1}{2k} \log\left(\frac{x+k}{x}\right) - \frac{k}{4} \psi'_k\left(\frac{x}{2} + \frac{5k}{6}\right) + \frac{k}{4} \psi'_k\left(\frac{k}{\sqrt{\frac{4k}{x}} - 2\log\left(\frac{x+2k}{x}\right)}\right),\end{aligned}$$

and the much simpler forms

$$\begin{aligned}& \frac{1}{2k} \log\left(\frac{x+k}{x}\right) + \frac{k}{4} \psi'_k\left(\frac{x}{2} + \frac{k}{3}\right) - \frac{k}{4} \psi'_k\left(\frac{x}{2} + \frac{5k}{6} - \frac{k}{6(x+k)+3}\right) \\ & < \beta_k(x) < \frac{1}{2k} \log\left(\frac{x+k}{x}\right) - \frac{k}{4} \psi'_k\left(\frac{x}{2} + \frac{5k}{6}\right) + \frac{k}{4} \psi'_k\left(\frac{x}{2} + \frac{k}{3} - \frac{k}{6x+3}\right).\end{aligned}$$

5 Conclusion

In the paper, we established some complete monotonicity properties involving generalized k -gamma and k -polygamma functions, and obtained some interesting inequalities. These conclusions generalized Sevli and Batir's results. As an application, we constructed new double inequalities for generalized k -digamma and k -polygamma functions. The results can be used to evaluate or estimate some integrals. Moreover, our conclusions would play an important role in the further study of these functions.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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