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On generalized strongly modified h-convex functions

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Abstract

We derive some properties and results for a new extended class of convex functions, generalized strongly modified h-convex functions. Moreover, we discuss Schur-type, Hermite–Hadamard-type, and Fejér-type inequalities for this class. The crucial fact is that this extended class has awesome properties similar to those of convex functions.

Keywords: h-convex function; Modified h-convex function; Schur-type inequality; Hermite–Hadamard inequality; Fejér-type inequality

1 Introduction

Nowadays, in science and modern analysis the convexity plays an important role in economics, statistics, management science, engineering, and optimization theory. For instance, Barani et al. [1] presented the Hermite–Hadamard inequality for functions with preinvex absolute values of derivatives. Characterizations of convexity via Hadamard's inequality has been studied in [2]. In 2003, Dragomir and Pearce [3] proposed some applications of Hermite–Hadamard inequalities. In 2015, Dragomir [4] presented inequalities of Hermite–Hadamard type for h-convex functions on linear spaces. Some other interesting results can be found in books [5, 6] and research papers [7, 8]. In the recent years, generalizations and extensions were made rapidly for convex functions; for a recent generalization, see [9–11].

Convexity in the classical sense for a function $g : L = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is defined as

$$g\big(ta_1 + (1-t)a_2\big) \leq tg(a_1) + (1-t)g(a_2),$$

where $a_1, a_2 \in L$ and $t \in [0, 1]$.

The work on the convexity is extended day by day by using some techniques; see [12–14]. The strongly extended convexity is widely used in optimization, economics, and nonlinear programming.

Convex functiosn satisfy several inequalities in which famous inequalities are of Schur type, Hermite–Hadamard-type, and Fejér-type inequalities. The Hermite–Hadamard-type inequality introduced by Jaques Hadamard for classical convex functions $g : L =$

$[a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ as

$$g\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} g(x)\,dx \le \frac{g(a_1) + g(a_2)}{2}.$$

For extended versions of this inequality, see [12] and [13]. For further reading, see [15–19].

Lipot Fejér presented an extended version of the Hermite–Hadamard inequality, known as the Fejér inequality or a weighted version of the Hermite–Hadamard inequality. If $f : I \to \mathbb{R}$ is a convex function, then

$$g\left(\frac{a_1 + a_2}{2}\right)\int_{a_1}^{a_2} w(x)\,dx \le \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} w(x)g(x)\,dx \le \frac{g(a_1) + g(a_2)}{2}\int_{a_1}^{a_2} w(x)\,dx,$$

where $a_1 \le a_2$, and $w : I \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. For further extended versions and development, see [20] and [8].

In this paper, we first present some preliminaries and basic results. In the next section, we investigate Schur-type, Hermite–Hadamard-type, and Fejér-type inequalities for the newly introduced class of functions.

2 Preliminaries

In this section, we investigate a new class of convexity by using a basic result. There is no loss of generality in the extended version of convexity. To get asymptotic results, it is necessary to put some restrictions: L is an interval in $\mathbb{R}$, and $\eta : A \times A \to B \subseteq \mathbb{R}$ is a bifunction.

Definition 1 (h-convex function [21]) Let $g, h : L \subset \mathbb{R} \to \mathbb{R}$ be nonnegative functions. Then g is called an h-convex function if

$$g\bigl(ta_1 + (1-t)a_2\bigr) \le h(t)g(a_1) + h(1-t)g(a_2)$$

for all $a_1, a_2 \in L$ and $t \in [0,1]$.

Definition 2 (Modified h-convex function [13]) Let $g, h : L \subset \mathbb{R} \to \mathbb{R}$ be nonnegative functions. Then g is called a modified h-convex function if

$$g\bigl(ta_1 + (1-t)a_2\bigr) \le h(t)g(a_1) + \bigl(1 - h(t)\bigr)g(a_2) \tag{1}$$

for all $a_1, a_2 \in L$ and $t \in [0,1]$.

Definition 3 (Generalized modified h-convex function) Let functions g, h: $J \subset \mathbb{R} \to \mathbb{R}$ be nonnegative functions. Then $g : I \subset \mathbb{R} \to \mathbb{R}$ is called a generalized modified h-convex function if

$$g\bigl(ta_1 + (1-t)a_2\bigr) \le g(a_2) + h(t)\eta\bigl(g(a_1), g(a_2)\bigr) \tag{2}$$

for all $a_1, a_2 \in I$ and $t \in [0,1]$.

Definition 4 (Wright-convex function [20]) A function $g : L \subset \mathbb{R} \to \mathbb{R}$ is said to be Wright-convex if

$$g\big((1-t)a_1 + ta_2\big) + g\big(ta_1 + (1-t)a_2\big) \le g(a_1) + g(a_2)$$

for all $a_1, a_2 \in L$ and $t \in [0,1]$.

Definition 5 (Additivity) A function η is said to be additive if $\eta(x_1,y_1) + \eta(x_2,y_2) = \eta(x_1 + x_2, y_1 + y_2)$ for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$; see [22] for more detail.

Definition 6 (Nonnegative homogeneity) A function η is said to be nonnegatively homogeneous if $\eta(\lambda a_1, \lambda a_2) = \lambda\eta(a_1,a_2)$ for all $a_1, a_2 \in \mathbb{R}$ and $\lambda \ge 0$.

Definition 7 (Supermultiplicativity [23]) A function $g : L \subset \mathbb{R} \to \mathbb{R}_+$ is said to be a supermultiplicative function if $g(a_1a_2) \ge g(a_1)g(a_2)$ for all $a_1, a_2 \in L$, $t \in [0,1]$.

Definition 8 (Similar-order functions [24]) Functions f and g are said to be of similar order on $L \subseteq \mathbb{R}$ if $\langle f(x) - f(y), g(x) - g(y)\rangle \ge 0$ for all $x, y \in L$.

Now we are going to introduce a new extended definition of convexity.

Definition 9 (Generalized strongly modified h-convex function) Let $g, h : L \subset \mathbb{R} \to \mathbb{R}$ be nonnegative functions. Then g is called a generalized strongly modified h-convex function if

$$g\big(ta_1 + (1-t)a_2\big) \le g(a_2) + h(t)\eta\big(g(a_1), g(a_2)\big) - \mu t(1-t)(a_1 - a_2)^2 \tag{3}$$

for all $a_1, a_2 \in L$ and $t \in [0,1]$.

Remark 1

1. Inequality (3) reduces to inequality (1) if $\mu = 0$ and $\eta(x,y) = x - y$.
2. Definition (9) becomes the definition of a classical convex function when $\mu = 0$, $\eta(x,y) = x - y$, and $h(t) = t$.
3. Inequality(3) reduces to inequality (2) when $\mu = 0$.
4. If $h(t) = t$, then definition (9) reduces to the definition of a strongly generalized convex function [12].

Example 1 A function $g : L = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2$, $\eta(a_1, a_2) = 2a_1 + a_2$, and $h(t) \ge t$, then g is a generalized strongly modified h-convex function.

3 Main results

This section contains some basic and straightforward results. The following proposition shows the linearity of the class of generalized strongly modified h-convex functions.

Proposition 1 *Let f and g be generalized strongly modified h-convex functions where η is additive and nonnegatively homogeneous. Then for all $a, b \in \mathbb{R}$, $af + bg$ is also a generalized strongly modified h-convex function.*

Proposition 2 *Let h_1, h_2 be nonnegative functions on L such that $h_2(t) \leq h_1(t)$. If g is a generalized strongly modified h_2-convex function, then g is also a generalized strongly modified h_1-convex function.*

Proof As g is generalized strongly modified h-convex function, for all $a_1, a_2 \in L$ and $t \in [0,1]$, we have

$$\begin{aligned} g\bigl(ta_1 + (1-t)a_2\bigr) &\leq g(a_2) + h_2(t)\eta\bigl(g(a_1), g(a_2)\bigr) - \mu t(1-t)(a_1 - a_2)^2 \\ &\leq g(a_2) + h_1(t)\eta\bigl(g(a_1), g(a_2)\bigr) - \mu t(1-t)(a_1 - a_2)^2. \end{aligned}$$

This completes the proof. □

Remark 2 If g is a generalized strongly modified h_1-convex and $h_1(t) \leq h_2(t)$, then g is a generalized strongly modified h_2-convex function.

Proposition 3 *Let f be a linear function such that $f(x) - f(y) = x - y$, and let g be a generalized strongly modified h-convex function. Then $g \circ f$ is also a generalized strongly modified h-convex function.*

Proof As f is a linear function such that $f(x) - f(y) = x - y$ and g is a generalized strongly modified h-convex function, for all $a_1, a_2 \in L$ and $t \in [0,1]$, we get

$$\begin{aligned} (g \circ f)\bigl(ta_1 + (1-t)a_2\bigr) &= g(tf(a_1) + (1-t)f(a_2) \\ &\leq (g \circ f)(a_2) + h(t)\eta\bigl((g \circ f)(a_1), (g \circ f)(a_2)\bigr) \\ &\quad - \mu t(1-t)\bigl(f(a_1) - f(a_2)\bigr)^2 \\ &= (g \circ f)(a_2) + h(t)\eta\bigl((g \circ f)(a_1), (g \circ f)(a_2)\bigr) \\ &\quad - \mu t(1-t)(a_1 - a_2)^2. \end{aligned}$$

This shows that $g \circ f$ is a generalized strongly modified h-convex function. □

Proposition 4 *Let functions $g_j : L \subset \mathbb{R} \to \mathbb{R}$ be generalized strongly modified h-convex functions, $\sum_{j=1}^{m} \lambda_j = 1$, and let η be additive non-negatively homogeneous function. Then their linear combination $f : \mathbb{R} \to \mathbb{R}$ is also a generalized strongly modified h-convex function.*

Proof As $g_j : L \subset \mathbb{R} \to \mathbb{R}$ be generalized strongly modified h-convex functions, for $a_1, a_2 \in L$ and $t \in [0,1]$, let

$$f(x) = \sum_{j=1}^{m} \lambda_j g_j(x).$$

Set $x = (ta_1 + (1-t)a_2)$. Then

$$\begin{aligned} f\bigl(ta_1 + (1-t)a_2\bigr) &= \sum_{j=1}^{m} \lambda_j g_j\bigl(ta_1 + (1-t)a_2\bigr) \\ &\leq \sum_{j=1}^{m} \lambda_j g_j(a_2) + h(t) \sum_{j=1}^{m} \lambda_j \eta\bigl(g_i(a_1), g_i(a_2)\bigr) \\ &\quad - \mu t(1-t)(a_1 - a_2)^2 \sum_{j=1}^{m} \lambda_j \\ &= f(a_2) + h(t)\eta\left(\sum_{j=1}^{m} \lambda_j g_i(a_1), \sum_{j=1}^{m} \lambda_j g_i(a_2)\right) \\ &\quad - \mu t(1-t)(a_1 - a_2)^2 \\ &= f(a_2) + h(t)\eta\bigl(f(a_1), f(a_2)\bigr) - \mu t(1-t)(a_1 - a_2)^2. \end{aligned}$$

This completes the proof. □

Corollary 1 *Every generalized strongly modified h-convex function is a generalized modified convex function.*

Proof Let g be a generalized modified h-convex function. Then

$$\begin{aligned} g\bigl(ta_1 + (1-t)a_2\bigr) &\leq g(a_2) + h(t)\eta\bigl(g(a_1), g(a_2)\bigr) - \mu t(1-t)(a_1 - a_2)^2 \\ &\leq g(a_2) + h(t)\eta\bigl(g(a_1), g(a_2)\bigr) \end{aligned}$$

for all $a_1, a_2 \in L \subset \mathbb{R}$. □

Corollary 2 *If g is generalized strongly convex function and $t \leq h(t)$, then g is a generalized strongly modified h-convex function.*

Theorem 1 (Schur-type inequality) *Let $g : L \to \mathbb{R}$ be a generalized strongly modified h-convex function, let h be a supermultiplicative function, and let $\eta : N \times N \to M$ be a bifunction for appropriate $A, B \subseteq \mathbb{R}$. Then for $a_1, a_2, a_3 \in L$ such that $a_1 < a_2 < a_3$ and $a_3 - a_1, a_3 - a_2, a_2 - a_1 \in L$, we have the inequality*

$$\begin{aligned} h(a_3 - a_1)g(a_2) &\leq h(a_3 - a_1)g(a_3) + h(a_3 - a_2)\eta\bigl(g(a_1), g(a_2)\bigr) \\ &\quad - \mu(a_3 - a_2)(a_2 - a_1)h(a_3 - a_1) \end{aligned} \tag{4}$$

if and only if g is a generalized strongly modified h-convex function.

Proof Let $a_1, a_2, a_3 \in L \subset \mathbb{R}$ be such that $\frac{(a_3-a_2)}{(a_3-a_1)} \in (0,1) \subseteq L$, $\frac{(a_2-a_1)}{(a_3-a_1)} \in (0,1) \subseteq L$, and $\frac{(a_3-a_2)}{(a_3-a_1)} + \frac{(a_2-a_1)}{(a_3-a_1)} = 1$. Then

$$h(a_3 - a_1) = h\left(\frac{a_3 - a_1}{a_3 - a_2}(a_3 - a_2)\right) \geq h\left(\frac{a_3 - a_1}{a_3 - a_2}\right)h(a_3 - a_2)$$

as h is supermultiplicative.

Suppose $h(a_3 - a_2) \geq 0$. Then by the definition of g we have

$$g\big(tx + (1-t)y\big) \leq g(y) + h(t)\eta\big(g(x), g(y)\big) - \mu t(1-t)(x-y)^2. \tag{5}$$

Inserting $\frac{(a_3-a_2)}{(a_3-a_1)} = t$, $x = a_1$, and $y = a_3$ into inequality (5), we obtain

$$\begin{aligned} g\left(\frac{(a_3-a_2)}{(a_3-a_1)}a_1 + \left(1 - \frac{(a_3-a_2)}{(a_3-a_1)}\right)a_3\right) &\leq g(a_3) + h\left(\frac{(a_3-a_2)}{(a_3-a_1)}\right)\eta\big(g(a_1), g(a_3)\big) \\ &\quad - \mu(a_3-a_2)(a_2-a_1) \\ &\leq g(a_3) + \frac{h(a_3-a_2)}{h(a_3-a_1)}\eta\big(g(a_1), g(a_3)\big) \\ &\quad - \mu(a_3-a_2)(a_2-a_1), \\ g(a_2)h(a_3-a_1) \leq h(a_3-a_1)g(a_3) & \\ + h(a_3-a_2)\eta\big(g(a_1), g(a_3)\big) & \\ - \mu(a_3-a_2)(a_2-a_1)h(a_3-a_1). & \end{aligned} \tag{6}$$

Conversely, suppose inequality (4) holds and insert $a_1 = x$, $a_2 = tx + (1-t)y$, and $a_3 = y$ into inequality (4). Then we get

$$\begin{aligned} h(y-x)g\big(tx + (1-t)y\big) &\leq h(y-x)g(y) + h(y-x)h(t)\eta\big(g(x), g(y)\big) \\ &\quad - \mu h(y-x)t(y-x)(1-t)(y-x), \\ g\big(tx + (1-t)y\big) &\leq g(y) + h(t)\eta\big(g(x), g(y)\big) - \mu t(1-t)(x-y)^2. \end{aligned}$$

This completes the proof. □

Remark 3

1. By taking $h(t) = t$ in (4) it is reduced to aSchur-type inequality for generalized strongly convex functions.
2. If $\mu = 0$ and $\eta(x, y) = x - y$, then (4) is reduced to a Schur-type inequality for modified h-convex functions; see [13].

Further, we will discuss the Hermite–Hadamard-type inequality for generalized strongly modified h-convex functions.

Theorem 2 (Hermit–Hadamard-type inequality) *Let function $g : L \to \mathbb{R}$ be a generalized strongly modified h-convex function on $[a_1, a_2]$ with $a_1 < a_2$. Then*

$$\begin{aligned} g\left(\frac{a_1+a_2}{2}\right) - h\left(\frac{1}{2}\right)M_\eta + \frac{\mu}{12}(a_2-a_1)^2 &\leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g((x)\,dx \\ &\leq g(a_2) + N_\eta - \frac{\mu}{6}(a_2-a_1)^2. \end{aligned} \tag{7}$$

Proof Choosing $w = ta_1 + (1-t)a_2$ and $z = (1-t)a_1 + ta_2$, we have

$$\begin{aligned} g\left(\frac{a_1+a_2}{2}\right) &= g\left(\frac{w+z}{2}\right) \\ &= g\left(\frac{ta_1+(1-t)a_2+(1-t)a_1+ta_2}{2}\right). \end{aligned}$$

Now by the definition of g we have

$$\begin{aligned} g\left(\frac{a_1+a_2}{2}\right) \le{}& g\big((1-t)a_1+ta_2\big) + h\left(\frac{1}{2}\right)\eta\big(g\big(ta_1+(1-t)a_2\big), g\big((1-t)a_1+ta_2\big)\big) \\ &- \mu\frac{1}{2}\left(1-\frac{1}{2}\right)(a_2-a_1)^2(2t-1)^2. \end{aligned}$$

Integrating with respect to t on [0,1], we get

$$\begin{aligned} g\left(\frac{a_1+a_2}{2}\right) \le{}& \int_0^1 g\big((1-t)a_1+ta_2\big)\,dt \\ &+ h\left(\frac{1}{2}\right)\int_0^1 \eta\big(g\big(ta_1+(1-t)a_2\big), g\big((1-t)a_1+ta_2\big)\big)\,dt \\ &- \frac{\mu}{4}(a_2-a_1)^2\int_0^1 (2t-1)^2\,dt. \end{aligned}$$

Putting $x = (1-t)a_1 + ta_2$, we get

$$\begin{aligned} &g\left(\frac{a_1+a_2}{2}\right) \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)\,dx + h\left(\frac{1}{2}\right)M_\eta - \frac{\mu}{12}(a_2-a_1)^2, \\ &g\left(\frac{a_1+a_2}{2}\right) - h\left(\frac{1}{2}\right)M_\eta + \frac{\mu}{12}(a_2-a_1)^2 \le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g((x)\,dx. \end{aligned} \tag{8}$$

In the right-hand side of inequality (8), we set $x = ta_1 + (1-t)a_2$, and using the definition of g, we get

$$\begin{aligned} &\int_{a_1}^{a_2} g(x)\,dx \le (a_2-a_1)g(a_2) + (a_2-a_1)\int_0^1 h(t)\eta\big(g\big(a_1, g(a_2)\big)\,dt - \frac{\mu}{6}(a_2-a_1)^2, \\ &\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} g(x)\,dx \le g(a_2) + N_\eta - \frac{\mu}{6}(a_2-a_1)^2. \end{aligned} \tag{9}$$

Now from inequalities (8) and (9) we get

$$\begin{aligned} g\left(\frac{a_1+a_2}{2}\right) - h\left(\frac{1}{2}\right)M_\eta + \frac{\mu}{12}(a_2-a_1)^2 &\le \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g((x)\,dx \\ &\le g(a_2) + N_\eta - \frac{\mu}{6}(a_2-a_1)^2. \end{aligned} \tag{10}$$

This completes the proof. □

Remark 4

1. If we take $\mu = 0$ and $\eta(x,y) = x - y$, then the Hermite–Hadamard-type inequality (10) is reduced to Hermite–Hadamard-type inequality for modified h-convex functions; for details, see [13].
2. If we put $h(t) = t$ in (10), then we get a Hermite–Hadamard-type inequality for generalized strongly convex functions; see [12].
3. If we take $\mu = 0$, $\eta(x,y) = x - y$ and $h(t) = t$, then inequality (10) is reduced to a Hemite–Hadard-type inequality for classical convex functions.

Now we prove the following lemma by using technique of [25]. This lemma has the crucial fact that generalized strongly modified h-convex functions behave like classic convex functions.

Lemma 1 *Let g be a generalized modified h-convex function, and suppose that $\eta(x,y) = -\eta(y,x)$. Then*

$$g(a_1 + a_2 - x) \leq g(a_1) + g(a_2) - g(x) \quad \forall x \in [a_1, a_2],$$

where $x = ta_1 + (1-t)a_2$ and $t \in [0,1]$.

Proof As g is generalized modified h-convex function, for $x = ta_1 + (1-t)a_2$, we get

$$\begin{aligned} g(a_1 + a_2 - x) &= g\big((1-t)a_1 + ta_2\big) \\ &\leq g(a_1) + h(t)\eta\big(g(a_2), g(a_1)\big) \\ &= g(a_1) + g(a_2) - g(a_2) - h(t)\eta(g(a_1), g(a_2)) \\ &= g(a_1) + g(a_2) - \big[g(a_2) + h(t)\eta(g(a_1), g(a_2))\big] \\ &\leq g(a_1) + g(a_2) - g(x). \end{aligned}$$

This completes the proof. □

Lemma 2 *Let g be q the generalized strongly modified h-convex function, and suppose that $\eta(x,y) = -\eta(y,x)$. Then*

$$g(a_1 + a_2 - x) \leq g(a_1) + g(a_2) - g(x) \quad \forall x \in [a_1, a_2], \tag{11}$$

where $x = ta_1 + (1-t)a_2$ and $t \in [0,1]$.

Proof Let g be a generalized strongly modified h-convex function. Then for $x = ta_1 + (1-t)a_2$, we get

$$\begin{aligned} g(a_1 + a_2 - x) &= g\big((1-t)a_1 + ta_2\big) \\ &\leq g(a_1) + h(t)\eta\big(g(a_2), g(a_1)\big) - \mu t(1-t)(a_1 - a_2)^2 \\ &\leq g(a_1) + g(a_2) - g(a_2) - h(t)\eta(g(a_1), g(a_2)) \\ &\quad - \mu t(1-t)(a_1 - a_2)^2 + 2\mu t(1-t)(a_1 - a_2)^2 \end{aligned}$$

$$\begin{aligned}&\leq g(a_1)+g(a_2)-\left[g(a_2)+h(t)\eta(g(a_1),g(a_2)-\mu t(1-t)(a_1-a_2)^2\right]\\&\leq g(a_1)+g(a_2)-g(x).\end{aligned}$$

This completes the proof. □

It is very interesting that when g is a modified h-convex function [13], generalized modified h-convex, or generalized strongly modified h-convex function, then inequality (11) holds.

Theorem 3 (Fejér-type inequality) *Let $g:[a_1,a_2]\to\mathbb{R}$ be a generalized strongly modified h-convex, and let $w:[a_1,a_2]\to\mathbb{R}$ be nonnegative, integrable, and symmetric with respect to $\frac{a_1+a_2}{2}$. Then*

$$\begin{aligned}&g\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}w(x)\,dx+\frac{\mu}{4}\int_{a_1}^{a_2}(a_1+a_2-2x)w(x)\,dx-N_\eta(a_1,a_2)\\&\quad\leq\int_{a_1}^{a_2}g(x)w(x)\,dx\\&\quad\leq\frac{g(a_1)+g(a_2)}{2}\int_{a_1}^{a_2}w(x)\,dx+T_\eta(a_1,a_2)-\mu\int_{a_1}^{a_2}(x-a_2)(a_1-x)w(x)\,dx,\end{aligned}\tag{12}$$

where

$$\begin{aligned}N_\eta(a_1,a_2)&=h\left(\frac{1}{2}\right)\int_{a_1}^{a_2}\eta\bigl(g(a_1+a_2-x),g(x)\bigr)w(x)\,dx,\\T_\eta(a_1,a_2)&=\frac{\eta(g(a_1),g(a_2))}{2}\int_{a_1}^{a_2}h\left(\frac{x-a_2}{a_1-a_2}\right)w(x)\,dx.\end{aligned}$$

Proof Let g be a generalized strongly modified h-convex function. Then

$$\begin{aligned}g\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}w(x)\,dx&=\int_{a_1}^{a_2}g\left(\frac{a_1+a_2-x+x}{2}\right)w(x)\,dx\\&\leq\int_{a_1}^{a_2}g(x)w(x)\,dx\\&\quad+h\left(\frac{1}{2}\right)\int_{a_1}^{a_2}\eta\bigl(g(a_1+a_2-x),g(x)\bigr)w(x)\,dx\\&\quad-\int_{a_1}^{a_2}\mu\frac{1}{2}\left(1-\frac{1}{2}\right)(2x-a_1-a_2)^2w(x)\,dx,\end{aligned}\tag{13}$$

$$\begin{aligned}&g\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}w(x)\,dx+\frac{\mu}{4}\int_{a_1}^{a_2}(a_1+a_2-2x)^2w(x)\,dx-N_\eta(a_1,a_2)\\&\quad\leq\int_{a_1}^{a_2}g(x)w(x)\,dx.\end{aligned}$$

In the right hand-side of inequality (13), put $x = ta_1 + (1-t)a_2$. Then

$$\begin{aligned}
&\int_{a_1}^{a_2} g(x)w(x)\,dx = (a_2-a_1)\int_0^1 g\big(ta_1+(1-t)a_2\big)w\big(ta_1+(1-t)a_2\big)\,dt,\\
&\frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)w(x)\,dx \le \int_0^1 g(a_2)w\big(ta_1+(1-t)a_2\big)\,dt\\
&\qquad + \eta\big(g(a_1),g(a_2)\big)\int_0^1 h(t)w\big(ta_1+(1-t)a_2\big)\,dt\\
&\qquad - \mu(a_2-a_1)^2\int_0^1 t(1-t)w\big(ta_1+(1-t)a_2\big)\,dt.
\end{aligned} \tag{14}$$

Similarly, if we put $x = ta_2 + (1-t)a_1$ in the right-hand side of inequality (13), then we get the inequality

$$\begin{aligned}
\frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)w(x)\,dx &\le \int_0^1 g(a_1)w\big(ta_2+(1-t)a_1\big)\,dt\\
&\quad + \eta\big(g(a_2),g(a_1)\big)\int_0^1 h(t)w\big(ta_2+(1-t)a_1\big)\,dt\\
&\quad - \mu(a_2-a_1)^2\int_0^1 t(1-t)w\big(ta_2+(1-t)a_1\big)\,dt.
\end{aligned} \tag{15}$$

Adding inequalities (14) and (15), where w is symmetric, we get

$$\begin{aligned}
&\frac{2}{a_2-a_1}\int_{a_1}^{a_2} g(x)w(x)\,dx\\
&\le \big(g(a_1)+g(a_2)\big)\int_0^1 w\big(ta_1+(1-t)a_2\big)\,dt\\
&\quad + \big[\eta\big(g(a_1),g(a_2)\big)+\eta\big(g(a_2),g(a_1)\big)\big]\int_0^1 h(t)w\big(ta_1+(1-t)a_2\big)\,dt\\
&\quad - 2\mu(a_2-a_1)^2\int_0^1 t(1-t)w\big(ta_1+(1-t)a_2\big)\,dt.
\end{aligned} \tag{16}$$

Putting $x = ta_1 + (1-t)a_2$ in the right-hand side of inequality (16), we have

$$\begin{aligned}
\int_{a_1}^{a_2} g(x)w(x)\,dx &\le \frac{(g(a_1)+g(a_2))}{2}\int_{a_1}^{a_2} w(x)\,dx\\
&\quad + \frac{[\eta(g(a_1),g(a_2))+\eta(g(a_2),g(a_1))]}{2}\int_{a_1}^{a_2} h(\frac{x-a_2}{a_1-a_2}w(x)\,dx\\
&\quad - \mu\int_{a_1}^{a_2}(x-a_2)(a_1-x)w(x)\,dx,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\int_{a_1}^{a_2} g(x)w(x)\,dx &\le \frac{(g(a_1)+g(a_2))}{2}\int_{a_1}^{a_2} w(x)\,dx + T_\eta(a_1,a_2)\\
&\quad -\mu\int_{a_1}^{a_2}(x-a_2)(a_1-x)w(x)\,dx.
\end{aligned}$$

Now from inequalities (13) and (17) we get Fejér-type inequality (12) for generalized strongly modified h-convex functions. □

Remark 5

1. If $h(t) = t$, then inequality (12) reduced to Fejér type inequality for generalized strongly convex functions, see [12].
2. If we put $\mu = 0$ and $\eta(x,y) = x - y$ then inequality (12) becomes a Fejér-type inequality for modified h-convex functions; see [13].
3. If we put $\mu = 0$, $\eta(x,y) = x - y$, and $h(t) = t$, then inequality (12) is reduced to a Fejér-type inequality for classical convex functions.

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